

QUANTUM STATISTICAL PHYSICS

REFS

SIMONS & ALTMAN

SHANKAR, KADANOFF

FRADKIN

FIRST: CLASS $\Rightarrow$ QUANTUM

LATER: PATH INTEGRALS
INSTANTONS } NOT THIS
YEAR

GENERIC PROPS OF CORR. FUNCTIONS
KUBO, PDT, ETC.

MATHAN, SIMONS ALTMAN

FINAL

QUANTUM PHASE TRANSITIONS
(AT STRICT ZERO TEMP.)

SACHDEV

CLASSICAL - QUANTUM CORRESPONDENCE

CANON. EQUILIBRIUM OF CLASS. SYSTEM
 $d+1$ DIMENSIONS AT
INVERSE TEMP β

|||

CANONICAL EQUIL. QUANTUM SYST
 d DIMENSIONS AT
INVERSE TEMP β_q

NON-TRIVIAL RELATIONS BTW PARAMETERS IN
THE TWO MODELS
(INCLUDING β AND β_q)

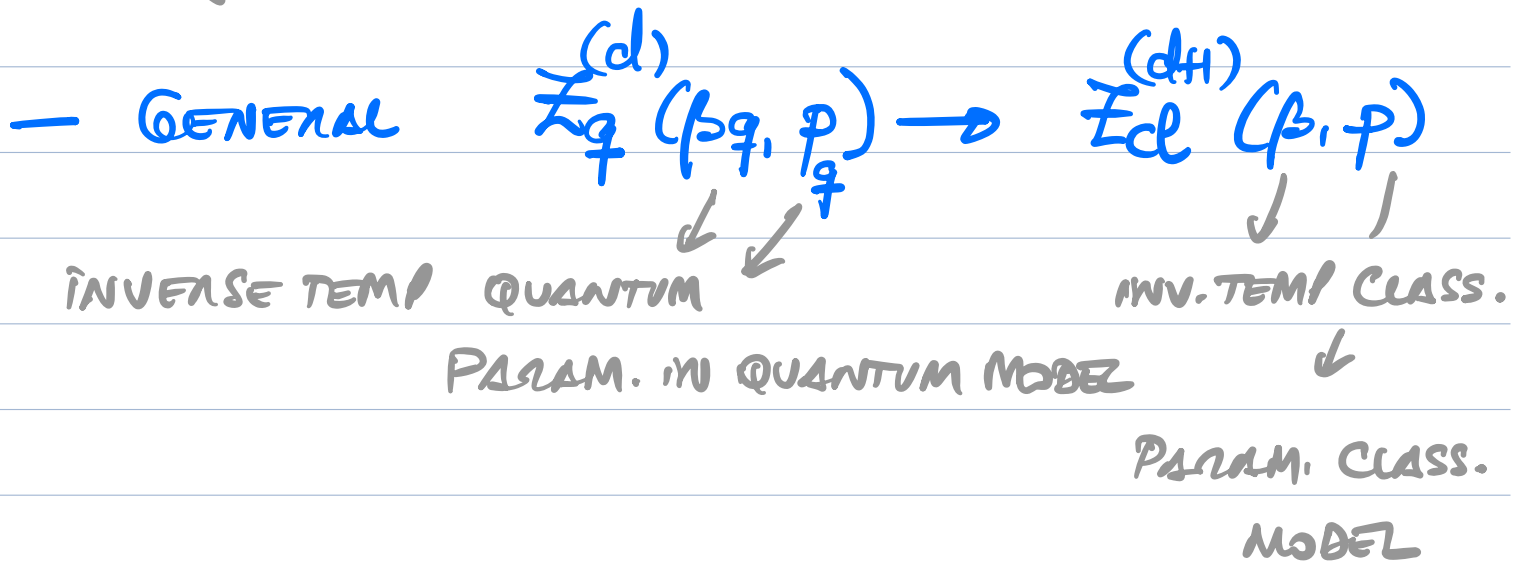
EVERYTHING WE SAID ABOUT CLASSICAL
EQUIL. PHASE TRANS.

APPLIES TO THE QUANTUM PROBLEMS

METHODS

TRANSFER MATRIX CLASSICAL $\rightarrow$ QUANTUM

eg. 1D ISING CHAIN $\rightarrow$ 1 QUANTUM SPIN



EVOL. OPERATORS

QUANTUM REAL TIME DYNAMICS t
 $\rightarrow$ CLASSICAL EQUIL $-i\beta\hbar$

PATH INTEGRALS

SEC. 3.1

THE CLASSICAL ONE DIM ISING CHAIN

- TRANSFER MATRIX

$$Z = \sum_{\{s_i\}} e^{-\beta H}$$

$$= \sum_{\{s_i\}} e^{k \sum_i s_i s_{i+1} + H \sum_i s_i}$$

$$k = \beta J$$

$$H = \beta h$$

TBC

$$s_{N+1} = s_1$$

CONVENIENCE: SYMMETRIZE THE WEIGHTS

$$e^{-\beta W(s_i, s_{i+1})}$$

$$\text{WITH } W(s_i, s_{i+1}) = W(s_{i+1}, s_i)$$

1st TERM IS ALREADY SYMM

$$\text{2nd TERM } \frac{h}{2} \sum_i (s_i + s_{i+1}) \quad \text{SYMMETRIZED}$$

$$-\beta W(s_i, s_{i+1}) = \underbrace{+\beta J}_{k} s_i s_{i+1} + \underbrace{\beta h}_{\tilde{h}} \frac{1}{2} (s_i + s_{i+1})$$

$$-\beta W(s_i, s_{i+1}) = K s_i s_{i+1} + \frac{H}{2} (s_i + s_{i+1})$$

WHY CAN ONE WRITE THE Z AS A PRODUCT OF MATRICES?

LOOK AT THREE SUBSEQ SPINS $i-1, i, i+1$:

$$\sum_{\{s_i\}} \underbrace{e^{-\beta W(s_{i-1}, s_i)}}_{M_L} \underbrace{e^{-\beta W(s_i, s_{i+1})}}_{M_R} \quad \text{TWO TERMS CONTRIB.}$$

M_L M_R
2x2 2x2

1st VARIABLE s_{i-1} , ITS VALUES ORGANIZED IN THE ROWS

2nd VARIABLE s_i , ITS VALUES ORG. IN COLUMNS

$$M_L = \begin{pmatrix} M(s_{i-1}=1, s_i=1) & M(s_{i-1}=1, s_i=-1) \\ M(s_{i-1}=-1, s_i=1) & M(s_{i-1}=-1, s_i=-1) \end{pmatrix}$$

$$M_R = \begin{pmatrix} M(s_i=1, s_{i+1}=1) & M(s_i=1, s_{i+1}=-1) \\ M(s_i=-1, s_{i+1}=1) & M(s_i=-1, s_{i+1}=-1) \end{pmatrix}$$

$-\beta W(s_i, s_{i+1})$ CAN TAKE 4 VALUES $\{s_i, s_{i+1} = \pm 1\}$

$\Rightarrow$ ORGANIZE THEM IN A 2×2 MATRIX

$$-\beta \begin{pmatrix} \overset{s_{i+1}=1}{\downarrow} w(1,1) & \overset{s_{i+1}=-1}{\downarrow} w(1,-1) \\ w(-1,1) & w(-1,-1) \end{pmatrix} = \begin{matrix} \leftarrow s_i = 1 \text{ ROW} \\ \leftarrow s_i = -1 \text{ ROW} \end{matrix}$$

$$= \begin{pmatrix} +k + H & -k \\ -k & k - H \end{pmatrix}$$

BUT WE REALLY HAVE $e^{-\beta W(s_i, s_{i+1})}$
IN THE MATRICES M

$$\Rightarrow \begin{pmatrix} e^{k+H} & e^{-k} \\ e^{-k} & e^{k-H} \end{pmatrix} = M$$

THIS IS THE TRANSFER MATRIX
KEEP THIS ONE IN BLACKBOARD

COMMENT: WE NOTE IT'S THE SAME
FOR ALL LINK PAIRS
(s_i, s_{i+1})

INDEX OF i

IN OTHER HETEROGENEOUS MODELS

COULD BE M_i

FOCUS ON HOMOGENEOUS ONES

$$Z = \sum_{\{s_i\}} e^{K \sum_i s_i s_{i+1} + H \sum_i s_i}$$

$\begin{matrix} \text{ROWS} & & \text{COLUMNS} \\ \downarrow & & \downarrow \end{matrix}$

$$= \sum_{\{s_i\}} \prod_i e^{-\beta w(s_i, s_{i+1})}$$

$$= T_r \prod_{i=1}^N M$$

T_r FROM PBC

$$= T_r M^N$$

$$= \sum_{a=1}^n \lambda_a^N$$

EIGENVALUES OF M

$$= \lambda_1^N + \lambda_2^N$$

FOR THE $n=2$ CASE
("INTERNAL DIM" OF
VARIABLE)

$$\mathcal{Z} = \lambda_1^N \left[1 + \left(\frac{\lambda_2}{\lambda_1} \right)^N \right] \underset{N \gg 1}{\approx} \lambda_1^N$$

$\lambda_1 > \lambda_2$

$$-\beta f = \frac{1}{N} \ln \mathcal{Z} = \ln \lambda_1$$

THE LARGEST EIGENVALUE OF M DOMINATES

$$-\beta f = \ln \lambda_1$$

PERRON - FROBENIUS -

SYMMETRIC MATRIX M w/ R ELEM

$\lambda_1 > 0$, NON DEGENERATE

THE TRANSFER MATRIX M is $\begin{pmatrix} e^{k+H} & e^{-k} \\ e^{-k} & e^{k-H} \end{pmatrix}$

CHARACTERISTIC POLYN.

$$P(\lambda) = (e^{k+H} - \lambda)(e^{k-H} - \lambda) - e^{-2k}$$

ROOTS $\Rightarrow$ EIGENVALUES

$$e^{2k} - \lambda e^{k+H} - \lambda e^{k-H} + \lambda^2 - e^{-2k} = 0$$

$$\lambda^2 - \lambda e^k 2\cosh H + 2\sinh 2k = 0$$

$$2\lambda_{1,2} = e^k 2\cosh H \pm \left[e^{2k} 4\cosh^2 H - 4 2\sinh 2k \right]^{1/2}$$

$$\begin{aligned} \boxed{\lambda_{1,2} = e^k \cosh H \pm \left[e^{2k} \cosh^2 H - 2\sinh 2k \right]^{1/2}} \\ = e^k \cosh H \pm e^k \left[\cosh^2 H - (e^{2k} - e^{-2k}) e^{-2k} \right]^{1/2} \\ = e^k \cosh H \pm e^k \left[\cancel{1} + \sinh^2 H - \cancel{1} + e^{-4k} \right]^{1/2} \\ \boxed{= e^k \cosh H \pm e^k \left[\sinh^2 H + e^{-4k} \right]^{1/2}} \end{aligned}$$

$$\lambda_1 = e^k \cosh H + e^k [\sinh^2 H + e^{-4k}]^{1/2}$$

$$H=0: \lambda_1 = e^k + e^{-k} = 2 \cosh k$$

$$-\beta f = \ln \lambda_1 = \ln(2 \cosh k)$$

AS WE KNEN ALREADY

dk

$$\text{IN TD} \Rightarrow \beta = \left(\ln \frac{\lambda_1}{\lambda_2} \right)^{-1}$$

EXERCISE

COMPLETE THIS CALC.

$$M(k, H) \begin{cases} \longrightarrow 1 \\ H \rightarrow \infty \\ \longrightarrow 0 \\ H \rightarrow \infty \end{cases}$$

END SOLUTION OF 1D ISING CHAIN WITH TRANSFER
MATRIX.

QUANTUM : PAULI MATRICES

$$\mathbb{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \in \mathbb{C}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

BASIS of
2x2 HERMITIAN
MATRICES

PROPERTIES

HERMITIAN

$$\sigma_a^\dagger = \sigma_a \quad a = 1, 2, 3$$

$$\dagger : (t, *)$$

$$\sigma_a^2 = \mathbb{I}$$

IDENTEMPOTENT

$$[\sigma_a, \sigma_b] = 2i \epsilon_{abc} \sigma_c$$

COMM.

RELATIONS

$$\{\sigma_a, \sigma_b\} = 0$$

ANTI-COMMUTE

IDEA

THINK OF

AS THE EXPECTATION VALUE
OF A QUANTUM OPERATOR

$$e^{-\beta \mathcal{H}(s_i, s_{i+1})} = \langle s_{i+1} | \hat{T} | s_i \rangle$$

$$= \langle s_{i+1} | e^{-\epsilon \hat{\mathcal{H}}_b} | s_i \rangle$$

DIRAC NOTATION BRA-KETS

$\hat{\mathcal{H}}_b$

QUANTUM HAMILT (HERMITIAN

1 OPERATOR)

TASK: FIND $\hat{T}$ AND HENCE $\hat{\mathcal{H}}_b$

STRATEGY FROM CLASSICAL TO QUANTUM

- WE KNOW M FROM THE CLASSICAL TRANSFER MATRIX

- WE ASK $M = \langle S_{i+1} | \hat{T} | S_i \rangle$ WITH

$$\hat{T} = a_0 + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z$$

AND FIND $\{a_0, a_1, a_3\}(k, \hbar)$

- WE ASK $\hat{T} = e^{-\varepsilon \hat{H}_\ell}$ WITH

$$\hat{H}_\ell = b_0 + b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z$$

AND FIND $\{b_0, b_1, b_3\}(k, \hbar)$

- ONCE FOUND THE INFINITESIMAL TRANSFER

GO TO THE MACROSCOPIC LIMIT

ε MICROSC

$\beta_f = N\varepsilon$ FINITE

$$e^{-\beta_f \hat{H}_\ell}$$

MODEL OF A SINGLE QUANTUM SPIN AT $\beta_f = N\varepsilon$

SYST SIZE IN β_f , PARAM. $(k, \hbar)$ IN (Δ_x, Δ_z)

NON-TRIVIAL $\varepsilon \rightarrow 0, N \rightarrow \infty$ LIMIT

FIRST identify $\hat{T}$ FROM

$$\begin{pmatrix} e^{k+\hbar} & e^{-k} \\ e^{-k} & e^{k-\hbar} \end{pmatrix} = M = \langle s_{i+1} | \hat{T} | s_i \rangle$$

REWRITE $\hat{T}$ AS A LINEAR COMBINATION
OF PAULI MATRICES
OPERATORS ACTING ON

A VECT
HILBERT

SPACE

$$\hat{T} = a_0 \hat{I} + a_1 \hat{\sigma}_x + a_2 \hat{\sigma}_y + a_3 \hat{\sigma}_z$$

||
0

SINCE $M \in \mathbb{R}$

$\Rightarrow a_0, a_1, a_3$?

4 eqs $\rightarrow$ 3 eqs
(symm)

$\geq$ 3 unknowns
 a_0, a_1, a_3

EXPLICIT CALCULATION

$$\langle S_{i+1} | \hat{T} | S_i \rangle$$

$$\hat{T} = a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_2 \hat{\sigma}_y + a_3 \hat{\sigma}_z$$

$\begin{smallmatrix} \parallel \\ 0 \end{smallmatrix}$

NO NEED SINCE
 $T_{ab} \in \mathbb{R}$

SANDWICH $\hat{T}$ WITH $\langle + | \dots | + \rangle$:

$$a_0 + a_1 \underbrace{\langle + | \hat{\sigma}_x | + \rangle}_{\langle + | - \rangle = 0} + a_3 \underbrace{\langle + | \hat{\sigma}_z | + \rangle}_1$$

$$= a_0 + a_3 \quad \text{SIMILARLY, OTHER SANDW.}$$

$$\langle + | a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z | - \rangle = a_1$$

$$\langle - | a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z | + \rangle = a_1$$

$$\langle - | a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z | - \rangle = a_0 - a_3$$

ASKING EQUALITY WITH ELEMENTS IN M :

$$a_0 + a_3 = e^{k+it} \quad (1)$$

$$a_1 = e^{-k} \Rightarrow \boxed{a_1 = e^{-k}}$$

$$a_0 - a_3 = e^{k-it} \quad (2)$$

$$\text{ADD (1) + (2)} : 2a_0 = e^{k+it} + e^{k-it}$$

$$\boxed{a_0 = e^k \cosh t}$$

SUBTRACT (1) - (2) :

$$\boxed{a_3 = e^k \sinh t}$$

$$\hat{T} = a_0 \mathbb{I} + a_1 \hat{\sigma}_x + a_3 \hat{\sigma}_z$$

$$\boxed{\hat{T} = e^k \cosh t \mathbb{I} + e^{-k} \hat{\sigma}_x + e^k \sinh t \hat{\sigma}_z}$$

TRANSFER MATRIX in Pauli Matrix Notation

TRANSFER OPERATOR AS THE EXP OF A QUANTUM HAMILTONIAN

$$\hat{T} = e^k \cosh \mathbb{I} + e^{-k} \hat{\sigma}_x + e^k \sinh \hat{\sigma}_z$$
$$= e^{-\epsilon \hat{H}} \quad \text{DEFINITION OF A QUANTUM HAMILT.}$$

$$\epsilon \hat{H} = \left[b_0 \mathbb{I} - b_1 \hat{\sigma}_x - \underbrace{b_2 \hat{\sigma}_y}_{\parallel 0} - b_3 \hat{\sigma}_z \right] \epsilon$$

WHICH is $\hat{H}$? || 0 SAME REASON

$$e^{-\epsilon \hat{H}} = e^{\epsilon b_0 \mathbb{I}} \underbrace{e^{\epsilon [b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z]}}_{\hat{\sigma}_x, \hat{\sigma}_z \text{ DON'T COMMUTE}}$$

↙

$$e^{\epsilon b_0 \mathbb{I}}$$

TAYLOR EXPAND

$$\sum_{n=0} \frac{(b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^n}{n!} \epsilon^n$$

WE NEED EXPRESSIONS FOR EVEN/ODD POWERS

WE WILL SEE THE SERIES OF $\cosh y$, $\sinh y$ APPEARING

$$\cosh y = \frac{e^y + e^{-y}}{2}$$

$$= \frac{1}{2} \sum_n \frac{y^n}{n!} + \frac{1}{2} \sum_n \frac{(-y)^n}{n!}$$

$$= \sum_{n \text{ even}} \frac{y^n}{n!} \quad (\text{ODD CANCEL})$$

$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$= \frac{1}{2} \sum_n \frac{y^n}{n!} - \frac{1}{2} \sum_n \frac{(-y)^n}{n!}$$

$$= \sum_{n \text{ odd}} \frac{y^n}{n!} \quad (\text{EVEN CANCEL})$$

$$\begin{aligned} (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^2 &= \overset{\text{I}}{b_1^2 \hat{\sigma}_x^2} + \overset{\text{I}}{b_3^2 \hat{\sigma}_z^2} + \\ &\quad + b_1 b_3 \underbrace{(\hat{\sigma}_x \hat{\sigma}_z + \hat{\sigma}_z \hat{\sigma}_x)}_{=0} \end{aligned}$$

$$= (b_1^2 + b_3^2) \mathbb{I}$$

$\Rightarrow$ All EVEN POWERS ARE PROP TO $\mathbb{I}$

$$\mathbb{I} \left\{ 1 + \frac{(b_1^2 + b_3^2)}{2!} + \frac{(b_1^2 + b_3^2)^2}{4!} + \dots \right\}$$

$$= \mathbb{I} \operatorname{ch} \left[(b_1^2 + b_3^2)^{1/2} \varepsilon \right]$$

RESUMMATION OF
EVEN POWERS:
PROPORTIONAL
TO $\mathbb{I}$

ODD POWERS

FIRST ONE IS $b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z$

NEXT

$$(b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^3 = (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z) \cdot$$

$$(b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^2$$

SO THEN

$$(b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^5 = (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z) \cdot$$

$$(b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^4$$

SO WE CAN USE RESULTS FROM THE
EVEN CASE TO WRITE

$$(b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z) \left[\mathbb{I} + (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^2 \frac{1}{3!} + (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)^4 \frac{1}{5!} + \dots \right]$$

$$= (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z) \mathbb{I} \left[1 + (b_1^2 + b_3^2) \frac{1}{3!} + (b_1^2 + b_3^2)^2 \frac{1}{5!} + \dots \right]$$

$$\Rightarrow (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z) \mathbb{I} \frac{1}{\sqrt{b_1^2 + b_3^2}} \operatorname{sh} \left[(b_1^2 + b_3^2)^{1/2} \varepsilon \right]$$

$$= \frac{\operatorname{sh} \left[(b_1^2 + b_3^2)^{1/2} \varepsilon \right]}{\sqrt{b_1^2 + b_3^2}} (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z)$$

RESUMMATION OF ODD
POWERS PROP. TO PAULI OPER.

$$e^{-\epsilon \hat{H}} = e^{\epsilon b_0} \left\{ \text{II} \text{ ch}[(b_1^2 + b_3^2)^{1/2} \epsilon] + \frac{\text{sh}[(b_1^2 + b_3^2)^{1/2} \epsilon]}{\sqrt{b_1^2 + b_3^2}} (b_1 \hat{\sigma}_x + b_3 \hat{\sigma}_z) \right\}$$

THIS IS THE EXPRESSION OF $\hat{T}$ IN TERMS OF THE THREE PARAMETERS IN $\hat{H}$

NOW, COMPARE TO $\hat{T}$

$$\hat{T} = e^k \text{ch} t \text{ II} + e^{-k} \hat{\sigma}_x + e^k \text{sh} t \hat{\sigma}_z$$

$$\underline{\text{II}}: e^{\epsilon b_0} \text{ch}[\sqrt{b_1^2 + b_3^2} \epsilon] = e^k \text{ch} t \quad (1)$$

$$\underline{\hat{\sigma}_x}: \frac{e^{\epsilon b_0} \text{sh}[\sqrt{b_1^2 + b_3^2} \epsilon]}{\sqrt{b_1^2 + b_3^2}} b_1 = e^{-k} \quad (2)$$

$$\underline{\sigma_2}: \quad e^{\epsilon b_0} \frac{\text{sh}[\sqrt{b_1^2 + b_3^2} \epsilon]}{\sqrt{b_1^2 + b_3^2}} b_3 = e^k \text{sh} t \quad (3)$$

THREE EQS & THREE UNKNOWN'S

$$\frac{(2)}{(3)} \Rightarrow \left[\frac{b_1}{b_3} = e^{-2k} \frac{1}{\text{sh} t} \right] \quad (4)$$

TAKING $(1)^2 - (2)(B/b_1 \epsilon)$ $B \equiv \sqrt{b_1^2 + b_3^2} \epsilon$

$$\begin{aligned} e^{2b_0 \epsilon} (\text{ch}^2 B - \text{sh}^2 B) &= \\ &= e^{2k} \text{ch}^2 t - \frac{(b_1^2 + b_3^2)}{b_1^2} e^{-2k} \end{aligned}$$

SINCE $\text{ch}^2 B - \text{sh}^2 B = 1 \Rightarrow$

$$\begin{aligned} e^{2b_0 \epsilon} &= e^{2k} \text{ch}^2 t - \left(1 + \frac{b_3^2}{b_1^2}\right) e^{-2k} \\ &= e^{2k} \text{ch}^2 t - \left(1 + e^{4k} \text{sh}^2 t\right) e^{-2k} \end{aligned}$$

USING (4)

$$= e^{2k} \underbrace{(\cosh^2 H - \sinh^2 H)}_1 - e^{-2k}$$

$$e^{2b_0 \varepsilon} = 2 \sinh 2k$$

FIXES

$$\varepsilon b_0 = \frac{1}{2} \ln(2 \sinh 2k)$$

WE CAN NOW GO BACK TO EQ. (2) AND
REPLACE $e^{2b_0 \varepsilon}$ FROM THIS RESULT

$$\sqrt{2 \sinh 2k} \frac{\sinh[\sqrt{b_1^2 + b_3^2} \varepsilon] b_1}{\sqrt{b_1^2 + b_3^2}} = e^{-k}$$

$$\Rightarrow b_1 = \frac{e^{-k} \sqrt{b_1^2 + b_3^2}}{\sqrt{2 \sinh 2k} \sinh[\sqrt{b_1^2 + b_3^2} \varepsilon]}$$

