TD : Quantum Ising Model

$$\mathcal{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z - \Gamma \sum_i \hat{\sigma}_i^x$$

I Ground state properties

(a) $J = 0$ $\qquad \mathcal{H} = -\Gamma \sum_i \hat{\sigma}_i^x$

Ground state $|0\rangle = |\rightarrow\rightarrow\rightarrow\rightarrow \dots \rightarrow\rangle = \prod_i |\rightarrow\rangle_i$

non degenerate

In the basis of the eigenstates of $\hat{\sigma}_i^z$

$$|\rightarrow\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle) = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$$

$$\langle 0|\mathcal{H}|0\rangle = -\Gamma N$$

$$\langle 0|\sigma_i^x|0\rangle = 1 \qquad\qquad \langle 0|\sigma_i^z|0\rangle = 0$$

Excitations $\quad |\rightarrow\rangle_i \longrightarrow |\leftarrow\rangle_i = \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle)$

N-degenerate 1-st excited states

with energy $\quad -\Gamma N + 2\Gamma$

(b) $\Gamma = 0$ $\qquad \mathcal{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z$

$$|0\rangle = |\uparrow \dots \uparrow\rangle = \prod_i |\uparrow\rangle_i \quad \text{or} \quad |0\rangle = \prod_i |\downarrow\rangle_i$$

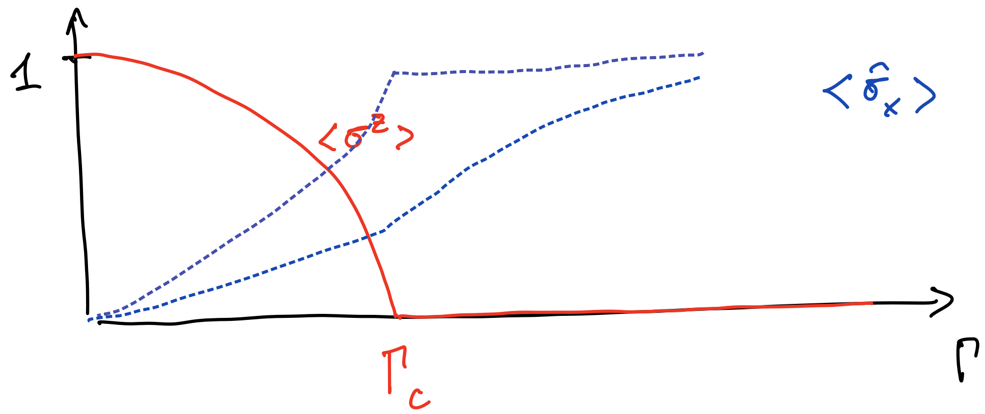
$$\langle 0 | H | 0 \rangle = -JdN$$

$$\langle 0 | \sigma_i^z | 0 \rangle = \pm 1 \qquad \langle 0 | \sigma_i^x | 0 \rangle = 0$$

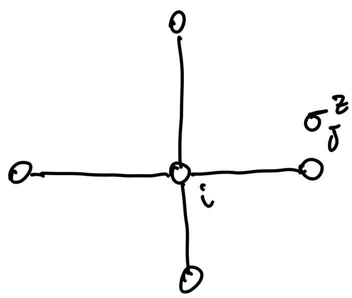
Excited states $|\uparrow\rangle_i \longrightarrow |\downarrow\rangle_i$

N-degenerate 1-st excited states with energy $-JdN + 4dJ$

(c)



II Curie-Weiss mean-field approach



$$\sigma_j^z \longrightarrow \langle \sigma_j^z \rangle \equiv m$$

$$\hat{H}_{mf} = -\sum_i \left(2Jdm\,\sigma_i^z + \Gamma \sigma_i^x \right)$$

$$\vec{h}_{mf} = (\Gamma, 0, 2Jdm)$$

$$\sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}^z = \sum_i \left(\sigma_i^z - m\mathbb{1} + m\mathbb{1}\right)\left(\sigma_j^z - m\mathbb{1} + m\mathbb{1}\right) =$$

$$\hat{\sigma}_i^z - m\mathbb{1} \equiv \delta\hat{\sigma}_i^z$$

$$= \sum_{\langle ij \rangle} \left[\delta\hat{\sigma}_i^z \, \delta\hat{\sigma}^z + m\,\delta\hat{\sigma}_i^z + m\,\delta\hat{\sigma}_j^z + m^2\mathbb{1} \right]$$

