

2 - CLASSICAL PHASE TRANSITIONS

2.1 PHENOMENA

- USUALLY, SYSTEM IN CONTACT W/
ENVIRONMENT
(BUT ALSO CLOSED POSSIBLE)
- GLOBAL OBSERVABLES
eg. $\frac{\text{ENERGY}}{N}$, $\frac{\text{MAGN.}}{N}$, ETC
DENSITIES
- CONTROL PARAMETERS
eg. TEMP., MAGN FIELDS, ETC

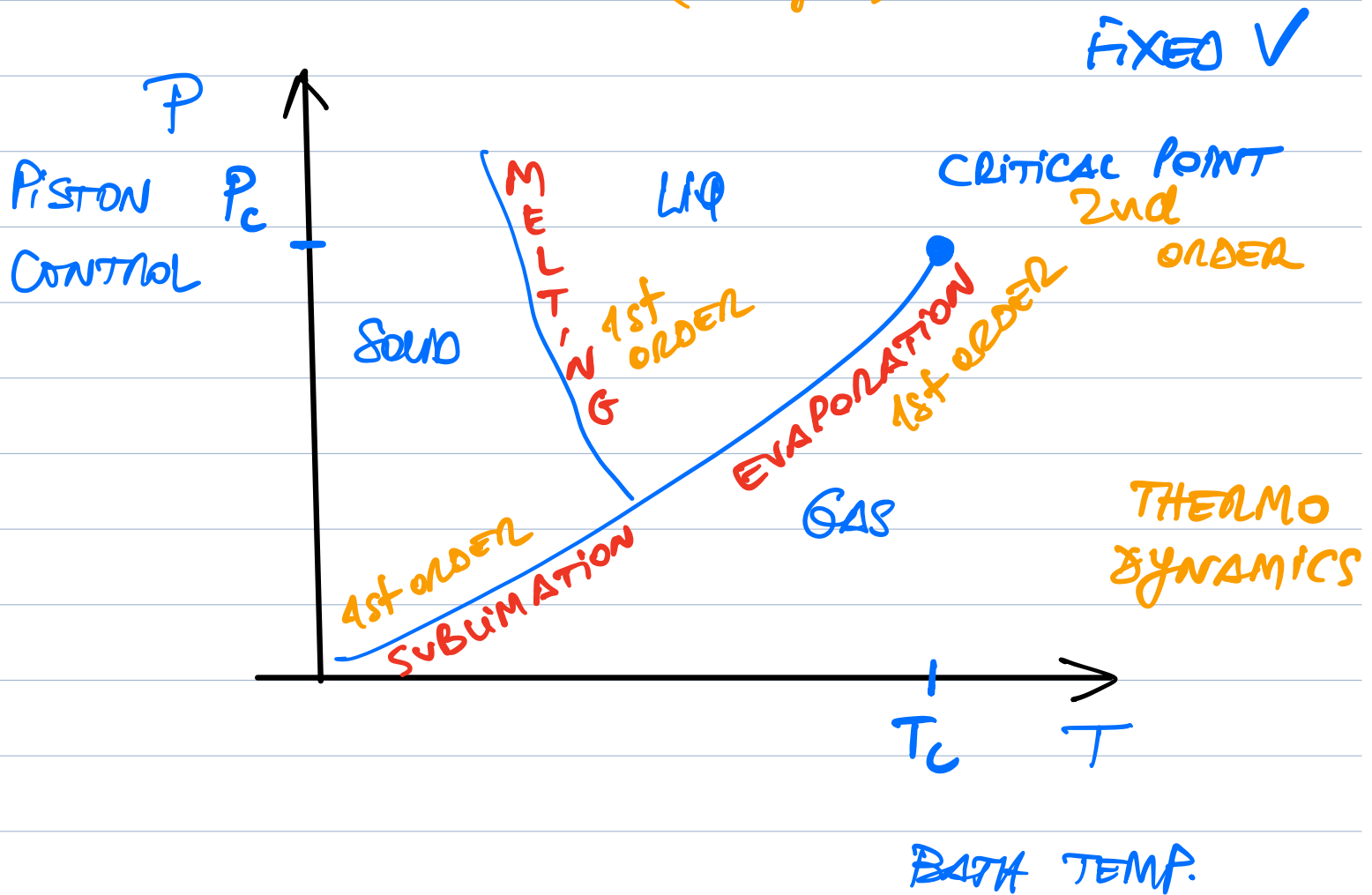
TUNE CONTROL PARAM $\longrightarrow$ MEASURE OBS

PHASE TRANS $\equiv$ SHARP CHANGES IN
MACROSC BEHAV.

COLLECT PHENOMENA

IN $N \rightarrow \infty$

PHASE DIAGRAMS (3d syst)



IN ISOLATION, E FIXED (REPLACES T)
VOLUME ALSO FIXED
CHANGES IN PARAM IN HAMILTONIAN

PHASES IDENTIFIED BY PARAM DEP. OF
ORDER PARAMETERS



GLOBAL OBS. JUDICIOUSLY CHOSEN

PHASE TRANSITIONS

SHARP CHANGES (INTUITIVE)

NON-ANALYTICITIES OF RELEVANT
THERMODYN POTENTIALS
WITH RESPECT TO CONTROL PARAM.
(FORMAL)

e.g. FREE-ENERGY

$$[\beta T] = 1$$

$$-\beta f(\beta T) = \frac{1}{N} \ln Z(\beta T)$$

$$Z(\beta T) = \sum_c e^{-\beta H_T(c)}$$

LOCATE w/ THEM THE CRITICAL CURVES

GOALS

- ORDER PARAM AS fcts of CONTROL PARAM
- CRITICAL CURVES IN PH. DIAG.
- CRITICAL BEHAVIOUR
- THERMODYN PROPS $PV = nRT$

METHODS

— MEAN FIELD

QUANTITATIVE UNDERSTANDING ✓

QUANTITATIVE APPROX ONLY ~

CRIT BEH TYP. WRONG X

— SCALING TH

PHENOM OF CRIT PHEN ✓

— RANDOM. GROUP

CRIT BEH ✓

BEYOND STANDARD PH. TRANS

TOPOLOGICAL ONES

QUANTUM PHASE TRANS (LATER)

DYN PHASE TRANS. (NOT
DISCUSSED)

CONTEXT

PART: IN INTERACT

SPINS IN INTERACT

EASIER TO TREAT

2.2 ISING MODELS & EXTENSIONS

SIMPLIFIED REPRESENTATION OF A
MAGN. SYST

MAGN MOM. $\nearrow$ $\vec{S}_i$ LOCALIZED
ARROW w/
N-COMP.

$$H[\{\vec{S}_i\}] = -\frac{J}{2} \sum_{ij} \vec{S}_i \cdot \vec{S}_j - \sum_i \vec{h} \cdot \vec{S}_i$$

$J > 0$

FM

INTERACTIONS

//

$J < 0$

AF

ANTI- //

$\sum_{ij}$

DECIDE WHETHER SPINS SIT ON A
LATTICE / A GRAPH

$\vec{h}$

UNIFORM MAGN FIELD

d

dim. of SPACE

n

dim of SPIN

FACTS

SIMPLEST MODEL ISING $J > 0$

- SOLVED IN $d=1, 2$ ($k=0$), ∞
UNSOLVED ANALYTICALLY IN $d=3$
- $d \geq d_p=2$ PH. TRANS AT $0 < \beta_c J < \infty$ ($k=0$)
BTW FM & PM
- d_u UPPER CRIT DIM: MFK $d > d_u=4$
FAILS BELOW.
- SCALING & RG AT WORK ✓

OTHER CASES, SAY STILL $J > 0 \forall j$

- $n=1$ ISING

$n \rightarrow \infty \quad \Theta(n)$

$n=2$ XY

$n=3$ HEISENBERG

$$\sum_{a=1}^n \left(s_i^a \right)^2 = n$$

MORE DIFFICULT / INTERESTING CASES

- FRUSTRATION AF ON Δ LATT.

- QUENCHED DISORDER $J \rightarrow J_{ij}$ pdfs
 $h \rightarrow h_i$

eg. RANDOM BOND FM $J_{ij} > 0$ SPIN-GLASS $J_{ij} \geq 0$

- STOCHASTIC DYNAMICS / KINETIC ISING OR
N COMP MODELS / DOMAIN GROWTH

DYNAMICS - WE'LL NOT TREAT THEM

2.3 CONCEPTS

THERMODYN LIMIT

$N \rightarrow \infty$ NECESSARY TO GET NON-ANAL
 $f(\beta J, \beta h)$

OTHERWISE JUST $\sum_{\text{config.}}$ POSIT. TERM = ∞

AND THIS IS CONTRADICTORY IN THE FERMION.
NO NON-ANALYTICITIES

STRESS

ROLE OF ONSAGER'S SOL 2dIM
A PROOF IN A SOLVABLE CASE

ORDER PARAMETERS

A MEASURE OF THE DEGREE OF ORDER

STATISTICAL AVERAGE OF AN OBSERVABLE

$$\langle \phi \rangle$$

WHICH TAKES DIFF VALUES IN $\neq$ PHASES

eg. $\tilde{\rho}(\vec{k}) = \int d^d r \rho(\vec{r}) e^{i\vec{k} \cdot \vec{r}}$

FOR SOLID LIQUID

LOCAL PART DENSITY

$$\vec{m} \equiv \left\langle \frac{1}{N} \sum_i \vec{s}_i \right\rangle \text{ FOR FM-PM}$$

SUBTLETY ABOUT SYMM TO BE DISCUSSED LATER

TRANSITION CONTINUOUS VS. DISCONT.

↓
CONT DEPARTS FROM 0

↓
JUMP IN $\langle \phi \rangle$

SYMMETRIES

CONTINUOUS

$O(n)$ symmetry

$$\vec{h}=0$$

UNIFORM ROTATION OF ALL SPINS

$$\begin{aligned} H[\{\vec{s}_i'\}] &= -\frac{J}{2} \sum_{ij} \vec{s}_i' \cdot \vec{s}_j' \\ &= -\frac{J}{2} \sum_{ij} \sum_{abc} R^{ab} s_i^b R^{ac} s_j^c \\ &= -\frac{J}{2} \sum_{ij} \sum_{bc} \underbrace{\sum_a (R^t)^{ba} R^{ac}}_{\delta^{bc}} s_i^b s_j^c \end{aligned}$$

$$R R^t = R^t R = \mathbb{1} \quad \Rightarrow \quad \delta^{bc}$$

$$= -\frac{J}{2} \sum_{ij} \sum_b s_i^b s_j^b = H[\{\vec{s}_i\}]$$

DISCRETE

$\mathbb{Z}_2$

UP-DOWN SYMM

FOR $n=1$, THE ISING CASE EVEN SIMPLER

$$\begin{aligned} H[\{s'_i\}] &= -\frac{J}{2} \sum_{ij} s'_i s'_j \\ &= -\frac{J}{2} \sum_{ij} (-s_i)(-s_j) \\ &= H[\{s_i\}] \end{aligned}$$

• NOTE THAT THESE SYMM HOLD FOR ALL LATTICES / GRAPHS

• BROKEN BY $\vec{h} \neq 0$ OR $h \neq 0$

FOR PARTICLE SYST AND SOLID-LIQUID INVARIANCE IS SPACE TRANSLATIONS.

$$\vec{r}' = \vec{r} + \vec{\Delta}$$

SINCE, TYPICALLY, $V(\vec{r}_i, \vec{r}_j) = V(|\vec{r}_i - \vec{r}_j|)$

CHECK CANDIDATE ORDER PARAM

$$m = \frac{1}{N} \sum_i \langle s_i \rangle$$

PINNING FIELD

$$\langle s_i \rangle = \sum_{\{s_j = \pm 1\}} \frac{s_i e^{-\beta H(\{s_j\})}}{Z(\beta)}$$

CHANGE
of

$$= \sum_{\{s_j' = \pm 1\}} \frac{s_i' e^{-\beta H(\{s_j'\})}}{Z(\beta)}$$

NAME

$$\{s_i \rightarrow s_i'\}$$

$$= \sum_{\{s_j = \pm 1\}} \frac{(-s_i) e^{-\beta H(\{s_j\})}}{Z(\beta)}$$

CHANGE VAR.

$$s_j' = -s_j$$

NOTE THAT

$$H(\{s_j\}) = H(\{-s_j\})$$

$$= -\langle s_i \rangle$$

$$\Rightarrow \langle s_i \rangle = 0 \quad \forall i$$

So, How CAN m BE $\neq 0$?

PINNING FIELD, WHEN GOING ACROSS
CRITICALITY

RESIDUAL OR INTENTIONAL

THE ACTUAL PROCEDURE IS

$$\langle S_i \rangle = \lim_{h \rightarrow 0^+} \lim_{N \rightarrow \infty} \langle S_i \rangle_h$$

(or $h \rightarrow 0^-$) AND THIS FORM YIELDS

$$\langle S_i \rangle \geq 0 \quad \text{AT } (T/J) < \text{CRIT } h \rightarrow 0^\pm$$

$$\langle S_i \rangle = 0 \quad \text{AT } (T/J) > \text{CRIT}$$

BROKEN ERGODICITY

WE DEFINED ERGOD FROM

$$\overline{A} = \langle A \rangle$$

FOR GLOBAL OBSERV.

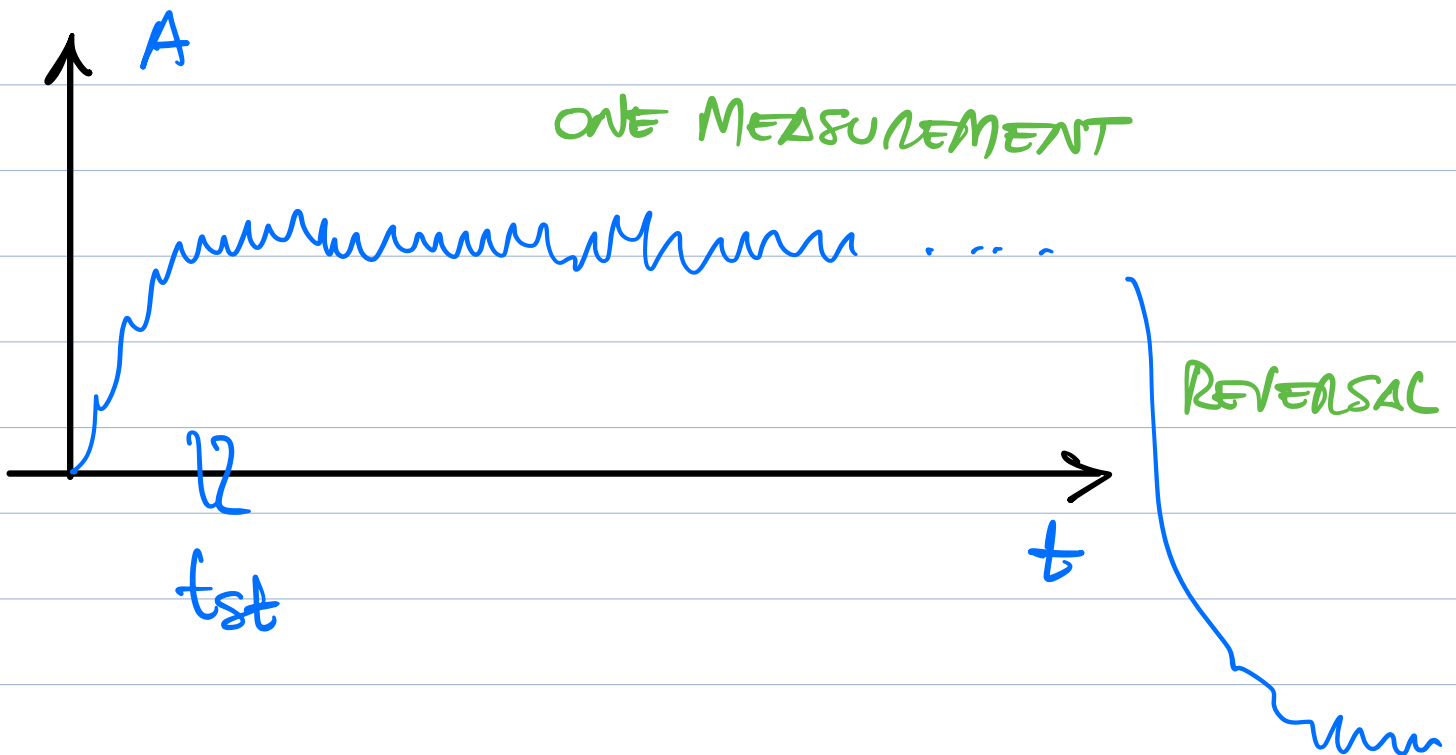
NOW, WHAT HAPPENS W/ THIS EQUATION.
IS A QUESTION OF TIME SCALES AND
DEFINITION OF $\langle \cdot \rangle$

EXPERIMENT

$t_{st} + \tau$

$$\bar{A} = \lim_{\tau \gg t_0} \lim_{t > t_{st}} \frac{1}{\tau} \int_{t_{st}}^{t_{st} + \tau} dt' A(t')$$

IN THE MAGNETIC (NICKEL) EXAMPLE $A =$ GLOBAL MAGN.



IF WE COMPARE :

- $\overline{A}$ WITH $\tau < \tau_{\text{rev}}$ TO $\langle A \rangle$ WITH NO PINNING FIELD

$$\overline{A} \neq \langle A \rangle = 0$$

- $\overline{A}$ WITH $\tau \gg \tau_{\text{rev}}$ TO $\langle A \rangle$ WITH NO PINNING FIELD

$$\overline{A} = \langle A \rangle = 0$$

- $\overline{A}$ WITH $\tau < \tau_{\text{rev}}$ TO $\langle A \rangle$ W/ h

$$\overline{A} = \langle A \rangle \neq 0$$

- $\overline{A}$ WITH $\tau \gg \tau_{\text{rev}}$ TO $\langle A \rangle$ W/ h

$$0 = \overline{A} \neq \langle A \rangle \neq 0$$

ERGODICITY DEPENDS ON TIME SCALES
AND HOW THE STAT AVERAGES ARE
COMPUTED

ORDER OF LIMITS
 $\hbar \rightarrow 0^+$ AND $N \rightarrow \infty$

MATTERS

SPONTANEOUS SYMM BREAKING

By CHOOSING ONE AMONG TWO (OR MORE) EQUIV. PHASES WHILE GOING ACROSS THE PH. TRANS

$\Rightarrow$ SPONTANEOUS SYMM BROKEN

$$\lim_{h \rightarrow 0^+} \lim_{N \rightarrow \infty} \langle S_i \rangle = - \lim_{h \rightarrow 0^-} \lim_{N \rightarrow \infty} \langle S_i \rangle$$

$$\neq 0$$

ENTROPY VS. ENERGY - PEIERLS

FROM THERMODYN WE KNOW THAT IN EQUIL

- ENERGY TENDS TO BE MINIMIZED
- ENTROPY MAXIMIZED.