IMPLICIT EXPRESSION IN TERMS OF $b = b_1^2 + b_3^2$ IN RHS

AND NOW IN (3)

$$\sqrt{2 \operatorname{sh} 2k} \frac{\operatorname{sh} \left[\sqrt{b_1^2 + b_3^2} \epsilon \right]}{\sqrt{b_1^2 + b_3^2}} b_3 = e^k \operatorname{sh} t$$

ALSO IMPLICIT IN TERMS OF b IN RHS

$$\Rightarrow b_3 = \frac{e^k \operatorname{sh} t \sqrt{b_1^2 + b_3^2}}{\sqrt{2 \operatorname{sh} 2k} \operatorname{sh} \left[\sqrt{b_1^2 + b_3^2} \epsilon \right]}$$

TAKE $t \rightarrow 0$ AS A SIMPLER CASE

$$b_3 = 0$$

AND TAKE $\frac{(2)}{(1)}$:

$$\text{th } b_1 \varepsilon = e^{-2k}$$

$$\frac{e^{b_1 \varepsilon} - e^{-b_1 \varepsilon}}{e^{b_1 \varepsilon} + e^{-b_1 \varepsilon}} = \frac{1 - e^{-2b_1 \varepsilon}}{1 + e^{-2b_1 \varepsilon}} = e^{-2k}$$

$$1 - e^{-2b_1 \varepsilon} = e^{-2k} (1 + e^{-2b_1 \varepsilon})$$

$$1 - e^{-2k} = e^{-2b_1 \varepsilon} (1 + e^{-2k})$$

$$\frac{1 - e^{-2k}}{1 + e^{-2k}} = e^{-2b_1 \varepsilon}$$

$$\text{th } k = e^{-2b_1 \varepsilon}$$

$$-2b_1 \varepsilon = \ln \text{th } k$$

$$b_1 \varepsilon = -\frac{1}{2} \ln \text{th } k$$

$$\text{Case } H=0 \\ b_3=0$$

IF I FOLLOW THIS ROUTE FOR $H \neq 0$
 (I SET $\epsilon = 1$ IN THIS PART,
 IT'S RECOVERED BELOW)

$$\cosh \sqrt{b_1^2 + b_3^2} = e^k \cosh t$$

$$\frac{\sinh \sqrt{b_1^2 + b_3^2}}{\sqrt{b_1^2 + b_3^2}} b_1 = e^{-k}$$

$$\frac{\sinh \sqrt{b_1^2 + b_3^2}}{\sqrt{b_1^2 + b_3^2}} b_1 = e^{-2k} \cosh t$$

USE NOW $b_3 = b_1 e^{2k} \sinh t$

$$\frac{\sinh b_1 \sqrt{1 + e^{4k} \sinh^2 t}}{\cancel{b_1} \sqrt{1 + e^{4k} \sinh^2 t}} \cancel{b_1} = e^{-2k} \cosh t$$

can $y = b_1 \sqrt{1 + e^{4k} \sinh^2 t}$

$$\frac{e^y - e^{-y}}{e^y + e^{-y}} = e^{-2k} \operatorname{ch} t \sqrt{\dots}$$

$$1 - e^{-2y} = (1 + e^{-2y}) e^{-2k} \operatorname{ch} t \sqrt{\dots}$$

$$(1 - e^{-2k} \operatorname{ch} t \sqrt{\dots}) =$$

$$= (1 + e^{-2k} \operatorname{ch} t \sqrt{\dots}) e^{-2y}$$

$$e^{-2y} = \frac{1 - e^{-2k} \operatorname{ch} t \sqrt{\dots}}{1 + e^{-2k} \operatorname{ch} t \sqrt{\dots}}$$

$$y = -\frac{1}{2} \ln \frac{1 - e^{-2k} \operatorname{ch} t \sqrt{\dots}}{1 + e^{-2k} \operatorname{ch} t \sqrt{\dots}}$$

$$= b_1 \sqrt{1 + e^{4k} \sinh^2 H}$$

ϵ RECOVERED HERE

$$\epsilon \dot{b} = -\frac{1}{2} \frac{1}{\sqrt{1 + e^{4k} \sinh^2 H}}$$

$$\ln \left(\frac{1 - e^{-2k} \cosh H \sqrt{1 + e^{4k} \sinh^2 H}}{1 + e^{-2k} \cosh H \sqrt{1 + e^{4k} \sinh^2 H}} \right)$$

CHECK $H = 0$

$$b_1 = -\frac{1}{2\epsilon} \ln \frac{1 - e^{-2k}}{1 + e^{-2k}} = -\frac{1}{2\epsilon} \ln \frac{1 + k}{1 - k}$$

AND WE KNOW
ALSO THE OTHER
TWO PARAMETERS

$$\epsilon b_3 = \epsilon b_1 e^{2k} \sinh H$$

$$\epsilon b_0 = \frac{1}{2} \ln(2 \sinh 2k)$$

H=0 SIMPLIFIED EXPRESSIONS

SHOW THESE ONES ON BOARD.

MAYBE ALSO

SHOW THE
GENERIC

ONES.

$$b_0 = \frac{1}{2\epsilon} \ln [2 \sinh 2u]$$

$$b_1 = -\frac{1}{2\epsilon} \ln \hbar k$$

$$b_3 = 0$$

QUANTUM HAMILTONIAN IN THE CASE $H=0$

$$-\epsilon \overset{1}{\mathcal{H}} = \frac{1}{2} \ln [2 \sinh(2u)] \mathbb{I}$$

$$- \frac{1}{2} \underbrace{\ln \hbar k}_{\delta} \overset{1}{\sigma}_x$$

"TRANSVERSE
FIELD"

THIS IS THE QUANTUM HAMILTONIAN FOR
THE "ELEMENTARY" TRANSFER MATRIX TAKING
FROM SITE i TO ITS NEIGHBOUR $i+1$ ON THE
CLASSICAL CHAIN.

TO GO TO THE "MACROSCOPIC" LIMIT OF N SITES
AND M^N OR 7^N WE NEED ϵ SMALL

$$e^{-\beta_g \hat{H}}$$

$$\beta_g = N\epsilon$$

↑

CLASSICAL CHAIN LENGTH

$$-\beta_g \hat{H} = -N\epsilon \hat{H} = A \mathbb{I} - \Delta \hat{\sigma}_x$$

$$\beta_g = N\epsilon$$

$$\Delta = \frac{N}{2} \ln h k$$

MAPPING FOR

CLASSICAL CHAIN



QUANTUM SPIN

CLASSICAL SYSTEM SIZE PROP. TO β_g

k is in Δ

● NOTE THAT $\epsilon \rightarrow 0$ & $N \rightarrow \infty$: β_g FINITE

Δ FINITE & $N \rightarrow \infty \Rightarrow k \rightarrow \infty$

BACK TO THIS DISCUSSION LATER.

IF WE WANT $\varepsilon \rightarrow 0$ & $N \rightarrow \infty$ THEN

LIMIT OF CLASSICAL CHAIN WITH $\beta \varphi = \varepsilon N$ FINITE

TO KEEP $\Delta = \frac{N}{2} \ln \tanh k$ FINITE $(\Rightarrow)$

$$\ln \tanh k \sim \frac{2\Delta}{N} \quad (\Leftrightarrow)$$

$$\tanh k \sim e^{2\Delta/N} \sim 1 + \frac{2\Delta}{N} + \dots$$

$\tanh k$ CLOSE TO 1 MEANS $k \rightarrow \infty$

$$\tanh k = \frac{e^k - e^{-k}}{e^k + e^{-k}} = \frac{1 - e^{-2k}}{1 + e^{-2k}}$$

$$\tanh k \simeq (1 - e^{-2k}) (1 - e^{-2k} + \dots) \simeq 1 - 2e^{-2k} + \dots$$

$$\frac{2\Delta}{N} \simeq -2e^{-2k} \Rightarrow -\frac{1}{2} \ln\left(-\frac{\Delta}{N}\right) = k$$

$$k = \frac{1}{2} \ln\left(-\frac{N}{\Delta}\right)$$

A GENERAL WAY OF SEEING THE EQUIVALENCE FROM QUANTUM TO CLASSICAL

$$Z_q = \text{Tr} e^{-\beta_q \hat{\mathcal{H}}}$$

$\mathbb{R}$ FOR QUANTUM

GENERAL
MODEL

$\vec{s} \rightarrow \begin{cases} \text{SPACE \&} \\ \text{INTERNAL} \\ \text{INDICES} \end{cases}$

SAN NOTATION JUST TO CHOOSE A NOTATION

$$= \sum_{|\vec{s}_0\rangle} \langle \vec{s}_0 | e^{-\beta_q \hat{\mathcal{H}}} | \vec{s}_0 \rangle$$

$\nwarrow$ SUM OVER A COMPLETE SET, BASIS OF
HILBERT SPACE

$\hat{\mathcal{H}}$ COMMUTES WITH ITSELF

(CAREFUL IF ONE WANTS TO SPLIT
IT IN TWO TERMS WHICH TYPICALLY
DON'T COMMUTE $\hat{\mathcal{H}}_1 + \hat{\mathcal{H}}_2$)

WRITE THE FULL EXPONENTIAL AS

$$e^{-\beta g \vec{p}} = \underbrace{e^{-\epsilon \vec{p}} \dots e^{-\epsilon \vec{p}}}_{N\epsilon = \beta g}$$

↓
NUMBER OF FACTORS

INTRODUCE RESOLUTIONS OF UNITY

$$1 = \sum_{\vec{s}_n} |\vec{s}_n\rangle \langle \vec{s}_n|$$

IN BETWEEN THOSE FACTORS

$$= \sum_{\vec{s}_0} \dots \sum_{\vec{s}_{N-1}} \langle \vec{s}_0 | e^{-\epsilon \vec{p}} | \vec{s}_{N-1} \rangle \dots \langle \vec{s}_1 | e^{-\epsilon \vec{p}} | \vec{s}_0 \rangle$$

RECALL $|\vec{s}_N\rangle = |\vec{s}_0\rangle$ PBC

Call

ADIMENSIONAL

NB : COULD BE $\mathbb{C}$ BUT, IF $\mathbb{R}$
AND POSITIVE $\Rightarrow$ RESULTING
OK

$$e^{-\beta \mathcal{E}(\vec{s}_n, \vec{s}_{n-1})} \xleftarrow{\text{ANOTHER } \beta} = \langle \vec{s}_n | e^{-\epsilon \mathcal{H}} | \vec{s}_{n-1} \rangle \Rightarrow$$

$$\mathcal{Z}_q = \sum_{\{\vec{s}_n\}} e^{-\beta \sum_{n=1}^N \mathcal{E}(\vec{s}_n, \vec{s}_{n-1})}$$

AND RECALL PBC

CLASSICAL PART. FUNCT $\mathcal{Z}$ of a
MODEL in $(d+1)$ DIM.

EXTRA DIM $\Rightarrow$ THE ONE USED TO SANDWICH $n=0, \dots, n-1$

(IN THE $\vec{s}$ IN $\vec{s}$ THERE'S THE DEF.
ON THE d -DIM. AS WELL AS THE POSS.
INTERNAL DIM of s)

HERE WE MAPPED A QUANTUM $\mathcal{Z}_q$
ON A CLASSICAL ONE (in $d+1$ dim)

IN NOTES $\Rightarrow$ QUANTUM ISING SPIN $\Leftrightarrow$ CHAIN 164M.

THE RELATION BETWEEN PARAM. HAS TO
BE FOUND FOR EACH SPECIFIC PROBLEM

SEC. 3.1.2 MAPPING VIA THE EVOLUTION OPERATOR

FROM REAL TO IMAGINARY TIME

SCHRÖDINGER'S
EQUATION

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle$$

$$|\psi(t)\rangle = \hat{U}(t, t') |\psi(t')\rangle$$

START FROM
QUANTUM &

GO CLASSICAL

$$\hat{U}(t, t') = e^{-i \frac{\hat{H}(t-t')}{\hbar}}$$

EVOLUTION OPERATOR

NOTE $\frac{-i(t-t')}{\hbar}$

$$\hat{U} \hat{U}^\dagger = \mathbb{I} \quad \text{if} \quad \hat{H} = \hat{H}^\dagger$$

$$\langle \psi(t) | \psi(t) \rangle = \langle \psi(t') | \psi(t') \rangle$$

WAVE FUNCTION

$$\psi(x, t) = \langle x | \psi(t) \rangle = \langle x | \hat{U}(t, t') | \psi(t') \rangle$$

$$= \int dx' \langle x | \hat{U}(t, t') | x' \rangle \langle x' | \psi(t') \rangle$$

$$= \int dx' \underbrace{U(x, t; x', t')} \psi(x', t')$$

WAVE FUNCTION
EVOLUTION -
PROPAGATION.

EVOLUTION KERNEL
(IT'S A FUNCTION)

QUANTUM PARTITION FCT - CANONICAL EQ.

$$Z_q = \text{Tr} e^{-\beta_q \hat{H}}$$
$$= \int dx \langle x | e^{-\beta_q \hat{H}} | x \rangle$$

SUM OVER
STATES OF AN ORTH.
BASIS OF HILBERT
SPACE

eg. EIGENVECTOR OF $\hat{x}$

IT'S LIKE A
PERIODIC EVOLUTION
IN IMAGINARY TIME

$$t - t' = -i\beta_q \hbar$$

FROM $|x\rangle$
BACK TO
 $\langle x|$

$$Z_q = \int dx \mathcal{U}(x, \overbrace{-i\beta_q \hbar}^t; x, 0)$$

AND COMING BACK TO SAME x
BECAUSE OF TRACE PBC

QUANTUM STAT PHYS. IS LIKE
IMAGINARY TIME EVOLUTION

$$Z_g = \int dx \underbrace{U(x, \overbrace{-i\beta_g^t h}^t; x, 0)}$$

EVOLUTION OPERATOR
DIAGONAL MATRIX
ELEMENT, WITH
REAL TIME REPLACED BY

THE IMAGINARY TIME: $-i\beta_g^t h$
REAL

$$t = -i\tau = -i(\beta_g^t h)$$



3/ PATH INTEGRALS

THE PROPAGATOR FROM x' TO x IS

$$\langle x | \hat{U}(t, t') | x' \rangle = U(x, t; x', t')$$

GOAL WRITE $U(x, t; x', t')$

AS A SUM OVER CLASSICAL PATHS

LINKING (x', t') & (x, t)

↓
TRAJECTORIES

LET'S DISCUSS WIENER'S DESCRIPTION OF LANGEVIN PROCESSES

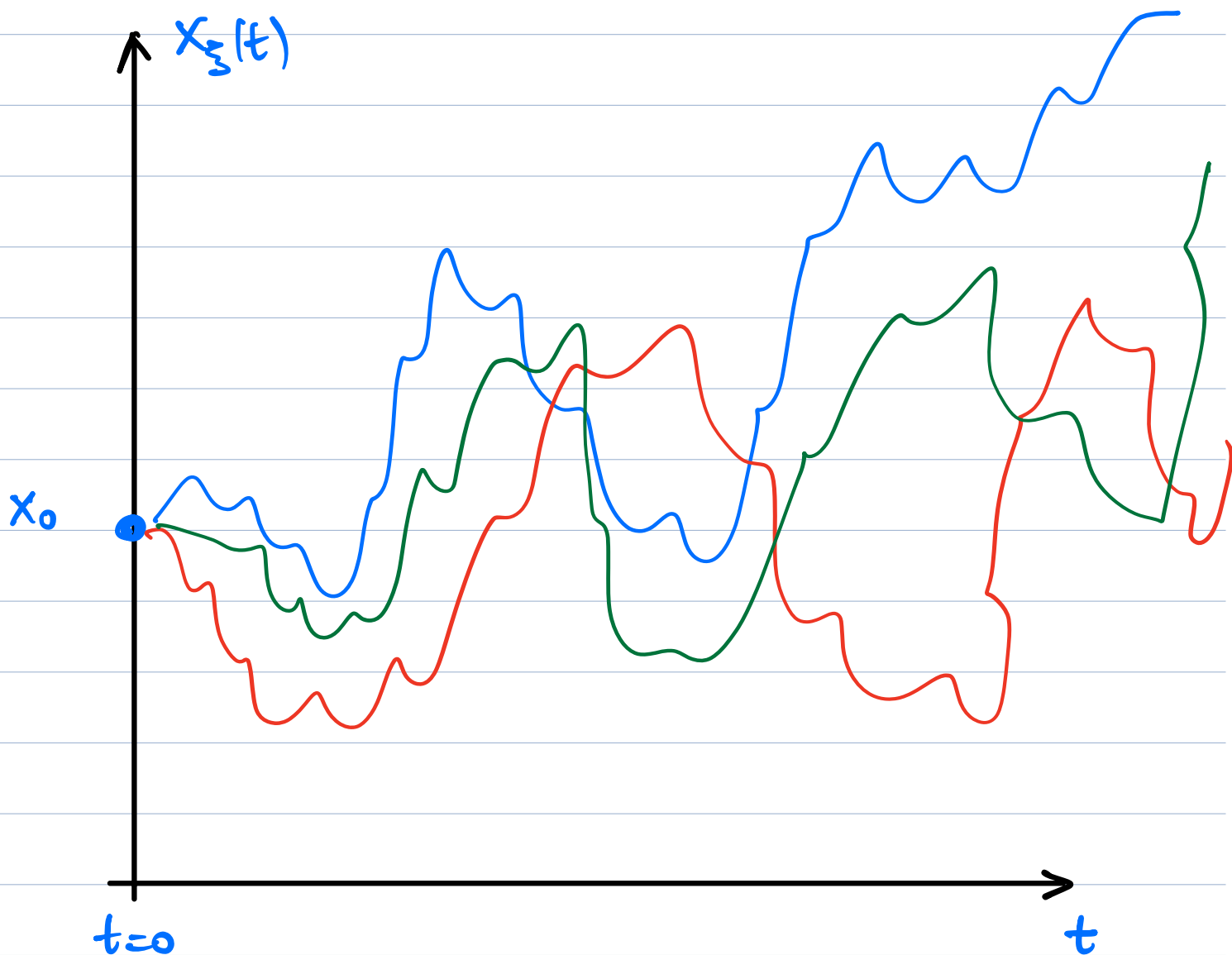
eg. THE SIMPLEST PROCESS

$$\dot{x} = \xi$$

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t) \xi(t') \rangle = 2k_B T \delta(t - t')$$

BROWNIAN MOTION, LANGEVIN



TRAJECTORIES GENERATED BY DIFF NOISES $\xi(t)$

$$x(t) = x_0 + \int_0^t dt'' \xi(t'') \Rightarrow \text{AVERAGES.}$$

$$\langle x(t) \rangle = x_0 + \int_0^t dt'' \langle \xi(t'') \rangle = x_0$$

$$\langle x^2(t) \rangle = x_0^2 + \int_0^t dt'' \int_0^t dt''' \underbrace{\langle \xi(t'') \xi(t''') \rangle}_{2k_B T \delta(t'' - t''')}$$

$$\langle x^2(t) \rangle - x_0^2 = 2k_B T t \quad \text{diffusion}$$

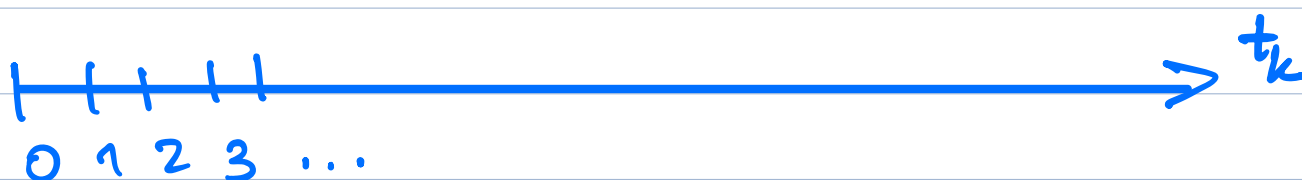
ALTERNATIVE DESCRIPTION

$$P(x_t | x'_t)$$

FIND AN EXPRESSION FOR THE
TRANSITION PROB. FROM
 $(x'_t) \rightarrow (x_t)$

DISCRETIZE TIME

N-DISCRETE VALUES



$$t_k = t_0 + k \delta t \quad k=0, \dots, N$$

$$\delta t = \frac{t - t'}{N} \quad t_0 = t' \quad t_N = t$$

$$x(t') = x_0 = x'$$

$$x(t) = x_N = x$$

$$P(x_t | x'_t) = \int dx_1 \dots \int dx_{N-1} p(x_t | x_{N-1}, t_{N-1}) \dots \\ \dots p(x_1, t_1 | x', t')$$

MARKOV CHARACTER OF PROCESSES

$$P(\mathbf{z}) = \prod_k p(S_k) = \prod_k \frac{1}{\sqrt{2\pi \cdot 2k_B T / \delta t}} e^{-\frac{S_k^2}{2 \cdot 2k_B T / \delta t}}$$

GAUSSIAN pdf

JOINT pdf of MEMORY-LESS NOISE

$\Rightarrow$

$$\langle S_k \rangle = 0 \quad \forall k$$

AVERAGES.

$$\langle S_k S_{k'} \rangle = \frac{2k_B T}{\delta t} \delta_{kk'}$$

DISCRETE VERSION OF DIRAC'S DELTA

$$X_{k+1} - X_k = \delta t S_k \quad \text{SOL. TO Lange Eq (170)}$$

$$S_k = \mathcal{O}(\delta t^{-1/2}) \Rightarrow \delta X_k = \mathcal{O}(\delta t^{1/2})$$

INFINITESIMAL $p(X_k, t_k | X_{k-1}, t_{k-1})$:

$$\begin{aligned} p(X_k, t_k | X_{k-1}, t_{k-1}) &= \int dS_{k-1} p(S_{k-1}) \delta(X_k - X_{k-1} - \underbrace{\delta t S_{k-1}}_{\bar{S}_{k-1}}) \\ &= \int \frac{d\bar{S}_{k-1}}{\delta t} p(\bar{S}_{k-1} / \delta t) \delta(\bar{S}_{k-1} - X_k + X_{k-1}) \\ &= \frac{1}{\delta t} p\left(\frac{X_k - X_{k-1}}{\delta t}\right) \end{aligned}$$

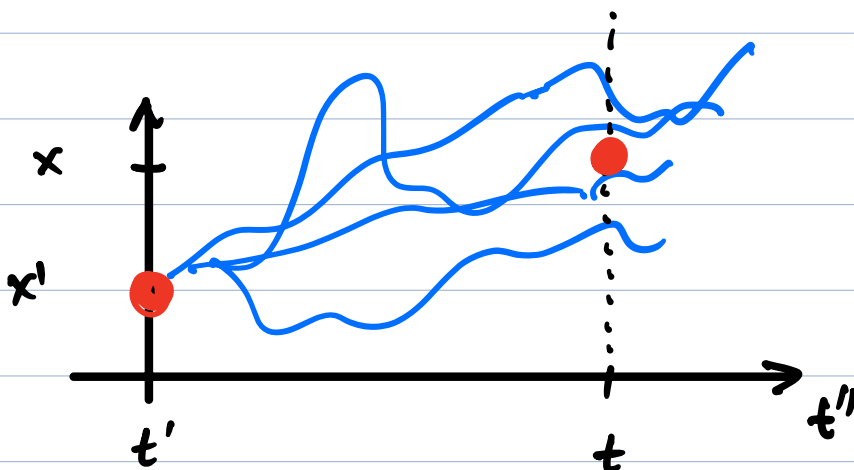
$$= \frac{1}{\sqrt{2\pi \cdot 2\hbar\epsilon T \delta t}} e^{-\frac{\delta t}{2 \cdot 2\hbar\epsilon T} \left(\frac{x_n - x_{n-1}}{\delta t} \right)^2}$$

NOW, BACK in THE FULL TRANS. PROB

$$p(x, t | x', t') = \underbrace{\int \mathcal{D}x}_{\text{sum over all trajectories}} e^{-\frac{1}{4\hbar\epsilon T} \int_{t'}^t dt'' (\dot{x}(t''))^2}$$

$$\int_{-\infty}^{\infty} \frac{dx_1}{A} \dots \int_{-\infty}^{\infty} \frac{dx_{N-1}}{A} \quad \left\{ \begin{array}{l} x(t) = x \quad \text{BOUND} \\ x(t') = x' \quad \text{COND.} \end{array} \right.$$

$$\frac{1}{(2\pi \cdot 2\hbar\epsilon T \delta t)^{1/2}} = \frac{1}{A}$$



SUM OVER ALL
TRAJECTORIES
GENERATED BY
LANGE. GOING
 $x' \rightarrow x$

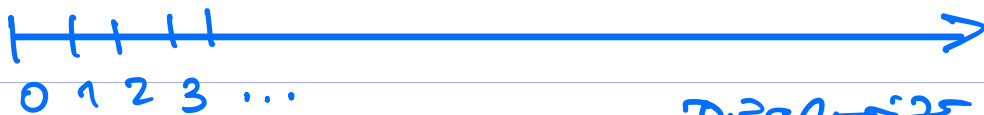
QUANTUM PROBLEM

THE PROPAGATOR FROM x' TO x IS

$$\langle x | \hat{U}(t, t') | x' \rangle = U(x, t; x', t')$$

WE FOLLOW A SIMILAR ROUTE TO WHAT DONE ABOVE
FOR THE STOCHASTIC EQ.