this therm will be neglected

$$\sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z \simeq \sum_{\langle ij \rangle} \left(m\sigma_i^z + m\sigma_j^z - m^2\mathbb{1} \right) = -Ndm^2 + 2md\sum_i \sigma_i^z$$

$$\hat{H} \simeq NdJm^2 - 2mdJ\sum_i \sigma_i^z - \Gamma\sum_i \sigma_i^x = C - \sum_i \vec{h}_{mf} \cdot \vec{\sigma}_i$$

$$\vec{h}_{mf} = \begin{pmatrix} \Gamma \\ 0 \\ 2Jdm \end{pmatrix}$$

the constant will be omitted in the following but it is necessary to find the correct values of the ground-state energy and of the free-energy

(b) $\hat{H}_{mf} = \begin{pmatrix} -2Jdm & -\Gamma \\ -\Gamma & 2Jdm \end{pmatrix}$ $\boxed{T=0}$

Eigenvalues $\lambda^2 - 4J^2d^2m^2 - \Gamma^2 = 0$

$$\lambda_\pm = \pm\sqrt{4J^2d^2m^2+\Gamma^2}$$

Ground state $H_{mf}|\psi\rangle = \lambda_-|\psi\rangle$

$|\psi\rangle = (\sin\theta, \cos\theta)$

$$\begin{pmatrix} -a & b \\ b & a \end{pmatrix}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = -\sqrt{a^2+b^2}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$a = 2Jdm$
$b = -\Gamma$

$$\begin{cases} -a\sin\theta + b\cos\theta = -\sqrt{a^2+b^2}\sin\theta \\ b\sin\theta + a\cos\theta = -\sqrt{a^2+b^2}\cos\theta \end{cases}$$

$$a^2\sin^2\theta + b^2\cos^2\theta - 2ab\sin\theta\cos\theta = (a^2+b^2)\sin^2\theta$$

$$b^2(\cos^2\theta - \sin^2\theta) = 2ab\sin\theta\cos\theta$$

$$b\cos(2\theta) = a\sin(2\theta)$$

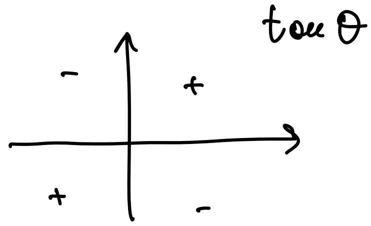
$$\tan(2\theta) = b/a$$

$$b^2\cos^2(2\theta) = a^2\sin^2(2\theta) = a^2(1-\cos^2(2\theta))$$

$$\cos(2\theta) = \pm\frac{a}{\sqrt{a^2+b^2}} \qquad \sin(2\theta) = \pm\frac{b}{\sqrt{a^2+b^2}}$$

$b = -\Gamma \ (<0)$

$a = 2Jdm \ (\gtrless 0)$



$$\langle 0 | \sigma^z | 0 \rangle = (\sin\theta, \cos\theta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$= (\sin\theta, \cos\theta) \begin{pmatrix} \sin\theta \\ -\cos\theta \end{pmatrix} = \sin^2\theta - \cos^2\theta$$

$$= -\cos(2\theta)$$

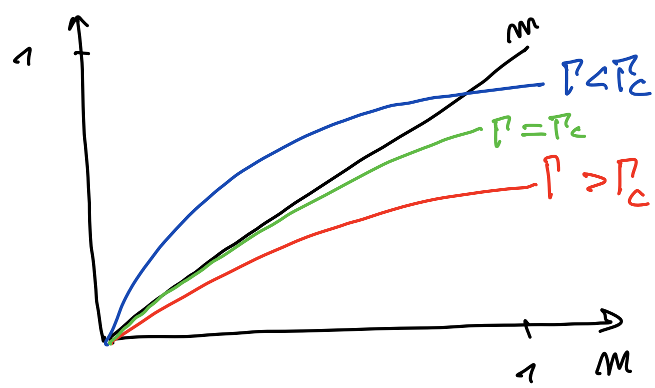
If $\quad m > 0 \qquad m = \dfrac{2Jdm}{\sqrt{4J^2d^2m^2 + \Gamma^2}}$

$$\langle \hat{\sigma}_x \rangle = (\sin\theta, \cos\theta) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

$$= (\sin\theta, \cos\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = 2\sin\theta\cos\theta = \sin(2\theta)$$