IN OPEN CANONICAL CONDITIONS THE RELEVANT THERMODYN. POTENTIAL TO BE MINIMIZED IS THE FREE-ENERGY

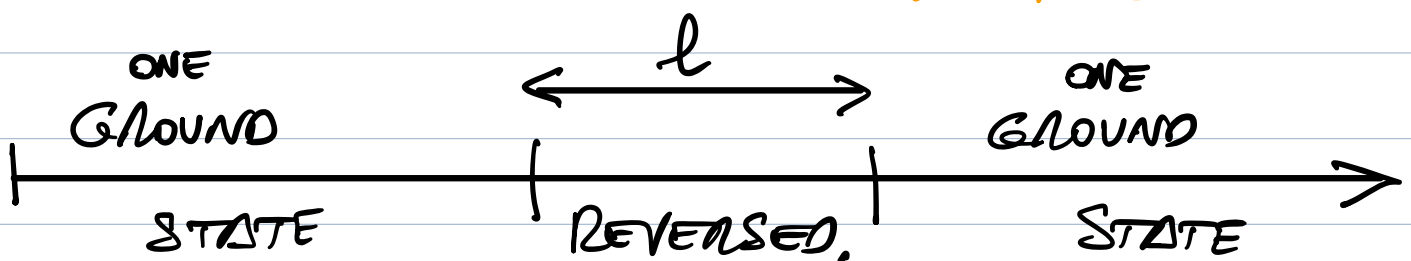
$$F = U - TS$$

$U = \langle H \rangle$ INTERNAL ENERGY

S ENTROPY

$$H = -J \sum_i S_i S_{i+1} \quad \text{EACH BOND ONCE}$$

EX. 1D ISING CHAIN W/ PBC



RE THE OTHER GROUND STATE

WE STUDY WHETHER $T=0$ GROUND STATE
 $\left\{ S_i = +1 \right\}$ IS STABLE
AGAINST TEMP FLUCT

LOWEST ENERGY EXCITATION :
ONE DOMAIN, TWO DOMAIN WALLS

TO SIMPLIFY DISCUSSION TAKE ONE
BACKGROUND - SAY ALL UP - AND
REVERSE A DOMAIN TO DOWN WITHIN

$$F_0 = U_0 - T S_0 \quad \text{GROUND STATE}$$

$$F_2 = U_2 - T S_2 \quad \text{LOWEST EXC.}$$

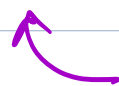
(THE SUBSCRIPT 2 $\equiv$ 2 DWs)

$$\Delta F = F_2 - F_0 \geq 0 \quad ?$$

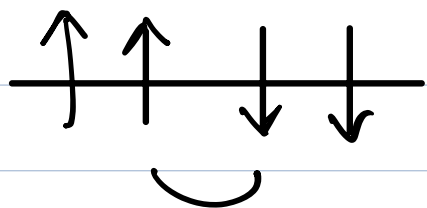
IF $< 0 \Rightarrow$ ORDERED WOULD
BY FLUCT

$$\Delta F = \Delta U - T \Delta S$$

$$\Delta U = U_2 - U_0 = 2 \cdot 2J = 4J$$



THE TWO WALLS OF
INTERVAL l



$$\begin{array}{lcl} \text{DW} & \Rightarrow & +J \\ \text{NO DW} & \Rightarrow & -J \end{array} \left. \vphantom{\begin{array}{l} \text{DW} \\ \text{NO DW} \end{array}} \right\} \text{DIFF: } 2J$$

ON ALL OTHER UNITS $U_2 - U_0 = 0$ locally

ΔS ONE CONF WITHOUT DW $S_0 = k_B \ln 1$
 N CONFS W/ ONE DOMAIN
 OF LENGTH l

$$S_2 = k_B \ln N$$

$$\Delta S = k_B \ln N - k_B \ln 1$$

$$\longrightarrow k_B \ln N$$

$T \neq 0$

$$\Delta F = 4J - k_B T \ln N \longrightarrow -k_B T \ln N$$

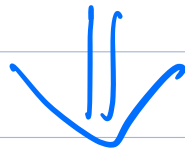
$N \gg 1$

J FINITE

if $T \neq 0$ ΔF IS NEGATIVE AND LOGAR
LARGE IN N

$\Rightarrow$ IT'S CONVENIENT TO HAVE
THE REVERSED DOMAIN

NEXT REPEAT THE ARGUMENT W/
ANOTHER REVERSED DOMAIN



1dIM DISORDERS AS SOON AS
 $T > 0$

BUT THERE CAN BE ORDER AT FINITE T :

1dIM W/ POWER LAW DECAYING
INTERACTIONS & FBC

RECALL $\alpha > d = 1$ ARGUED IN CH. 1
WITH ENERGY INTEGRAL
TO HAVE SHORT-RANGE INTER.

SEE CALC. IN THE NOTES

FOR SUFF. LONG-RANGE INT $\alpha < 1$

LONG-RANGE ORDER IS POSSIBLE

MANY MATH PAPERS w/ COND
FOR ABSENCE OF ORDER

2.4 OTHER METHODS

(WE WILL NOT TREAT THEM)

• LOW TEMP EXPANSIONS

— IDENTIFY GROUND STATES
(MINIMA OF THE ENERGY)
AND 1st EXCITED STATES

2 SUM OVER THESE CONFS ONLY

— MORE SYSTEMATIC

$$e^{-\beta H}$$

• SADDLE POINT APPROX $\beta \rightarrow \infty$ ($T \rightarrow 0$)

• EXPAND AROUND THE MINIMUM
2 TREAT NON-QUAD. TERMS
AS PERTURB w/ WICK.

• HIGH TEMP EXPANSION

EXPLOIT SOME RE-WRITING

$$e^{\beta J S_i S_j} = a (1 + b S_i S_j)$$

AND ORGANIZE THE PARTITION SUM

$\Rightarrow$ CLUSTER EXP

- DUALITY MAPPING $g \rightarrow \frac{1}{g}$

STRONG $\leftrightarrow$ WEAK COUPLING

- TRANSFER MATRIX SPECIFIC TO 2d-

WE WILL SEE IT WHEN DISCUSSING
QUANTUM CHAINS AND THEIR MAPPING
TO CLASSICAL 2d MODELS.

2.5 MEAN-FIELD

WE USE SPIN MODELS SINCE THEY ARE SIMPLER TO TREAT THAN PARTICLE ONES

YOU NEED TO LEARN CONCEPTS AND METHODS, AND HOW TO USE THEM

BACK TO SPIN MODELS, THE ONES WITH FM INTERACTIONS ARE EASIER.

ONE CAN GUESS THEIR HIGH / LOW T PHASES

BUT SOLVING THEM, e.g. FINDING ANALYT. EXPRESSIONS FOR T_c , ORDER PARAM. CRITICAL BEHAVIOUR : DIFFICULT

DONE IN $d = 1, 2, \infty$

MF METHODS GET APPROX RESULTS FOR T_c , ORDER PARAMETERS, CRITICAL BEHAVIOUR

THESE RESULTS ARE GENERICALLY
CORRECT

QUANTITATIVELY

AND TO A CERTAIN EXTENT ALSO

QUANTITATIVELY

BUT, e.g. CRITICAL BEHAVIOUR WRONG
BELOW d_u
SOMETIMES PHASE TRANS. PREDICTED
WHICH $\neq$, etc.

GENERIC MODEL

p-SPIN WITH FM INTER.

$$H(\{s_i\}) = -J \sum_{i_1 \dots i_p} s_{i_1} \dots s_{i_p} - h \sum_i s_i$$

FOCUS ON $p=2$, BUT $p \geq 2$ ALSO INTERESTING

POPULAR, EXTENSIONS $p > 2$

GLASSY PHYSICS
OPTIM. PBM

EVEN BLACK HOLE PHYSICS
IF QUANTUM

$p=2$ TWO BODY INTERACTION

AND KEEP J_{ij} & h_i LOCAL
IN DERIVATION BELOW

$$H = - \frac{1}{2} \sum_{\langle ij \rangle} J_{ij} s_i s_j$$

DOUBLE SUM NO SELF-COUPLING

KEEP IT GENERAL

$$\frac{1}{2} \sum_{\langle ij \rangle} \bullet = \frac{1}{2} \sum_i \sum_{j \neq i} \bullet$$

EQUIVALENT NOTATIONS

$1/2$ ENSURES THAT EACH PAIR $\Rightarrow$ ONCE

FACTORS AND NOTATION

$$\sum_{i \neq j} \mapsto \sum_{i \neq j} \quad \text{SINCE THERE'S NO SELF-INTERACTION}$$

$i \neq j$
only
once

$$\frac{1}{2} \sum_{i \neq j} \dots = \frac{1}{2} \sum_i \sum_{i \neq j}$$

TO AVOID SUMMING TWICE EACH PAIR

eg $d=1$

$$\sum_{i \neq j} s_i s_j = \sum_i s_i s_{i+1} = \dots s_1 s_2 + s_2 s_3 + \dots$$

$$\frac{1}{2} \sum_i \sum_{j \neq i} s_i s_j = \frac{1}{2} \left[\begin{aligned} &s_1 (s_0 + \sum_{i \neq 1} s_i) + \\ &+ \sum_{i \neq 2} s_2 (s_1 + s_3) + \\ &+ \sum_{i \neq 3} s_3 (s_2 + s_4) \end{aligned} \right] +$$

TERMS WITH s_2

$s_1 s_2$ APPEARS TWICE IN THE SUM

STANDARD INTRODUCTION OF MF

$$s_i = m_i + \delta s_i$$

$$\delta s_i \ll m_i$$

NON FLUCT

FLUCT

WILL ASK $m_i = \langle s_i \rangle$

CANDIDATE ORDER PARAMETER

eg. TWO BODY INTERACTION

$$s_i s_j = (m_i + \delta s_i) (m_j + \delta s_j)$$

$$\simeq m_i m_j + m_i \delta s_j + m_j \delta s_i$$

2nd ORDER IN δs DROPPED

USING AGAIN $\delta s_i = s_i - m_i$

$$s_i s_j \simeq m_i m_j + [m_i (s_j - m_j) + i \leftrightarrow j]$$

$$= m_i s_j + m_j s_i - m_i m_j$$

USE GENERIC TWO BODY COUP. J_{ij} & FIELDS ψ_i

$$-\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} s_i s_j \approx$$

← APPROXIMATION

$$-\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \left(\underbrace{m_i s_j + m_j s_i}_{\text{SPINS IN A FIELD}} - \underbrace{m_i m_j}_{\text{CONST WRT } \{s_i\}} \right)$$

IN BOLTZMANN FACTOR $\exp \left\{ -\beta \left[\left(-\frac{1}{2} \right) \sum_{i \neq j} J_{ij} s_i s_j - \sum_i h_i s_i \right] \right\}$

$$Z(\beta J_{ij}, \beta h_i) = e^{-\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} m_i m_j} \sum_{\{s_i = \pm 1\}} e^{+\frac{\beta}{2} \sum_{\langle ij \rangle} J_{ij} m_i s_j} e^{+\beta \sum_i h_i s_i}$$

JUST INDEPENDENT SPINS UNDER LOCAL FIELDS

$$h_i^{\text{loc}} = \underbrace{\sum_{j \in \partial_i} J_{ij} m_j}_{\text{GENERATED BY THE OTHER ONES}} + h_i$$


 ∂_i : NEIGHBOURS OF i
 THE EXT. FIELD PINNING SOURCE

IT IS AS IF THE HAMILTONIAN BECAME AN EFFECTIVE ONE

$$H_{\text{eff}} = - \sum_i h_i^{\text{loc}} s_i \quad \text{AND THE } \{s_i\} \text{-DEP}$$

BOLTZMANN WEIGHT

$$e^{\beta \sum_i h_i^{\text{loc}} s_i}$$

USING NOW $\sum_{s_i = \pm 1} e^{\beta h_i^{\text{loc}} s_i} = 2 \cosh(\beta h_i^{\text{loc}})$

FOR EACH i

$$\mathcal{Z}(\beta J_{ij}, \beta h_i) = e^{-\beta \sum_i \sum_{j \in \partial i} J_{ij} m_i m_j} \prod_i 2 \cosh \beta \left(\sum_{j \in \partial i} J_{ij} m_j + h_i \right)$$

RECALL
NOTATION

$\sum_{j \in \partial i}$: SUM
OVER
NEIGHB
OF SITE i

THE FREE-ENERGY DENSITY IS

$$-\beta f = \frac{1}{N} \ln \mathcal{Z}$$

SO TAKING THE $\ln$:

(1)

$$-\beta f(\beta J_{ij}, \beta h_i) = -\frac{1}{N} \frac{\beta}{2} \sum_i \sum_{j \neq i} J_{ij} m_i m_j + \frac{1}{N} \sum_i \ln \left[2 \cosh \beta \left(\sum_{j \neq i} J_{ij} m_j + h_i \right) \right]$$

THE SELF-CONSISTENCY IS

$$\langle s_i \rangle = m_i$$

WITHIN THIS
APPROX

QUANTITY
 $\Theta(1)$, SINGLE
SITE ONE

$$\langle s_i \rangle = \frac{\sum_{\{s_j = \pm 1\}} s_i e^{\beta \sum_j J_{ij} s_j}}{\sum_{\{s_j = \pm 1\}} e^{\beta \sum_j J_{ij} s_j}}$$

IN J_{ij} THERE'S J_{ij} AS ONE TERM

IN GENERAL $\forall H - \sum_i h_i s_i$

$$Z = \sum_{\{s_i\}} e^{-\beta H + \beta \sum_i h_i s_i}$$

$$\frac{\partial \ln Z}{\partial h_i} = \frac{1}{Z} \sum_{\{s_i\}} e^{-\beta H + \beta \sum_i h_i s_i} \beta s_i$$

$$= \beta \langle s_i \rangle$$

$$-\beta f = \frac{1}{N} \ln Z$$

$$\frac{\partial (-\cancel{\beta} f N)}{\partial h_i} = \cancel{\beta} \langle s_i \rangle$$

$$- \frac{\partial (f N)}{\partial h_i} = \langle s_i \rangle$$

ONLY ASSUMPTION h_i IN 2ND TERM BUT NOT IN FIRST H TERM.

METHOD OF SOURCES

ADD TO $H \rightarrow H + J \cdot \Theta$

SOURCE J

OBSERVABLE Θ

$$Z = \sum_{\text{conf}} e^{-\beta H} \rightarrow$$

$$Z[J] = \sum_{\text{conf}} e^{-\beta H + \beta J \Theta}$$

$$\frac{1}{\beta} \frac{\partial \ln Z[J]}{\partial J} = \frac{1}{Z} \sum_{\text{conf}} e^{-\beta H + \beta J \Theta} \Theta$$

$$= \langle \Theta \rangle \quad \swarrow \text{CALCULATED WITH } J$$

$$-\beta f = \ln Z \Rightarrow - \frac{\partial f}{\partial J} \Big|_{J=0} = \langle \Theta \rangle_{\text{no } J}$$

$$\frac{1}{\beta} \frac{\partial \ln Z(\{\beta J_{ij}, \beta h_i\})}{\partial h_i} = \langle s_i \rangle$$

THE $\ln$ IS HERE TO GET THE NORM. & $1/\beta$ TO GET RID OF $(\beta) s_i$

USING NOW

ACTING AS A SOURCE

$$-\beta f = \frac{1}{N} \ln Z$$

$$\Rightarrow \boxed{-\frac{\partial (Nf)}{\partial h_i} = \langle s_i \rangle}$$

$\mathcal{O}(1)$
 AVERAGE UNDER h_i
 IF WANTED AT $h_i = 0$
 $\Rightarrow$ EVALUATE $h_i = 0$
 AFTER TAKING $\partial/\partial h_i$

WE THEREFORE HAVE

$$m_i = \langle s_i \rangle = \frac{1}{\beta} \frac{\partial \ln Z}{\partial h_i} = - \frac{\partial (Nf)}{\partial h_i}$$

NOW, APPLY THIS TO THE MEAN-FIELD $(-\beta f)$

BACK TO THE EXPRESSION ABOVE FOR $(-\beta f)$ (1)

THE 1ST TERM IS INDEP. OF h_i

THE 2ND TERM DOES DEP. ON h_i

WE FIND *

$$m_i = \frac{1}{\beta} \frac{\partial}{\partial h_i} \sum_k \ln \left[2 \cosh \beta \left(\sum_{j \in \partial k} J_{kj} m_j + h_k \right) \right]$$

$$= \frac{1}{\beta} \sum_k \frac{2 \sinh \beta \left(\sum_{j \in \partial k} J_{kj} m_j + h_k \right)}{2 \cosh \beta \left(\sum_{j \in \partial k} J_{kj} m_j + h_k \right)} \delta_{hi} \beta$$

$$m_i = \tanh \beta \left(\sum_{j \in \partial i} J_{ij} m_j + h_i \right)$$

IN THE CONTEXT OF DISORDERED
SYSTEMS, WHERE J_{ij} DEPENDS ON ij
AND TAKEN FROM A PDF :
NAIVE MF EQS (TAP)

A GENERALIZATION OF THE USUAL
FORM, RECOVERED FOR $J_{ij} = J$ FOR $m_i = m$
 $h_i = h$ WHICH

$$m = \tanh \left(\beta J \sum_{\partial i} m + \beta h \right) \quad (2)$$

$\sum_{\partial i} \bullet = z$ # NEIGHBOURS OF SITE i DEP ON
LATTICE, e.g. $z=4$ 2d $\square$.

* COMMENT / CLARIFICATION

WE GET AN EQ FOR m_i THE SOL. OF WHICH DEPENDS ON h_i .