FIRST STEP



DISCRETIZE TIME

$$t_k = t_0 + k \delta t \quad k=0, \dots, N$$

$$\delta t = \frac{t - t'}{N} \quad t_0 = t' \quad t_N = t$$

ADD RESOLUTIONS OF THE IDENTITIES

$$\mathbb{1} = \int dx |x\rangle \langle x|$$

POSITION REPRESENTATION

$$\langle x | \hat{U}(t, t') | x' \rangle =$$

$$= \int dx_1 \dots \int dx_{N-1} \langle x | \hat{U}(t_N, t_{N-1}) | x_{N-1} \rangle$$

$$\langle x_{N-1} | \hat{U}(t_{N-1}, t_{N-2}) | x_{N-2} \rangle \dots \langle x_1 | \hat{U}(t_1, t_0) | x' \rangle$$

VERY SIMILAR TO THE CLASSICAL CHAIN WITH TRANS. PROB. $p(x_{k+1}, t_{k+1} | x_k, t_k)$

WE NEED TO COMPUTE THE GENERIC FACTOR

$$\begin{aligned} \langle x_k | \hat{U}(t_k, t_{k-1}) | x_{k-1} \rangle &= \\ &= \langle x_k | e^{-\frac{i}{\hbar} \hat{H}_0(t_k - t_{k-1})} | x_{k-1} \rangle \end{aligned}$$

WITH $t_k - t_{k-1} = \delta t$ VERY SMALL

$$\text{NOW WE USE } \hat{H}_0 = \frac{\hat{p}^2}{2m} + V(\hat{x})$$

BUT WE RECALL THAT 1ST & 2ND TERM DO NOT COMMUTE. SO WE NEED TO EVALUATE THE AMPLITUDE IN THE LIMIT δt SMALL.

CAMPBELL - HAUSSDORF FORMULA

$$e^{\hat{A}} e^{\hat{B}} = e^{\hat{A} + \hat{B} + \frac{1}{2} [\hat{A}, \hat{B}] + \dots}$$

WHICH, if $\hat{A} \propto \varepsilon$
 $\hat{B} \propto \varepsilon$ } $\Rightarrow$ 2nd TERM
IN EXP

$\ll$ 1st TERM

& so on & so forth

$$\Rightarrow e^{\hat{A}} e^{\hat{B}} \approx e^{\hat{A} + \hat{B}}$$

USED BELOW

$$\text{THANKS TO } \begin{cases} \hat{A} \propto \delta t \\ \hat{B} \propto \delta t \end{cases}$$

APPLY IT TO THE FACTOR TO BE CALCULATED

$$\langle x_k | e^{-\frac{i}{\hbar} \hat{p}_k^1 (t_k - t_{k-1})} | x_{k-1} \rangle$$

$$\sim \langle x_k | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \delta t} e^{-\frac{i}{\hbar} V(\hat{x}) \delta t} | x_{k-1} \rangle$$

$$\sim \langle x_k | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \delta t} | x_{k-1} \rangle e^{-\frac{i}{\hbar} V(x_{k-1}) \delta t}$$

INSERT AN IDENTITY RESOLVED IN THE MOMENTUM REPRESENTATION

$$\mathbb{1} = \int dp_{k-1} |p_{k-1}\rangle \langle p_{k-1}|$$

$$\langle x_k | e^{-\frac{i}{\hbar} H(t_k - t_{k-1})} | x_{k-1} \rangle \approx$$

$$e^{-\frac{i}{\hbar} V(x_{k-1}) \delta t}$$

$$\int dp_{k-1} \langle x_k | e^{-\frac{i}{\hbar} \frac{\hat{p}^2}{2m} \delta t} | p_{k-1} \rangle \langle p_{k-1} | x_{k-1} \rangle$$



$$\langle x_k | p_{k-1} \rangle e^{-\frac{i}{\hbar} \frac{p_{k-1}^2}{2m} \delta t}$$

$$\langle x_n | e^{-\frac{i}{\hbar} H(t_n - t_{n-1})} | x_{n-1} \rangle \simeq$$

$$e^{-\frac{i}{\hbar} V(x_{n-1}) \delta t}$$

$$\int dp_{n-1} e^{-\frac{i}{\hbar} \frac{p_{n-1}^2}{2m} \delta t} \langle x_n | p_{n-1} \rangle \langle p_{n-1} | x_{n-1} \rangle$$

THE FREE PARTICLE WAVE FUNCTIONS ARE

$$\langle p_{n-1} | x_{n-1} \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{-i}{\hbar} p_{n-1} x_{n-1}}$$

$$\langle x_n | p_{n-1} \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_{n-1} x_n}$$

THEN

$$\langle x_n | e^{-\frac{i}{\hbar} H(t_n - t_{n-1})} | x_{n-1} \rangle \simeq e^{-\frac{i}{\hbar} V(x_{n-1}) \delta t}$$

$$\int \frac{dp_{n-1}}{2\pi\hbar} e^{-\frac{i}{\hbar} \frac{p_{n-1}^2}{2m} \delta t} e^{\frac{i}{\hbar} p_{n-1} \frac{(x_n - x_{n-1})}{\delta t} \delta t}$$

WHICH CAN BE REWRITTEN AS

$$\langle x_k | e^{-\frac{i}{\hbar} \hat{H}_0(t_k - t_{k-1})} | x_{k-1} \rangle \simeq$$

$$\int \frac{dp_{k-1}}{2\pi\hbar} e^{\frac{i}{\hbar} [p_{k-1} \dot{x}_{k-1} - H_0(p_{k-1}, x_{k-1})] \delta t}$$

THIS IS A GAUSSIAN INTEGRAL OVER THE MOMENTA
WHICH READILY YIELDS

$$\langle x_k | e^{-\frac{i}{\hbar} \hat{H}_0(t_k - t_{k-1})} | x_{k-1} \rangle \simeq e^{-\frac{i}{\hbar} V(x_{k-1}) \delta t}$$

$$\left(\frac{-im}{2\pi\hbar \delta t} \right)^{1/2} e^{\frac{im}{2} \left(\frac{x_k - x_{k-1}}{\delta t} \right)^2 \frac{\delta t}{\hbar}}$$

$$\propto e^{\frac{i}{\hbar} \left[\frac{m \dot{x}_{k-1}^2}{2} - V(x_{k-1}) \right] \delta t}$$

PUTTING ALL TOGETHER

$$U(x, t; x', t') = \int_{x'}^x \mathcal{D}x \ e^{\frac{i}{\hbar} S}$$

WITH THE ACTION

$$S = \int_{t'}^t dt'' \left[\frac{m}{2} \dot{x}^2(t'') - V(x(t'')) \right]$$

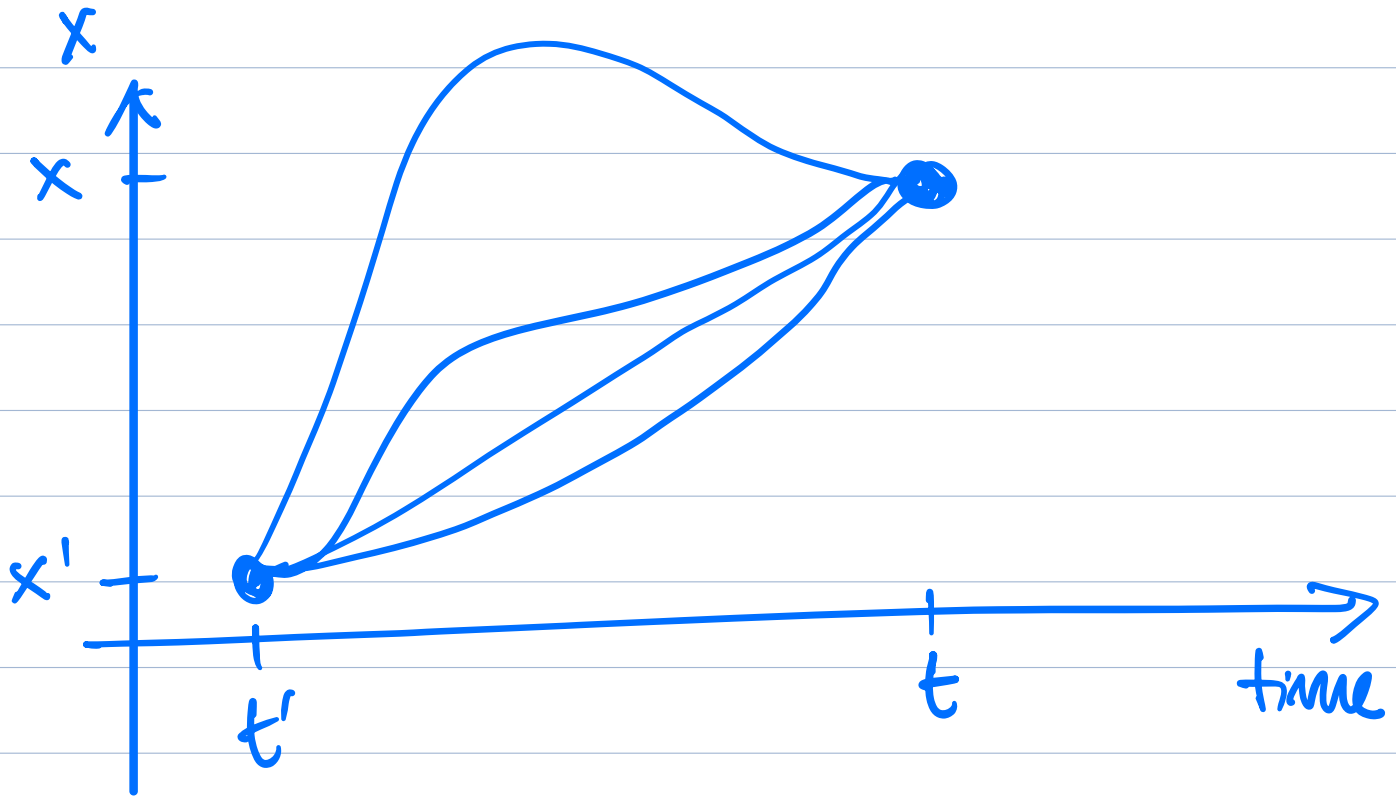
THE MEAS. FACTOR GOES IN $\mathcal{D}x$

$$S = \int_{t'}^t dt'' \mathcal{L}(x, \dot{x})$$

LAGRANGIAN

LIKE IN CLASSICAL MECHANICS.

FEYNMAN'S PATH INTEGRAL



SUM OVER ALL PATHS LINKING
 x' AND x
FROM t' TO t

$$t \rightarrow -i\tau$$

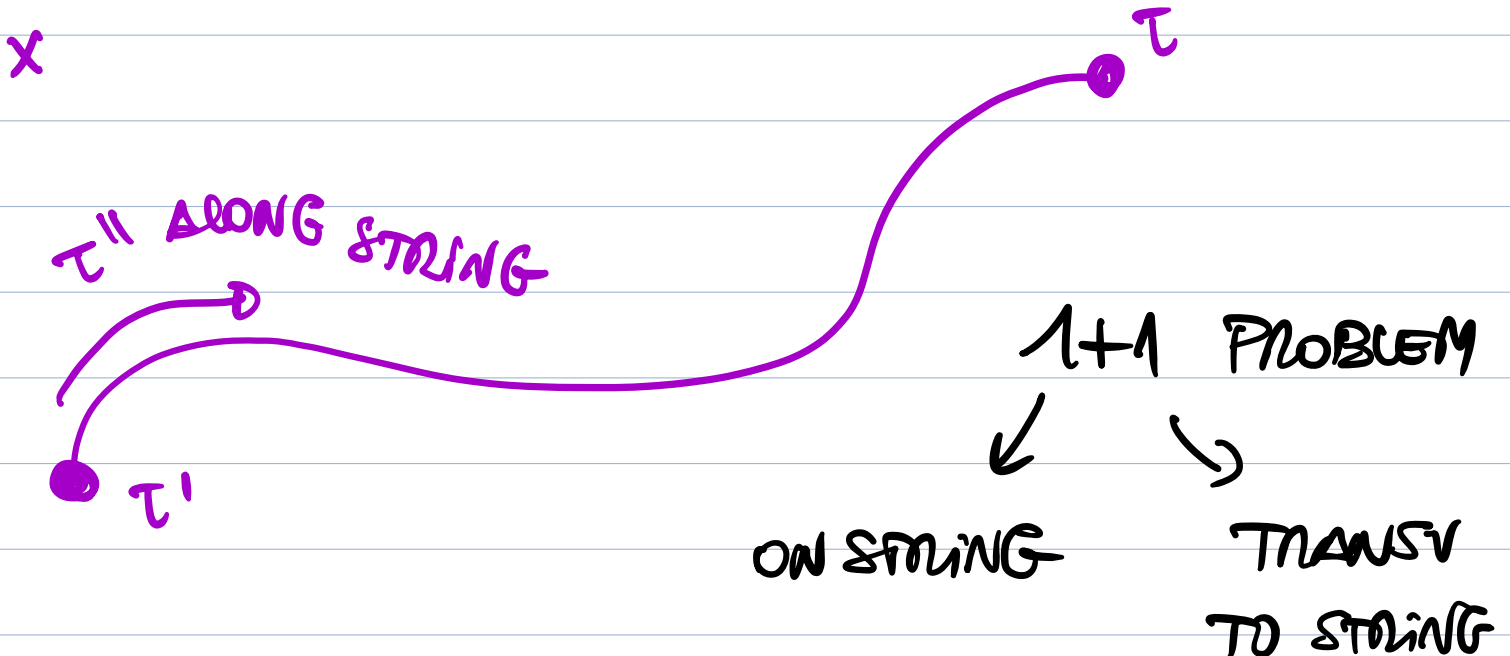
WICK ROTATION

$$U(x, -i\tau; x', -i\tau') =$$

$$\int \mathcal{D}x \, e^{-\frac{1}{\hbar} \int_{\tau'}^{\tau} d\tau'' \left[\frac{m}{2} \dot{x}^2(\tau'') + V(x(\tau'')) \right]}$$

TO MAKE β APPEAR $\Rightarrow \tau''$ FROM 0 UP TO $\beta\hbar$

IT'S THE CLASSICAL STATISTICAL
MECHANICS PART SUM OF A STRING



THE BOUNDARY CONDITIONS ARE

$$x(\tau') = x'$$

$$x(\tau) = x$$

PARTITION FUNCTION

$$Z = \text{Tr } e^{-\beta \hat{H}}$$

$$= \int dx \langle x | e^{-\beta \hat{H}} | x \rangle$$



THIS IS THE EVOLUTION OPERATOR
IN IMAGINARY TIME τ OVER AN INTERVAL
OF DURATION $\beta \hbar$

$$e^{-\frac{i}{\hbar} \hat{H} t}$$

WITH $t \rightarrow -i\tau$ ONE HAS

$$e^{-\frac{i}{\hbar} \hat{H} (-i\tau)} = e^{-\hat{H} \frac{\tau}{\hbar}}$$

NOW $\tau \rightarrow \beta \hbar \Rightarrow e^{-\beta \hat{H}}$

THE OPERATOR IN Z

SO, IT'S THE EVOLUTION OPERATOR OVER

$$\tau = \beta \hbar$$

WITH PERIODIC BOUNDARY CONDITIONS.

ZERO TEMP LIMIT & GROUND STATE E_0

$$\langle x | e^{-\beta \hat{H}} | x' \rangle$$

$$= \sum_n \langle x | e^{-\beta \hat{H}} | n \rangle \langle n | x' \rangle$$

$$= \sum_n \langle x | n \rangle e^{-\beta E_n} \langle n | x' \rangle$$

$$= \sum_n e^{-\beta E_n} \psi_n^*(x) \psi_n(x')$$

$$\langle x | e^{-\beta \hat{H}} | x \rangle = \sum_n e^{-\beta E_n} |\psi_n(x)|^2$$

$$\mathcal{Z} = \int dx \langle x | e^{-\beta \hat{H}} | x \rangle$$

$$= \sum_n e^{-\beta E_n} \int dx |\psi_n(x)|^2$$

$$= \sum_n e^{-\beta E_n}$$

THEN,

$$E_0 = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \ln \text{Tr} e^{-\beta \hat{H}}$$

$$E_0 = - \lim_{\beta \rightarrow \infty} \frac{1}{\beta} \ln Z$$

CLASSICAL LIMIT

NOTE THAT BOTH IN THE $\mathbb{R}$ -TIME AEP.
AS IN THE IMAGINARY TIME ONE, THERE'S A

$$\frac{1}{\hbar}$$

IN THE EXPONENTIAL

$\Rightarrow$ SADDLE-POINT $\hbar \rightarrow 0$

$x_{cl}(t)$ or $x_{cl}(\tau)$

DOMINATES THE SUM OVER PATHS.

$$\frac{\delta S}{\delta x(t'')} = 0$$

STATIONARY TRAJECTORY

$$\frac{\partial L}{\partial x(t'')} - \frac{d}{dt''} \frac{\partial L}{\partial \dot{x}(t'')} = 0$$

EULER-LAGRANGE EQUATION

FIXES $x_{cl}(t'')$

WITH BOUNDARY VALUES $\left\{ \begin{array}{l} x_{cl}(t') = x' \\ x_{cl}(t) = x \end{array} \right.$

STABILITY

$$\frac{\delta^2 S}{\delta x(t) \delta x(t')}$$

POS.
DEF.
IN IMAG
TIME

x_{cl}

QUANTUM CORRECTIONS

EXPAND

$$S[x] = S[x_{cl} + \delta x]$$

$$= S[x_{cl}]$$

$$+ \int dt'' \frac{\delta S}{\delta x(t'')} \bigg|_{x_{cl}} \delta x(t'')$$

$$+ \int dt'' \int dt''' \frac{\delta S}{\delta x(t'') \delta x(t''')} \bigg|_{x_{cl}} \delta x(t'') \delta x(t''')$$

+ ...

GAUSSIAN
CORRECTIONS

$$\Rightarrow \left[\det \frac{\delta S}{\delta x(t'') \delta x(t''')} \right]^{-1/2} \text{ TO CALCULATE}$$

CHECK SIGN OF POWER $-1/2$

$$\int \frac{dz}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2} \frac{z^2}{\sigma^2}} = 1 \Rightarrow$$

$$\int dz e^{-\frac{z^2}{2\sigma^2}} = \sqrt{2\pi\sigma^2}$$

FACTORS 2 AND 2π : DON'T WORRY ABOUT THEM

NOTE THAT $1/\sigma^2$ IN EXP GOES TO $(\sigma^2)^{1/2}$ IN RESULT

$\Rightarrow -1/2$ IS OK ABOVE.

QUANTUM HARMONIC etc

CANONICAL EQUIL.

$$Z = \text{Tr } e^{-\beta \hat{H}}$$

FROM PATH - INTEGRAL CALC.

WITHOUT DOING ALL CALCULATION
WOULD TAKE TOO LONG

FEYNMAN - HIBBS OR ANY OTHER
PATH - INT. BOOK

$$U(x, t | x', 0) = \sqrt{\frac{m\omega}{2\pi i \hbar \sin \omega t}}$$

$$\exp \left\{ \frac{i m \omega}{2 \hbar \sin \omega t} \left[(x^2 + x'^2) \cos \omega t - 2 x x' \right] \right\}$$

TAKE NOW $t = -i\tau$, $t' = 0$ & $x = x'$

$$U(x - i\tau | x, 0) = \sqrt{\frac{m\omega}{2\pi\hbar \sin \omega\tau}}$$

$$\exp \left\{ -\frac{m\omega}{\hbar \sin \omega\tau} [\cosh \omega\tau - 1] x^2 \right\}$$

MOREOVER $\tau = \beta\hbar$ & INT OVER x

$$Z = \frac{1}{\sqrt{2 [\cosh \beta\hbar\omega - 1]}}$$

$$= \frac{e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}}$$

INTEGRAL OF
 $U(x, -i\tau; x, 0)$
OVER x AND
 $\tau = \beta\hbar$

$$= \sum_{n=0} e^{-\beta \left(n + \frac{1}{2}\right) \hbar \omega}$$

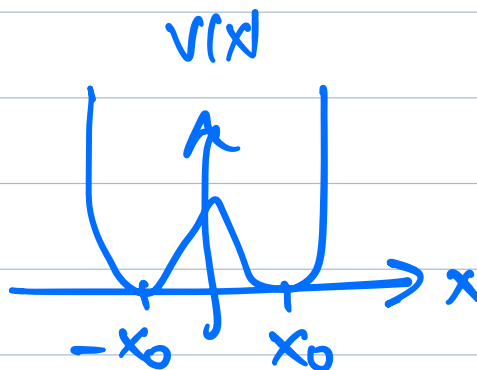
THE KNOWN RESULT

ONE OF THE MOST SPECTACULAR APPLIC OF PATH INTEGRALS

3.2.8 TUNNELING & INSTANTONS

PROBLEM

- DOUBLE WELL
- TUNNELING



QUARTIC POTENTIAL

$$r < 0$$

$$\frac{r^2}{4u} + \frac{r}{2} x^2 + \frac{u}{4} x^4$$

$$x_0 = \pm \left(-\frac{\Gamma}{u}\right)^{1/2}$$

$$V(x_0) = 0 \quad V(x_{\max}) = V(0) = \frac{\Gamma^2}{4u}$$

CLASSICAL

$$E < V(x_{\max})$$

OSCILLATIONS IN A WELL

QUANTUM

$$P(\text{TUNNELING}) > 0$$

CALCULATE THE TRANSITION AMPITUDE
WITH THE PATH INTEGRAL METHOD.
(IMAGINARY TIME)

TRANSITION AMPLITUDE IN PATH INT

$$U(x, \tau; x', \tau') = \int_{x'}^x \mathcal{D}x \ e^{-\frac{i}{\hbar} S_E}$$

SUM OVER ALL TRAJECTORIES; BUT IN THE

SEMI-CLASSICAL LIMIT $\hbar \rightarrow 0$

SADDLE POINT EVALUATION $\frac{1}{\hbar} \rightarrow \infty$

(PLAYS THE ROLE OF N IN STAT PHYS)

$$\underbrace{\sim e^{-\frac{i}{\hbar} S_{E_{cl}}}}_{U_{cl}} \underbrace{\left[\det \frac{\delta^2 S_E}{\delta x(\tau) \delta x(\tau')} \right]_{x_{cl}}^{-1/2}}_{U_q}$$

$$S_{E_{cl}} = S_E(x_{cl}(\tau))$$

↪ "CLASSICAL TRAJECT"

MINIMIZER of $S(x_{cl})$.

$$\frac{\delta S_E}{\delta x(\tau)} = 0 \quad \text{AT} \quad \underline{x_{cl}(\tau)}$$

THE INSTANTON

GOES FROM ONE WELL TO THE OTHER

2nd FACTOR U_q GAUSSIAN FUNCT.