$$\langle \hat{\sigma}_x \rangle = \frac{\Gamma}{\sqrt{4J^2d^2m^2 + \Gamma^2}}$$

(c) $\quad m = \dfrac{2Jdm}{\sqrt{(2Jdm)^2 + \Gamma^2}}$



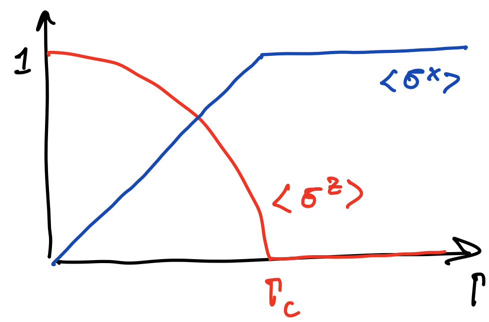
$m \simeq 0$

$$m = \frac{2Jd\, m}{\Gamma\left(1+\left(\frac{2Jd\,m}{\Gamma}\right)^2\right)^{1/2}} \simeq \frac{2Jd}{\Gamma} m \left(1 - \frac{1}{2}\left(\frac{2Jd\,m}{\Gamma}\right)^2\right)$$

$$\boxed{\frac{2Jd}{\Gamma_c} = 1} \qquad \Gamma_c = 2Jd$$

$$m = \frac{(\Gamma_c/\Gamma)\, m}{\sqrt{1+(\Gamma_c/\Gamma\, m)^2}} \Rightarrow 1 + \left(\frac{\Gamma_c}{\Gamma} m\right)^2 = \left(\frac{\Gamma_c}{\Gamma}\right)^2 \Rightarrow m^2 = 1 - \left(\frac{\Gamma}{\Gamma_c}\right)^2$$

$$\boxed{m = \pm\sqrt{\frac{\Gamma_c^2 - \Gamma^2}{\Gamma_c^2}}}$$



$\Gamma > \Gamma_c \qquad \langle\sigma^x\rangle = 1$

$$\langle\sigma^x\rangle = \frac{1}{\sqrt{1+\left(\frac{\Gamma_c m}{\Gamma}\right)^2}} = \begin{cases} 1 & \text{for } \Gamma > \Gamma_c \\ \Gamma/\Gamma_c & \text{for } \Gamma < \Gamma_c \end{cases}$$

(d) $Z = \mathrm{Tr}\, e^{-\beta \hat{H}_{mf}} = e^{-\beta\lambda_-} + e^{-\beta\lambda_+} = 2\,\mathrm{ch}(\beta\lambda)$

$$Z = 2\,\mathrm{ch}\left(\beta\sqrt{(2Jd\,m)^2 + \Gamma^2}\right)$$

$$\langle\hat{\sigma}^z\rangle = \frac{1}{Z}\mathrm{Tr}\, \hat{\sigma}^z e^{-\beta H} = \frac{1}{Z}\Big(\langle -|\sigma^z e^{-\beta H}|-\rangle + \langle +|\sigma^z e^{-\beta H}|+\rangle\Big)$$

$$\langle \sigma^z \rangle = \frac{1}{Z}\left(\frac{2Jdm}{\sqrt{(2Jdm)^2+\Gamma^2}} e^{\beta\lambda} - \frac{2Jdm}{\sqrt{(2Jdm)^2+\Gamma^2}} e^{-\beta\lambda} \right)$$

$$\langle \sigma^z \rangle = \frac{2Jdm}{\sqrt{(2Jdm)^2+\Gamma^2}} \tanh(\beta\lambda)$$

$\Gamma_c = 2Jd \longrightarrow$ critical transverse field at $T=0$

$$m = \frac{\Gamma_c m}{\sqrt{(\Gamma_c m)^2+\Gamma^2}} \, \mathrm{th}\left(\beta\sqrt{(\Gamma_c m)^2+\Gamma^2}\right)$$

$m=0$ is always a solution of this equation

Another solution for $m \neq 0$?

$$\Gamma\left(1+\left(m\frac{\Gamma_c}{\Gamma}\right)^2\right)^{1/2} \simeq \Gamma\left(1+\frac{1}{2}\left(\frac{\Gamma_c}{\Gamma}m\right)^2\right)$$

$$\mathrm{th}\left(\beta\Gamma + \frac{\beta\Gamma}{2}\left(\frac{\Gamma_c}{\Gamma}m\right)^2\right) \simeq \mathrm{th}(\beta\Gamma) + \frac{1}{\mathrm{ch}^2(\beta\Gamma)}\frac{\beta\Gamma}{2}\left(\frac{\Gamma_c}{\Gamma}m\right)^2$$