WHY HAVEN'T WE CONSIDERED $\frac{\partial m_j}{\partial h_i}$ AS WELL WHEN TAKING $\frac{\partial f}{\partial m_i}$?

$$-\beta f(\beta J_{ij}, \beta h_i) = -\frac{1}{N} \beta \sum_i \sum_j J_{ij} m_i m_j + \frac{1}{N} \sum_i \ln \left[2 \cosh \beta \left(\sum_j J_{ij} m_j + h_i \right) \right]$$

$$\frac{\partial f}{\partial h_i} = \underbrace{\frac{\partial f}{\partial h_i}}^{\text{explicit}} + \sum_k \frac{\partial f}{\partial m_k} \frac{\partial m_k}{\partial h_i}$$

WHAT WE
CALCULATED

$$N \frac{\partial f}{\partial m_k} = \beta \sum_{i=k} J_{ki} m_i$$

$$-\beta \sum_i \frac{\cancel{2} \cosh \beta \left(\sum_j J_{ij} m_j + h_i \right)}{\cancel{2} \cosh \beta \left(\sum_j J_{ij} m_j + h_i \right)} \frac{\partial}{\partial m_k} \left(\sum_j J_{ij} m_j \right)$$

$$\sum_{\partial i} J_{ij} \delta_{jk}$$

CUTS THE SUM OVER
 $j = \partial i$ TO BE ONLY k
 $\Rightarrow$ CUTS SUM OVER i TO
 BE CONNECTED TO k

$$N \frac{\partial f}{\partial m_k} = \beta \sum_{i=\partial k} J_{ki} m_i$$

$$- \beta \sum_{i=\partial k} J_{ik} + \beta \left(\sum_{\partial i} J_{ij} m_j + h_i \right)$$

NB. IT'S $\beta \sum_{i=\partial k}$ APPLIED TO WHAT WE GET FROM

$$\frac{\partial f}{\partial h_i} \text{ EXPLICIT DERIVATIVE}$$

SO, ONE CAN TAKE

$$-N \frac{\partial f}{\partial h_i}^{\text{EXPLICIT}} = m_i$$

TO GIVE THE EQ. FIXING m_i . OTHER WAYS TO
 JUSTIFY IT TOO.

WE REPLACED THE TWO-BODY INTER BY A ONE
SITE PROBLEM DECOUPLED FROM OTHERS, SEEN
"ON AVERAGE"

THE NUMERICAL FACTORS. — A CHECK

- LET'S CHECK WHETHER WE GET $m = \pm 1$
FOR $T \rightarrow 0$ AND $R = 0$ FROM THE EQ.

$$m = \lim_{\beta J \rightarrow \infty} (\beta J z m + \beta h) \rightarrow 0$$

$$\begin{aligned} \text{— FOR } \beta J \rightarrow \infty \quad h > 0 \quad \lim_{\beta J \rightarrow \infty} \beta h &\rightarrow 1 \\ \implies m &= 1 \quad \underline{\text{OK}} \end{aligned}$$

$$\begin{aligned} \text{— FOR } \beta J \rightarrow \infty \quad h < 0 \quad \lim_{\beta J \rightarrow \infty} \beta h &\rightarrow -1 \\ \implies m &= -1 \quad \underline{\text{OK}} \end{aligned}$$

- DO WE GET $m \rightarrow 0$ FOR $\beta \rightarrow 0$ AND ANY h ?

$$m = \tanh(\beta J z m + \beta h)$$

$$= \tanh 0 = 0 \Rightarrow m = 0 \quad \underline{\text{OK}}.$$

- m IS IN $[-1, 1]$ SINCE $\tanh$ IS IN $[-1, 1]$
- NOTE THAT THE "TRICKY" FACTOR

$$\beta J z$$

IS HARD TO CHECK. ONE MEMOTECHNIQUE RULE

$$z \rightarrow N-1 \quad J \rightarrow J/N$$

$$\beta J z \rightarrow \beta J \Rightarrow T_c = J$$

FULLY CONNECTED MODEL WITH EACH PAIR COUNTED ONCE.

FREE-ENERGY LANDSCAPE

ONE CAN DEF. AN ORDER PARAM.
AND CONTROL PARAM. DEP. FUNCTION

$$f(m; \beta J, \beta h)$$

FROM WHICH THE EQ. (2) COMES TO

$$\frac{\partial f(m; \beta J, \beta h)}{\partial m} = 0$$

$$f(m) = \frac{J_2 m^2}{2} - \frac{1}{\beta T} \ln 2 \cosh(\beta J_2 m + \beta h)$$

I WRITE ONLY THE
ORDER PARAM. DEP. IN f

IN THE FOLLOW.
 $J_i = J$ & $h_i = h$
TO SIMPLIFY
DISCUSSION.

LET'S CHECK IT:

$$\frac{\partial f(m)}{\partial m} = 0 =$$

$$\cancel{\frac{2Jzm}{2}} - \cancel{\ln 2} \frac{\cancel{2} \cancel{8h} (\beta J m z + \beta h)}{\cancel{2} \cancel{ch} (\beta J m z + \beta h)} \cancel{\beta J z}$$

$$\Rightarrow m = \tanh(\beta J m z + \beta h)$$

AND IT IS OK

IT CAN BE GENERALIZED TO

$$f(\{m_i\}; \beta J_{ij}, \beta h_i)$$

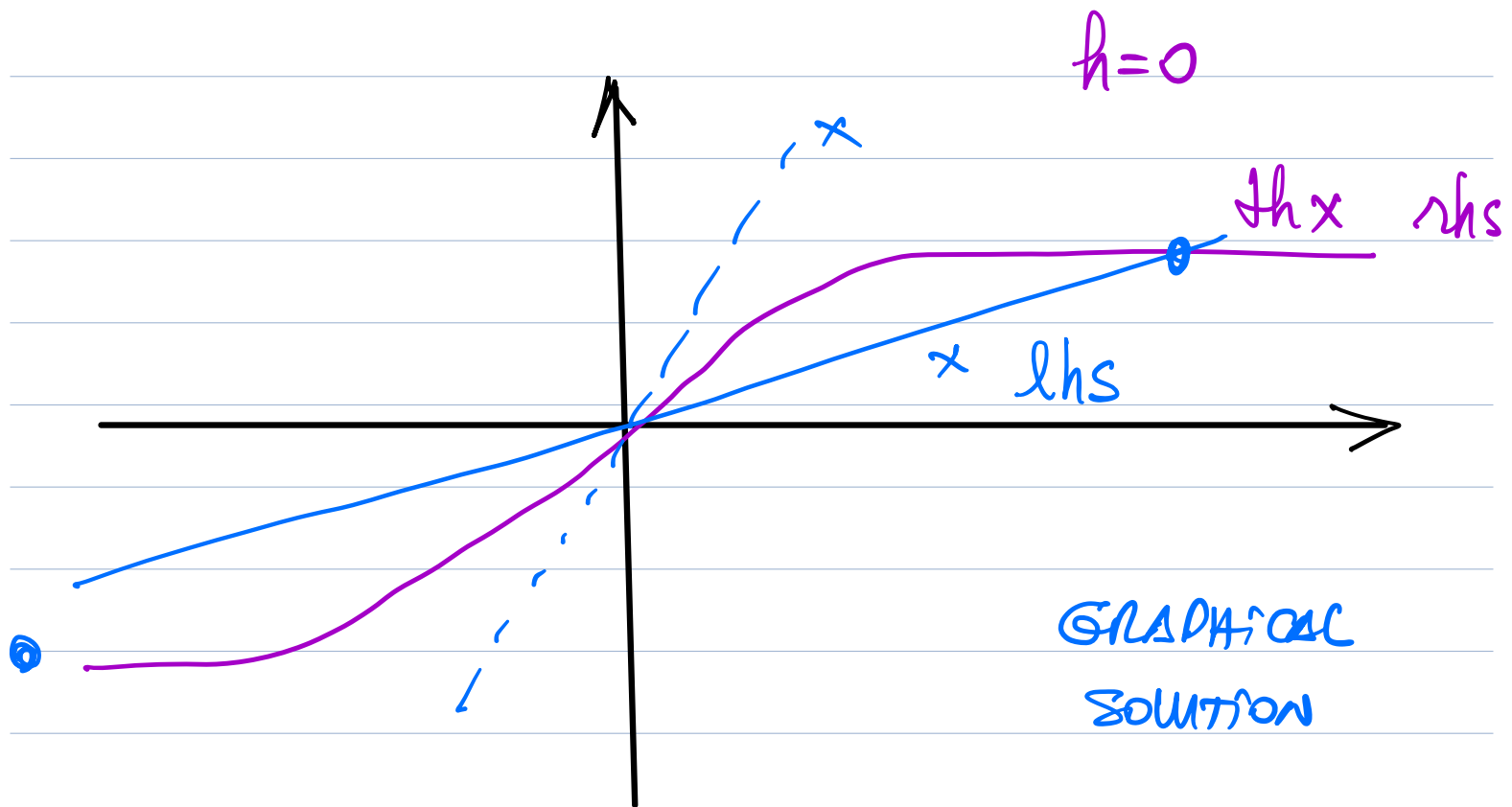
MEAN-FIELD EQ

ANALYSIS



MEAN FIELD EQUATION ANALYSIS

$$m = \tanh(\beta z J m + \beta h)$$



CHANGE IN # SOLS WHEN SLOPE AT $m \approx 0$ OF lhs = SLOPE rhs

- ONE SOLUTION $m=0$ if $zJ\beta < 1$
- THREE SOLUTIONS $m=0, m_+ = -m_-$ OTHERWISE

$$zJ\beta_c = 1$$

NB NUM.

FACTORS DEP
ON LATTICE (Z)
AND CONNECTION

PROPERTIES

- HIGH T only $m=0$ NO ORDER
- LOW T symm $m \leftrightarrow -m$

WHICH ONE WINS? $|m| \neq 0$ OR $m=0$
WHO IS STABLE?

ITERATIVE SOLUTION & CONVERGENCE

$$m = \tanh(\beta J z m) \quad h=0$$

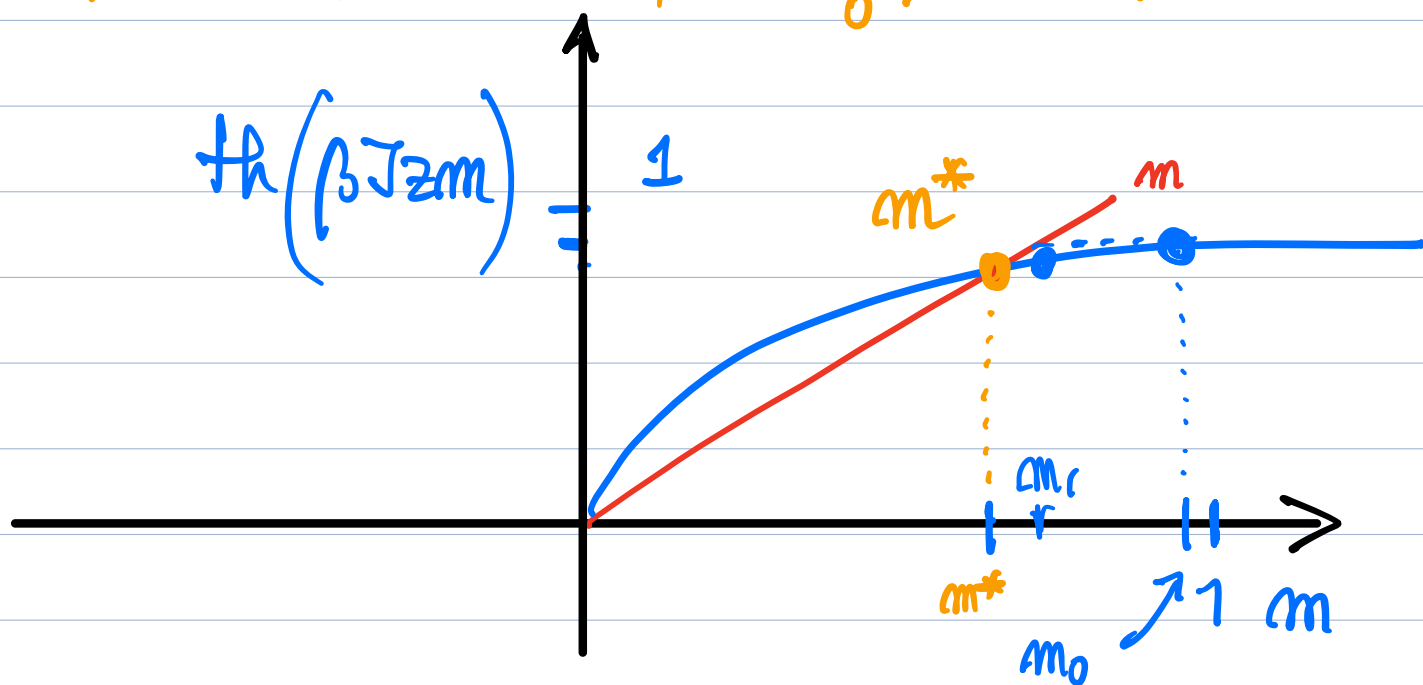
ITERATIVE SOLUTION

SAY YOU START FROM $m_0 = 1$

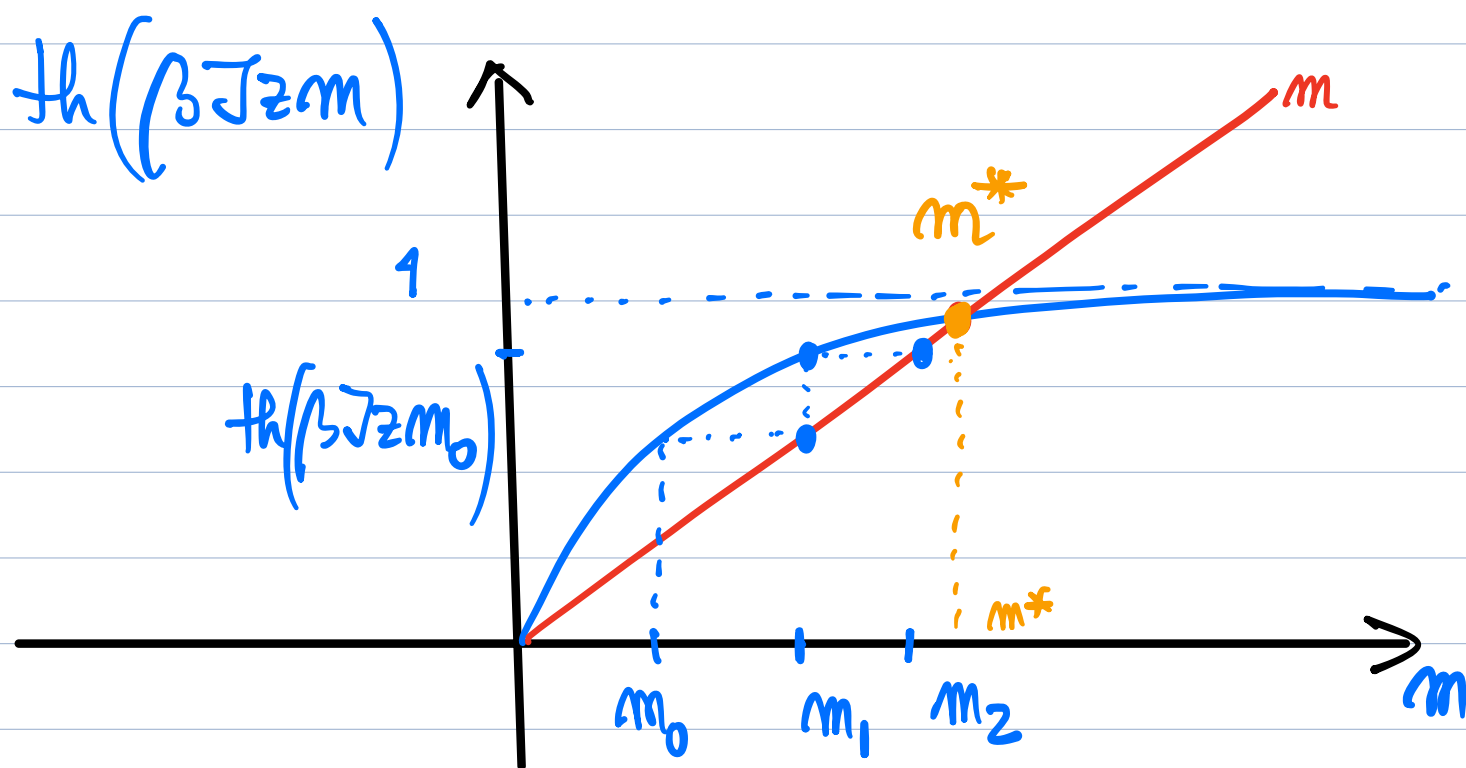
$$m_1 = \tanh(\beta J z m_0) < 1 \quad \text{IF } \beta J z > 1$$

$m_2 = th(\beta J_2 m_1) < m_1$ EX UNTIL
FIXED POINT REACHED

EX STARTING FROM $m_0 > m^* > 0$

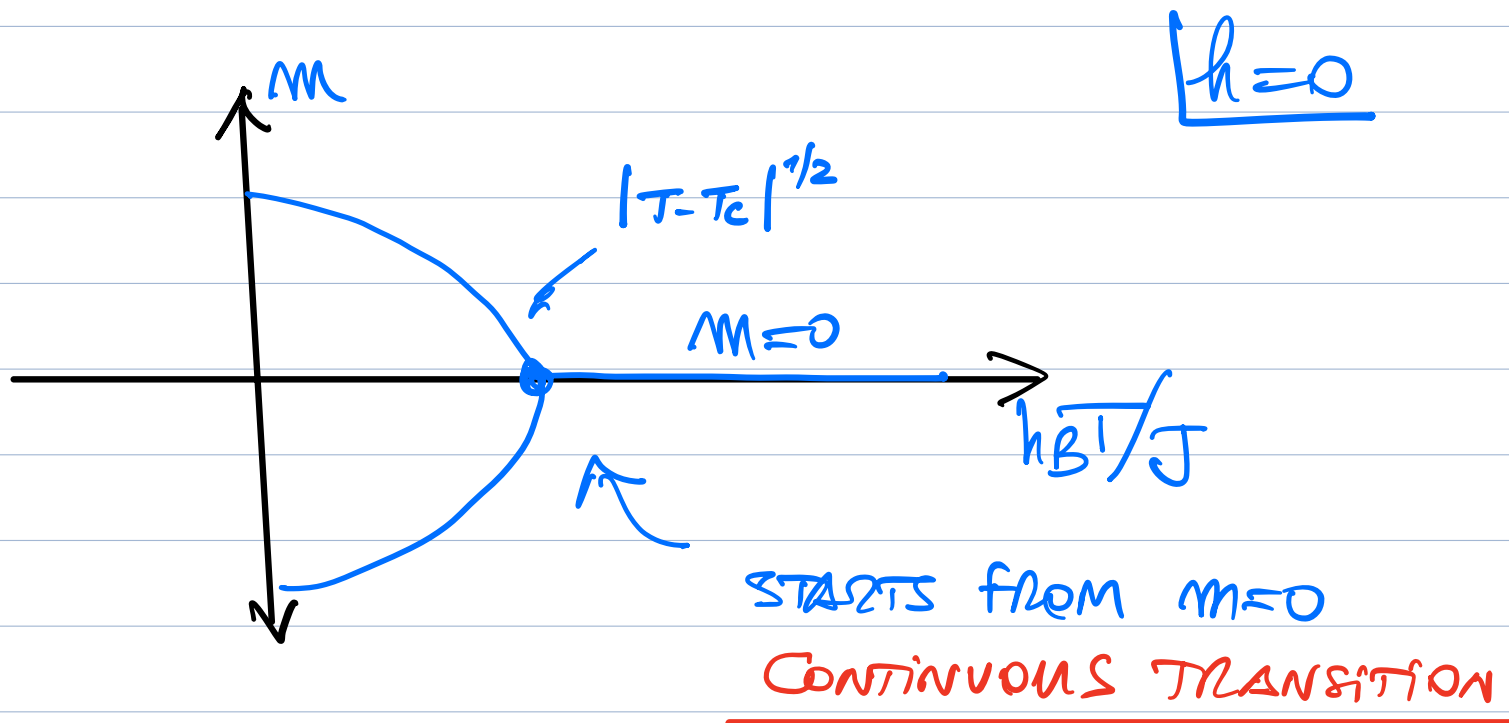


NOW, FROM $0 < m_0 < m^*$



STARTING FROM A NEGATIVE VALUE $m_0 < 0 \Rightarrow m^* < 0$

THE CONTROL PARAM. DEP. OF THE ORDER
PARAM. m



IF WE INTERPRET CHANGE IN # SOL AS
THE PHASE TRANSITION
BUT IF WE SUM OVER TWO BRANCHES
NEEDED AT LOW $T \Rightarrow 0$ AGAIN!

