TD 4 : PERCOLATION TRANSITION + RG

Percolation: geometric type of phase transition occurring when nodes or links are added to a network.

cluster : group of nearest neighbouring occupied sites.

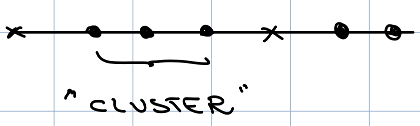
Percolation threshold p_c : occupation probability at which an ∞ - size cluster appears as $L \to \infty$.

cluster number $n_s(p)$: number of s - clusters per lattice site.

The average number of clusters of size s in a hypercubic lattice of linear size L is $L^d n_s(p)$, where d is the dimensionality. Defining $n_s(p)$ per lattice site guarantees the independence of L.

(A)

1 dimensional chain : analytically solvable

1) 

$i = 1, 2 \dots N$

Sites occupied with probab. p

Sites empty with probab. $(1-p)$.

L is assumed to be sufficiently large to ignore boundary effects.

The probability that there exists a cluster of size $s < L$ is:

$n_s(p) = p^s (1-p)^2$

Because s occupied sites are needed surrounded by 2 empty sites:

this result relies on the independence of the occupancy probab.

In 1d, we can rewrite $n_s(p) = (1-p)^2 p^s = (1-p)^2 \exp[\ln(p^s)]$

$$= (1-p)^2 \exp[s \ln(p)] =$$

$$= (p_c - p)^2 \exp\left[-\frac{s}{s_\xi}\right].$$

s_ξ : characteristic cluster size

$$s_\xi = -\frac{1}{\ln(p)} = -\frac{1}{\underbrace{\ln[p_c - (p_c - p)]}_{\ln(1-x) \simeq -x - x^2/2 + O(x^3)}} \simeq (p_c - p)^{-1} \quad \text{for } p \to p_c$$

2) What is S, the average size of a finite cluster?

$$S = \frac{\overbrace{\sum_s s\, n_s(p)}^{N}}{\underbrace{\sum_s n_s(p)}_{D}}$$

$$D: \sum_{s=1}^{\infty} n_s(p) = \sum_{s=1}^{\infty} (1-p)^2 p^s = (1-p)^2 \frac{p}{1-p} = p(1-p)$$

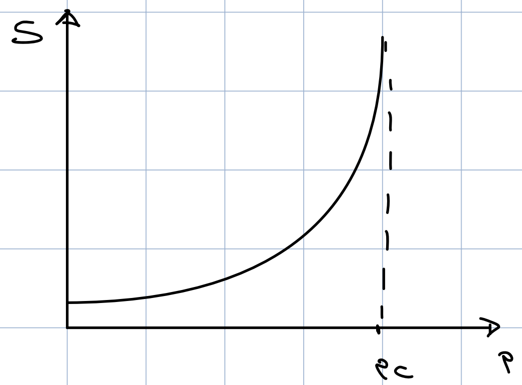
$$N: \sum_{s=1}^{\infty} s\, n_s(p) = \sum_{s=1}^{\infty} (1-p)^2 s\, p^s = (1-p)^2 [p + 2p^2 + 3p^3 + \dots] =$$

$$= p(1-p)^2 [\underbrace{1 + 2p + 3p^2 + \dots}_{\frac{d}{dp}\left(\sum_{s=0}^{\infty} p^s\right)}]$$

$N: \ p(1-p)^2 \dfrac{1}{(1-p)^2} = p$

$$S = \frac{\sum_s n_s(p) \cdot s}{\sum_s n_s(p)} = \frac{p}{p(1-p)} = \frac{1}{1-p}$$

S diverges as $p \to p_c = 1$.



3) Calculate the correlation function $g(r)$.

Let $r = |\vec{r}|$, $g(\vec{r} = 0) = 1$ meaning that the site 0 is occupied by definition.

In 1d, for a site at distance r to be occupied and belong to the same cluster, this one as well as the previous $(r-1)$ intermediate ones must be occupied:

$$g(r) = p\,p^{r-1} = p^r \qquad \forall p$$

$$g(r) = \exp[\ln(p^r)] = \exp[r \ln(p)] = \exp\left(-\frac{r}{\xi}\right),$$

4) where again $\underset{=}{\xi} = -\dfrac{1}{\ln(p)} = -\dfrac{1}{\ln[p_c - (p_c - p)]} \sim (p_c - p)^{-1}$ for $p \to p_c$

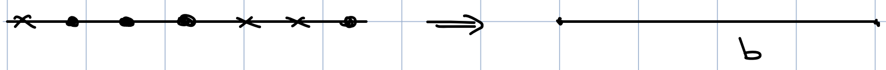
CORRELATION LENGTH

diverging as well for $p \to p_c$.

The critical exponent $\nu = 1$ (exact in 1d).

(B) One-dimensional chain: RG approach.

Replace a group of sites by a coarse-grained super-site, with linear dimension b $(1 < b < \xi)$.



$$\xi(p) = b\,\xi'(p')$$

$p' = p'(p)$ The concentration of occupied super-sites p' will be different from that of the original model.

Only at the critical point - for self-similarity - $p' = p = p_c$.

1) Take the exact transformation:

$$b\,\xi'(p') = \xi(p)$$

$$b\,\xi' = \cancel{A}\, b\,(p' - p_c)^{-\nu} = \cancel{A}(p - p_c)^{-\nu}$$

Linearize: $p' = p'(p_c) + \left.\frac{dp'}{dp}\right|_{p_c} \cdot \varepsilon$

Replace in the above expression: $b\left(\left.\frac{dp'}{dp}\right|_{p_c} \cdot \varepsilon\right)^{-\nu} = \varepsilon^{-\nu}$

Take the log of LHS and RHS:

$$\ln(b) - \nu \ln\left(\left.\frac{dp'}{dp}\right|_{p_c}\right) = 0$$

$$\nu = \frac{\ln(b)}{\ln\left(\left.\frac{dp'}{dp}\right|_{p_c}\right)}$$

2) Group the sites and find p'.

A b-site cell is occupied if we can get across it.

The probability of having a spanning cluster in a block of b sites is : $p' = \underbrace{R_b(p)}_{} = p^b$

"recursion relation" or
"renormalization transformation"

3) Fixed points : obtained by imposing $p' = p$.

$R_b(p^*) = {p^*}^b = p^* \implies p^* = \begin{cases} 0 & \text{stable fixed point} \\ 1 & \text{unstable} \end{cases}$

- If we start from an empty lattice ($p=0$), the renormalized lattice will also be empty ($p^*=0$).

- If we start from a fully occupied lattice ($p=1$), the RG lattice will also be fully occupied ($p^*=1$).

- If we start with $p<1$, the "new" RG lattice will contain even more empty site (because $p^b < p$).

4) Computation of the exponent ν.

$$\left.\frac{dR_b(p)}{dp}\right|_{p^*=1} = b p^{b-1} \Big|_{p^*=1} = b$$

$\nu = \ln(b) / \ln(b) = 1$ exact in 1d !