WE NOTE THAT FIRST "CLASSICAL"
FACTOR IS EXPONENTIAL WHILE U_q
APPEARS IN NUMERATOR $\rightarrow$ LESS
IMPORTANT BUT TO BE CONSIDERED!

$$S_{cl} = \int d\tau'' \left[\frac{(\dot{x}(\tau''))^2}{2} + V(x(\tau'')) \right]$$

$$\frac{\delta S_{cl}}{\delta x(\tau)} = 0 = -\ddot{x}(\tau) + V'(x(\tau))$$

NON TRIVIAL (NON CONST) SOLUTION

INSTANTON :

SAME EQ AS DOMAIN WALL IN
GINZBURG LANDAU $\lambda\phi^4$ FIELD TH.

CLASSICAL TRAJECTORY IN IM. TIME

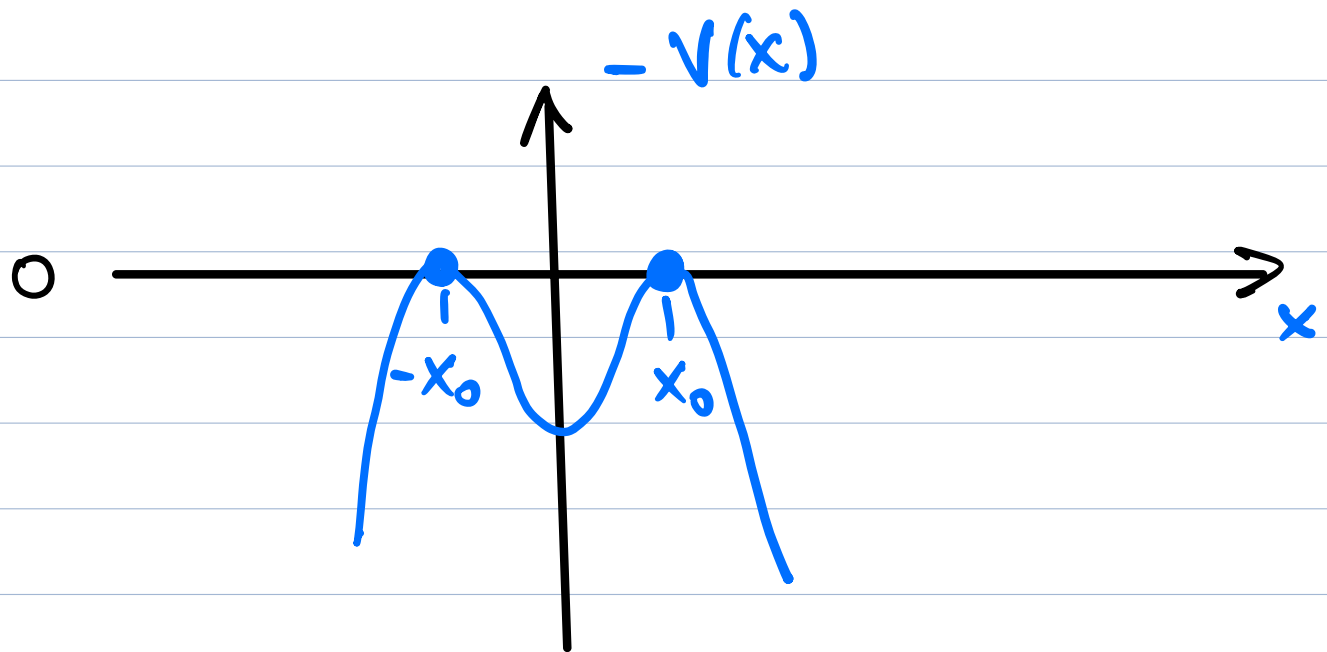
MOTION IN THE INVERTED POT $-V(x)$

REMARK : A BIT LIKE THE VORTICES IN
2d XY.

TOPOLOGICAL SOL. (FIXED BY
BOUNDARY CONDITIONS)
ARE IMPORTANT

NOTATION IN THE EQS BELOW I CALL τ
THE TIME WHICH WILL RUN FROM
 τ' TO τ . TO SIMPLIFY THE
NOTATION

THE EQ. TO SOLVE : $\frac{d^2 x(\tau)}{d\tau^2} = V'(x(\tau))$



PARTICLE ROLLING FROM $-x_0$ TO x_0
WITH $m=1$

$$x' = x(\tau') = -x_0 \quad \text{TOWARDS}$$

$$x = x(\tau) = x_0$$

$$\ddot{x} = V'(x) = \Gamma x + \mu x^3$$

$$\Gamma < 0$$

SOLUTION TO THE EQ. MOTION $\forall \tau$

$$X(\tau) = X_0 \tanh\left(\frac{\tau - \tau_0}{\tau_w}\right)$$

CHECK IT'S A SOLUTION $\&$ FIX τ_w

$$\ddot{x} = \frac{X_0}{\tau_w} \left[\frac{\cosh y \cosh y - \sinh y \sinh y}{\cosh^2 y} \right]$$

$$\frac{d \cosh y}{dy} = \frac{e^y - e^{-y}}{2} = \sinh y$$

WHERE WE
USED

$$\frac{d \sinh y}{dy} = \frac{e^y + e^{-y}}{2} = \cosh y$$

$$\Rightarrow \ddot{x} = \frac{X_0}{\tau_w} \frac{1}{\cosh^2 \frac{\tau - \tau_0}{\tau_w}} \quad \text{TAKING } \frac{d}{d\tau} \text{ AGAIN}$$

$$X'' = \frac{X_0}{\tau_w} \frac{(-2)ch \frac{\tau-\tau_0}{\tau_w} \frac{1}{\tau_w} sh \frac{\tau-\tau_0}{\tau_w}}{ch^3 \frac{\tau-\tau_0}{\tau_w}}$$

$$= -\frac{2X_0}{\tau_w^2} sh \frac{\tau-\tau_0}{\tau_w} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_w}}$$

BACK IN EQ :

$$\frac{-2X_0'}{\tau_w^2} sh \frac{\tau-\tau_0}{\tau_w} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_w}} = ?$$

$$r \cancel{X_0} sh \frac{\tau-\tau_0}{\tau_w} + u X_0^2 sh^2 \frac{\tau-\tau_0}{\tau_w}$$

$$\Rightarrow \frac{-2}{\tau_w^2} \frac{1}{ch^2 \frac{\tau-\tau_0}{\tau_w}} = ? \quad r + u X_0^2 sh^2 \frac{\tau-\tau_0}{\tau_w}$$

$$? = \frac{\Gamma ch^2 \frac{\tau - \tau_0}{\tau_w} + u x_0^2 sh^2 \frac{\tau - \tau_0}{\tau_w}}{ch^2 \frac{\tau - \tau_0}{\tau_w}}$$

if $\boxed{-\Gamma = u x_0^2}$ $\Rightarrow$

RECALL $\Gamma < 0$

WHICH IT IS, WE CHOSE THE INITIAL AND FINAL POSITION IN THIS WAY

NUM = Γ

DEN CANCELS WITH THE ONE IN LHS

$\Rightarrow$

$$\frac{2}{\tau_w^2} = -\Gamma \Rightarrow$$

$$\boxed{\tau_w^2 = \frac{-2}{\Gamma}}$$

IF WE FURTHER REQUIRE

$$X(\tau') = -x_0 = x_0 \tanh\left(\frac{\tau' - \tau_0}{\tau_w}\right)$$



WE NEED THIS FACTOR
TO BE -1

$$X(\tau) = x_0 = x_0 \tanh\left(\frac{\tau - \tau_0}{\tau_w}\right)$$



SHOULD BE 1

SO, WE NEED

$\tau' \rightarrow -\infty$
$\tau \rightarrow \infty$

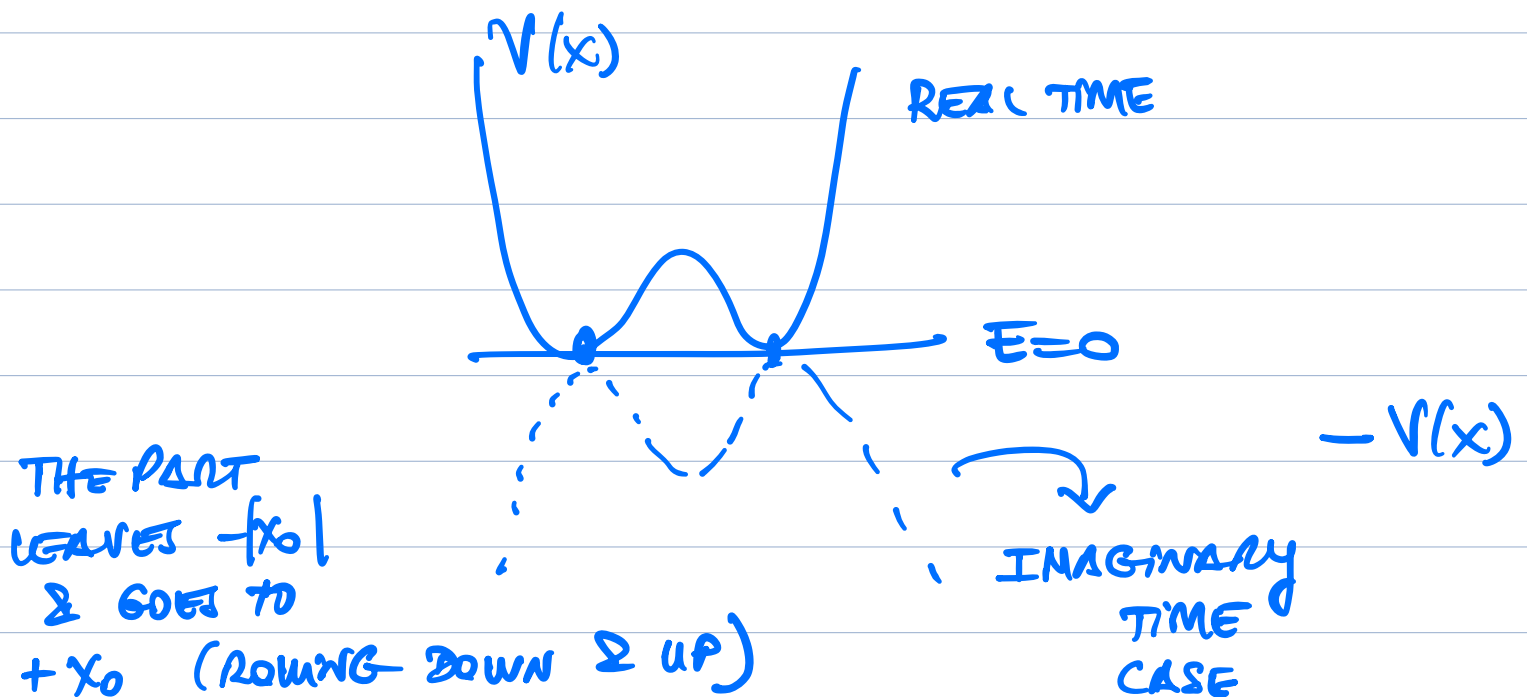
RECALL WHAT IS $x_0^2 = -\frac{\Gamma}{u}$

$$V(x) = \frac{\Gamma x^2}{2} + \frac{u}{4} x^4 + \text{CONST}$$

$$V'(x) = \Gamma x + u x^3 = 0 \Rightarrow$$

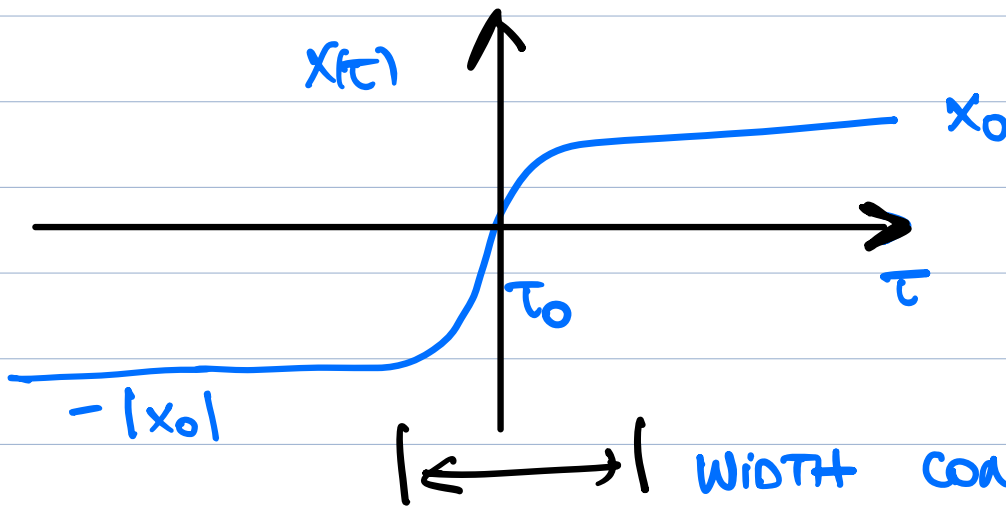
$$x=0 \text{ OR } x^2 = -\frac{\Gamma}{u}$$

$\Rightarrow$ THEY ARE THE MINIMA OF
 $V(x)$ OR THE MAXIMA OF
THE INVERTED
 $-V(x)$



THE INSTANTON CONF

NB τ_0 is FREE
SYMM UNDER TRANS OF
-INSTANTS



⇒ SHOULD INTEG
OVER τ_0 &
ALL # OF TUNNEL.

$$\tau_w^2 = -\frac{2}{r}$$

WIDTH CONTROLED BY

THE PARTICLE MOVES FROM LEFT TO RIGHT
WELL IN AN "INSTANT" OF DURATION $\sqrt{-2/r}$
(TOP OF BARRIER)

THE CLASSICAL
ACTION OF THE
INSTANTON IS :

JUST REPLACE
 x_{cl} IN $S(x_{cl})$

$$S_{cl} = \frac{4x_0^2}{3\tau_w}$$

- INDEP OF τ_0
- FINITE

IMPORTANT
PROPERTIES

CALCULATE S_{cl}

$$S_{Ecl} = \int d\tau'' \left[\frac{1}{2} (\dot{x}_{cl}(\tau)) ^2 + V(x_{cl}(\tau)) \right]$$

SHOW $\frac{1}{2} \dot{x}_{cl}^2(\tau) = V(x_{cl}(\tau))$

EQ NOTION

$$\ddot{x}_{cl}(\tau) = V'(x_{cl}(\tau))$$

MULT BY $\dot{x}_{cl}(\tau)$

$$\ddot{x}_{cl} \dot{x}_{cl} = V'(x_{cl}) \dot{x}_{cl}$$

INTEG. OVER τ

$$\frac{1}{2} \dot{x}_{cl}^2 = V(x_{cl}) + \text{CONST}$$

$\rightarrow 0$

$$S_{Ecl} = \int d\tau'' (\dot{x}_{cl}(\tau))^2$$

SET
 $x_{cl} = \pm x_0$

THE TWO TERMS ARE =

Now, For $X_Q(\tau) = x_0 \tanh\left(\frac{\tau - \tau_0}{\tau_w}\right)$

$$\dot{X}_Q(\tau) = \frac{x_0}{\tau_w} \cdot \frac{\cosh^2 \frac{\tau - \tau_0}{\tau_w} - \sinh^2 \frac{\tau - \tau_0}{\tau_w}}{\cosh^2 \left(\frac{\tau - \tau_0}{\tau_w} \right)}$$

$$= \frac{x_0}{\tau_w} \frac{1}{\cosh^2 \frac{\tau - \tau_0}{\tau_w}}$$

REPLACE:

$$S_{EQ} = \int_{\tau'}^{\tau} d\tau'' \left(\frac{x_0}{\tau_w} \right)^2 \frac{1}{\cosh^4 \frac{\tau - \tau_0}{\tau_w}}$$

$$= \left(\frac{x_0}{\tau_w} \right)^2 \int_{u' \rightarrow -\infty}^{u \rightarrow \infty} du'' \tau_w \frac{1}{\cosh^4 u''}$$

$$= \frac{x_0^2}{\tau_w} \cdot \frac{4}{3} \quad \begin{array}{l} u = \frac{\tau - \tau_0}{\tau_w} \\ du = \frac{d\tau}{\tau_w} \end{array}$$

THE INSTANTON HAS FINITE ACTION.

RECALL THAT IN A SADDLE-POINT APPROX IF WE HAVE MANY MINIMA,

WE HAVE TO SUM OVER ALL

$\Rightarrow$ INTEGRAL OVER τ_0

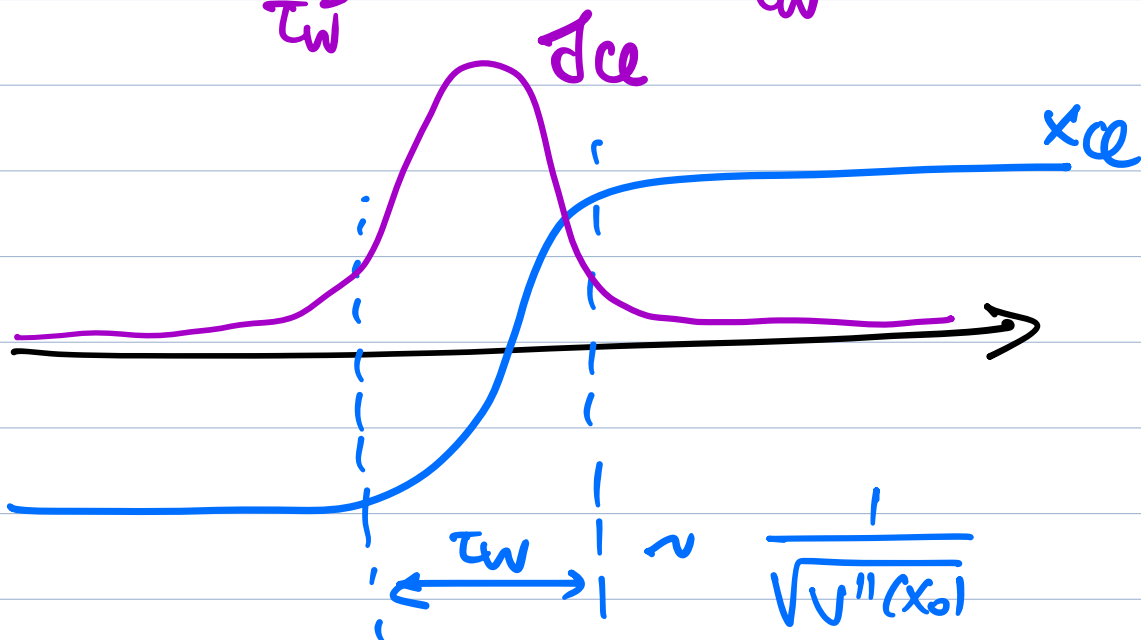
Summary

$$x_{cl}(\tau) = x_0 \tanh \frac{\tau - \tau_0}{\tau_w} \quad \begin{array}{l} \tau' \rightarrow -\infty \\ \tau \rightarrow \infty \end{array}$$

$$\tau_w^2 = -\frac{2}{\Gamma}$$

S_{cl} : DENSITY OF ACTION = $(\dot{x}_{cl})^2$

$$= \frac{x_0^2}{\tau_w^2} \operatorname{sech}^4 \frac{\tau - \tau_0}{\tau_w}$$



$$S_{cl} = \frac{4}{3} \frac{x_0^2}{\tau_w} = \frac{4}{3} x_0^2 \sqrt{-\frac{\Gamma}{2}}$$

FOR
QUANTIC
POT.

WE IDENTIFIED A TOPOLOGICAL CONF.
THE INSTANTON

FINITE S_{cl}

CANNOT BE SMOOTHLY DEFORMED
TO $X(\tau) = ct$.

LARGELY DETERMINED BY
BORDER COND $X(-\infty) = -x_0$
 $X(\infty) = x_0$

RECALL VORTICES
IN $2d \times y$

ZERO MODE TO IRRELEVANT

S_{cl} DOESN'T DEP. ON T_0

WE CONSIDERED ONE INSTANTON. WHY
NOT MANY?

They "LOCALIZED" IN THE SENSE OF
SMALL WIDTH T_W

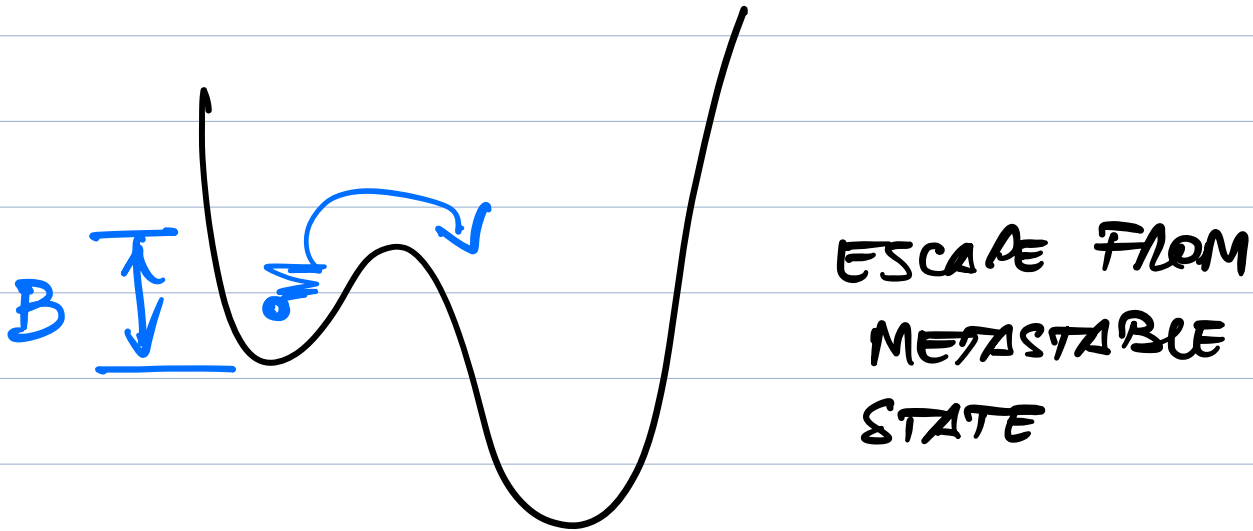
HAVE TO SUM OVER ALL SADDLE POINTS
WHICH COULD BE MADE OF MANY
INSTANTONS

eg. A GAS OF INSTANTONS

ALSO CONSIDER GAUSSIAN FLUCT AROUND
EACH. S.P.

ARRHENIUS LAW

JUMP OVER A BARRIER WITH
CLASSICAL STOCHASTIC DYN.



$$t_A \sim \tau_0 e^{\frac{B}{k_B T}}$$

B = HEIGHT BARRIER

$k_B T$ = PLAYS ROLE OF $\hbar$

SEMICLASSICAL $\approx$ LOW T

τ_0 = MICROSC ATTEMPT TIME

NB NOISE IS NEEDED TO KICK THE
PART OVER THE BARRIER

PROBLEM SOLVED BY KRAMERS W/
ARGUMENTS FROM FP-LIKE EQ.