SEE PINNING FIELD BELOW

EXPERIMENTS WERE SHOWING MEASUREMENTS
WITH SMALLER VALUES OF β IN $d=3$
SAMPLES

CASTING DOUBTS ABOUT MF THEORY
CONNECTEDNESS CLOSE TO T_c

THE EXTERNAL PINNING FIELD

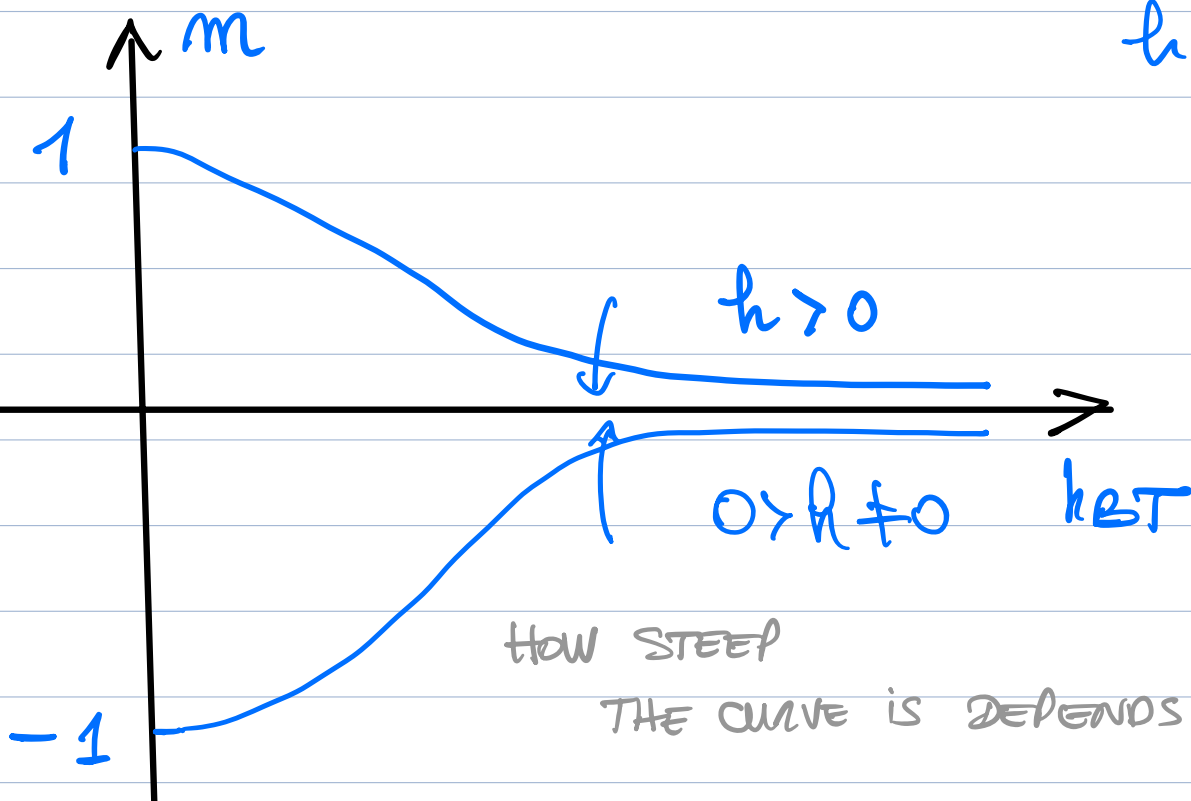
$$m = \tanh \beta (Jz m + h)$$

IS $m=0$ A SOLUTION IF $h \neq 0$? NO!

$$0 = \tanh \beta h \neq 0$$

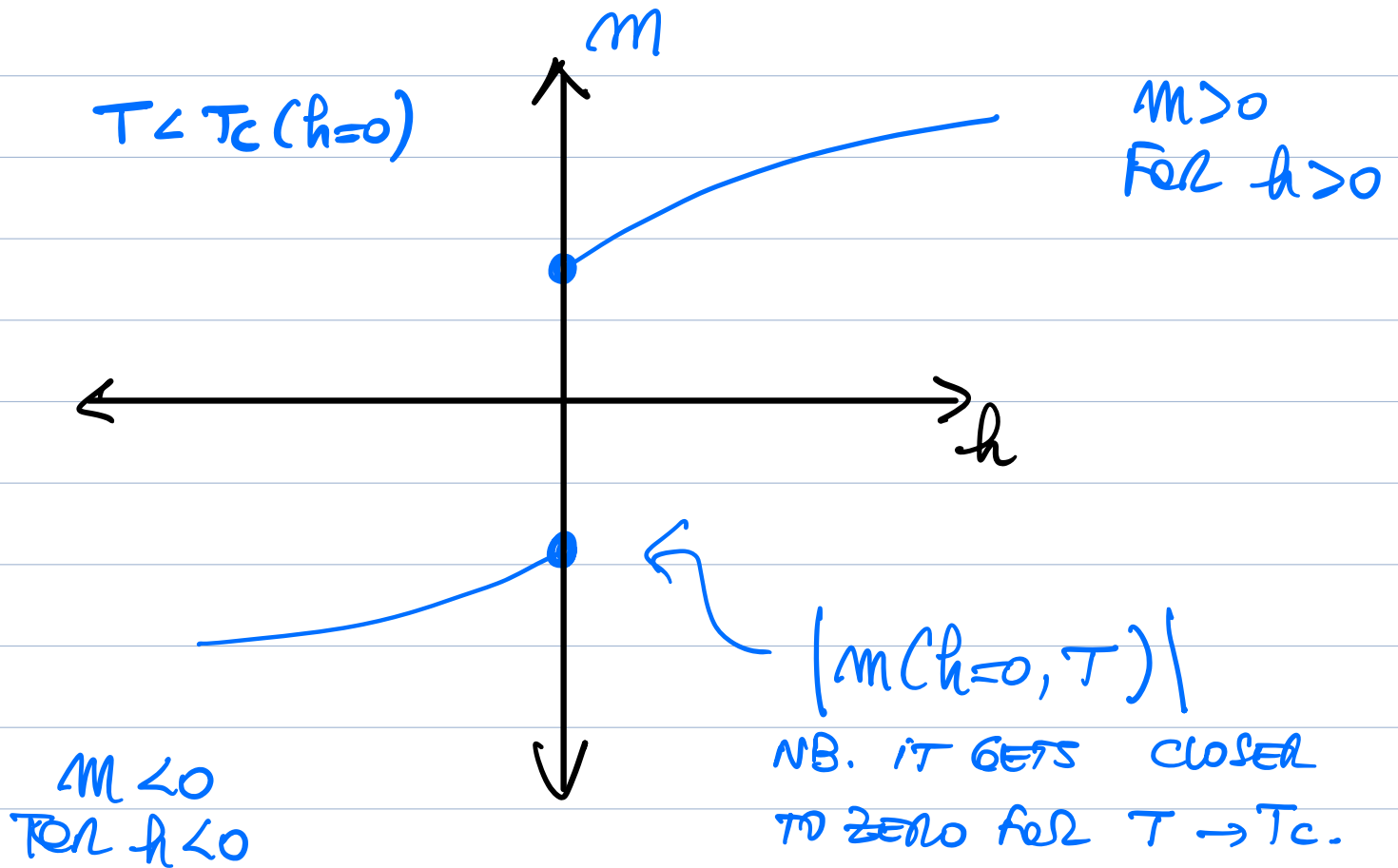
$$\Rightarrow \boxed{m \neq 0} \quad \forall k_B T < \infty, h \neq 0$$

only for
 $k_B T \rightarrow \infty$
 h finite
 $m \rightarrow 0$



$$\lim_{h \rightarrow 0^+} m(h) = m > 0 \quad T < T_c(h=0)$$

$$\text{S.M.L. For } h \rightarrow 0^- \quad m < 0$$



NOTE DISCONTINUITY OF m AT $h=0$

How does $m \rightarrow 0$ with $h \rightarrow 0$
AT PRECISELY T_c ? LATER

MEANING OF THE EQUILIBRIUM STATE

WITH A POSITIVE PINNING FIELD

- AT $T=0$ THERE IS A SINGLE CONF
w/ $S_i = +1$ AND $m = 1$
WHICH IS THE GROUND STATE
- AT $T = \epsilon > 0$ FLUCTUATIONS ARE
SWITCHED ON & $m \lesssim 1$

ALREADY N CONFS CONTRIBUTE TO
 $m = 1 - 2/N$

THE NON-VANISHING T EQUIL. STATES
ARE LARGE ENSEMBLES OF
MICRO CONFS.

MACROSTATES

VS.

MICROSTATES

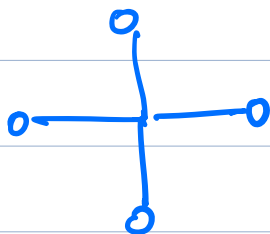
HOW DOES T_C DEPEND ON LATTICE?

$$k_B T_C = zJ$$

↑ if J ↑ ok

if MODEL ON A SQUARE LATTICE

$$z = 4$$



T_C ↑ if z ↑ WHICH IS EXPECTED

⇒ ORDERED FAVOURED BY MORE INTER.

BUT $z = 2$ in $d=1$ ⇒ $k_B T_C = 2J$
⇒ PROBLEM !

MF FAILS AT LOW d !

THE ORDER PARAM (h=0) - β EXPONENT

$$m = \frac{1}{\beta} \ln \left(\beta J z m + 1 \right)$$

$$m \simeq \beta J z m - \frac{1}{3} \left(\beta J z m \right)^3$$

$\xrightarrow{0}$
 $\xrightarrow{m^2}$

$$m^2 \simeq \frac{3}{(\beta J z)^3} \left[1 - \beta J z \right]$$

$\beta = \beta_c \rightarrow$

$$\frac{1}{k_B T} (k_B T - k_B T_c)$$

$$m \sim \sqrt{3} \beta_c \left| k_B T - k_B T_c \right|^{1/2}$$

THE NUMERICAL
PREFACTOR $\sqrt{3}$ IS

NOT OF INTEREST
[m] = 1 $\Rightarrow$ ok

CRITICAL EXPONENT
CONTROLS HOW

THE ORDER PARAM VANISHES

$$\beta = 1/2$$

$$\ln m \sim \text{ct.} + \frac{1}{2} \ln (k_B T_c - k_B T) = \text{ct.}' + \frac{1}{2} \ln (1 - T/T_c)$$

THE MAGN. SUSCEPT - γ EXPONENT

VARIATION OF THE ORDER PARAM
WRT CONJUGATE FIELD

$$m = \frac{1}{h} \beta (\gamma z m + h)$$

THE SOLUTION IS $m(h)$

COMPUTE $\frac{\partial m(h)}{\partial h}$ AND TAKE $h \rightarrow 0$

CALLING $m(h) = m_h$ & TAKING $\frac{\partial m_h}{\partial h}$

$$\frac{\partial m_h}{\partial h} = \frac{\beta \left(\gamma z \frac{\partial m_h}{\partial h} + 1 \right)}{h^2 \beta (\gamma z m_h + h)} \Rightarrow$$

$$\frac{\partial m_h}{\partial h} = \frac{\beta}{\left[h^2 \beta (\gamma z m_h + h) - \beta \gamma z \right]}$$

DIMENSIONS $[m] = 1$ $[h] = [k_B T] \Rightarrow [\gamma] = [\beta]$

RECALL $h_{BTc} = J_2 = 1/\beta_c$

• $T > T_c$ χ_+ $m_h = 0, h = 0$

$$\chi_+ = \left. \frac{\partial m_h}{\partial h} \right|_{h=0} = \frac{\cancel{\beta_c} \cancel{h_{BTc}}^1}{(h_{BT} - h_{BTc})}$$

DIMENSION OF FACTOR IS 1

$\chi_+ = 1$

CAREFUL THIS IS THE EXPONENT $\rightarrow$

• $T < T_c$ χ_- $m_h \propto (T_c - T)^\beta$

$$\chi_- = \left. \frac{\partial m_h}{\partial h} \right|_{h=0} \simeq \text{EXPAND } m_h \simeq 0$$

$$\cosh x \simeq \frac{e^x + e^{-x}}{2} =$$

$$= \frac{1}{2} \left[\left(1 + x + \frac{x^2}{2} \right) + \left(1 - x + \frac{x^2}{2} \right) \right] + \dots$$

$$= 1 + \frac{x^2}{2} + \dots$$

$$ch^2x \simeq \left(1 + \frac{x^2}{2} + \dots\right)^2 \simeq 1 + x^2 + \dots$$

$$\chi_- \simeq \frac{\beta_c}{1 + \beta^2 (J_z m_h + h)^2 - 1}$$

$\nearrow$ AT $\beta = \beta_c$

SET $h=0$ AND $m_{h \rightarrow 0} \simeq \sqrt{3\beta_c} |k_B T - k_B T_c|^\beta$
 $\beta = \beta_c$

$$\chi_- = \frac{\beta_c}{\beta_c^2 \left(\frac{J_z}{2}\right)^2 (k_B T_c - k_B T)^{2\beta} \frac{3}{2}}$$

$$= \frac{1}{3} (k_B T_c - k_B T)^{-2\beta - 1}$$

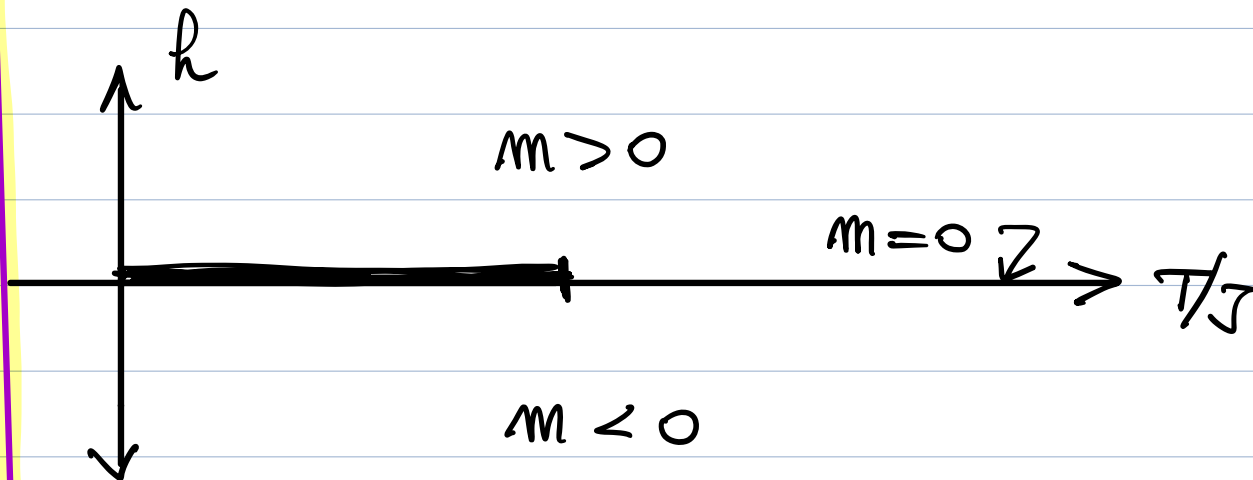
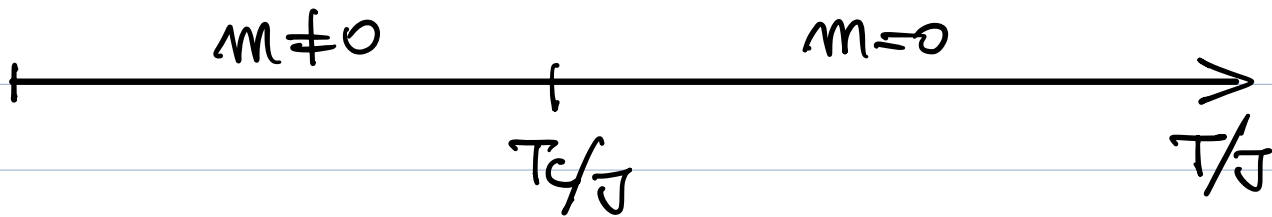
$[\chi_-] = [\beta]$ ok.