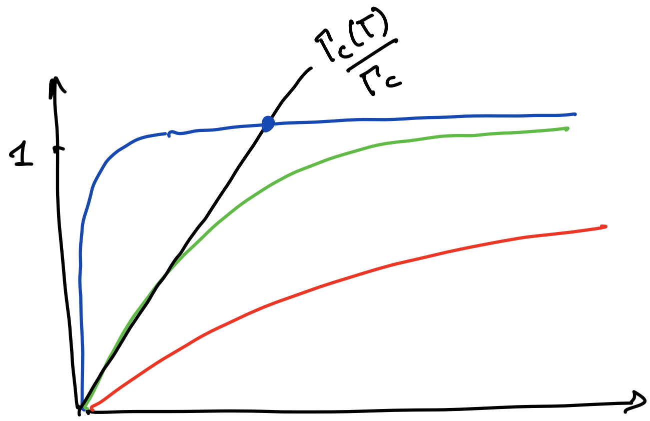
$$\cancel{m} \simeq \frac{\cancel{m}\Gamma_c}{\Gamma}\left(1-\frac{1}{2}\left(\frac{\Gamma_c}{\Gamma}m\right)^2\right)\left(\mathrm{th}(\beta\Gamma) + \frac{1}{\mathrm{ch}^2(\beta\Gamma)}\frac{\beta\Gamma}{2}\left(\frac{\Gamma_c}{\Gamma}m\right)^2\right)$$

$$1 \simeq \frac{\Gamma_c}{\Gamma}\left(\text{th}(\beta\Gamma) + \left(\frac{\beta\Gamma}{2\,\text{ch}^2(\beta\Gamma)} - \frac{1}{2}\right)\left(\frac{\Gamma_c}{\Gamma} m\right)^2 \right)$$

$$\boxed{\frac{\Gamma_c}{\Gamma_c(T)}\,\text{th}(\beta\Gamma_c(T)) = 1} \iff \frac{\Gamma_c(T)}{\Gamma_c} = \text{th}(\beta\Gamma_c(T))$$

$T \to 0$ $\quad \text{th}(\beta\Gamma_c(T)) \simeq 1 - 2e^{-2\beta\Gamma_c(T)}$

$$\Gamma_c(T) \simeq \Gamma_c - 2\Gamma_c e^{-2\beta\Gamma_c}$$

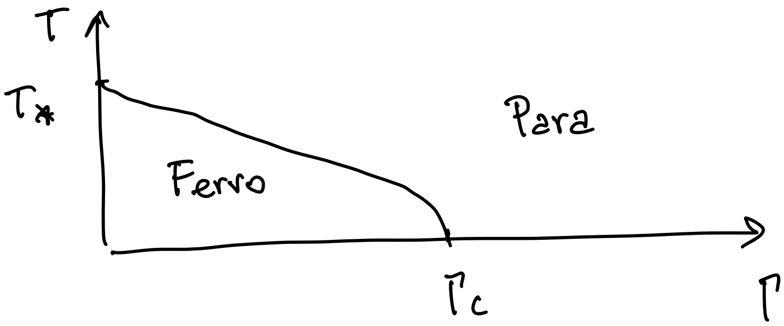


No transition for $T > T_*$

$$\frac{\Gamma_c(T_*)}{\Gamma_c} \simeq \beta_*\Gamma_c(T_*)$$

$$\beta_* = 1/\Gamma_c$$

Phase diagram



$$T_* = \frac{2Jd}{k_B} = T_c^{(\text{Curie-Weiss})}$$

$$\langle \sigma^x \rangle = \frac{1}{2}\,\text{Tr}\,\sigma^x e^{-\beta H}$$

$$\langle \sigma^x \rangle = \frac{1}{Z}\left(\frac{\Gamma}{\sqrt{(2J\alpha m)^2+\Gamma^2}}\, e^{\beta\lambda} - \frac{\Gamma}{\sqrt{(2J\alpha m)^2+\Gamma^2}}\, e^{-\beta\lambda} \right)$$

$$\langle \sigma^x \rangle = \frac{\cancel{\Gamma}}{\cancel{\Gamma}\sqrt{\left(\frac{\Gamma_c}{\Gamma} m\right)^2+1}}\ \mathrm{th}\left(\beta\Gamma\sqrt{\left(\frac{\Gamma_c}{\Gamma} m\right)^2+1}\right)$$