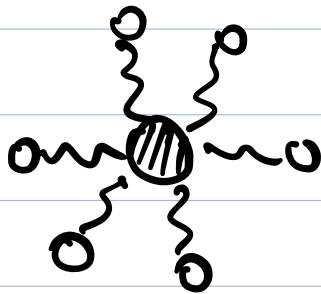
PATH INTEGRAL CALCULATION
(À LA WIENER)

CAROLI, CAROLI, ROUET (89)

3.2.9. REDUCED SYSTEM

INFLUENCE FUNCTIONAL

RECALL DERIVATION OF LANGEVIN EQ
IN 1ST TD



PARTICLE

m



POSITION x , MOMENTUM $p = mv$

COUPLED TO AN ENSEMBLE OF N
HARMONIC OSCILLATORS

$m_a, \omega_a, q_a, \pi_a$

ALL IN $d=1$ TO SIMPLIFY NOTATION

$$H = H_{\text{sys}} + H_{\text{BATH}} + H_{\text{INT}} + H_{\text{COUNTER}}$$

$\downarrow$
 PART

$\downarrow$
 OSC

$\downarrow$
 INTERACTION

- H_{COUNTER} TO AVOID PART QUAD. TERM "RENORM."
 $\propto x^2$ WITH $\propto (m_a \omega_a^2, c_a)$
- WE SET NEWTON'S EQ FOR THE COUPLED SYSTEM
- WE SOLVED FOR OSC
- WE INTRODUCED SOL IN PARTICLE'S EQ.

ALL DETERMINISTIC 'TIL HERE

IMPORTANT

WE CLAIMED OSC'S INITIAL COND
 DRAWN FROM BOLTZMANN'S WEIGHT
 AT TEMP T IN PRESENCE OF
 PART'S INITIAL STATE (COUNTER)

$\Rightarrow$ STOCHASTICITY

WE OBTAINED

GENERALIZED LANGEVIN EQS.

WITH MEMORY - RETARDED FRICTION &
COLOURED NOISE

MULTIPLICATIVE NOISE - DEPENDING ON
COUPLING.

BI-LINEAR	$\times q_a \Rightarrow$	ADDING
GENERAL	$f(x) q_a \Rightarrow$	MULTIPLICATIVE

NB WE COULD HAVE BEEN INTERESTED IN

$$Z_{\text{TOT}} = \sum_{\{\text{SYST}\}} \sum_{\{\text{BATH}\}} e^{-\beta H_{\text{TOT}}}$$

$$Z_{\text{RED}} = \sum_{\{\text{SYST}\}} e^{-\beta H_{\text{RED}}}$$

"REDUCED" PART FUNCTION

QUANTUM MECHANICS ?

WE CAN ASK THE DYN. QUESTION

REAL TIME EVOL. OF SYST COUPLED TO BATH

OR THE EQUIL QUESTION

Z_{RED}

IN THESE LECTURES, MOSTLY EQ $\rightarrow Z_{RED}$

QUANTUM PARTICLE $\hat{x}, \hat{p}$ (m)

QUANTUM OSC. $\hat{q}_a, \hat{\pi}_a$ (ma, ω_a)

$$[\hat{\pi}_a, \hat{q}_b] = -i\hbar \delta_{ab}$$

$$[\hat{x}, \hat{p}] = +i\hbar$$

$$\begin{aligned} \rho_{TOT}(x, q_a; x', q'_a) &= \\ &= \frac{1}{Z_{TOT}} \langle x, q_a | e^{-\beta \hat{H}_{TOT}} | x', q'_a \rangle \end{aligned}$$

DENSITY
OPERATOR

$$Z_{TOT} = Tr e^{-\beta \hat{H}_{TOT}}$$

USE THE PATH INTEGRAL LAGRANGIAN
REPRESENTATION OF S_{TOT}

$$S_{TOT}(x, q_a; x', q'_a) = \frac{1}{Z_{TOT}} \int_{x(0)=x'}^{x(\beta\hbar)=x} \mathcal{D}x \int_{q_a(0)=q'_a}^{q_a(\beta\hbar)=q_a} \mathcal{D}q_a e^{-\frac{1}{\hbar} S_E}$$

$$S_E = S_E^{sys} + S_E^{BATH} + S_E^{INT} + S_E^{COUNTER}$$

THE ACTION OF THE BATH IS

$$S_E^{BATH} = \int_0^{\beta\hbar} d\tau \left\{ \frac{m\dot{q}_a^2(\tau)}{2} + \frac{m\omega_a^2}{2} q_a^2(\tau) \right\}$$

THE INT. ONE

$$S_E^{INT} = \int_0^{\beta\hbar} d\tau x(\tau) \frac{1}{\hbar} \sum_{\alpha} C_{\alpha} q_{\alpha}(\tau)$$

$$N^{1/2} \quad a=1$$



COUPLING CONST SCALED W/ N SO AS
TO HAVE AN INTERESTING RESULT

SIMPLEST CHOICE, $f(x)$ ALSO POSSIBLE

ONE CAN NOW SIMPLY INTEGRATE AWAY THE OSC.
TO FIND

$$S_E^{\text{RED}} = S_E^{\text{SYST}} + S_E^{\text{BATH EFFECT}}$$

RECALL THAT $q_a(\tau)$ ARE PERIODIC
FUNCTIONS OF IMAGINARY TIME
OVER $\beta\hbar$

$$q_a(0) = q_a(\beta\hbar) \quad \forall a$$

FOURIER REPRESENTATION - MATSUBARA FREQ.

$$q_a(\tau) = \sum_{n=-\infty}^{\infty} q_a^{(n)} e^{i\omega_n \tau}$$

$$\omega_n = \frac{2\pi n}{\beta\hbar}$$

AFTER SOME STRAIGHTFORWARD BUT LENGTHY STEPS

$$S_{\text{red}}(x, x') = \text{Tr}_{\text{env.}} S_{\text{TOT}}(x, q_a; x', q'_a)$$

$$= \frac{1}{Z_{\text{RED}}} \int_{x(0)=x'}^{x(\beta\hbar)=x} \mathcal{D}x \, e^{-\frac{1}{\hbar} S_{\text{E}}^{\text{SYST}}}$$

$$e^{-\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int_0^{\beta\hbar} d\tau' \, x(\tau) K(\tau-\tau') x(\tau')}$$

$$Z_{\text{RED}} = Z_{\text{TOT}} / Z_{\text{ENV}}$$

→ PART FUNCT. OF
ISOLATED ENS. OF
OSC

$$K(\tau) = \frac{2}{\pi\beta\hbar} \sum_{n=-\infty}^{\infty} \int_0^{\infty} d\omega \, \frac{S(\omega)}{\omega} \frac{\gamma_n^2}{\gamma_n^2 + \omega^2} e^{i\gamma_n \tau}$$

SPECTRAL DENSITY

$$S(\omega) = \frac{1}{N} \sum_{a=1}^N \frac{c_a^2}{m_a \omega_a^2} \delta(\omega - \omega_a)$$

$$\frac{S(\omega)}{\omega} = 2\gamma_0 \left(\frac{|\omega|}{\tilde{\omega}} \right)^{\alpha-1} f_c \left(\frac{|\omega|}{\wedge} \right)$$

$\alpha = 1$ OHMIC

$\alpha < 1$ SUB OHMIC

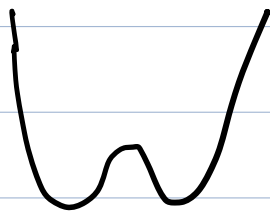
$\alpha > 1$ SUPER OHMIC

TYPICALLY $k(\tau)$ LONG-RANGE IN τ

STRONG EFFECTS ON PARTICLE'S BEHAVIOUR!

eg. LOCALIZATION

BRAY MOORE
CALDEIRA-
LEGGETT



$\alpha = 1$ γ_0

↑
SAY PART INITIALLY HERE

TUNNELING SUPPRESSED FOR SUFFICIENTLY
STRONG COUPLING TO OHMIC BATH

3.3 GENERIC PROPERTIES OF QUANTUM OBSERV. CORRELATIONS & LINEAR RESPONSES

EXP. MEASUREMENTS

LINEAR RESP.

ELECT CONDUCTIVITY
THERMAL CONDUCTIVITY
etc.

GENERIC PROPS. VALID $\forall$ SYSTEMS

VERY IMPORTANT TO KNOW THEM
& DON'T BREAK THEM

NUMERICALLY OR WITH APPROX.

FOCUS ON SYST WITH

NO EXPLICIT TIME DEP.

$$\hat{H} \neq \hat{H}(t)$$

PICNUP: USE HEISENBERG'S

OPERATOR DEPEND ON TIME

EXPECTATION VALUES

FIRST, "ONE TIME" OBSERVABLES

$$\hat{A}(t) = e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar}$$

NO EXPLICIT
t DEP. IN $\hat{A}$

EXPECTATION OF $\hat{A}$ AT TIME t

$$\langle \hat{A}(t) \rangle \equiv \text{Tr} \hat{A}(t) \hat{\rho}_0$$

$$\text{Tr} \hat{\rho}_0 = 1 \quad \text{ASSUMED HENCEFORTH}$$

$$\hat{\rho}_0 = \frac{e^{-\beta \hat{H}}}{Z} \quad \text{TAKE BOLTZMANN EQ.}$$

$$\langle \hat{A}(t) \rangle = \frac{1}{Z} \text{Tr} e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} e^{-\beta \hat{H}}$$



COMMUTE

TRACE ALLOWS US TO
MOVE IT HERE

$$\langle \hat{A}(t) \rangle = \frac{1}{Z} \text{Tr} \hat{A} e^{-\beta \hat{H}}$$

INDEP. OF TIME

IF SYST IN EQ. $\Rightarrow$ TIME INDEP. OBSERV.

HAVE EXP. VALUES WHICH ARE INDEP OF t

ANY OTHER $\rho_0(\vec{H}) \Rightarrow$ SAME t -INDEP OF $\langle \hat{A}(t) \rangle$

GREEN FUNCTIONS VS. CORRELATION FUNCTIONS



USED TO NAME
CORR FUNCTIONS OF
 $a, a^\dagger$ say or
 $\hat{\phi}, \hat{\phi}^\dagger$



MORE GENERAL
CORR ANY OBSERV.

TWO TIME OBSERV.

$$C_{AB}(t, t') \equiv \frac{\text{Tr} \mathcal{T} \hat{A}(t) \hat{A}(t') e^{-\beta \hat{H}}}{\text{Tr} e^{-\beta \hat{H}}}$$

$\mathcal{T}$ TIME ORDERED OPERATOR

$$\mathcal{T} \hat{A}(t) \hat{A}(t') = \hat{A}(t) \hat{A}(t') \theta(t-t') + \hat{A}(t') \hat{A}(t) \theta(t-t')$$

CAREFUL WITH $t=t'$ $\theta(0) = 1/2$

STATIONARITY

TAKE $t > t'$

$$C_{AB}(t, t') = \frac{1}{Z} \text{Tr} e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} e^{i\hat{H}t'/\hbar} \hat{B} e^{-i\hat{H}t'/\hbar} e^{-\beta\hat{H}}$$

AGAIN, USING CYCLIC PROPS OF Tr

$$\begin{aligned} C_{AB}(t, t') &= \\ &= \frac{1}{Z} \text{Tr} e^{i\hat{H}(t-t')/\hbar} \hat{A} e^{-i\hat{H}(t-t')/\hbar} e^{-\beta\hat{H}} \end{aligned}$$

ALL TIME DEP. IS OF THE FORM $t-t'$

$$= \bar{C}_{AB}(t-t')$$

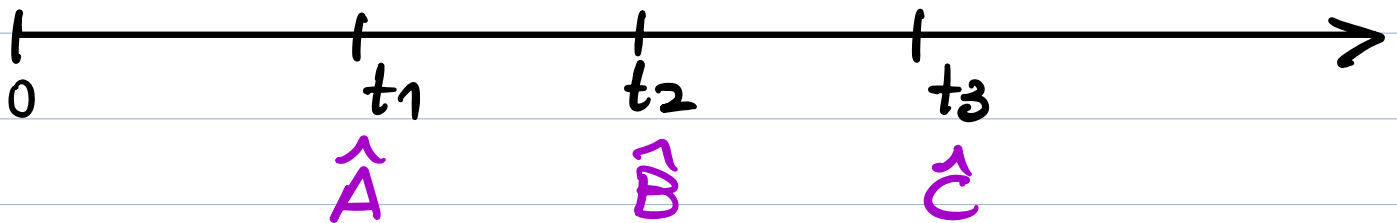
STATIONARITY OR

TIME TRANSLATION INVARIANCE

$$C_{AB}(t+st, t'+st) = C_{AB}(t, t')$$

TRANSLATE ALL TIMES BY THE SAME st

THREE (OR MORE) TIMES



$$C_{ABC}(t_1, t_2, t_3) = \text{Tr } \hat{C}(t_3) \hat{B}(t_2) \hat{A}(t_1) \hat{\rho}_0 =$$

$$= \text{Tr } e^{i\hat{H}t_3/\hbar} \hat{C} e^{-i\hat{H}t_3/\hbar} e^{i\hat{H}t_2/\hbar} \hat{B} e^{-i\hat{H}t_2/\hbar} e^{i\hat{H}t_1/\hbar} \hat{A} e^{-i\hat{H}t_1/\hbar} \hat{\rho}_0$$

USE $\rho_0 = e^{-\beta \hat{H}} / Z$

$$= \text{Tr} \ e^{i\hat{H}t_3/\hbar} \hat{C} e^{-i\hat{H}(t_3-t_2)/\hbar} \hat{B}$$

$$e^{-i\hat{H}(t_2-t_1)/\hbar} \hat{A} e^{-i\hat{H}t_1/\hbar} e^{-\beta \hat{H}}$$

$$= \text{Tr} \ \hat{C} e^{-i\hat{H}(t_3-t_2)/\hbar} \hat{B} e^{-i\hat{H}(t_2-t_1)/\hbar}$$

$$\hat{A} e^{-i\hat{H}(t_1-t_3)/\hbar} e^{-\beta \hat{H}}$$

$$= \overline{C}_{ABC}(t_3-t_2, t_2-t_1, t_1-t_3)$$

$$= \overline{C}_{ABC}(\underbrace{t_3-t_2}_{\tau_{32}}, \underbrace{t_2-t_1}_{\tau_{21}}, t_1-t_2+t_2-t_3)$$

$$\tau_{32} \quad \tau_{21} \quad -\tau_{32}-\tau_{12}$$

$$= \bar{C}_{ABC}(t_3 - t_2, t_2 - t_1)$$

TIME-TRANS INV.

$$t_3 \rightarrow t_3 + \delta t$$

$$t_2 \rightarrow t_2 + \delta t$$

$$t_1 \rightarrow t_1 + \delta t$$

NO CHANGE
UNDER THESE
TRANSLATIONS

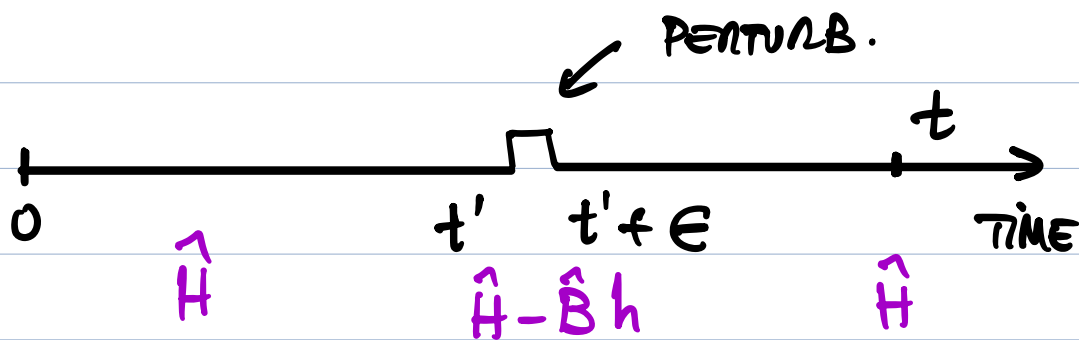
KUBO FORMULA

LINEAR RESPONSE THEORY

WRITING THE LINEAR RESPONSE AS A
CORRELATION FUNCTION

LINEAR RESPONSE

VARIATION OF AN EXPECTATION OF AN OBSERVABLE DUE TO A PERTURBATION



$$\hat{U}_h(t, 0) = e^{-i\hat{H}(t-t'-\epsilon)/\hbar} e^{-i(\hat{H}-h\hat{B})(t'+\epsilon-t')/\hbar} e^{-i\hat{H}t'/\hbar}$$

$$\hat{U}_h^\dagger(t, 0) = e^{i\hat{H}t'/\hbar} e^{i(\hat{H}-h\hat{B})\epsilon/\hbar} e^{i\hat{H}(t-t'-\epsilon)/\hbar}$$

$$\hat{H} = \hat{H}^\dagger \quad \hat{B} = \hat{B}^\dagger$$

NO PROBLEM WITH NON-COMMUTATIVITY OF $\hat{H}$ AND $\hat{B}$ BELOW SINCE BOTH $h \rightarrow 0$
 $\epsilon \rightarrow 0$

$$\left. \frac{1}{\epsilon} \frac{\delta \hat{U}_h(t_0)}{\delta h} \right|_{h=0} = e^{-i\hat{H}(t-t')/\hbar} \frac{i}{\hbar} \hat{B} e^{-i\hat{H}t'/\hbar}$$

$$\left. \frac{1}{\epsilon} \frac{\delta \hat{U}_h^\dagger(t_0)}{\delta h} \right|_{h=0} = -e^{i\hat{H}t'/\hbar} \frac{i}{\hbar} \hat{B} e^{i\hat{H}(t-t')/\hbar}$$

INSTANTANEOUS LINEAR RESPONSE

$$R_{AB}(t, t') = \left. \frac{\delta \langle \hat{A}(t) \rangle}{\delta h} \right|_{h=0} \quad t > t'$$

$$= \frac{\delta}{\delta h} \text{Tr} \left(\hat{U}_h^\dagger(t_0) \hat{A} \hat{U}_h(t_0) \hat{\rho}_0 \right) \Big|_{h=0}$$

$$= -\frac{i}{\hbar} \langle \hat{B}(t') \hat{A}(t) \rangle + \frac{i}{\hbar} \langle \hat{A}(t) \hat{B}(t') \rangle$$

$$= \frac{i}{\hbar} \langle [\hat{A}(t), \hat{B}(t')] \rangle$$

$$R_{AB}(t|t') = \frac{i}{\hbar} \langle [\hat{A}(t), \hat{B}(t')] \rangle \theta(t-t')$$

KUBO FORMULA

NOTE THAT WE HAVE NOT USED $\hat{\rho}_0 = \frac{e^{-\beta \hat{H}}}{Z}$
 $\hat{\rho}_0$ WAS KEPT GENERIC HERE

VALID EVEN OUT OF EQ.

JUST LINEAR RESPONSE USED

$$\underbrace{h \rightarrow 0 \quad \Sigma \rightarrow 0}$$

$$h\epsilon \rightarrow 0$$

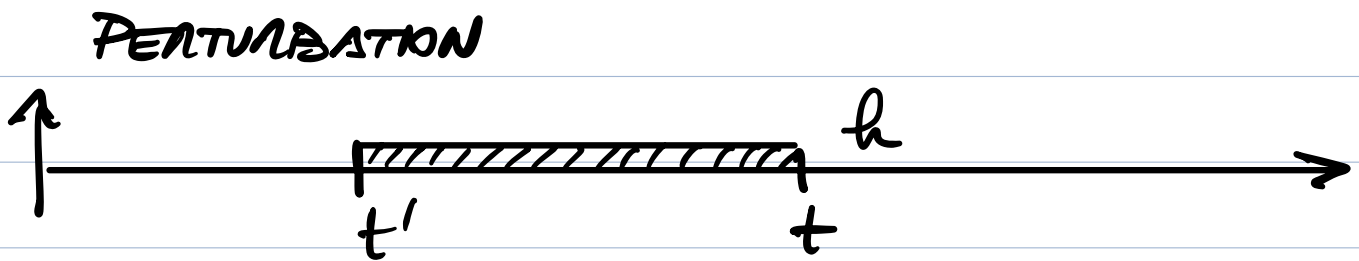
CAUSALITY

THERE CANNOT BE A RESPONSE
BEFORE THE PERTURBATION

INTEGRATED LINEAR RESPONSE

$$X(t, t') = \int_{t'}^t dt'' R(t, t'')$$

SUM OF INFINITESIMAL RESPONSES
OVER A FINITE TIME INTERVAL



KMS (PROP OF CYCLIC TRACE & EQUIL.)

$$C_{AB}(t, t') = \langle A(t) B(t') \rangle =$$

$$\frac{1}{\text{Tr } e^{-\beta \hat{H}}} \text{Tr} \left[e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} e^{i\hat{H}t'/\hbar} \hat{B} e^{-i\hat{H}t'/\hbar} e^{-\beta \hat{H}} \right]$$

WE INTRODUCE IDENTITIES $e^{-\beta \hat{H}} e^{\beta \hat{H}}$
AND USE CYCLIC PROPS
OF TRACE

• MOVE $\hat{B}(t') e^{-\beta \hat{H}}$

$$C_{AB}(t, t') = \frac{1}{\text{Tr } e^{-\beta \hat{H}}} \hat{B}(t')$$

$$\text{Tr} \left[e^{i\hat{H}t'/\hbar} \hat{B} e^{-i\hat{H}t'/\hbar} e^{-\beta \hat{H}} e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} e^{\beta \hat{H}} e^{-\beta \hat{H}} \right]$$

$$= \frac{1}{\text{Tr} e^{-\beta \hat{H}}} \text{Tr} \hat{B}(t') \hat{A}(t + i\beta \hbar) e^{-\beta \hat{H}}$$

using $\frac{it}{\hbar} - \beta = \frac{i}{\hbar} (t + i\beta \hbar)$

$$-\frac{it}{\hbar} + \beta = -\frac{i}{\hbar} (t + i\beta \hbar)$$

$$C_{AB}(t, t') = C_{BA}(t', t + i\beta \hbar)$$

FOURIER TRANSFORM

FIRST, STATIONARY

$$C_{AB}(t, t') \rightarrow C_{AB}(t - t')$$

$$C_{BA}(t', t + i\beta \hbar) \rightarrow C_{BA}(t' - t - i\beta \hbar)$$

$$\tilde{C}_{AB}(\omega) = \int_{-\infty}^{\infty} dt (t-t') e^{i\omega(t-t')} C_{AB}(t-t')$$

KMS

$$= \int_{-\infty}^{\infty} dt (t-t') e^{i\omega(t-t')} C_{BA}(t'-t-i\beta\hbar)$$

$$= \int dt (t-t'+i\beta\hbar) e^{i\omega(t-t'+i\beta\hbar)}$$

INSERT IDENTITY

TIME "TRANSL."

$$e^{-i\beta\hbar\omega} C_{BA}(t'-t-i\beta\hbar)$$

CAU

$$u = t-t'+i\beta\hbar$$

$$e^{i\beta\hbar\omega} = e^{-\beta\hbar\omega}$$

$$= e^{\beta\hbar\omega} \underbrace{\int du e^{i\omega u} C_{BA}(-u)}_{\tilde{C}_{BA}(-\omega)}$$

$u \rightarrow -u$

$$\tilde{C}_{AB}(\omega) = e^{\beta\hbar\omega} \tilde{C}_{BA}(-\omega)$$

KMS
in
FOURIER

FDT

$$\text{DEF } C_{[A,B]}(t,t') \equiv \frac{1}{2} \langle [\hat{A}(t), \hat{B}(t')] \rangle$$

$$C_{\{A,B\}}(t,t') \equiv \frac{1}{2} \langle \{ \hat{A}(t), \hat{B}(t') \} \rangle$$

FOURIER TRANSF, if STAT

$$2 \tilde{C}_{[A,B]}(\omega) = \mathcal{F} \langle \hat{A}(t) \hat{B}(t') \rangle - \mathcal{F} \langle \hat{B}(t') \hat{A}(t) \rangle$$

$\mathcal{F}$ • FOURIER TRANSFORM

$$\mathcal{F} \langle \hat{A}(t) \hat{B}(t') \rangle = \mathcal{F} C_{AB}(t-t') = \tilde{C}_{AB}(\omega)$$

$$\mathcal{F} \langle \hat{B}(t') \hat{A}(t) \rangle = \mathcal{F} C_{BA}(t'-t)$$

$$\mathcal{F} C_{BA}(t'-t) = \int \underbrace{d(t-t')}_{\tau} e^{i \underbrace{\omega(t-t')}_{(-\tau)}} C_{BA}(\underbrace{t'-t}_{\tau})$$

$$= \tilde{C}_{BA}(-\omega)$$

$$2\tilde{C}_{[A,B]}(\omega) = \tilde{C}_{AB}(\omega) - \tilde{C}_{BA}(-\omega)$$

$$= \tilde{C}_{AB}(\omega) - e^{-\beta\hbar\omega} \tilde{C}_{AB}(\omega)$$

KMS

$$= (1 - e^{-\beta\hbar\omega}) \tilde{C}_{AB}(\omega)$$

$$2\tilde{C}_{\{A,B\}}(\omega) = (1 + e^{-\beta\hbar\omega}) \tilde{C}_{AB}(\omega)$$

TAKE THE RATIO BETWEEN THE TWO

$$\tilde{C}_{[A,B]}(\omega) = \frac{\hbar\beta\hbar\omega}{2} \tilde{C}_{\{A,B\}}(\omega)$$

THIS IS FDT IN FOURIER

USE KUBO NOW TO WRITE IT IN
TERMS OF THE LINEAR RESPONSE

BUT CAREFULLY, BECAUSE OF θ !