NB: THE NUM. PREF. IS $\neq$. $a_+ \neq a_-$

$$2\beta = \chi_- = 1$$

DEPENDENCE ON FIELD - δ EXPONENT

$h=0$



WHAT HAPPENS GOING ACROSS $h=0$
AT FIXED $T < T_c (h=0)$?

A JUMP IN $m > 0$ TO $m < 0$
OR $m < 0$ TO $m > 0$

DISCONTINUOUS
TRANSITION

$$m = \frac{h}{J} \rho(Jzm + h)$$

EXPAND m, h SMALL

$$m \sim \beta (Jz_{m+h}) - \frac{\beta^3 (Jz_{m+h})^3}{3} \quad \beta = \beta_c$$

$$0 \sim h - \frac{\beta_c^2 (Jz_{m+h})^3}{3}$$

if $m \sim h^{1/8}$ AND $1/8 < 1 \Rightarrow$ DROP h WRT m !

$$h \sim \frac{\beta_c^2}{3} (Jz)^3 m^3 \Rightarrow$$

$$m \sim h^{1/3}$$

$$\delta = 3$$

CONSISTENTLY

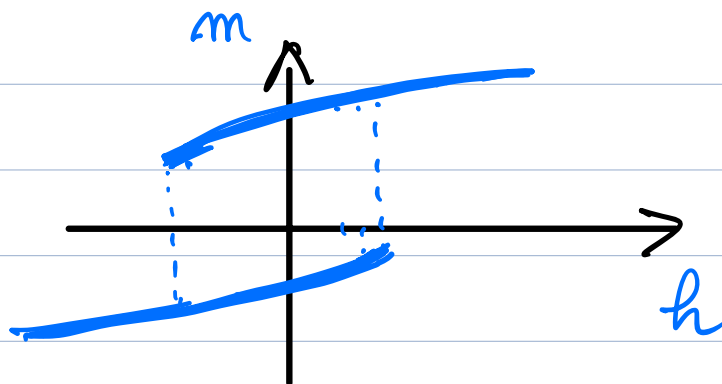
$$1/8 < 1$$

PREFACTOR $(3\beta_c)^{1/3}$

$$[m] = [\beta_c h]^{1/3} = 1 \text{ ok.}$$

HYSTERESIS

IN AN ACTUAL EXP., SINCE THE
"INITIAL" STATE IS STILL STABLE*
AFTER GOING ACROSS THE TRANS. $\Rightarrow$
ABRUPT CHANGE CAN OCCUR LATER



* LATER f

SUMMARY OF MEAN-FIELD (SO FAR)

- MEAN-FIELD FOR UNIFORM m

(IN NOTES, DONE FOR POSSIBLY
LOCAL m_i IF J_{ij} NOT ALL THE
SAME, IE HETEROGENEOUS)

$$m_i + \delta s_i = s_i$$

$$\langle \delta s_i \rangle = 0$$

DROP $\delta s_i \delta s_j$ TERM IN BOITZMANN WEIGHT

- $f(m) = \frac{Jzm^2}{2} - hm - k_B T \ln [2 \cosh \beta (Jzm + h)]$

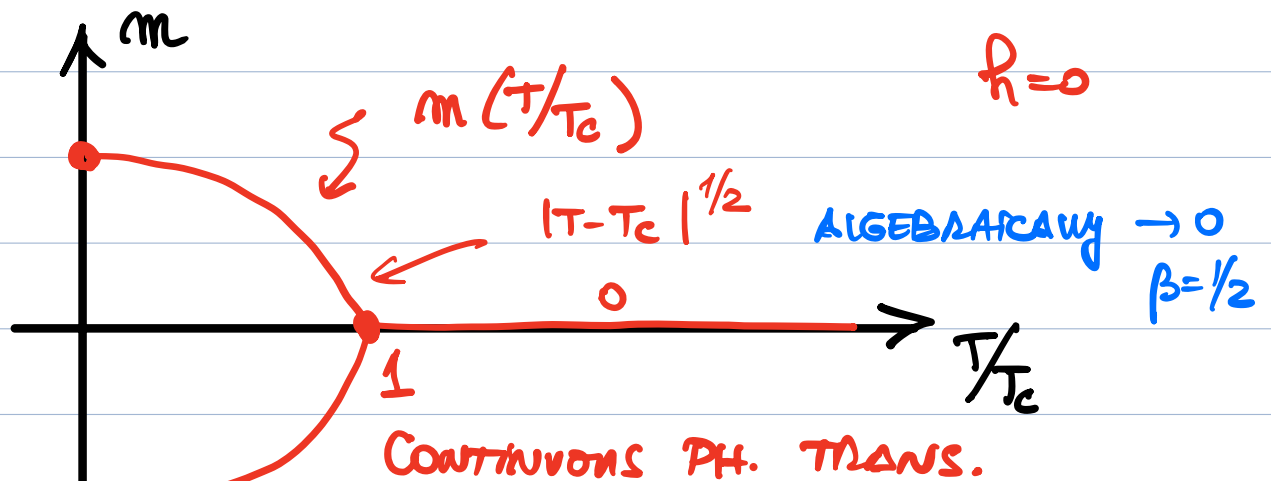
- MF EQ FROM $\frac{\partial f(m)}{\partial m} = 0$

SO FAR WE SAW $f(m)$ AS DETERM BY  CONDITION

- MF EQ $m = \tanh(\beta J z m + \beta h)$

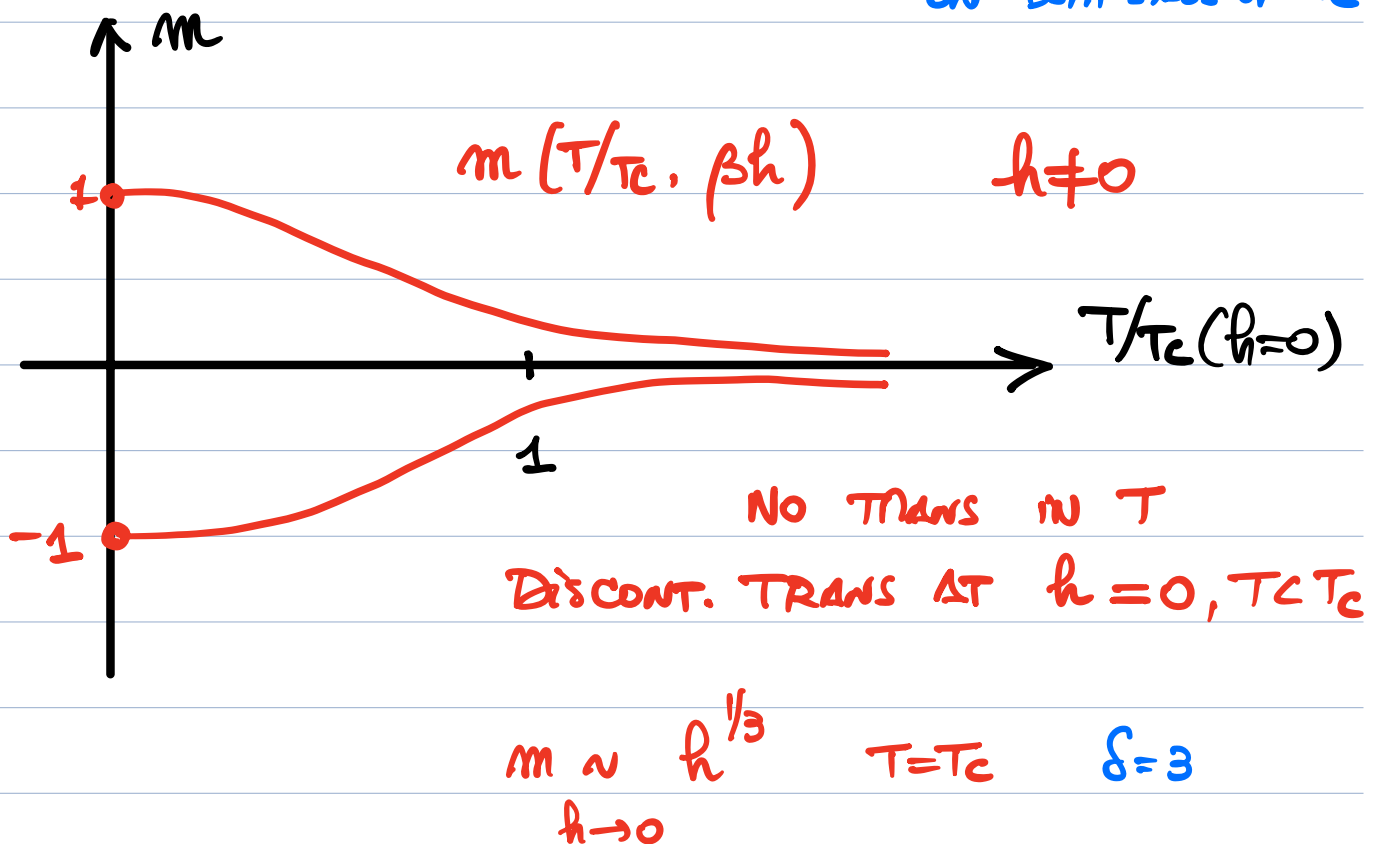
2.5.3 ORDER PARAMETER

$$h_B T_C(h=0) = J z$$

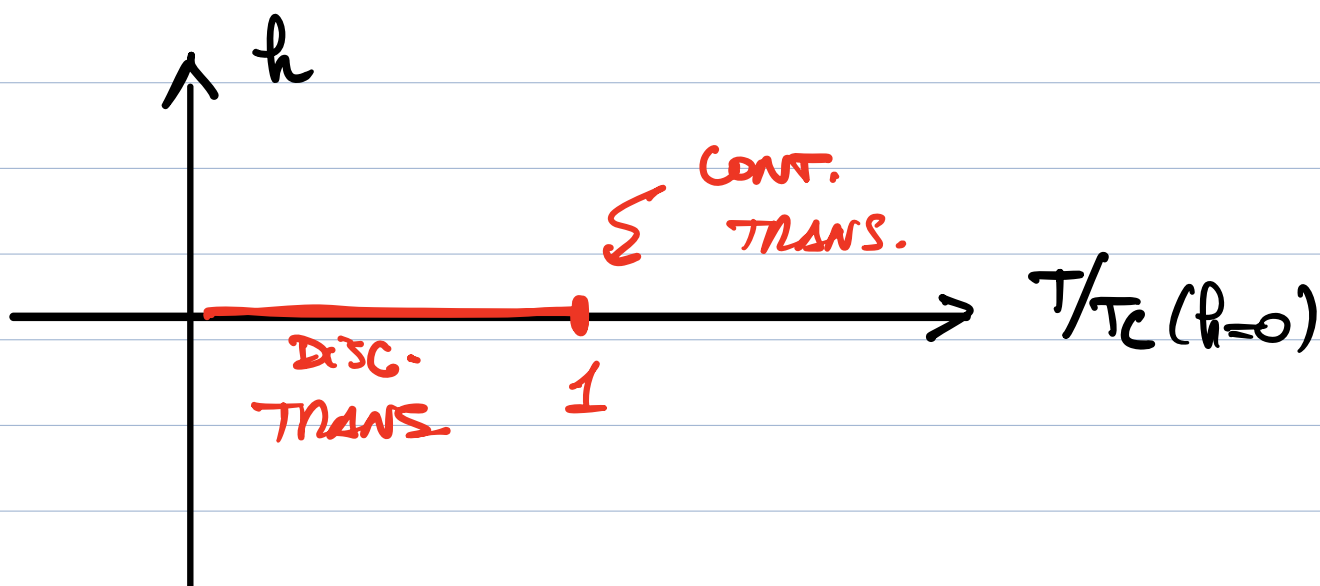


$$\chi = \left. \frac{\partial m_h}{\partial h} \right|_{h \rightarrow 0} \propto |T-T_c|^{-1}$$

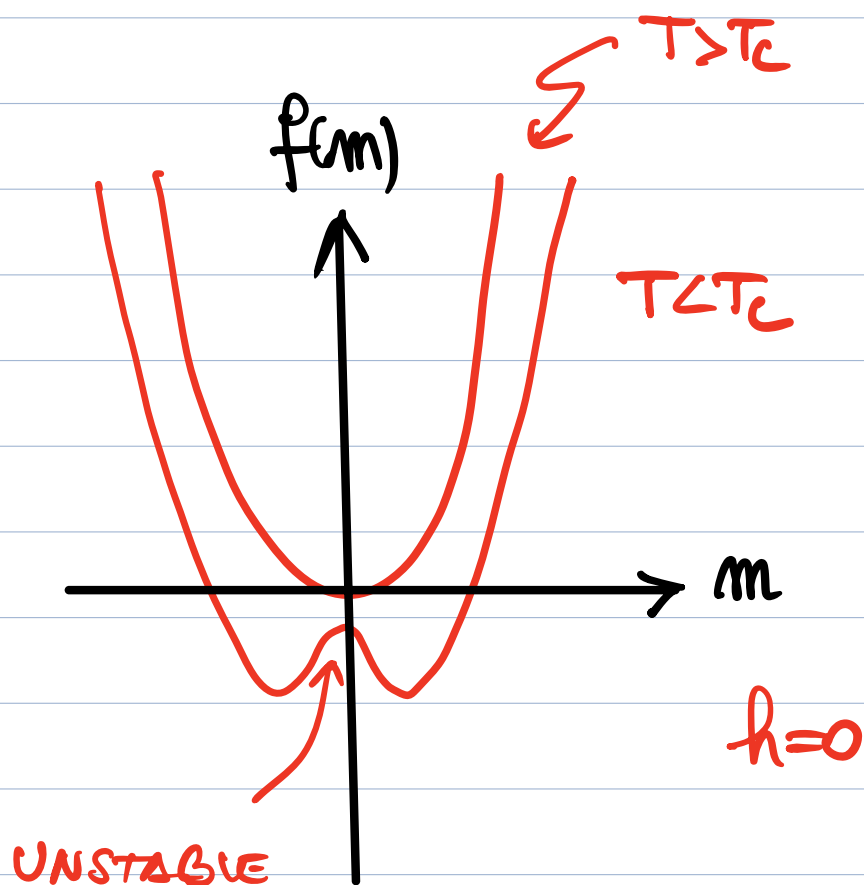
ALGEBRAIC DIVERGENCE $\gamma=1$
ON BOTH SIDES OF T_c