$T > T_c \qquad m = 0 \qquad \langle \sigma^x \rangle = \mathrm{th}(\beta\Gamma)$

III Fully-connected mean-field model

$$\hat{H} = -\frac{J}{N^{p-1}} \sum_{i_1 \neq i_2 \neq \dots \neq i_p} \hat{\sigma}^z_{i_1} \cdots \hat{\sigma}^z_{i_p} - \Gamma \sum_i \hat{\sigma}^x_i$$

$$\hat{m}_{x,y,z} = \frac{1}{N} \sum_i \sigma_i^{x,y,z}$$

(a) $$[\hat{m}_\alpha, \hat{m}_\beta] = \frac{1}{N^2}\left[\sum_i \sigma_i^\alpha, \sum_j \sigma_j^\beta\right] = \frac{1}{N^2}\sum_i [\sigma_i^\alpha, \sigma_i^\beta]$$

$$= \frac{1}{N^2}\sum_i 2i\,\epsilon_{\alpha\beta\gamma}\,\sigma_i^\gamma = \frac{1}{N}\, 2i\,\epsilon_{\alpha\beta\gamma}\,\hat{m}_\gamma$$

For $N \to \infty$ $\hat{m}^{x,y,z}$ commute and the system behaves as a classical one

(b) $$\sum_{i_1\neq i_2\neq\ldots\neq i_p} \hat\sigma^z_{i_1}\hat\sigma^z_{i_2}\cdots\hat\sigma^z_{i_p} \simeq \left(\sum_i \sigma^z_i\right)^p$$

$$\hat{\mathcal{H}} = -JN(\hat m_z)^p - \Gamma N\,\hat m_x$$

(c) $$Z = \mathrm{Tr}\, e^{-\beta\mathcal{H}} = \sum_{\substack{\{\vec S(1)\}\\ \{\vec S(2)\}\\ \vdots \\ \{\vec S(M)\}}} \langle \vec S(1)| e^{-\epsilon\hat{\mathcal{H}}} |\vec S(2)\rangle \ldots \langle \vec S(M)| e^{-\epsilon\hat{\mathcal{H}}} |\vec S(1)\rangle$$

$$\epsilon = \frac{\beta}{M}$$

$$Z = \sum_{\{\vec S(1)\ldots\vec S(M)\}} \prod_{\alpha=1}^{M} e^{\epsilon JN\left(\frac{1}{N}\sum_i s_i(\alpha)\right)^p} \langle \vec S(\alpha)| e^{+\epsilon\Gamma N\hat m_x} |\vec S(\alpha+1)\rangle$$

(d) $$m(\alpha) = \frac{1}{N}\sum_i s_i(\alpha)$$

$$\int_{-\infty}^{+\infty} dx\,\delta(x-x_0) = 1$$

$$\prod_\alpha \int_{-\infty}^{+\infty} \delta\left(N m(\alpha) - \sum_i s_i(\alpha)\right) dm(\alpha) = 1$$

$$\prod_\alpha \iint_{-\infty}^{+\infty} \frac{dm(\alpha)\,d\lambda(\alpha)}{2\pi}\, e^{i\lambda(\alpha)\left(N m(\alpha) - \sum_i s_i(\alpha)\right)} = 1$$

$$Z = \int \prod_\alpha \left[\frac{dm(\alpha)\, d\lambda(\alpha)}{2\pi} e^{\epsilon J N (m(\alpha))^p + i N \lambda(\alpha) m(\alpha)} \right] \times$$

$$\times \sum_{\{\vec{S}(1) \dots \vec{S}(M)\}} \prod_\alpha \langle \vec{S}(\alpha) | e^{-i\lambda(\alpha) \sum_i \hat{\sigma}_i^z + \epsilon\Gamma \sum_i \hat{\sigma}_i^x} | \vec{S}(\alpha+1) \rangle$$