WE PAUSE TO DISCUSS LINEAR RESPONSE
IN FOURIER DOMAIN

$$R_{AB}(t-t') = \int \frac{d\omega}{2\pi} e^{-i\omega(t-t')} \tilde{R}_{AB}(\omega)$$

$\hookrightarrow \propto \theta(t-t')$ CAUSALITY

BECAUSE OF CAUSALITY $\Rightarrow z = z_R + i z_I$

$\tilde{R}_{AB}(z)$ ANALYTIC FOR $z_I > 0$

IN UPPER HALF COMPLEX PLANE

(NO POLES)

SKETCH OF A PROOF

• IMAGINE $t-t' < 0$

• EXTEND $\tilde{R}_{AB}(w) \rightarrow \tilde{R}_{AB}(z)$

• $\int_C dz e^{-iz(t-t')} \tilde{R}_{AB}(z)$

↳ CLOSE ON UPPER HALF PLANE

$$z = z_R + i z_I, \quad z_I > 0$$

$$e^{-iz(t-t')} = e^{-i z_R(t-t') + \underbrace{z_I(t-t')}_{\downarrow 0} \underbrace{}_{\uparrow 0}}$$

$$\xrightarrow{z_I \rightarrow \infty} 0$$

if NO POLES of $\tilde{R}_{AB}(z) \Rightarrow$ INT. = 0

$\Rightarrow R_{AB}(t, t') = 0$ if $t-t' < 0$

ANALYTICITY of $\tilde{R}_{AB}(z) \Rightarrow R_{AB}(t, t') = 0$
FOR $z_I > 0$ FOR $t-t' < 0$

KRAMERS - KRÖNIG RELATIONS

$$\operatorname{Re} \tilde{\chi}_{AB}(\omega) = -\mathcal{P} \int \frac{d\omega'}{\pi} \frac{\operatorname{Im} \tilde{\chi}_{AB}(\omega')}{\omega - \omega'}$$

$$\operatorname{Im} \tilde{\chi}_{AB}(\omega) = \mathcal{P} \int \frac{d\omega'}{\pi} \frac{\operatorname{Re} \tilde{\chi}_{AB}(\omega')}{\omega - \omega'}$$

PROOF IN LECTURES NOTES 3.3.4

BACK TO FDT

$$\tilde{C}_{[A,B]}(\omega) = \frac{\hbar \beta \hbar \omega}{2} \tilde{C}_{\{A,B\}}(\omega)$$

HOW SHOULD ONE RELATE

$$\tilde{C}_{[A,B]}(\omega) \text{ TO } \tilde{R}_{AB}(\omega) ?$$

$$R_{AB}(t, t') \stackrel{\text{KUBO}}{=} \frac{i}{\hbar} \theta(t-t') \langle [A(t), B(t')] \rangle$$

$$\stackrel{\text{DEF.}}{=} \frac{i}{\hbar} \theta(t-t') \underbrace{2 C_{[A,B]}(t, t')}$$

WRITE IN FOURIER

$$\stackrel{\text{FOURIER}}{=} \frac{i}{\hbar} \theta(t-t') \underbrace{2 \int \frac{d\omega'}{2\pi} e^{-i\omega'(t-t')} \tilde{C}_{[A,B]}(\omega')}$$

$$\stackrel{\text{KMS}}{=} \frac{i}{\hbar} \theta(t-t') \underbrace{2 \int \frac{d\omega'}{2\pi} e^{-i\omega'(t-t')} \frac{\hbar \beta \hbar \omega'}{2} \tilde{C}_{\{A,B\}}(\omega')}$$

APPLY IDENTITY

$$\int_0^{\infty} dt e^{i\omega t} = \pi \delta(\omega) + i \frac{P}{\omega}$$

BECAUSE
0 INSTEAD OF $-\infty$

TO FOURIER TRANSFORM $R_{AB}(t-t')$:

$$\int_0^{\infty} d(t-t') e^{i\omega(t-t')} R_{AB}(t-t') = \tilde{R}_{AB}(\omega) \in \mathbb{C}$$

$$= \int_0^{\infty} d(t-t') e^{i\omega(t-t')} \frac{2i}{\hbar} \times$$

THE θ ALREADY TAKEN INTO ACCOUNT

$$\int \frac{d\omega'}{2\pi} e^{-i\omega'(t-t')} \frac{\hbar \beta \hbar \omega'}{2} \tilde{C}_{\{A,B\}}(\omega')$$

$$= \frac{2i}{\hbar} \left\{ \frac{1}{2\pi} \int d\omega' \delta(\omega - \omega') \frac{\hbar \beta \hbar \omega'}{2} \tilde{C}_{\{A,B\}}(\omega') + \frac{1}{2\pi} i P \int d\omega' \frac{1}{\omega - \omega'} \frac{\hbar \beta \hbar \omega'}{2} \tilde{C}_{\{A,B\}}(\omega') \right\}$$

$\tilde{C}_{\{A,B\}}(\omega') \in \mathbb{R}$

$\Rightarrow$ TWO RELATIONS

CONDITION FOR

$$\tilde{C}_{\{A,B\}}(\omega) \in \mathbb{R}$$

$$\begin{aligned}\tilde{C}_{\{A,B\}}(\omega) &= \mathcal{F} \underbrace{C_{\{A,B\}}(t,0)}_{\text{STATIONARY}} \\ &= \int_{-\infty}^{\infty} dt e^{i\omega t} C_{\{A,B\}}(t,0)\end{aligned}$$

$$\begin{aligned}\tilde{C}_{\{A,B\}}^*(\omega) &= \int_{-\infty}^{\infty} dt e^{-i\omega t} \underbrace{C_{\{A,B\}}^*(t,0)}_{\in \mathbb{R}} \\ &= \int_{\infty}^{-\infty} (-dt) e^{i\omega t} C_{\{A,B\}}(-t,0) \\ &= \int_{-\infty}^{\infty} dt e^{i\omega t} C_{\{A,B\}}(-t,0)\end{aligned}$$

$$\text{IF } C_{\{A,B\}}(t,0) = f(t-0) = f(t)$$

$\Rightarrow$

$$C_{\{A,B\}}(-t-0) = f(-t)$$

$$\tilde{C}_{\{A,B\}}(\omega) = \tilde{C}_{\{A,B\}}^*(\omega)$$

$$\text{Im } \tilde{R}_{AB}(\omega) = \frac{1}{\hbar} \tanh \frac{\beta \hbar \omega}{2} \tilde{C}_{\{A,B\}}(\omega)$$

$$\text{Re } \tilde{R}_{AB}(\omega) = -\frac{2}{\hbar} \mathcal{P} \int \frac{d\omega'}{2\pi} \frac{1}{\omega - \omega'} \tanh \frac{\beta \hbar \omega'}{2} \tilde{C}_{\{A,B\}}(\omega')$$

BECAUSE $\tilde{R}_{AB}(\omega)$ HAS REAL & IM. PARTS, AND $\tilde{C}_{\{A,B\}}(\omega)$ WE TAKE IT REAL

FDT

LOOK AT $\text{Im } \tilde{R}_{AB}(\omega)$ IN THE $\hbar \rightarrow 0$ LIMIT

$$\lim_{\hbar \rightarrow 0} \text{Im } \tilde{R}_{AB}(\omega) = \frac{1}{\hbar} \frac{\beta \hbar \omega}{2} \tilde{C}_{AB}(\omega)$$

AND THIS IS FINE.

BACK IN TIME DOMAIN

$$R_{AB}(t-t') = -\beta \frac{d}{d(t-t')} C_{AB}(t-t') \Theta(t-t')$$

MODEL INDEP. RELATION VALID IN LINEAR RESPONSE & EQUILIBRIUM

GREEN FUNCTIONS

WE DEFINED

$$R_{AB}(t, t') = \frac{\delta \langle \hat{A}(t) \rangle}{\delta h_B(t')} \Big|_{h=0}$$

$$\hat{H} \rightarrow \hat{H} - h_B \hat{B}$$

$$\hat{B} = \hat{B}^\dagger \quad \hat{H} = \hat{H}^\dagger$$

IF WE WANT TO WORK WITH FERMIONIC OPERATIONS

$$G(t, t') = \langle T \hat{c}^\dagger(t) \hat{c}(t') \rangle$$

etc.  TIME ORDER

WE CAN DEFINE 4 KINDS OF GREEN FUNCTIONS COMBINING

$\hat{C}$ AND C IN DIFFERENT WAYS

AND RELATE THEM

$\Rightarrow$ EXTENSIONS OF FDTs TO FERM.
GREEN FUNCTIONS

BUT NOT REALLY LINEAR RESPONSES.
IN THE WAY INTRODUCED ABOVE

SEE E.G. KAMENEV'S

BOOKS.

MAHAN'S

3.4 QUANTUM PHASE TRANSITIONS

GENERIC

- STRICTLY AT $T=0$
- DRIVEN BY QUANTUM FLUCTUATIONS
- TUNE SOME PARAM IN HAMILTONIAN TO GO THROUGH A PHASE TRANS.

$$\text{eg } \hat{H} = -J \sum_{\langle ij \rangle} \hat{\sigma}_i^z \hat{\sigma}_j^z + \Gamma \sum_i \hat{\sigma}_i^x$$

$g = \Gamma/J$ ADIM CONTROL PARAM.

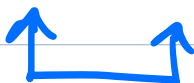
- ZERO TEMP $\Rightarrow$ GROUND STATE ENERGY PROPERTIES

NON-ANALYTICITY of $e_{\text{GS}}(g)$

- COULD BE 2nd ORDER OR 1st ORDER OR BKT.

FOCUS ON 2nd ORDER

- NB IN QUANTUM THERE IS REAL TIME

$$\hat{H} = \hat{K} + \hat{V}$$


DON'T COMMUTE

RECALL WHAT HAPPENS WITH TIME IN A CLASSICAL PHASE TRANS (THOUGH WE HAVEN'T

DISCUSSED THIS WHEN

CRITICAL SLOWING DOWN

DEALT W/ CLASS. PHASE TRANS.)

$$t_c \sim \xi^z \quad \xi \rightarrow \infty \Rightarrow t_c \rightarrow \infty$$

$$C(t, t') = \langle \phi(t) \phi(t') \rangle$$

$$\longrightarrow |t-t'|^{-\text{POWER}} e^{-\frac{|t-t'|}{t_c}}$$

z DYNAMIC CRITICAL EXPONENT

DIVERGING $t_c \Rightarrow$ VANISHING FREQ

$$t_c \rightarrow \infty \quad \omega_c \sim 1/t_c \sim \xi^{-z}$$

SUGGESTS THAT $\hbar\omega_c$ AN ENERGY GAP $\rightarrow 0$

- AT 2nd ORDER TRANSITIONS

AT CRITICAL POINT GAP

$$\Delta = E_{1st\ exc} - E_{G.S} \longrightarrow 0$$

$$\Delta \sim J |g - g_c|^{2\nu}$$

✓ CRITICAL EXPONENT OF SPATIAL
CORRELATION LENGTH

SPACE-TIME $d+1$ DIM. REPRESENTATION

d - QUANTUM $\Leftrightarrow$ $d+1$ CLASSICAL

↓

IMAG. TIME

$\hat{A}(\vec{r})$

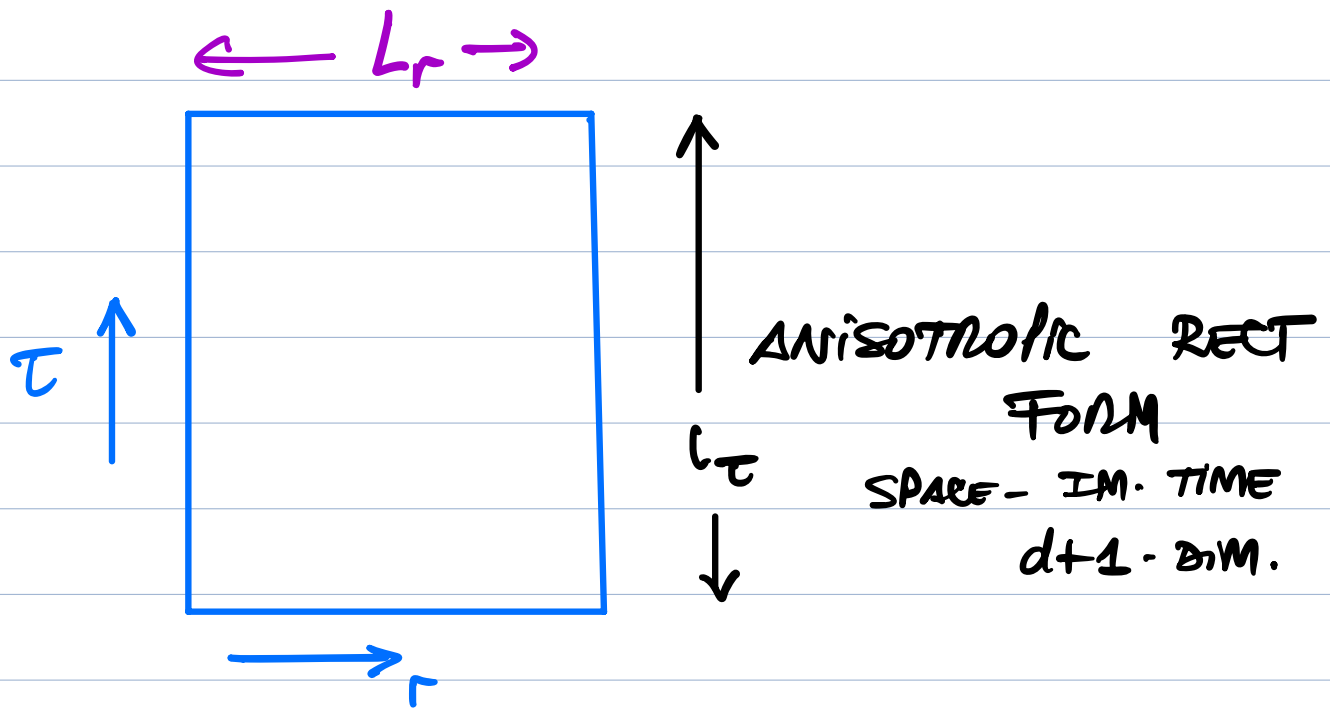
QUANTUM OBSERVABLE

$\Rightarrow$

$A(\tau, r)$

CLASSICAL

MAPPING



AT CRITICALITY

$$C(r, \tau) \sim r^{-\text{POWER}} e^{-r/\xi_r}$$

↑
FIXED

$$\xi_r \sim |g - g_c|^{-\nu}$$

z

DYNAMIC CRIT EXPONENT

REMEMBER EXTRA DIM IN

CLASS - QUANTUM IS AN IMAGINARY
TIME SO CORREL ALONG THIS

DIRECT. ARE TIME - CORREL

$$C(r, \tau) \sim \tau^{-\text{POWER}} e^{-\tau/\xi_\tau}$$

↑
FIXED

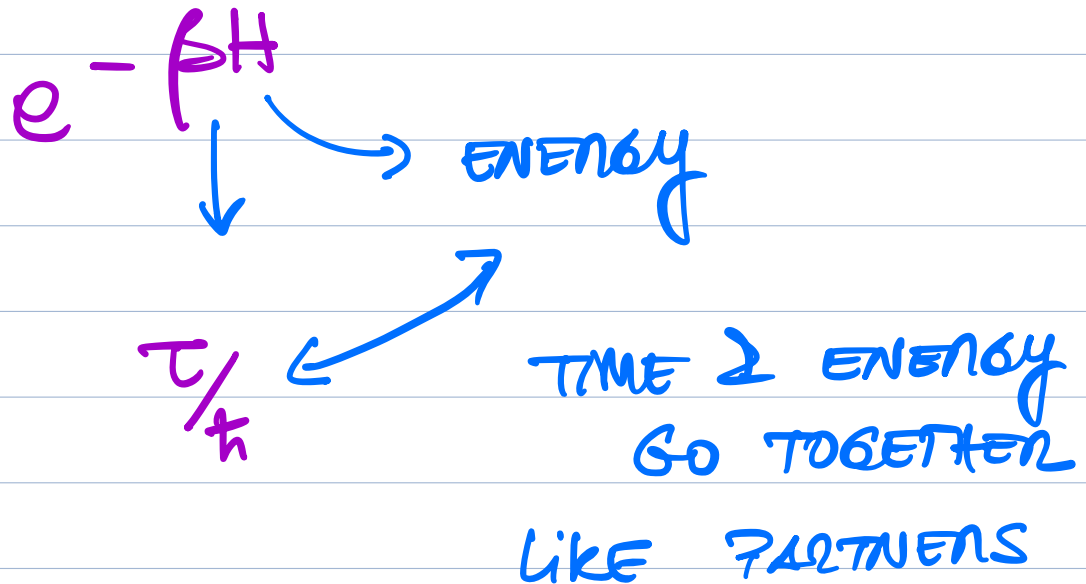
↑
IT'S
A TIME

$$\xi_\tau \sim |g - g_c|^{-\nu z}$$

NOT NECESSARILY THE SAME
POWER

$$\xi_r \sim \xi_\tau^{1/z}$$

A SIMPLE WAY TO SEE THAT Δ
IS CONTROLLED BY $\sum \tau$ DIVERGENCE

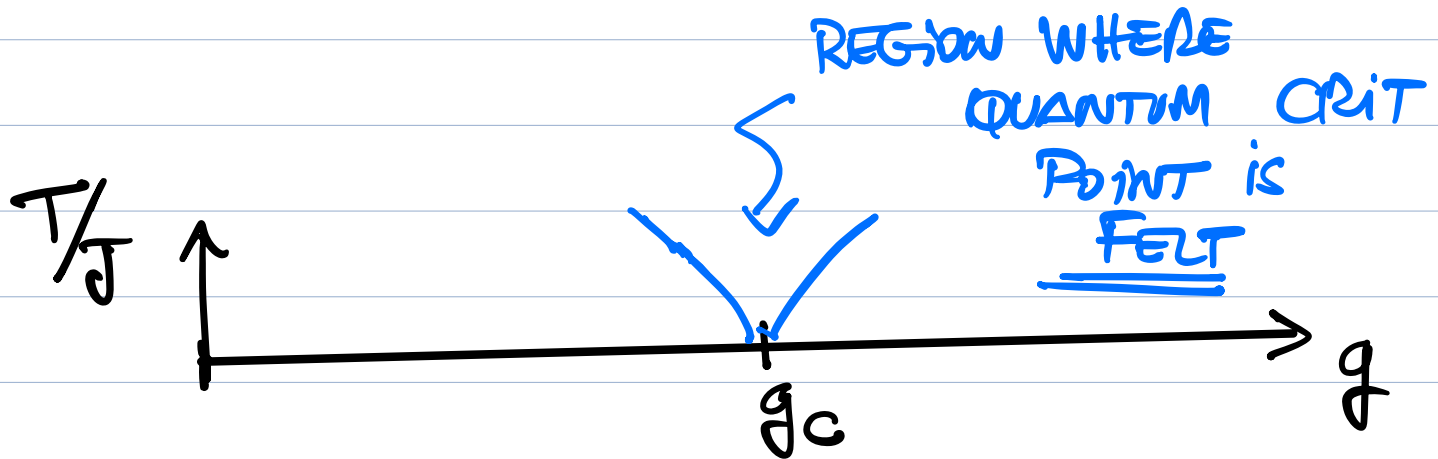


$\Rightarrow$

$$\Delta \sim |g - g_c|^{\gamma z}$$

ENERGY GAP CLOSES AT g_c

LOW TEMP. EFFECTS



HAND WAVING ARGUMENT

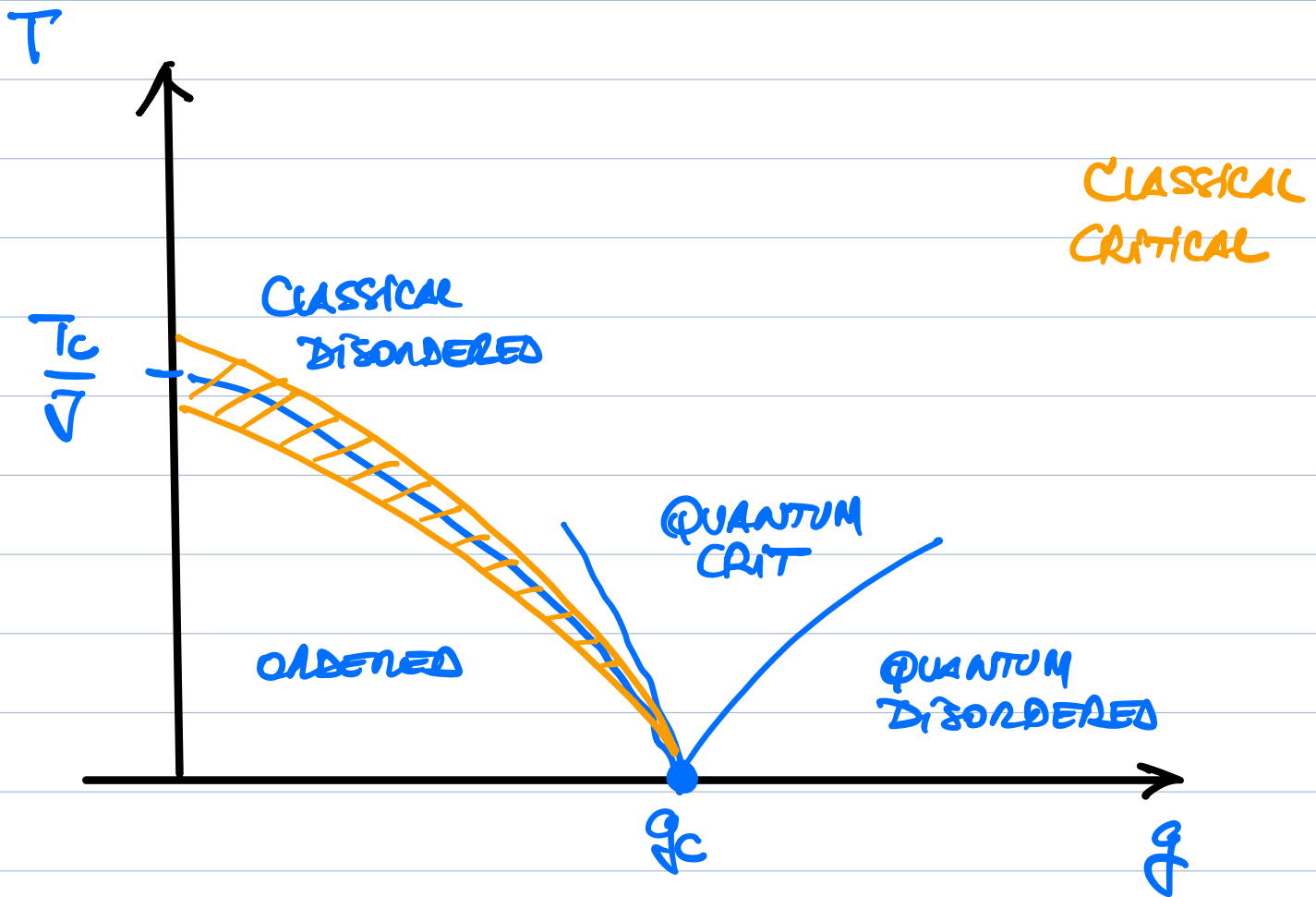
$$\hbar \omega_c \sim k_B T$$

QUANTUM
TYP
ENERGY
SCALE

THERMAL
ENERGY

if lhs > rhs $\Rightarrow$ QUANTUM

lhs < rhs $\Rightarrow$ CLASSICAL



• CLASSICAL - QUANTUM

IMAGINALLY TIME LENGTH

$\beta \rightarrow \infty$ EXTRA-DIMENSION DIVERGES $L_T \rightarrow \infty$
 $\frac{1}{T} \rightarrow 0$

$0 < \beta < \infty$ EXTRA-DIM. IS FINITE $L_T < \infty$
 $\frac{1}{T} > 0$

$L \rightarrow \infty$ REAL SPACE LENGTH

RELATION BTW GREEN'S FUNCTIONS FERMIONS

$$G^R(t, t') = -i \theta(t - t') \langle \{ \hat{c}(t), \hat{c}^\dagger(t') \} \rangle$$

$$G^A(t, t') = i \theta(t' - t) \langle \{ \hat{c}(t), \hat{c}^\dagger(t') \} \rangle$$

$$G^K(t, t') = -i \langle [\hat{c}(t), \hat{c}^\dagger(t')] \rangle$$

$$\hat{H} - \mu \hat{N} \quad \text{IN AVERAGE}$$

$$G^K(\omega) = \tanh\left(\frac{\beta \hbar \omega}{2}\right) (G^R(\omega) - G^A(\omega))$$

SEE eg KAMENEV'S OR MAHAN'S BOOKS

$$NB \quad \tanh \leftrightarrow \coth$$

QUANTUM ISING CHAIN

RECALL PAULI MATRICES

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

DON'T NEED σ_y

$$\hat{\sigma}_i = \underbrace{\mathbb{I} \otimes \dots \otimes \mathbb{I}}_{\text{I}} \hat{\sigma} \underbrace{\otimes \dots \otimes \mathbb{I}}_{\text{I}}$$

↑
SITE i

$$[\hat{\sigma}_i^a, \hat{\sigma}_j^b] = 0 \quad \forall \quad i \neq j, a, b.$$

$J > 0$ ENERGY SCALE, g CONTROL PARAM.