PHASE DIAGRAM



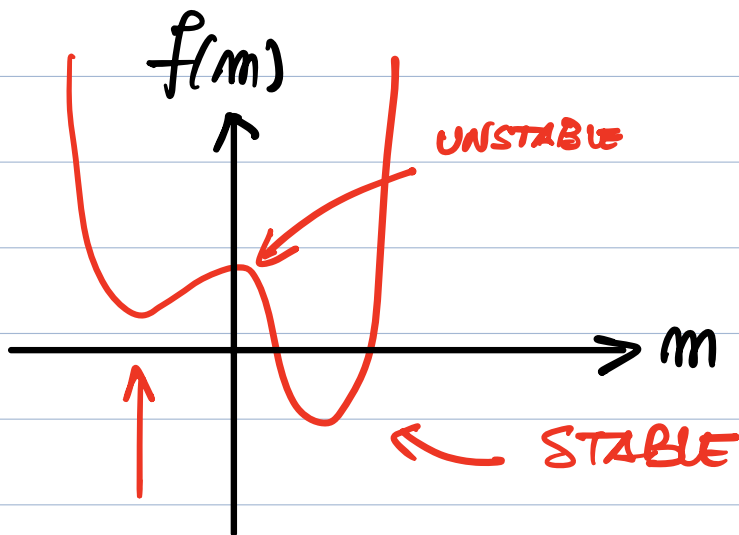
FREE-ENERGY DENSITY AS fct ORDER PARAM



AND GETS
TILTED BY h

EFFECT OF A FIELD

- DISCONTINUOUS TRANS
- HYSTERESIS

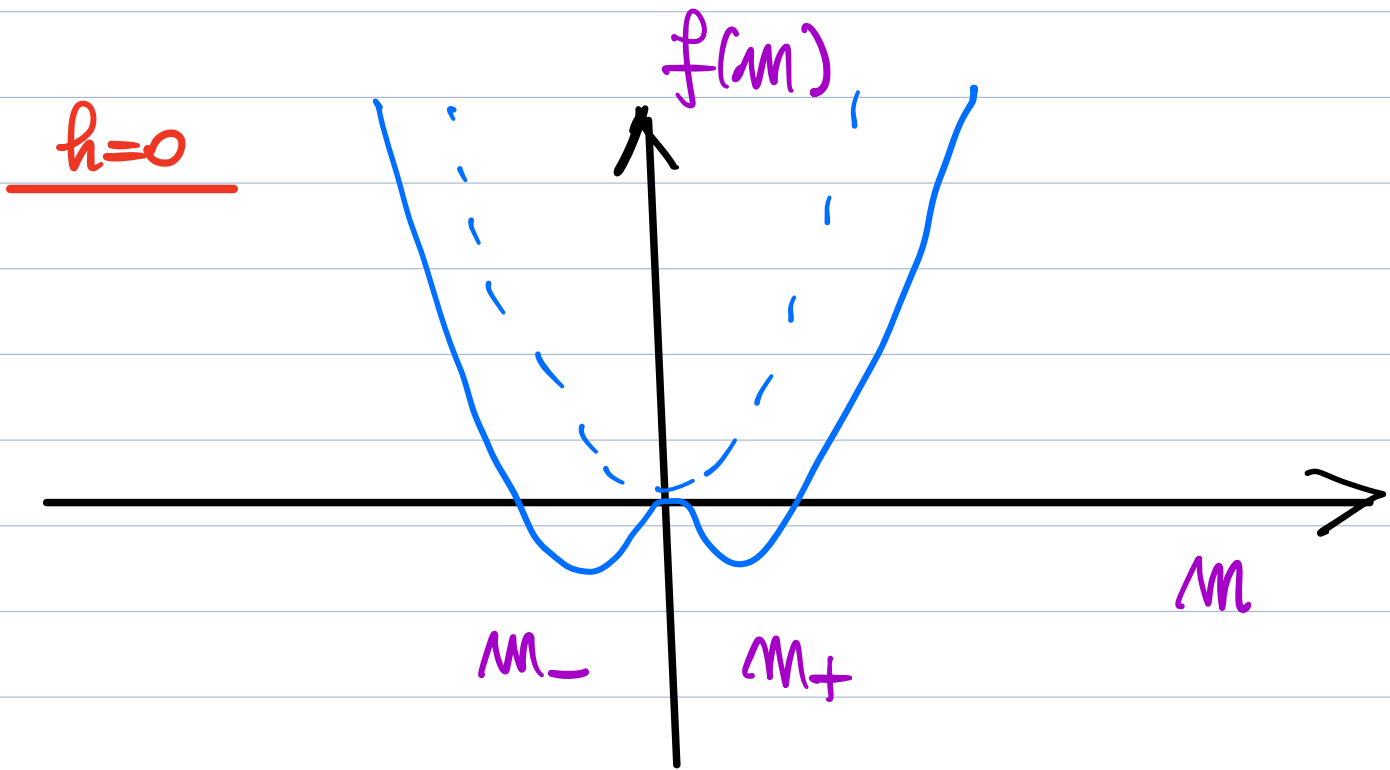


METASTABLE

THE SYST CAN BE NEGATV.
MAGN. UNDER $h > 0$ IF
PREPARED AT $m < 0$



THE FULL FREE-ENERGY LANDSCAPE



HERE $f(m)$ IS A FUNCT OF ALL POSSIB
M VALUES, ie $m \in [-1, 1]$

BELOW T_c $|m| \neq 0$ ARE BELOW $m=0$
 $\Rightarrow$ NON TRIVIAL EQ. SOLUTIONS

$$f(m) = \frac{J}{2} z m^2 - k_B T \ln 2 \cosh \beta (J z m + h)$$

THE PINNING FIELD ROLE

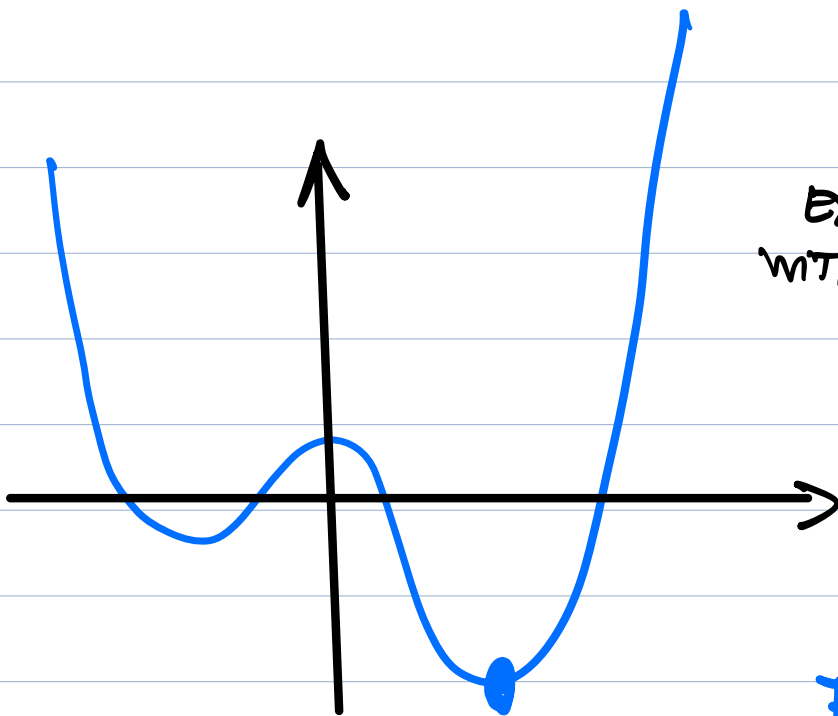
EFFECT OF FIELD - $h m \Rightarrow$ TILT
 $f(m)$ TO MAKE $m > 0$ LOWER IF $h > 0$
 $m < 0$ LOWER IF $h < 0$

$h \rightarrow 0$

SPONTANEOUS SYMM BREAKING

IT TILTS THE FREE-ENERGY

$f(m)$



EX. PLOT IT
WITH SOME GRAPHICAL
TOOL.

FAVOURABLE

THE $\lambda\phi^4$ POTENTIAL

$$f(m) = \frac{zJ}{2} m^2 - k_B T \ln 2 \cosh \beta (Jzm + h)$$

IF WE ASSUME $m \approx 0$ CLOSE TO T_c (ie $T \lesssim T_c, h=0$)

TAYLOR EXPAND 2ND TERM

$$\ln \left[\cosh y \approx 1 + \frac{y^2}{2} \right] \approx \frac{y^2}{2} - \frac{y^4}{12}$$

$$f(m) \approx \frac{zJ}{2} m^2 - \frac{k_B T}{2} (\beta J z m)^2 + \frac{k_B T}{12} (\beta J z m)^4 - k_B T \ln 2$$

$$= \frac{zJ}{2} m^2 \left[1 - \beta J z \right] + \frac{\beta^3 J^4 z^4}{12} m^4 - k_B T \ln 2$$

1 AT T_c

$$f(m) \approx \frac{z\beta J}{2} (k_B T - k_B T_c) m^2 + \frac{\lambda m^4}{4!}$$

THE CURVATURE CLOSE TO $m=0$
CHANGES AT T_c

+ ADDITIVE
CONST $-k_B T \ln 2$

WHAT HAPPENS IF WE KEEP h ?
ALSO CLOSE TO T_c OF COURSE

$\Rightarrow$ QUADRATIC TERM

$$- \frac{h\beta T}{2} \left(\beta (J_z m + h) \right)^2$$

$$\approx - \frac{h\beta T}{2} \underbrace{\beta^2 (J_z)^2}_{1} m^2$$

$$- \frac{h\beta T}{2} \underbrace{\beta^2 J_z}_{1} m h + \cancel{\theta(h^2)} \quad \text{Drop} \nearrow$$

$$= - \frac{h\beta T}{2} m^2 - m h$$

MAGN. FIELD CONTRIB.

THE SPECIFIC HEATIS CONTINUOUS AT T_c

$$C_v = \frac{\partial E}{\partial T} = -\frac{1}{k_B T^2} \frac{\partial^2 (\beta F)}{\partial \beta^2} = -2k_B \frac{\partial \beta F}{\partial T} - k_B T^2 \frac{\partial^2 (\beta F)}{\partial T^2}$$

$$\ln Z = -\beta F = -\beta N f \quad E = \langle H \rangle$$

$$f(m) \sim -k_B T \ln 2 + \frac{1}{2} (k_B T - k_B T_c) m^2$$

WHAT IS f AS A FUNCT. OF T ? DROPPED $O(m^4)$ TERMS

- $T > T_c$ $m=0$

$$\beta f(m(T)) = -\ln 2$$

$$\Rightarrow \frac{\partial^2 (\beta f(m(T)))}{\partial T^2} = 0$$

- $T \lesssim T_c$ $m \sim (T_c - T)^\beta$

$$\beta f(m(T)) \sim -\ln 2 - \frac{1}{2} \beta (T_c - T)^{1+2\beta} \quad \uparrow$$

$$1+2\beta=2 \quad \Leftarrow \quad \text{Exp. } \beta=1/2$$

$$\frac{\partial(\beta f)}{\partial T} = +\frac{1}{2} \frac{(T_c - T)^2}{(\ln 3 T)^2} - \cancel{\frac{1}{2} \beta} \cancel{2} (T_c - T) = 0 \quad T=T_c$$

$$\frac{\partial^2(\beta f)}{\partial T^2} = \cancel{\frac{1}{2}} \cancel{(-2)} \frac{(T_c - T)^2}{(\ln 3 T)^3} + \cancel{\frac{1}{2}} \cancel{2} \frac{(T_c - T)(-1)}{(\ln 3 T)^2}$$

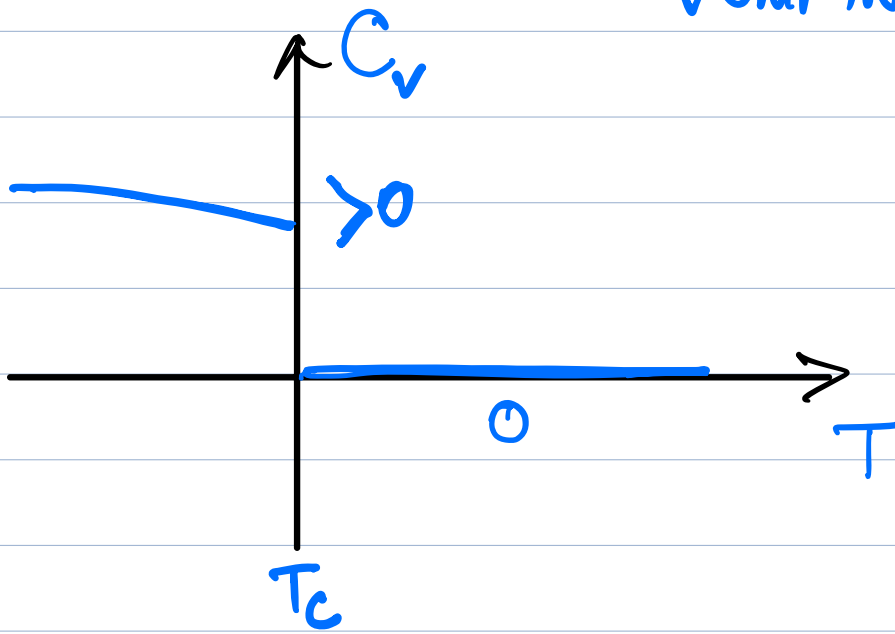
$$+ \frac{1}{(\ln 3 T)^2} (T_c - T) + \frac{1}{\ln 3 T} (-1)$$

$$\Delta T \lesssim T_c : \quad = -\beta_c$$

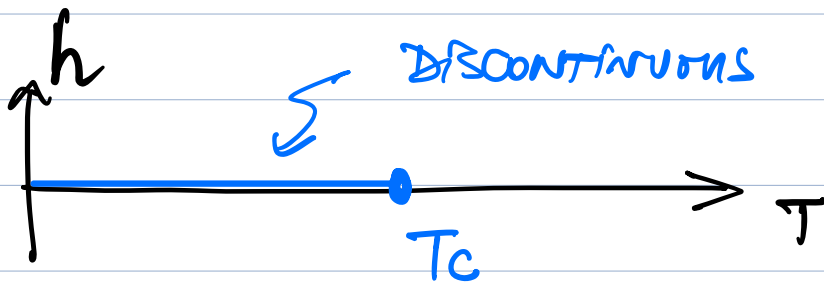
$$C_v = \beta_c^2 \frac{\partial^2 \ln z}{\partial T^2} = \beta_c^2 \frac{\partial^2 (-\beta N f)}{\partial T^2}$$

$$\downarrow \quad \Delta T \lesssim T_c = \begin{cases} 0 & T > T_c \\ \text{CONST} & T \lesssim T_c \end{cases}$$

JUMP IN C_v AT T_c



2.5.7 HINTS ON UNIVERSALITY



VERY SIMILAR

MONEDVER

SPIN MODEL

LATTICE GAS

$$S_i^0$$

$$n_i^0 = \frac{S_i^0 + 1}{2}$$

$$\pm 1$$

$$1, 0$$

VERY $\neq$ PROBLEMS w/ SIMILAR PHYSICS

$\Rightarrow$ UNIVERSALITY? SAME EXPONENTS $\alpha, \beta, \gamma, \delta$
MORE LATER

WHAT DO WE MISS WITH THE MF APPROX?

2.5.1 ABSENCE OF CORR

CONNECTED CORRELATIONS: $C_{ij}^{\text{CONN}} \equiv$

$$\langle (s_i - \langle s_i \rangle) (s_j - \langle s_j \rangle) \rangle = \langle \delta s_i \delta s_j \rangle$$

UNDER
THE ASSUMPTIONS MADE

RECALL WE DROPPED THIS QUAD. TERM

DO WE GET ZERO FOR THIS CORR?

SEVERAL WAYS TO DO IT. BY DEF: ($h_i = 0$):

$$C_{kl}^{\text{CONN}} = \sum_{\{s_i\}} s_k s_l e^{-\beta H(m_i, s_i)}$$

$$\sum_{\{s_i\}} e^{-\beta H(m_i, s_i)}$$

$$= \frac{\sum_{\{s_i\}} s_k e^{-\beta H(m_i, s_i)}}{\sum_{\{s_i\}} e^{-\beta H(m_i, s_i)}} \frac{\sum_{\{s_j\}} s_l e^{-\beta H(m_j, s_j)}}{\sum_{\{s_j\}} e^{-\beta H(m_j, s_j)}}$$

WITH $H(m_i, s_i) = -\frac{J}{2} \sum_{i \neq j} (m_i m_j - 2 s_i m_j)$

NOW, MANY CANCELLATIONS BETWEEN NUM
& DEN. FROM THE SUMS $\sum_{3s; y} \dots$ ONLY
TERMS WHERE

s_k, s_l
APPEAR WILL REMAIN.

MOREOVER, IF k AND l ARE $\neq$
AND DON'T SHARE NEIGHBORS

$\Rightarrow$ FACTORIZE 1st SUM

$\Rightarrow$ TWO TERMS IN SUM ARE THE
SAME AND RESULT
VANISHES IDENTICALLY

$$C_{kl}^{\text{CONN}} = 0$$

NO CORRELATION BEYOND
LATTICE DISTANCE

EVEN MORE CLEARLY SEEN IN THE FOLLOWING
WAY OF OBTAINING THE SAME DERIVATION:

2.5.2

ANOTHER WAY TO OBTAIN THE SAME EQS.

$$P(\{s_i\}) = \prod_i P_i(s_i)$$

$$P_i(s_i) = \frac{(1+m_i)}{2} \delta_{s_i,1} + \frac{(1-m_i)}{2} \delta_{s_i,-1}$$

NOTATION: WE KEEP A POSSIBLE i -DEP. IN m_i
IN CASE WE WANT TO CONSIDER J_{ij} HETEROG.

CHECK $\sum_{\{s_i = \pm 1\}} P_i(s_i) = 1$

NORM.

FOR USUAL ISING
 $J_{ij} = J$ &
 $m_i = m$

$\sum_{\{s_i = \pm 1\}} s_i P_i(s_i) = m_i$ DEF OF $\{m_i\}$

$$F = U - TS$$

FREE-ENERGY

WHAT IS U ?

$$\langle H \rangle_P$$

WHAT IS S ?

$$-k_B \langle \ln P \rangle_P$$

FIND $F(\{m_i\})$

sgn $P < 1$, $\ln P < 0$, $S > 0$

ASK IT TO BE A MINIMUM WRT $\{m_i\}$

$\Rightarrow$ FIND EQS ABOVE FOR $\{m_i\}$

$$\begin{aligned}\langle H \rangle_P &= -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \langle s_i s_j \rangle_P = -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} \langle s_i \rangle_P \langle s_j \rangle_P \\ &= -\frac{1}{2} \sum_{\langle ij \rangle} J_{ij} m_i m_j\end{aligned}$$

if ALSO LOCAL FIELDS $\langle \sum_i h_i s_i \rangle_P = \sum_i h_i \langle s_i \rangle_P = \sum_i h_i m_i$

$$\begin{aligned}\langle \ln P \rangle_P &= \left\langle \sum_i \ln \left(\frac{1+m_i}{2} \delta_{s_i,1} + \frac{1-m_i}{2} \delta_{s_i,-1} \right) \right\rangle_P = \\ &= \frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \quad (< 0)\end{aligned}$$

$$\begin{aligned}\beta f(\{m_i\}) &= -\frac{\beta}{2N} \sum_{\langle ij \rangle} J_{ij} m_i m_j - \frac{\beta}{N} \sum_i h_i m_i \\ &\quad + \frac{1}{N} \sum_i \left(\frac{1+m_i}{2} \ln \frac{1+m_i}{2} + \frac{1-m_i}{2} \ln \frac{1-m_i}{2} \right)\end{aligned}$$

LOOKS A BIT DIFF. BUT IT'S EQUIVALENT

SHOW

$$P_i(s_i) = \frac{e^{\beta h_i^{\text{loc}}(m_i) s_i}}{2 \cosh \beta h_i^{\text{loc}}(m_i)}$$

EXERCISE

$$h_i^{\text{loc}} = h_i^{\text{eff}} + h$$

ARE THE TWO EXPRESSIONS OF $F(m)$ THE SAME?

$$-\frac{1}{2} J_2 m^2 - h m + k_B T \left[\frac{1+m}{2} \ln \frac{1+m}{2} + \frac{1-m}{2} \ln \frac{1-m}{2} \right]$$

?

=

$$\frac{J_2}{2} m^2 - h m - k_B T \ln 2 \cosh(\beta J_2 m + \beta h)$$

• CHECK $\frac{\partial f(m)}{\partial m} = 0$ FROM 1st EXPRESSION

SET $h=0$ TO MAKE EXPRESSIONS SHORTER

$$0 = -J_2 m + k_B T \left[\frac{1}{2} \ln \frac{1+m}{2} - \frac{1}{2} \ln \frac{1-m}{2} \right]$$
$$+ k_B T \left[\cancel{\frac{1+m}{2}} \frac{\frac{1}{2}}{\cancel{\frac{1+m}{2}}} + \cancel{\frac{1-m}{2}} \frac{(-1/2)}{\cancel{\frac{1-m}{2}}} \right]$$

= 0

$$0 = -Jzm + \frac{k_B T}{z} \ln \frac{\frac{1+m}{z}}{\frac{1-m}{z}}$$

$$\frac{1+m}{1-m} = \exp\left(\frac{-2Jzm}{k_B T}\right)$$

$$1+m = (1-m) \exp(-2\beta Jzm)$$

$$1 - e^{-2\beta Jzm} = m(e^{-2\beta Jzm} + 1)$$

$$m = \frac{1 - e^{-2\beta Jzm}}{1 + e^{-2\beta Jzm}}$$

$$m = \tanh(\beta Jzm)$$

- Check now $\frac{\partial f(m)}{\partial m} = 0$ from 2nd exp.

$$\frac{J_z m^2}{2} - k_B T \ln 2 \cosh(\beta J_z m)$$

$$\Rightarrow \frac{J_z m - k_B T \ln 2 \cosh(\beta J_z m)}{2 \cosh(\beta J_z m)} \cdot \beta J_z = 0$$

$$m = \tanh(\beta J_z m) \quad \underline{\text{SAME EQ.}}$$

OK

CAN ALSO TRANSFORM ONE $f(m)$ INTO ANOTHER.