$$-i\lambda(\alpha) = \epsilon \bar{\lambda}(\alpha)$$

$$d\lambda(\alpha) = i\epsilon\, d\bar{\lambda}(\alpha)$$

$$Z = \int \prod_\alpha \left[\frac{dm(\alpha)\, d\bar{\lambda}(\alpha)\, (i\epsilon)}{2\pi} \right] e^{\sum_\alpha \left[\epsilon N J (m(\alpha))^p - \epsilon N \bar{\lambda}(\alpha) m(\alpha) \right]} \times$$

$$\times \sum_{\{\vec{S}(1) \dots \vec{S}(M)\}} \prod_\alpha \underbrace{\langle \vec{S}(\alpha) | e^{\epsilon \bar{\lambda}(\alpha) \sum_i \hat{\sigma}_i^z + \epsilon\Gamma \sum_i \hat{\sigma}_i^x} | \vec{S}(\alpha+1) \rangle}$$

$$= \prod_i \langle S_i(\alpha) | e^{\epsilon \bar{\lambda}(\alpha) \hat{\sigma}_i^z + \epsilon\Gamma \hat{\sigma}_i^x} | S_i(\alpha+1) \rangle$$

$$= \left(\mathrm{Tr} \left[\prod_\alpha e^{\epsilon \bar{\lambda}(\alpha) \hat{\sigma}^z + \epsilon\Gamma \hat{\sigma}^x} \right] \right)^N = e^{N \ln \mathrm{Tr} \left[\prod_\alpha e^{\epsilon \bar{\lambda}(\alpha) \hat{\sigma}^z + \epsilon\Gamma \hat{\sigma}^x} \right]}$$

$$Z = \int \prod_\alpha \left[\frac{dm(\alpha)\, d\lambda(\alpha)\, (i\epsilon)}{2\pi} \right] e^{N \left\{ \sum_\alpha \left[\epsilon J (m(\alpha))^p - \epsilon \bar{\lambda}(\alpha) m(\alpha) \right] + \ln \mathrm{Tr} \left[\prod_\alpha e^{\epsilon \bar{\lambda}(\alpha) \hat{\sigma}^z + \epsilon\Gamma \hat{\sigma}^x} \right] \right\}}$$

(e) $m(\alpha) = m$ $\quad$ constant trajectories
$\lambda(\alpha) = \lambda$ $\quad$ in imaginary time

$$\prod_\alpha e^{\epsilon \bar{\lambda}(\alpha)\hat{\sigma}^z + \epsilon \Gamma \hat{\sigma}^x} = e^{\beta\bar{\lambda}\hat{\sigma}^z + \beta\Gamma\hat{\sigma}^x}$$

$$\mathrm{Tr}\left[e^{\beta\bar{\lambda}\sigma^z + \beta\Gamma\hat{\sigma}^x}\right] = 2\,\mathrm{ch}\left[\beta\sqrt{\bar{\lambda}^2 + \Gamma^2}\right]$$

$$Z \propto e^{-\beta N f(\lambda, m)}$$

$$-\beta f = +\beta J m^p - \beta\bar{\lambda} m + \ln\left(2\,\mathrm{ch}\left[\beta\sqrt{\bar{\lambda}^2 + \Gamma^2}\right]\right)$$

$$f = -J m^p + \bar{\lambda} m - \frac{1}{\beta}\ln\left[2\,\mathrm{ch}\left[\beta\sqrt{\bar{\lambda}^2 + \Gamma^2}\right]\right]$$

(f) $\dfrac{\partial f}{\partial m} = 0$ $\qquad -Jp\,m^{p-1} + \bar{\lambda} = 0 \Rightarrow \bar{\lambda} = Jp\,m^{p-1}$

$\dfrac{\partial f}{\partial \bar{\lambda}} = 0$ $\qquad m - \dfrac{1}{\beta}\,\mathrm{th}\left[\beta\sqrt{\bar{\lambda}^2 + \Gamma^2}\right]\dfrac{\beta\bar{\lambda}}{\sqrt{\bar{\lambda}^2 + \Gamma^2}} = 0$

$$m = \mathrm{th}\left[\beta\sqrt{(Jp\,m^{p-1})^2 + \Gamma^2}\right]\frac{Jp\,m^{p-1}}{\sqrt{(Jp\,m^{p-1})^2 + \Gamma^2}}$$

$p = 2$ $\qquad m = \mathrm{th}\left[\beta\sqrt{(2Jm)^2 + \Gamma^2}\right]\dfrac{2Jm}{\sqrt{(2Jm)^2 + \Gamma^2}}$ $\qquad$ Same as Curie-Weiss