THE MODEL

PBC OR FBC

$$H = -J \sum_{i=1}^{L-1} \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - Jg \sum_i \hat{\sigma}_i^x$$

↑
DON'T COMMUTE

• $g=0 \Rightarrow$ CLASSICAL LIMIT

• $g \rightarrow \infty \Rightarrow$ JUST TRANSV. FIELD
NO SPIN INTERACT

TWO LIMITS WILL BE FURTHER
DISCUSSED LATER

RECALL ACTION OF SPIN OPERATORS

$$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma^z \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \sigma^z \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\sigma^x \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \underline{\text{FIPS}}$$

SYMMETRY

like U-DOWN IN
CLASSICAL ISING MODEL

$$P = \prod_j \sigma_j^x$$

(MATRICIAL NOTATION. NO HATS)

PROPERTIES OF P

- HERMITIAN $P = P^\dagger$

$$P^\dagger = \left(\prod_j \sigma_j^x \right)^\dagger = \prod_j (\sigma_j^x)^\dagger = \prod_j \sigma_j^x = P$$

- IDEM - POTENT $P^2 = \mathbb{I}$

$$P^2 = \left(\prod_j \sigma_j^x \right) \left(\prod_k \sigma_k^x \right)$$

$$\begin{matrix} j=1, \dots, L \\ k=1, \dots, L \end{matrix}$$

EXAMPLE $L=3$ $j, k = 1, 2, 3$

$$P^2 = (\sigma_1^x \sigma_2^x \sigma_3^x) (\sigma_1^x \sigma_2^x \sigma_3^x)$$

$$= (\sigma_1^x)^2 (\sigma_2^x)^2 (\sigma_3^x)^2 = \mathbb{I} \quad \begin{array}{l} \text{SINCE} \\ \sigma_i^x \sigma_j^x \\ \text{COMMUTE} \end{array} \quad i \neq j$$

$$P^2 = \prod_j (\sigma_j^x)^2 = \prod_j \mathbb{I} = \mathbb{I}$$

• UNITARY

$$P P^\dagger = P^\dagger P = \mathbb{I}$$

$$P P^\dagger = \prod_j \sigma_j^x \left(\prod_k \sigma_k^x \right)^\dagger =$$

$$= \sigma_1^x \sigma_2^x \dots \sigma_L^x \sigma_L^{x\dagger} \sigma_{L-1}^{x\dagger} \dots \sigma_1^{x\dagger}$$

$$= \sigma_1^x \sigma_2^x \dots \sigma_L^x \sigma_L^x \sigma_{L-1}^x \dots \sigma_1^x$$

$$= \sigma_L^x \sigma_{L-1}^x \dots \sigma_1^x \sigma_1^x \sigma_2^x \dots \sigma_L^x$$

$$= \sigma_L^{x\dagger} \sigma_{L-1}^{x\dagger} \dots \sigma_1^{x\dagger} \sigma_1^x \sigma_2^x \dots \sigma_L^x$$

$$= \left(\prod_k \sigma_k^x \right)^\dagger \prod_j \sigma_j^x = P^\dagger P$$

ACTION ON SPIN OPERATORS

$$\mathcal{P}^\dagger \sigma_i^z \mathcal{P} = -\sigma_i^z$$

$$\mathcal{P}^\dagger \sigma_i^x \mathcal{P} = \sigma_i^x$$

PROOFS

$$\mathcal{P}^\dagger \sigma_i^z \mathcal{P} = \left(\prod_j \sigma_j^x \right)^\dagger \sigma_i^z \left(\prod_k \sigma_k^x \right)$$

$$= \prod_{j,k} \sigma_j^x \sigma_i^z \sigma_k^x$$

$$= \sigma_1^x \dots \sigma_i^x \dots \sigma_L^x \sigma_i^z \sigma_1^x \dots \sigma_i^x \dots \sigma_L^x$$

$$= \sigma_1^x \dots \sigma_{i-1}^x \sigma_{i+1}^x \dots \sigma_L^x \underbrace{\sigma_i^x \sigma_i^z \sigma_i^x \sigma_i^x}_{\downarrow} \dots$$

$$\dots \sigma_{i-1}^x \sigma_{i+1}^x \dots \sigma_L^x$$

$$\begin{aligned}
 \sigma^x \sigma^z \sigma^x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \\
 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = -\sigma^z
 \end{aligned}$$

USING

$$\begin{aligned}
 (\sigma^x)^2 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
 &= \mathbb{I}
 \end{aligned}$$

$$\sigma^x \sigma^z \sigma^x = -(\sigma^x)^2 \sigma^z \quad \text{THUS}$$

$$\mathbb{P}^\dagger \sigma_i^z \mathbb{P} = \underbrace{\sigma_1^x \dots \sigma_L^x}_{\mathbb{P}} \underbrace{\sigma_1^x \dots \sigma_L^x}_{\mathbb{P}} (-\sigma_i^z)$$

$$= -\sigma_i^z \quad \checkmark$$

similarly for the other relation:

$$P^\dagger \sigma_i^x P = \sigma_i^x$$

$$\left(\prod_j \sigma_j^x \right)^\dagger \sigma_i^x \prod_k \sigma_k^x = \text{EVERYTHING COMMUTES} \Rightarrow$$

$$= P^\dagger P \sigma_i^x = I \sigma_i^x = \sigma_i^x$$

DONE $\checkmark$

ACTION ON STATES

$$\begin{aligned} \text{eg } \hat{P} |\uparrow\rangle &= |\downarrow\rangle \\ \hat{P} |\downarrow\rangle &= |\uparrow\rangle \end{aligned}$$

REVERSES
ALL SPINS

THIS IS CLEAR SINCE EACH FACTOR IN $\hat{P}$ DOES REVERSE ONE SPIN

COMMUTATION w/ $\hat{H}$

$$[\hat{H}, \hat{P}] = 0$$

Proof $\hat{H} = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x$

$$[\hat{H}, \hat{P}] = \left(-J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x \right) \prod_k \hat{\sigma}_k^x + \prod_k \hat{\sigma}_k^x \left(-J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x \right)$$

TAKE ONE TERM IN $\sum_i \hat{\sigma}_i^x$

$$\hat{\sigma}_i^x \hat{\sigma}_1^x \dots \hat{\sigma}_L^x = \hat{\sigma}_1^x \dots \hat{\sigma}_L^x \hat{\sigma}_i^x$$

SO IT COMMUTES

TAKE ONE TERM IN $\sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$

$$\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_1^x \dots \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \dots \hat{\sigma}_L^x =$$

$$= \hat{\sigma}_1^x \dots \underbrace{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z} \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \dots \hat{\sigma}_L^x =$$

$$= -\hat{\sigma}_1^x \dots \underbrace{\hat{\sigma}_i^z \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x}_{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z} \hat{\sigma}_{i+2}^x \dots \hat{\sigma}_L^x =$$

$$= \hat{\sigma}_1^x \dots \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \dots \hat{\sigma}_L^x \underbrace{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z}$$

BASICALLY TWO (-1) SIGNS FROM

ANTICOMMUTATION

OF PAULI MATRICES

AND IT ALSO COMMUTES

ANTI-COMM. OF PAULI MATRICES

$$\begin{aligned} \sigma_x \sigma_z + \sigma_z \sigma_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0 \quad \underline{\text{OK}} \end{aligned}$$

$$\Rightarrow [\hat{H}, \hat{P}] = 0$$

IF TWO OPERATORS COMMUTE $\Rightarrow$ THEY ARE

$\hat{H}$ & $\hat{P}$ DIAGON. IN SAME BASIS

$$\hat{H} |\psi_n\rangle = E_n |\psi_n\rangle$$

$$\hat{P} |\psi_n\rangle = \pm |\psi_n\rangle$$

ENERGETIC CONSIDERATIONS

● TRANSVERSE ALIGNMENT $g \gg 1$

$$\hat{H} \sim -Jg \sum_i \hat{S}_i^x$$

TWO EIGENSTATES TO FOCUS ON

GROUND STATE $|\rightarrow\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle + |\downarrow\rangle)$

QUANTUM PARAMAGNET
SINCE

WITH $|\rightarrow\rangle_i = \frac{1}{\sqrt{2}} (|\uparrow\rangle_i + |\downarrow\rangle_i)$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

↪ EQUAL
MIXTURE OF
 $|\uparrow\rangle_i$ & $|\downarrow\rangle_i$

$$\hat{\sigma}_i^x |\rightarrow\rangle_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = |\rightarrow\rangle_i$$

IT'S AN EIGENVECTOR
WITH EIGENVALUE
(+1)

$$\hat{\sigma}_i^x |\leftarrow\rangle_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = - \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= - |\leftarrow\rangle_i$$

IT'S ALSO AN EIGENVECTOR
BUT WITH EIGENVALUE (-1)

GROUND STATE ENERGY

$$E_0 = \langle \rightarrow | \hat{H} | \rightarrow \rangle$$

⇒ CONTRIB OF
EACH i :

$$= \frac{1}{2} \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

$$= -Jg \frac{1}{\sqrt{2}} (1 \ 1) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{\sqrt{2}} (1)$$

$$= -Jg \frac{1}{2} (1 \ 1) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -Jg \frac{1}{2} \underbrace{(1+1)}_2$$

$$E_0 = -Jg L$$

FOR EACH i

EXCITED STATE $|\rightarrow\rangle = \prod_{i(\neq j)} |\rightarrow\rangle_i \otimes |\leftarrow\rangle_j$

$$E_1^{(j)} = \langle \leftarrow | \hat{H} | \leftarrow \rangle_j$$

$\Rightarrow$

CONTRIBUTION OF
REVERSED SPIN j

$$= -Jg \frac{1}{\sqrt{2}} (1 \ -1) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= -\frac{Jg}{2} (1-1) \begin{pmatrix} -1 \\ 1 \end{pmatrix} = -\frac{Jg}{2} (-2) = Jg$$

$$E_1^{(j)} = Jg$$

DIFFERENCE IN TOTAL ENERGY

$$E_1 - E_0 = -Jg \cancel{(L-1)} + Jg + \cancel{JgL}$$

$$= Jg + Jg = 2Jg$$

FOR EACH REVERSED SPIN (HERE JUST ONE) WE PAY $\Delta E = 2Jg$ FINITE

IF ONE REVERSES THEM ALL

$$|\leftarrow\rangle = \prod_i \otimes |\leftarrow\rangle_i$$

$$\Rightarrow \Delta E = 2JgL \quad \text{MACROSCOPIC}$$

ONE NEEDS TO REVERSE A MACROSC.
NUMBER TO GET $\Delta E = \Theta(L)$

WHAT HAPPENS WITH THE 1ST TERM
IN $\hat{H}$? HOW DOES IT ACT ON $|0\rangle_{g \gg 1}$?

SANDWICH $\langle 0 | \hat{H}_e | 0 \rangle_{g \gg 1}$

$$\langle \rightarrow | -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z | \rightarrow \rangle =$$

RECALL $|\rightarrow\rangle_i = \pi_i \otimes |\rightarrow\rangle_i$

$$= \pi_j \otimes \frac{|\uparrow\rangle_j + |\downarrow\rangle_j}{\sqrt{2}}$$

$$\sigma_j^z \left(\frac{|\uparrow\rangle_j + |\downarrow\rangle_j}{\sqrt{2}} \right) = \frac{|\uparrow\rangle_j - |\downarrow\rangle_j}{\sqrt{2}} = |\leftarrow\rangle_j$$

THUS, $\langle\rightarrow| -J \sigma_i^z \sigma_{i+1}^z |\rightarrow\rangle =$

$$= \pi_k \otimes \langle\rightarrow|_k -J \prod_{j(\neq i, i+1)} \pi_j \otimes |\rightarrow\rangle_j \otimes |\leftarrow\rangle_i \otimes |\leftarrow\rangle_{i+1}$$

ALL SITES $\neq i, i+1$ YIELD 1; WHILE REMAINS

$$-J \langle\rightarrow|_i |\leftarrow\rangle_i \langle\rightarrow|_{i+1} |\leftarrow\rangle_{i+1} = 0$$

SINCE

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$$

ON THE PERFECT
GROUND STATE.

ON AN EXCITATION?

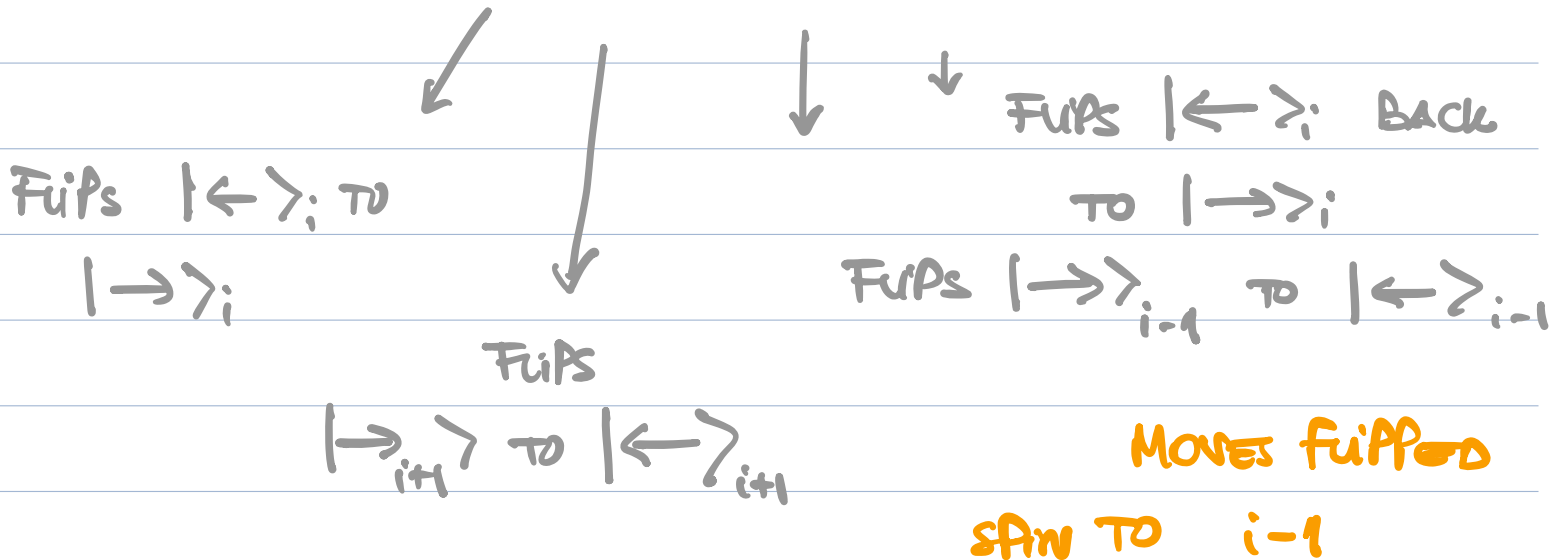
ONE SPIN
FLIP

$$\text{CALL } |i\rangle = \prod_{j(\neq i)} \pi_j \otimes |\rightarrow\rangle_j \otimes |\leftarrow\rangle_i$$

$$\langle j | \hat{H}_e | i \rangle = \langle j | (-J) \sum_k \hat{\sigma}_k^z \hat{\sigma}_{k+1}^z | i \rangle$$

RELEVANT

$$\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \hat{\sigma}_{i-1}^z \hat{\sigma}_i^z | i \rangle$$



MOVES FLIPPED SPIN
FROM i TO $i+1$

WHEN SANDWICHED WITH $\langle j | \Rightarrow$

$$-J (\delta_{j,i-1} + \delta_{j,i+1})$$

SO, THE PERTURBATION $\hat{H}_e$ WITH REF TO $\hat{H}_t$ MOVES THE LOCAL EXCITATIONS TO NEARBY SITES.

LINEAR SUPERPOSITION OF LOCAL EXCITATIONS

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_i e^{ikx_i} |i\rangle \quad \text{PLANE-WAVES}$$

WHAT IS THEIR ENERGY IN FULL HAMILTONIAN?

$$\langle k | \hat{H} | k \rangle = ?$$

$$\frac{1}{\sqrt{L}} \sum_i e^{-ikx_i} \frac{1}{\sqrt{L}} \sum_j e^{ikx_j} \underbrace{\langle i | \hat{H} | j \rangle}$$

WE CALCULATED
ALREADY

THE LONG. CONTRIBUTION

$$\langle i | \hat{H}_e | j \rangle = -J (\delta_{i,j-1} + \delta_{i,j+1})$$

AND, IN THE SUM

$$\langle k | \hat{H}_e | k \rangle = \frac{1}{L} \sum_{ij} e^{-ik(x_i - x_j)} (-J) (\delta_{i,j-1} + \delta_{i,j+1})$$

$$= -\frac{J}{L} \sum_i e^{-ik(x_i - x_{i+1})} + e^{-ik(x_i - x_{i-1})}$$

$$= -\frac{J}{L} \sum_i e^{+ika} + e^{-ika}$$

$$= -\frac{J}{L} \cancel{L} 2 \cos ka$$

$$= -2J \cos ka$$

THE TRANSVERSE CONTRIB

$$\langle i | \hat{H}_t | j \rangle = \langle i | Jg \sum_l \hat{\sigma}_l^x | j \rangle$$

ONE TERM

$$\langle i | Jg \hat{\sigma}_l^x | j \rangle = Jg \langle i | l, j \rangle$$

$$= Jg \delta_{l,j} \delta_{ij}$$

IN THE SUM

$$\langle k | \hat{H}_t | k \rangle = Jg$$

$$\frac{1}{\sqrt{L}} \sum_i e^{-ikx_i} \frac{1}{\sqrt{L}} \sum_j e^{ikx_j} \delta_{ij}$$

$$= \frac{1}{L} Jg \sum_i 1$$

$$= Jg$$

THE TWO TERMS TOGETHER

$$\Delta E_k = E_k^{\text{exc}} - E_{\text{G.S.}}$$

$$= 2Jg \left[1 - \frac{1}{g} \cos ka \right]$$

+ CORRECTIONS

LOWEST EXCIT $k=0$

$$\Delta E_0 = 2J (g-1)$$

$$\Delta Q_k \rightarrow 0 \quad \Delta T \quad k=0$$

$$g = g_c = 1$$

- LONGITUDINAL ALIGNMENT $g \ll 1$

$|0\rangle_{g=0} = \frac{1}{\sqrt{2}} |\uparrow\rangle$, AND $|0\rangle_{g=0} = \frac{1}{\sqrt{2}} |\downarrow\rangle$; TWO GROUND STATES

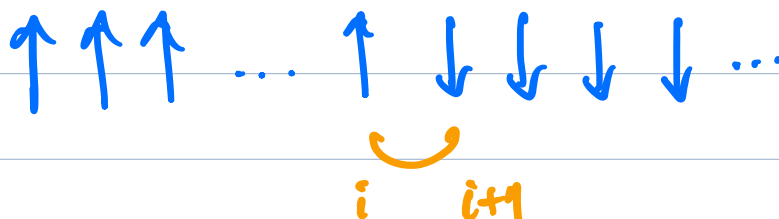
$$\begin{aligned} \langle 0 | \hat{H}_{g=0} | 0 \rangle_{g=0} &= \\ &= -J \sum_j \pi \langle \uparrow | \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \frac{\pi}{K} | \uparrow \rangle_K \end{aligned}$$

$$= -JL$$

$$\hat{\sigma}_{i+1}^z |\uparrow\rangle_{i+1} = |\uparrow\rangle_{i+1}$$

JUST THE SAME CONTRIBUTION
FROM EACH SITE

BASIC EXCITATIONS : DOMAIN WALLS



EFFECT OF $\hat{H}$ ON THESE DOMAIN WALL STATES

$$\langle i | \hat{H} | i+1 \rangle = -Jg$$

$$\left(\prod_{j < i} \pi \otimes \langle \uparrow | \prod_{k \geq i} \pi \langle \downarrow | \right) \hat{H}$$

CIRCULATION
FIELDS

$$\left(\prod_{j < i+1} \pi \otimes | \uparrow \rangle \prod_{k \geq i+1} \pi | \downarrow \rangle \right)$$

PLANE WAVES

$$|k\rangle = \frac{1}{\sqrt{L}} \sum_i e^{ikx_i} |i\rangle$$



DOMAIN WALL
AT $(i, i+1)$

WE IDENTIFIED THE GROUND STATES
AND THEIR E_0 AT $g \gg 1$
 $g = 0$

THEY ARE VERY $\neq$ STATES.