SAME $m \geq 0$ EXPANSION

2.5.5 THIRD WAY OF GETTING THE MEAN-FIELD EDS

- IN NOTES GENERALIZED TO P-SPIN MODELS

IN ISING SPIN MODEL

$$Z = \sum_{\{s_i = \pm 1\}} e^{\frac{\beta}{2} \sum_{ij} J_{ij} s_i s_j + \beta \sum_i h_i s_i}$$

$J_{ij} = J$ $\sum_{i < j}$ FULLY CONNECTED GRAPH
SUM OVER ALL PAIRS
ON COMPLETE GRAPH

$$J \rightarrow J/N \text{ OTHERWISE } H = \Theta(N^2)$$

$$\begin{aligned} \sum_{i < j} s_i s_j &= \sum_{ij} s_i s_j - \sum_i s_i^2 \\ &= \left(\sum_i s_i \right) \left(\sum_j s_j \right) - \sum_i 1 \\ &= N \left(\frac{1}{N} \sum_i s_i \right) \left(\frac{1}{N} \sum_j s_j \right) - N \end{aligned}$$

TRANSFORM SUM OVER $\{s_i = \pm 1\}$ IN SUM OVER

$$x = \frac{1}{N} \sum_i s_i \in [-1, 1], \quad \delta x = \frac{2}{N} \ll 1$$

$$x = -1, -1 + \frac{2}{N}, -1 + \frac{4}{N}, \dots, 1$$

$$\mathcal{Z} = \sum_x \mathcal{N}(\{s_i\} / \frac{1}{N} \sum_i s_i = x) e^{\frac{\beta N}{2} x^2}$$

WHAT IS THE DEGENERATION $\mathcal{N}$?

$$\mathcal{N} = \binom{N}{N_+} \quad N_+ : \# \text{ of spins}$$

$$= \frac{N!}{N_+! N_-!}$$

STIRLING $\Rightarrow$

$$\mathcal{N} = e^{N f(x)}$$

WITH

$$-f(x) = \frac{1+x}{2} \ln \frac{1+x}{2} + \frac{1-x}{2} \ln \frac{1-x}{2} \quad \begin{array}{l} \text{ENTROPY} \\ \text{OF ISING} \\ \text{SPINS} \end{array}$$

$$\mathcal{Z} \longrightarrow \int dx e^{-\beta N f(x)}$$

$$f(x) = -\frac{x^2}{2} - \ln 2 - k_B T f(x)$$

THE SADDLE-POINT VALUES EQUAL THE AVERAGE

$$Z \sim \sum_{\alpha} e^{-\beta N f(x_{sp}^{\alpha})}$$

LABELLING
THE SADDLE POINTS (if SEVERAL)

$$x_{sp} = - \frac{\partial f(x)}{\partial h} \bigg|_{x_{sp}} = \langle x \rangle$$

JUST FROM THE
LINEAR COUPLING
BTW h & x

USING h AS
A SOURCE WITHIN
PARTITION SUM

$\Rightarrow$

$$x_{sp} = \langle x \rangle$$

PROOF BELOW

if MORE THAN ONE SADDLE POINT $\Rightarrow$
AVERAGE OVER THE CORRESP "ERGODIC
COMPONENT" OR "STATE"

CAN SELECT THEM w/ PINNING FIELD

HOW DOES THE "SOURCE" METHOD WORK?

$$Z = \sum_{\alpha} e^{-\beta N f(x_{sp}^{\alpha})} \Rightarrow e^{-\beta N f(x_{sp})}$$

JUST ONE
TO SIMPLIFY
 $- \beta N f(x_{sp})$

$$\frac{\partial Z}{\partial h} = -\beta N \frac{\partial f(x_{sp})}{\partial h} e$$

$$\text{if } f(x_{sp}^{\alpha}) = \underbrace{f^{(0)}(x_{sp}^{\alpha})}_{\text{NOT DEP ON } h} - \underbrace{h x_{sp}}_{\text{THE FIELD DEP. TERM}}$$

NOT DEP ON h

THE FIELD
DEP. TERM

$$\frac{\partial f(x_{sp}^{\alpha})}{\partial h} = -x_{sp}$$

NB NO "CHAIN RULE" CONTRIB SINCE

$$\underbrace{\frac{\partial f(x_{sp}^{\alpha})}{\partial x_{sp}^{\alpha}}}_{\text{"0" by def of } x_{sp}^{\alpha}} \frac{\partial x_{sp}^{\alpha}}{\partial h} = 0$$

"0" by def of x_{sp}^{α} .

$$\text{THEN } \frac{1}{Z} \frac{\partial Z}{\partial h} = \frac{\beta N x_{sp} e^{-\beta N f(x_{sp})}}{e^{-\beta N f(x_{sp})}} = \beta N x_{sp}$$

THE LEFT-HAND-SIDE IS $\frac{\partial \ln Z}{\partial h}$

$$\text{BUT } Z = \int dx e^{-\beta N f(x)} + \beta N h x$$

BEFORE SADDLE-POINT

$$\Rightarrow \frac{\partial \ln Z}{\partial h} = \frac{1}{Z} \int dx e^{-\beta N f(x)} \beta N x$$

$$= \beta N \langle x \rangle$$

FINALLY, COMPARING

$$\frac{\partial \ln Z}{\partial h} = \beta N x_{sp} = \beta N \langle x \rangle$$

$$\Rightarrow x_{sp} = \langle x \rangle$$

IN THE TD YOU'LL SEE YET ANOTHER WAY OF
GETTING THE MF $f(m)$ & EQ. ON m .

PROBLEM THE EXPONENTS MEASURED ARE NOT
THE MEAN-FIELD ONES

SHOW $m \sim (T_c - T)^\beta$ $\beta \approx 0.3$
IN $d=3$
SAMPLES $C \sim (T_c - T)^\alpha$ $\alpha \approx 0.1$

ONSAAGER VALUES IN $d=2$

$$\beta = 1/8 \quad \alpha = 0$$

VS $\beta_{MF} = 1/2 \quad \alpha_{MF} = 0 \quad \forall d$

SO?

→ SOLUTION LATER

NOT COMPLETELY WRONG

ORDER OK

PHASE DIAGRAM OK

} $d=2,3$

$d \geq 4 = d_u$ EVERYTHING IS OK

WHAT HAVE WE DONE?

- MEAN-FIELD WITH NO SPATIAL STRUCT
FOUND PHASE TRANS



CRITICAL EXP
INDEX of d
SIMPLE FRACTIONS

SUMMARY 4th LECTURE

- FINISHED ANALYSIS OF MF EQS
& OBTAINED ALL EXPONENTS

$$\alpha, \beta, \gamma, \delta$$

$$S(m) = -k_B \ln P$$

NO CORRECTIONS TO T_C
 $T_C = J$

- COMPARED TO EXP. DATA AND
SAW THAT THE INTEGER
VALUES - INDEP. OF d -
WERE NOT MEASURED
- SHOWED $\langle \delta S_i \delta S_j \rangle = 0$
if $i \neq j$
- STARTED ADDING SPACE DEP.

5th LECTURE

- LANDAU'S PRESCRIPTIONS

- HOMOGENEOUS S.P.
DOMAIN WALL / KINKS

- CORR. OF FLUCT $\langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$

$$\text{CORR. LENGTH } \xi \sim |T - T_c|^{-\nu}$$

- GINZBURG CRITERION

$$\langle \delta\phi_{oh}^2 \rangle \text{ vs. } \langle \phi \rangle^2$$

- 1st ORDER

• WE NEED SOME SPACE DEPENDENCE



2.6 Ginzburg-Landau Theory

1 - LANDAU TH. $F(\phi)$

2 - CORR FCT. $C_{\text{corr}}(r)$

3 - GINZBURG CRIT

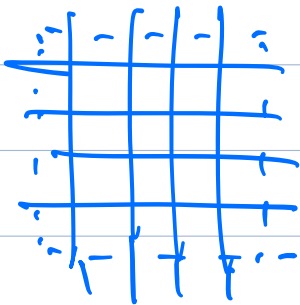
ESTIMATE "STRENGTH" OF
FLUCTUATIONS

$$d_u \leq d \text{ MF OK}$$

2.6 LANDAU THEORY

- WE HAD NO SPATIAL INFO SO FAR
- LANDAU $\rightarrow$ GO A BIT BEYOND, SIMILAR APPROACH TO WHAT DONE FOR FULLY-CONNECTED

- COARSE-GRAIN



$\longleftrightarrow$
 l

$$\phi(\vec{x}) = \frac{1}{n_{\vec{x}}} \sum_{i \in v_{\vec{x}}} s_i$$

SUM OVER SPINS IN BOX

$$a \ll l \ll L$$

$$\begin{aligned} n_{\vec{x}} &= \# \text{ SPINS IN BOX} \\ &= (l/a)^d \end{aligned}$$

$\phi(\vec{x})$ TAKES DISCRETE VALUES BUT IN THIS LIMIT $\Rightarrow$ CONTINUOUS FCT OF SPACE

OVERLAPPING BOXES

$$\delta x = a$$

NON-OVERL. BOXES

$$\delta x = l$$

$$Z = \sum_{\{s_i\}} e^{-\beta H(\{s_i\})}$$

$$= \sum_{\alpha} \sum_{\{s_i/\alpha\}} e^{-\beta H_{\alpha}(\{s_i\})}$$

SAME AS WHAT WE
DID FOR FC MODEL,
GROUP SPINS ACCORD.
TO THE VALUES OF
SOME AUX. VAR. &
SUM OVER THEM.
GENERIC α INDEX

APPLY TO α BEING ϕ

$$= \int \mathcal{D}\phi \sum_{\{s_i/\phi\}} e^{-\beta H_{\phi}(\{s_i\})}$$

ALL CONF OF SPINS
YIELDING THE
SAME ϕ

$$Z = \int \mathcal{D}\phi e^{-\beta F(\phi)}$$



WRITTEN AS A
FIELD DEP. (β FREE ENERGY)
GINZBURG-LANDAU
FREE ENERGY
(NB ITS MACROSCOPIC)

$$\int \mathcal{D}\phi ?$$

MEANS THAT ONE HAS TO SUM OVER ALL POSSIBLE VALUES OF ϕ AT EACH POINT IN SPACE

$\vec{x}$ LABELS THE BOXES

$$\int \mathcal{D}\phi = \prod_{\vec{x}} \int_{-\infty}^{\infty} d\phi(\vec{x})$$

WE LIFT THE CONSTRAINT $-1 \leq \phi(\vec{x}) \leq 1$

THE BEST WAY TO UNDERSTAND IT IS TO THINK ABOUT $\vec{x}$ AS A LABEL

$F(\phi)$ IS A MACROSC. FREE ENERGY FUNCTIONAL

(A FUNCTION OF A FUNCTION)
 F ϕ

THE INTEGRAL IS A FUNCTIONAL INTEGRAL

RECALL $x = \frac{1}{N} \sum_i s_i$ CURIE-WEISS

HERE $\phi(\vec{x}) = \frac{1}{N_{\vec{x}}} \sum_{i \in V_{\vec{x}}} s_i$ SPACE-DEP.

GUESSES FOR $F(\phi)$

1- Local $F(\phi) = \int d^d x f(\phi(\vec{x}))$
 $\supset$ Macroscopic

2- SPATIAL SYMMETRIES PRESERVED
 TRANSL $\supset$ ROTAT OF LATTICE
 $\Rightarrow$ CONT TH. TOO

3- SPIN REVERSAL $\Rightarrow \phi \rightarrow -\phi$
 SYMM.

$$F(\phi) = F(-\phi)$$

4- ANALYTIC FUNCTIONAL
 $F(\phi)$ TAYLOR EXPANDABLE IN
 POWERS OF $(\phi, \vec{\nabla}\phi)$

5- SMOOTH VARIATIONS IN SPACE
 KEEP ONLY $(\vec{\nabla}\phi)^2$

$$F(\phi) = \int d^d x \left[\frac{c}{2} (\vec{\nabla}\phi)^2 + \frac{\bar{a}}{2} \phi^2 + \frac{\lambda}{4!} \phi^4 \right]$$

a controls small ϕ BEHAVIOUR

$$\bar{a} > 0 \quad T > T_c$$

$$\bar{a} < 0 \quad T < T_c$$

$$\bar{a} = a \cdot \frac{T - T_c}{T_c}$$

MEAN-FIELD

cfr FULLY CONNECTED ISING MODEL
FOR SMALL ORDER PARAMETER (x)

THE SAME + $(\vec{\nabla} \phi)^2$ TERM

DIMENSIONS

$$[c] \frac{[\phi^2]}{L^2} = [\bar{a}] \frac{[\phi^2]}{L^2} = [\lambda] [\phi^2]$$

$\Downarrow$

$$[c] = [\bar{a}] L^2 \quad L = [x], \quad [\bar{a}] = [a]$$

$$[c] = [a] L^2$$

$$[\phi^2] [\lambda] = [a]$$

$$[c/a]^{1/2} = L$$

NB THE RÔLE PLAYED BY N IS NOW PLAYED BY
THE VOLUME OF SPACE $\int d^d x \rightarrow \text{Volume}$

SADDLE POINT

$$\frac{\delta F(\phi)}{\delta \phi(\vec{y})} = 0$$

HOW DO YOU CALCULATE FUNCTIONAL DERIV.?
HOW MANY OF YOU KNOW HOW TO DO IT?

MATH SUPPORT

$$F(\phi) = \int d^d x f(\phi(\vec{x}))$$

$$\frac{\delta F(\phi)}{\delta \phi(\vec{y})} = \int d^d x \frac{\delta f(\phi(\vec{x}))}{\delta \phi(\vec{y})}$$

if $f(\phi(\vec{x}))$
JUST A fct
OF $\phi(\vec{x})$

$$= \int d^d x \, f'(\phi(\vec{x})) \underbrace{\frac{\delta \phi(\vec{x})}{\delta \phi(\vec{y})}}_{\delta^d(\vec{x}-\vec{y})}$$

$$= f'(\phi(\vec{y}))$$

NB: IT SETS $\vec{x}$ TO $\vec{y}$
AND TAKES A FUNCTION
(NORMAL DERIV.)

WHAT HAPPENS WITH TERMS WITH $\vec{\nabla} \phi(\vec{x})$?

$$\text{eg. } F_{\text{grad}}(\phi) = \int d^d x \, \frac{1}{2} (\vec{\nabla} \phi(\vec{x}))^2$$

$$= \int d^d x \, \frac{1}{2} \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla} \phi(\vec{x})$$

$$\frac{\delta F_{\text{grad}}(\phi)}{\delta \phi(\vec{y})} = \int d^d x \left[\frac{1}{2} \vec{\nabla} \frac{\delta \phi(\vec{x})}{\delta \phi(\vec{y})} \cdot \vec{\nabla} \phi(\vec{x}) + \frac{1}{2} \vec{\nabla} \phi(\vec{x}) \cdot \vec{\nabla} \frac{\delta \phi(\vec{x})}{\delta \phi(\vec{y})} \right]$$

TWICE THE SAME TERM

$$= \int d^d x \vec{\nabla} \delta(\vec{x} - \vec{y}) \cdot \vec{\nabla} \phi(\vec{x})$$

$$= - \int d^d x \delta(\vec{x} - \vec{y}) \nabla^2 \phi(\vec{x}) + \text{BORDER TERMS}$$

$$= - \nabla^2 \phi(\vec{y}) \quad \propto \delta(\infty - \vec{y}) \quad \parallel \quad 0$$

ALL TOGETHER

$$0 = \frac{\delta F(\phi)}{\delta \phi(\vec{y})} = -c \nabla^2 \phi(\vec{y}) + \frac{\lambda}{3!} \phi^3(\vec{y}) + a \left(\frac{T - T_c}{T_c} \right) \phi(\vec{y})$$

$$\frac{\lambda}{3!} \phi_{sp}^3 + a \frac{T - T_c}{T_c} \phi_{sp} = 0$$

$$\phi(\vec{x}) = \phi_{sp} \text{ CONST}$$

ONE KIND OF SOLUTION

$$\phi_{sp} = 0 \quad \text{or}$$

$$\phi_{sp} = \pm \sqrt{\frac{3! a (T_c - T)}{\lambda T_c}}$$

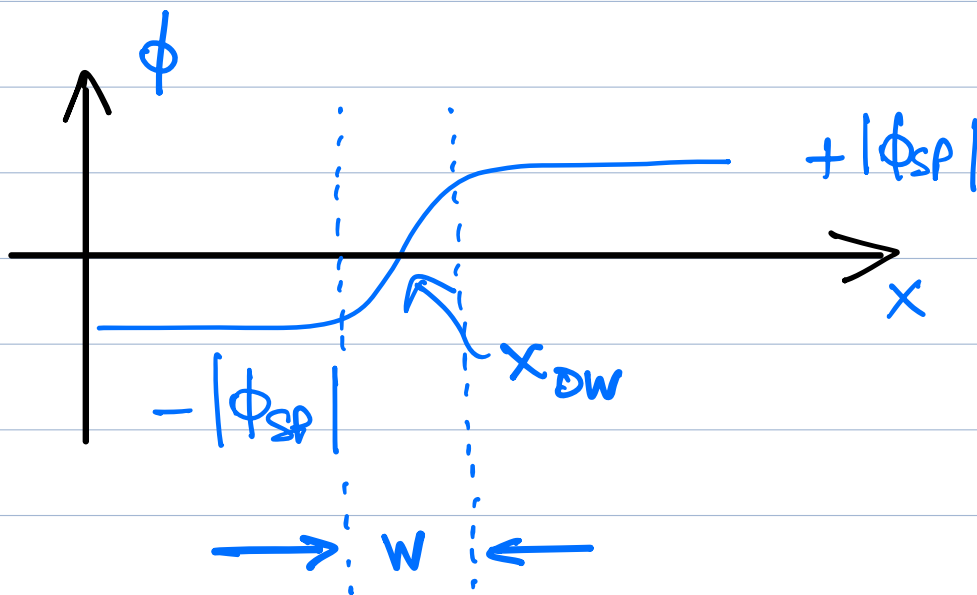
UNITS OK

UNTIL HERE $\rightarrow$ REALLY SIMILAR! SAME EXPONENTS

DOMAIN WALLS

SPATIAL VARIATION / info

EX. $d=1$



ADDING $\nabla^2 \phi$ TO THE SP. EQ
WITH $\lambda \phi^4$ "POTENTIAL"

$\Rightarrow$ SOLUTION VARIATION ALONG DIRECT. x

$$\phi_{DW}(x) = |\phi_{sp}| \tanh\left(\frac{x - x_{DW}}{W}\right)$$

χ_{DW} FREE, ϕ_{SP} THE SOL WITHOUT $\nabla^2 \phi$

$$W = \left(\frac{c T_c}{a(T_c - T)} \right)^{1/2} \text{ WIDTH}$$

$$\text{UNITS } \left(\frac{[c]}{[a]} \right)^{1/2} = \left(\frac{[a][L^2]}{[a]} \right)^{1/2} = [L^2]^{1/2} = [L]$$

OH, IT'S
A LENGTH.