WE DEVELOPED ABOVE
PERTURBATIVE ARGUMENTS TO SEE
HOW THEY CHANGE FOR $g \gtrsim 0$
AND $1/g \gtrsim 0$

MORE DETAILS IN NOTES

DUALITY

DEFINE

$$\hat{\tau}_i^z = \prod_{j \leq i} \hat{\sigma}_j^x \quad \& \quad \hat{\tau}_i^x = \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

STRING OPERATOR

ON A DUAL LATTICE

$$\tau = i + \frac{1}{2}$$

NEW SPIN OPERATORS

$\tau \hat{\sigma}_i^z \tau = \hat{\sigma}_i^z$

$$\{\hat{\tau}_{\bar{i}}^{\alpha}, \hat{\tau}_{\bar{i}}^{\beta}\} = 2\delta^{\alpha\beta}$$

ON THE SAME $\bar{i}$

$$[\hat{\tau}_{\bar{i}}^{\alpha}, \hat{\tau}_{\bar{j}}^{\beta}] = 0 \quad \text{FOR } \bar{i} \neq \bar{j}$$

PROOFS

RECALL
THE SPIN
ALGEBRA

$$\{\hat{\sigma}_i^x, \hat{\sigma}_i^x\} = 2(\hat{\sigma}_i^x)^2 = 2$$

$$\{\hat{\sigma}_i^z, \hat{\sigma}_i^z\} = 2(\hat{\sigma}_i^z)^2 = 2$$

$$\{\hat{\sigma}_i^x, \hat{\sigma}_i^z\} = \hat{\sigma}_i^x \hat{\sigma}_i^z + \hat{\sigma}_i^z \hat{\sigma}_i^x$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0$$

IF $\bar{i} = \bar{j}$ AND $\alpha = \beta = x$

$$\{\hat{\tau}_{\bar{i}}^x, \hat{\tau}_{\bar{i}}^x\} = \{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z, \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z\} =$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z + \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$= \underbrace{\hat{\sigma}_i^z \left(\hat{\sigma}_{i+1}^z \right)^2 \hat{\sigma}_i^z}_{\text{II}} + \underbrace{\hat{\sigma}_i^z \left(\hat{\sigma}_{i+1}^z \right)^2 \hat{\sigma}_i^z}_{\text{II}}$$

$$= 2 \left(\hat{\sigma}_i^z \right)^2 = 2$$

if $\bar{i} = \bar{j}$ AND $\alpha = x$ $\beta = z$

$$\{ \hat{\tau}_i^x, \hat{\tau}_i^z \} = \{ \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z, \prod_{j \leq i} \hat{\sigma}_j^x \}$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i-1}^x \hat{\sigma}_i^x \right) + \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i-1}^x \hat{\sigma}_i^x \right) \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$= \left(\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i-1}^x \right) \underbrace{\left(\hat{\sigma}_i^z \hat{\sigma}_i^x + \hat{\sigma}_i^x \hat{\sigma}_i^z \right)}_{\{ \hat{\sigma}_i^z, \hat{\sigma}_i^x \} = 0} \hat{\sigma}_{i+1}^z$$

$$\{ \hat{\sigma}_i^z, \hat{\sigma}_i^x \} = 0$$

$$= 0$$

if $\bar{i}, \bar{j} = \bar{i}+1$ AND $\alpha = \beta = x$

$$[\hat{\tau}_i^x, \hat{\tau}_{i+1}^x] =$$

$$= \underbrace{\hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_{i+1}^z \hat{\sigma}_{i+2}^z}_{\text{II}} - \underbrace{\hat{\sigma}_{i+1}^z \hat{\sigma}_{i+2}^z \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z}_{\text{II}} =$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+2}^z - \hat{\sigma}_{i+2}^z \hat{\sigma}_i^z$$

$$= [\hat{\sigma}_i^z, \hat{\sigma}_{i+2}^z] = 0 \quad \text{COMMUT ok.}$$

$$[\hat{\tau}_i^z, \hat{\tau}_{i+1}^z] =$$

$$\prod_{j \leq i} \hat{\sigma}_j^x \prod_{k \leq i+1} \hat{\sigma}_k^x - \prod_{k \leq i+1} \hat{\sigma}_k^x \prod_{j \leq i} \hat{\sigma}_j^x$$

$$= (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_i^x) (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i+1}^x) -$$

$$- (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i+1}^x) (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_i^x)$$

$$= \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - \hat{\sigma}_{i+1}^x \hat{\sigma}_i^x = 0 \quad \text{COMMUT ok}$$

$$[\hat{\tau}_i^x, \hat{\tau}_{i+1}^z] =$$

$$= \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \prod_{j \leq i} \hat{\sigma}_j^x - \prod_{j \leq i} \hat{\sigma}_j^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z$$

$$\begin{aligned}
&= (\hat{\sigma}_1^x \dots \hat{\sigma}_{i-1}^x) \left\{ \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x - \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z \right\} \\
&= (\hat{\sigma}_1^x \dots \hat{\sigma}_{i-1}^x) \left\{ \hat{\sigma}_i^z \hat{\sigma}_i^x \hat{\sigma}_{i+1}^z \hat{\sigma}_{i+1}^x - \hat{\sigma}_i^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^x \hat{\sigma}_{i+1}^z \right\} \\
&= (\hat{\sigma}_1^x \dots \hat{\sigma}_{i-1}^x) \left\{ -\hat{\sigma}_i^x \hat{\sigma}_i^z (-\hat{\sigma}_{i+1}^x \hat{\sigma}_{i+1}^z) - \hat{\sigma}_i^x \hat{\sigma}_i^z \hat{\sigma}_{i+1}^x \hat{\sigma}_{i+1}^z \right\} \\
&\quad \text{THEY CANCEL} \\
&= 0
\end{aligned}$$

THE HAMILTONIAN CAN BE RE-WRITTEN AS
(WITH PERIODIC BOUND. CONDITIONS)

$$\hat{H} = -J \sum_{i=1}^L \hat{\tau}_i^x - Jg \sum_{i=1}^L \hat{\tau}_i^z \hat{\tau}_{i+1}^z$$

SAME HAMILTONIAN WITH

$$g \xrightarrow{\text{1st}} 1/g \quad \text{AND} \quad J \xrightarrow{\text{next}} Jg$$

$\Rightarrow$

$$g_c = 1$$

DUALITY

CHECK

$$\bar{i} = i + 1/2$$

REPLACING τ 's
BY THE σ 's

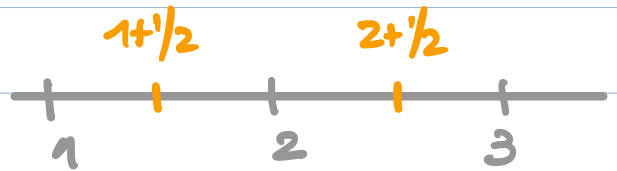
$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z$$

$$-J \sum_i \left(\prod_{j \leq i} \hat{\sigma}_j^x \right) \left(\prod_{k \leq i+1} \hat{\sigma}_k^x \right)$$

$$= -J \sum_i \sigma_i^z \sigma_{i+1}^z$$

$$-J \sum_i (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_i^x) (\hat{\sigma}_1^x \hat{\sigma}_2^x \dots \hat{\sigma}_{i+1}^x)$$

$$= -J \sum_i \sigma_i^z \sigma_{i+1}^z$$



$$-J \sum_i \underbrace{(\hat{\sigma}_i^x)^2}_{\text{II}} \underbrace{(\hat{\sigma}_2^x)}_{\text{II}} \dots \underbrace{(\hat{\sigma}_i^x)^2}_{\text{II}} (\hat{\sigma}_{i+1}^x)$$

$$= -J \sum_i \sigma_i^z \sigma_{i+1}^z - J \sum_i \hat{\sigma}_{i+1}^x$$

case $i+1 \rightarrow i$

THE FM PHASE OF ONE MODEL



THE PM OF THE OTHER

"FIXED POINT" $g_c = 1$

- THE FM PHASE OF THE γ MODEL $\Rightarrow$
DOMAIN WALL CONDENSATE

SOLVING THE TULL PROBLEM

1- JORDAN - WIGNER

$\Rightarrow$ FERMIONS

2- BOGOLIUBOV

$\Rightarrow$ FOURIER MODES

END WITH FREE FERMIONS

$$\epsilon_k(q)$$

GAP AT $k=0$ CLOSES AT

$$g = g_c = 1.$$

JORDAN - WIGNER

RECALL CLASSICAL ISING SPINS $\Leftrightarrow$
OCCUP. $\neq$

$$n_i = \frac{s_i + 1}{2}$$

QUANTUM ?

$$\hat{n}_i = \hat{c}_i^\dagger \hat{c}_i \quad \text{NUMBER OF FERMIONS}$$

$$\text{SO} \quad \hat{c}_i^\dagger \hat{c}_i = \frac{\hat{\sigma}_i^z + 1}{2} ?$$

$$\text{OR} \quad \boxed{\hat{c}_i^\dagger \hat{c}_i = \frac{\hat{\sigma}_i^z + 1}{2}} \quad \text{WITH RAISING* / LOWERING } \hat{\sigma}_i^\pm$$

NOT SO EASY SINCE PROBLEMS TO
SATISFY THE ANTI-COMMUTATION RELATIONS
AT SAME $\neq$ SITES. IN PARTICULAR,

$$\{\hat{c}_i^\dagger, \hat{c}_j\} = 0 \quad \forall i, j \quad \text{WITH CHOICE ABOVE}$$

(SHOULDN'T FOR $i=j$)

TRUE FERMION OPERATORS CAN BE BUILT
IN $d=1$ WITH STRING OPERATORS
(NON-LOCAL IN SPIN OPERATIONS)

*

RECALL

RAISING & LOWERING OPERATORS

$$\hat{\sigma}_i^+ = \frac{\hat{\sigma}_i^x + i\hat{\sigma}_i^y}{2}$$

$$\hat{\sigma}_i^- = \frac{\hat{\sigma}_i^x - i\hat{\sigma}_i^y}{2}$$

MATRIX

$$\sigma^\pm = \frac{\sigma^x \pm i\sigma^y}{2} = \frac{1}{2} \left[\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \pm i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right]$$

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

THEIR EFFECT

$$\sigma^- \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \text{REVERSES}$$

$$\sigma^+ \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow //$$

WITH NO SQU INVOLVED

ANTI-COMM $\{ \hat{\sigma}^+, \hat{\sigma}^- \} = ?$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}$$

$$\{\hat{\sigma}_i^+, \hat{\sigma}_i^-\} = \mathbb{I}$$

Locally
same i

$$\text{while } [\hat{\sigma}_i^+, \hat{\sigma}_j^-] = 0 \text{ if } i \neq j$$

COMMUTATOR AT SAME SITE

$$[\hat{\sigma}_i^+, \hat{\sigma}_i^-] = \hat{\sigma}_i^z$$

check

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^z$$

DEFINE MODIFIED RAISING / LOW.

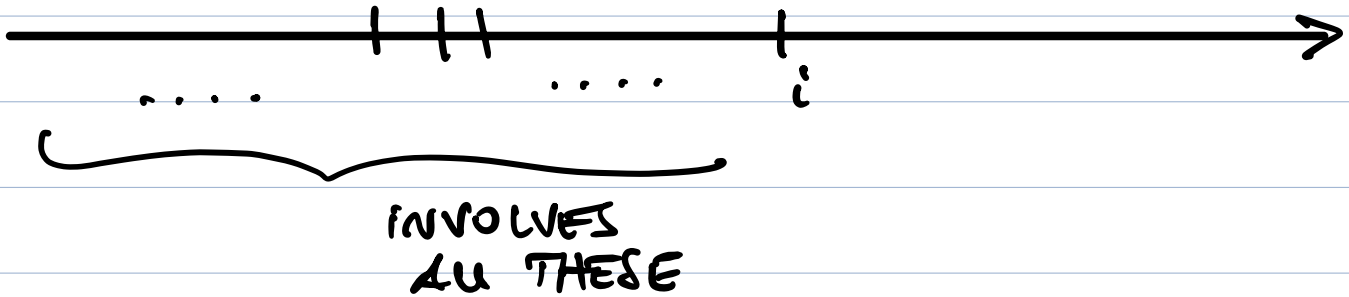
SHOW ONLY THE
FDS WITH

● $\hat{\sigma}_i^+ = \prod_{j < i} (1 - 2\hat{c}_j^+ \hat{c}_j) \hat{c}_i$

EQS WITH ●
EXPLAINING THAT
THESE ARE
"GOOD" FERMIONS

$\hat{S} \equiv$ STRING HERMITIAN OPERATOR

AND THE $\hat{\sigma}_i^-$ RELATES TO $\hat{c}_i^+$



$\hat{n}_j \equiv \hat{c}_j^+ \hat{c}_j$ ARE LOCAL # OPERATORS

$\Rightarrow$ THE $\hat{S} |\text{STATE}\rangle = \pm |\text{STATE}\rangle$

WHETHER EVEN # FERMIONS
ODD

TO THE LEFT OF SITE i IN $|\text{STATE}\rangle$

ONE CAN CHECK THAT

$\hat{\sigma}_i^+ \hat{\sigma}_k^- = \hat{\sigma}_k^- \hat{\sigma}_i^+ \quad k \neq i$

INVERSE RELATIONS

FROM SPINS TO FERMIONS

- $$\hat{c}_i^- = \left(\prod_{j < i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^-$$

SHOW THEM

- $$\hat{c}_i^+ = \left(\prod_{j < i} \hat{\sigma}_j^z \right) \hat{\sigma}_i^+$$

ONE CAN CHECK THAT

SHOW THEM

- $$\{ \hat{c}_i, \hat{c}_j^+ \} = \delta_{ij}$$

- $$\{ \hat{c}_i, \hat{c}_i \} = \{ \hat{c}_i^+, \hat{c}_i^+ \} = 0$$

} $\Leftrightarrow$

$$\begin{cases} [\hat{\sigma}_i^+, \hat{\sigma}_j^-] = \delta_{ij} \hat{\sigma}_i^z \\ [\hat{\sigma}_i^z, \hat{\sigma}_j^\pm] = \pm 2 \delta_{ij} \hat{\sigma}_i^\pm \end{cases}$$

COMMUTE AT $\neq$ SITES

CONVENIENT: PERFORM A

ROTATION AROUND THE y AXIS BY 90°

$$\begin{cases} \hat{\sigma}_i^z \longrightarrow \hat{\sigma}_i^x \\ \hat{\sigma}_i^x \longrightarrow -\hat{\sigma}_i^z \end{cases}$$

BUT NOT
NECESSARY FOR
THE JORDAN-
WIGNER TRANSF.

$$\hat{H} = -J \sum_i \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z - gJ \sum_i \hat{\sigma}_i^x \quad \text{OLD}$$

$$\hat{H} = -J \sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + gJ \sum_i \hat{\sigma}_i^z \quad \text{NEW}$$

THE "DUAL" $\hat{H}$ WITH $g \rightarrow -g$

BUT CAN IT AGAIN
g

● APPLY THE MAPPING PROPOSED ABOVE

$$\hat{\sigma}_i^x = \mathbb{I} - 2\hat{c}_i^\dagger \hat{c}_i$$

$$\hat{\sigma}_i^z = -\prod_{j < i} (1 - 2\hat{c}_j^\dagger \hat{c}_j) (\hat{c}_i + \hat{c}_i^\dagger)$$

AND IN TERMS OF $\{\hat{c}_i^\dagger, \hat{c}_i\}$ $\hat{H}$ READS

$$\hat{H} = -J \sum_i \left(\hat{c}_i^\dagger \hat{c}_{i+1} + \underbrace{\hat{c}_i^\dagger \hat{c}_i^\dagger + \text{h.c.}}_{\text{FERMION \# NOT CONSERVED}} \right) - gJ \sum_i (1 - 2\hat{c}_i^\dagger \hat{c}_i) \quad \text{NON-INTERACTING FERMIONS}$$

$[\hat{H}, \hat{P}] = 0$ PARITY IS CONSERVED
BUT FERMION \# NOT CONS.

HOW TO PROVE THE CONNECTION WITH FERMIONIC $\hat{H}$:

$$\text{eg. } \hat{\sigma}_i^z \hat{\sigma}_{i+1}^z = \prod_{j < i} (1 - 2\hat{c}_j^\dagger \hat{c}_j) (\hat{c}_i + \hat{c}_i^\dagger)$$

$$\prod_{k < i+1} (1 - 2\hat{c}_k^\dagger \hat{c}_k) (\hat{c}_{i+1} + \hat{c}_{i+1}^\dagger)$$

expand and go ahead.

BOGO L'UBOV TRANSF

FOURIER TRANSFORM

(BUILD COLLECTIVE MODES)

$$\hat{c}_k = \sum_j \hat{c}_j e^{-ik \cdot \underbrace{j a}_{x_j}}$$

POSITION
ON AXIS

AFTER MANY STEPS THAT YOU
CAN FOLLOW ON YOUR OWN

$$\hat{H} = 2J \sum_{k > 0} \left\{ (\hat{c}_k^\dagger \hat{c}_{-k}) A_k \begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{pmatrix} \right.$$

$$-g + (g - \cos(ka)) \}$$

$$+ 2J(g-1) \hat{c}_0^\dagger \hat{c}_0$$

$$A_k = (g - \cos(ka)) \sigma^z - \sin(ka) \sigma^y$$

NON-DIAGONAL.

DIAGONALIZABLE

$$D_k = U_k^\dagger A_k U_k$$

WITH ROTATION MATRIX

$$U_k = e^{-i\theta_k \sigma^x / 2} = \cos\left(\frac{\theta_k}{2}\right) \mathbb{I} - i \sin\left(\frac{\theta_k}{2}\right) \sigma^x$$

WITH $\tan \theta_k = \frac{\sin(ka)}{g - \cos(ka)}$

$$D_k = \left[(g - \cos(ka))^2 + \sin^2(ka) \right]^{1/2} \sigma^z$$

WHICH IS DIAGONAL

CONCERNING FERMIONS

$$\begin{pmatrix} \hat{c}_k \\ \hat{c}_{-k}^\dagger \end{pmatrix} = U_k \begin{pmatrix} \hat{\gamma}_k \\ \hat{\gamma}_{-k}^\dagger \end{pmatrix}$$

$\hat{\gamma}_k$ STILL FERMION OPERATORS

$$\{\gamma_k, \gamma_{k'}^\dagger\} = \delta_{kk'}$$

$\Rightarrow$ IN TERMS OF THE $\{\tilde{\gamma}'_s\}$:

$$\hat{H} = \sum_k \epsilon_k \left(\tilde{\gamma}_k \tilde{\gamma}_k^\dagger - \frac{1}{2} \right) + \text{CONST}$$

$$\epsilon_k = 2J \left[(g - \cos(ka))^2 + \sin^2(ka) \right]^{1/2}$$

ϵ_k ACHIEVES ITS MINIMUM AT $k=0$

SINCE EACH TERM MINIMIZES INDEP.

$$\epsilon_0 = 2J (g-1)^2 \cdot 1/2$$

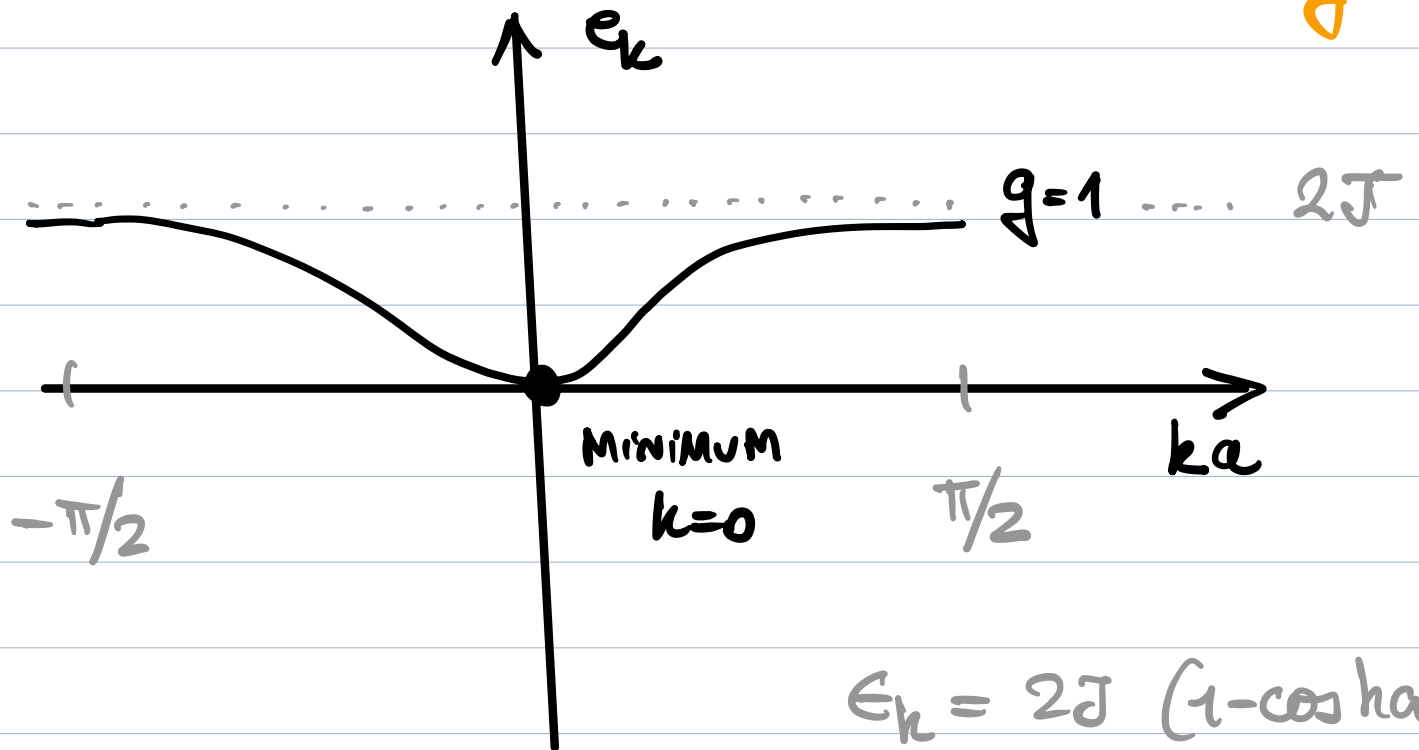
$$\epsilon_0 = 2J |g-1| \quad \text{valid } \forall g$$

$$g \ll 1 \quad \epsilon_k \simeq 2J [1 - g \cos(ka)] + \mathcal{O}(g^2)$$

$$g \gg 1 \quad \epsilon_k \simeq 2Jg \left[1 - \frac{1}{g} \cos(ha) \right] + \mathcal{O}\left(\frac{1}{g}\right)$$

AS WHEN WE STUDIED FLUCTUATIONS

NOTE
 $g \leftrightarrow 1/g$
 DUALITY



Minimum of E_k

$$E_k = 2J \sqrt{(g - \cos(ka))^2 + \sin^2 ka}$$

with respect to k :

Ask that each term minimizes:

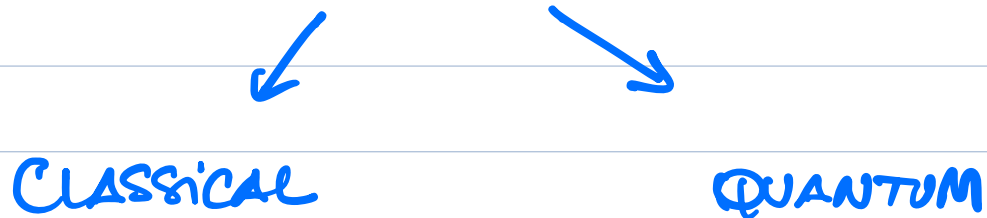
$$\sin^2 ka \Rightarrow h=0$$

$$g - \cos ka \Rightarrow +(\sin ka) a = 0$$

$$\Rightarrow h=0$$

SUMMARY

EQUILIBRIUM PROPERTIES



COLLECTIVE PHENOMENA

PHASE TRANSITIONS

(in) EQUIV. OF ENSEMBLES
(LONG) SHORT RANGE

1st ORDER	1d
2nd ORDER	MEAN-FIELD CURIE WEISS
TOPOLOGICAL	LANDAU-GINZBURG
	REAL SPACE RG METHODS

DOMAIN WALLS $\longleftrightarrow$ TUNNELING

COUPLING TO ENVIRONMENTS

LANGEVIN EQS

BATH $\rightarrow$ LONG-RANGE COUPL
IN IMAGINARY TIME

QUANTUM PHASE TRANSITIONS ($T=0$)
GROUND STATE PROPS.