W DIVERGES AT $T \rightarrow T_c^-$

DEPENDS ON c ELAST CONST.

DIVERGENCE OF $W \Rightarrow \phi_{DW} \rightarrow 0$

FLATTER DW CONF $\Rightarrow \phi_{DW} \sim \text{ZERO EVERYWHERE}$

$W \propto \xi$ CORRELATION LENGTH

WE'LL SEE IT LATER

FROM ITS DEF. FROM THE
CORR. FUNCTION.

WHICH SOLUTION IS BETTER?

ONE SHOULD COMPARE

$$F[\phi_{sp}] \geq F[\phi_{dw}]$$

THE EVALUATION YIELDS

$$F[\phi_{sp} = \text{const}] < F[\phi_{dw}]$$

CLERΩ FROM THE FACT THAT

$$(\vec{\nabla} \phi_{sp})^2 = 0 \quad \text{WHILE} \quad (\vec{\nabla} \phi_{dw})^2 > 0$$

BUT BOTH ARE STABLE, IN THE SENSE
THAT

$$\frac{\delta^2 F[\phi]}{\delta \phi(\vec{x}) \delta \phi(\vec{y})} \quad \text{POSITIVE DET AT} \quad \begin{cases} \phi_{sp} \\ \phi_{dw} \end{cases}$$

RECALL THAT WHEN DISCUSSING THE PEIERLS ALG.
WE ALSO MENTION DOMAIN WALLS $\uparrow\uparrow\uparrow \downarrow\downarrow\downarrow \uparrow\uparrow\uparrow$

CORRELATION FUNCTIONS

THE CONNECTED CORRELATION

$$C_{\text{conn}}(\vec{x}, \vec{y}) = \langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$$

$$\delta\phi(\vec{x}) = \phi(\vec{x}) - \langle \phi(\vec{x}) \rangle$$

OF THE FLUCTUATIONS CAN BE CALCULATED
ADDING A SPACE DEP. FIELD AS A SOURCE

$$\mathcal{Z}_h = \int \mathcal{D}\phi \quad e^{-\beta F(\phi) + \beta \int d^d x \, h(\vec{x}) \phi(\vec{x})}$$

RECALL PROPS OF CUMULANTS (MATH SUPPORT)

$$\left. \frac{\delta^2 \ln \mathcal{Z}_h}{\delta h(\vec{x}) \delta h(\vec{y})} \right|_{h=0} = \beta^2 \langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$$

HOW DO WE ISOLATE THE h -DEP IN $\mathcal{Z}_h$?

- WE DROP THE $\phi^4(\vec{x})$ TERM IN F_h
ASSUMING WE WORK CLOSE TO T_c

- WE FOURIER TRANSFORM THE FIELD TO "DECOUPLE" THE GRADIENT TERM

$$\phi(\vec{r}) = \int d^d x e^{i\vec{k} \cdot \vec{x}} \phi(\vec{x})$$

$$\phi(\vec{x}) \in \mathbb{R} \Rightarrow \phi(\vec{k}) \in \mathbb{C} \quad \text{BUT}$$

$$\phi^*(\vec{k}) = \phi(-\vec{k}) \quad \begin{array}{l} \text{NO} \\ \text{DOUBLING} \\ \text{OF d.o.f.} \end{array}$$

IN TERMS OF $\phi(\vec{k})$ THE FREE-ENERGY IS
($\phi(\vec{k}) \approx 0$)

$$F_h(\phi) = \int \frac{d^d k}{(2\pi)^d} \left[\frac{ck^2}{2} |\phi(\vec{k})|^2 + \frac{a(T-T_c)}{2 T_c} |\phi(\vec{k})|^2 \right]$$

$$- \int \frac{d^d k}{(2\pi)^d} h(\vec{k}) \phi^*(\vec{k})$$

FOR THE MOMENT SHORT-HAND NOTATION:

$$\tilde{\gamma}^{-2} \equiv a \cdot \frac{T-T_c}{T_c}$$

- WE IDENTIFY $\tilde{\gamma}$ AS A FACTOR OF THE QUADRATIC TERM

WE'LL SEE LATER IT'S THE CORRELATION LENGTH

$\nu = 1/2$
 $T \gtrsim T_c$ IMPLICITLY ASSUMED

$$F_h(\phi) = \int \frac{d^d k}{(2\pi)^d} \frac{1}{2} (ck^2 + \xi^{-2}) |\phi(k)|^2 \\ - \int \frac{d^d k}{(2\pi)^d} h(k) \phi^*(k)$$

NB EACH k ON ITS OWN $\Rightarrow$ DECOUPLED

- NOW, WE NEED TO EXTRACT THE $h(k)$ DEPENDENCE FROM THE FUNCT. INT. OVER $\phi(k)$

WE USE A TRICK:

TRANSLATE THE FIELD

$$\varphi(k) = \phi(k) - \frac{h(k)}{ck^2 + \xi^{-2}} \quad \begin{matrix} \text{A } \searrow \\ \text{NUM } \mathbb{R} \\ \text{CONST TO FIX} \\ \text{CONVENIENTLY} \end{matrix}$$

AND REWRITE F IN TERMS OF $\varphi(k)$:

$$F_h[\phi] = F[\varphi] + \text{TERM in } h^2$$

THE
SQUARE
TERM

$$|\phi(k)|^2 = \left| \varphi(k) + \frac{h(k) A}{ck^2 + \xi^{-2}} \right|^2$$

$$= |\varphi(k)|^2 + \left(\frac{\varphi(k) h^*(k)}{ck^2 + \xi^{-2}} + \text{c.c.} \right) A$$

$$+ \frac{|h(k)|^2 A^2}{(ck^2 + \xi^{-2})^2}$$

THE FIELD-DEP. TERM

$$h(k) \phi^*(k) = h(k) \left[\varphi(k) + \frac{h(k) A}{ck^2 + \xi^{-2}} \right]^*$$

$$= h(k) \varphi^*(k) + \frac{|h(k)|^2 A}{ck^2 + \xi^{-2}}$$

• PUTTING ALL TOGETHER: IN THE EXPONENTIAL

$$= (ck^2 + \xi^{-2}) |\phi(k)|^2 - h(k) \phi^*(k) =$$

$$= (ck^2 + \xi^{-2}) \left[|\varphi(k)|^2 + \left(\frac{\varphi(k) h^*(k) + \text{c.c.}}{ck^2 + \xi^{-2}} \right) A + \frac{|h(k)|^2 A^2}{(ck^2 + \xi^{-2})^2} \right]$$

$$- h(k) \varphi^*(k) - \frac{|h(k)|^2 A}{ck^2 + \xi^{-2}}$$

$$= (ck^2 + \xi^{-2}) |\varphi(k)|^2 - h(k) \varphi^*(k)$$

SAME EXP AS ORIGINAL ONE
BUT IN $\varphi(k)$

$$+ \left(\varphi(k) h^*(k) + \varphi^*(k) h(k) \right) A$$

$$\Rightarrow \varphi(-k) h(k)$$

$$\Rightarrow \varphi(k) h(-k) \rightarrow \text{UNDER}$$

$$= \varphi(k) h^*(k) \quad \text{INT. OVER}$$

$$\text{TWICE THE SAME} \quad \int d^d k$$

$$- \frac{|h(k)|^2 A}{ck^2 + \xi^{-2}}$$

IT DOESN'T DEP. ON $\varphi(k)$
 $\Rightarrow$ GOES OUTSIDE INTEGRAL

WE CAN CANCEL THE TERMS IN THE BUBBLES
TAKING $A = 1/2$

THOSE ARE THE ONES LINEAR IN $h(k)$
COUPLED TO $\varphi(k)$

THEN, WE HAVE

$$F_h(\varphi) = \underbrace{\frac{1}{2} \int d^d k \left[c k^2 |\varphi(k)|^2 + \tilde{\Sigma}^{-2} |\varphi(k)|^2 \right]}_{F[\varphi]} - \frac{1}{2} \int d^d k \frac{|h(k)|^2}{c k^2 + \tilde{\Sigma}^{-2}}$$

AND THE VARIATIONS WRT $h(\vec{x})$ ONLY TOUCH THE LAST TERM IN $F_h(\phi)$ WHICH IS DEPEND. OF φ

$$Z_h = \underbrace{\int \mathcal{D}\varphi e^{-\beta F(\varphi)}}_B e^{\underbrace{\frac{\beta}{2} \int d^d k \frac{|h(k)|^2}{c k^2 + \tilde{\Sigma}^{-2}}}_{\text{WHAT WE NEEDED}}}$$

- NOW, FOURIER TRANSFORM BACK TO $\vec{x}$

SPACE

$$Z_h = B e^{\frac{\beta}{2} \int d^d x \int d^d y h(\vec{x}) G(\vec{x}-\vec{y}) h(\vec{y})}$$

$$G(\vec{x}-\vec{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik(\vec{x}-\vec{y})}}{c k^2 + \tilde{\Sigma}^{-2}}$$

- Firstly,

$$\ln Z_h = \ln B + \frac{\beta}{2} \int d\vec{x} d\vec{y} h(\vec{x}) G(\vec{x}-\vec{y}) h(\vec{y})$$

$$\frac{\delta^2 \ln Z_h}{\delta h(\vec{x}) \delta h(\vec{y})} = \beta G(\vec{x}-\vec{y}) = \beta^2 \langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle$$

WE ALREADY KNOW IT

AND THEN

$$\langle \delta\phi(\vec{x}) \delta\phi(\vec{y}) \rangle = k_B T G(\vec{x}-\vec{y})$$

NOW WE HAVE TO GO BACK TO THE DEF OF

$$G(\vec{x}-\vec{y}) = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik(\vec{x}-\vec{y})}}{ck^2 + \tilde{\zeta}^{-2}}$$

AND TRY TO CALCULATE THE INTEGRAL
OVER k^2 EXPLICITLY

* HERE, FOLLOW THE CALC. IN THE LECTURE NOTES, VERY DETAILED ! ALSO IN TD.

• $r \gg \xi$ $G(r) \sim \exp(-r/\xi)$

• $r \ll \xi$

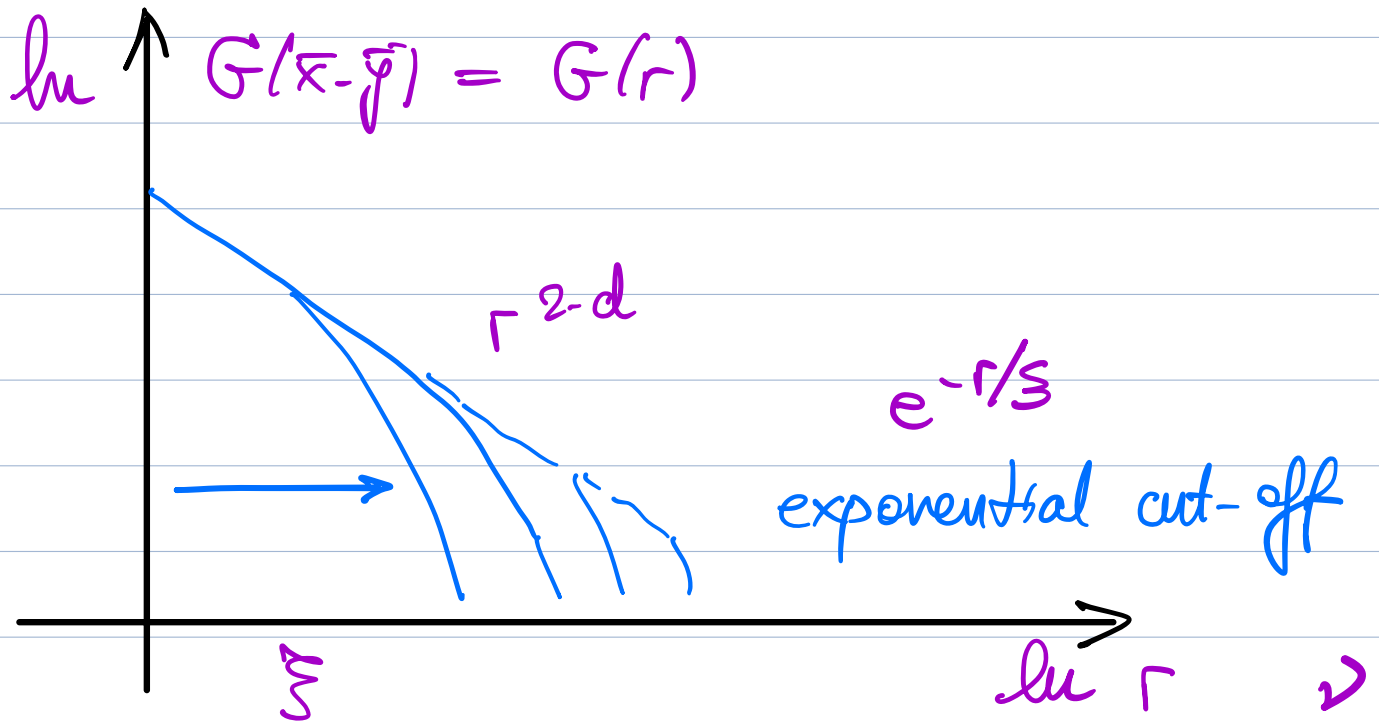
$G(\vec{x}-\vec{y}) \propto r^{2-d}$ $r \ll \xi$

AND NOW FOR BOTH
 $r \ll \xi$ AND
 $r \gg \xi$

PUTTING TOGETHER
THE TWO LIMITS

$G(\vec{x}-\vec{y}) \sim r^{2-d} e^{-r/\xi}$

DISCUSS THE FORM OF $G(\vec{x}-\vec{y})$



ξ DEPENDS ON $T-T_c$

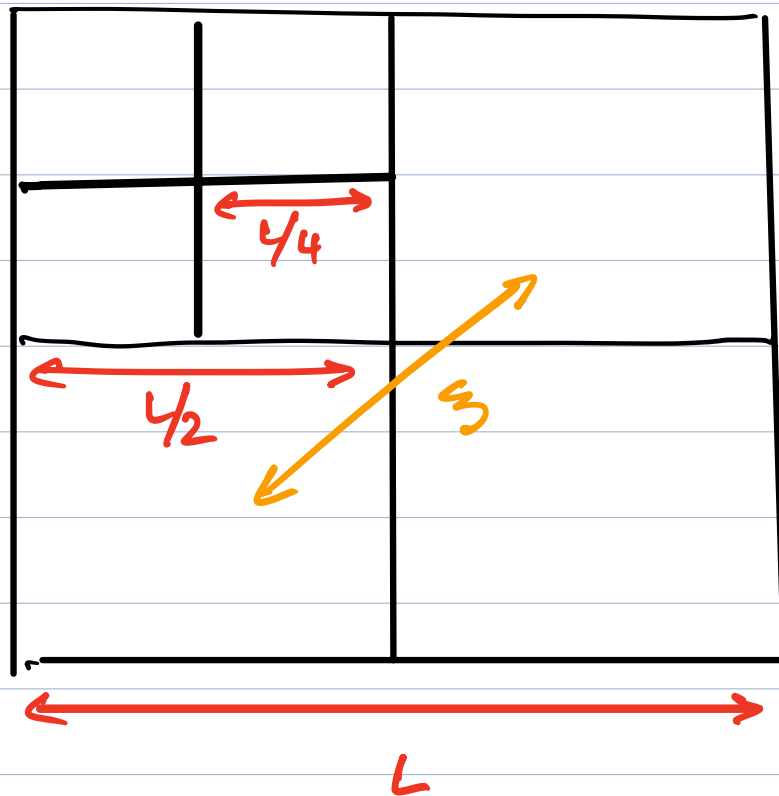
$$\xi \sim (T-T_c)^{-1/2}$$

- FROM LEFT TO RIGHT T GETS CLOSER TO T_c
- FOR $T \rightarrow T_c^+$ JUST THE POWER LAW r^{2-d}
- WE SEE THAT $d=2$ PLAYS A SPECIAL ROLE IN THIS THEORY.
THE POWER LAW IS A CONST.

THE CORRELATION LENGTH - MEANING

DISTANCE OVER WHICH THE
FLUCT. OF THE ORDER PARAMETER
ARE CORRELATED $\delta\phi$

SAY $\xi \ll L$



MEASURE OBSERV IN $L^d, (L/2)^d, \dots$
SAME RESULT UNTIL

$$L \sim \xi$$

THEN, RESULTS SHOULD CHANGE.

FROM N INDEP BOXES, TO CORRELATED THINGS

WE SEE THERMAL FLUCTUATIONS IN SPIN
CONF - SNAPSHOTS

SEE SLIDES & LECTURE NOTES

THE AVERAGED SIZE OF THESE
REVERSED DROPLETS GIVE AN IDEA OF

ξ

ALTHOUGH IT'S NOT EXACTLY EQUAL TO IT

CONSEQUENTLY

SYST WITH $\xi \rightarrow \infty$ HAVE DROPLETS
OF ALL SIZES.

TRANSLATIONAL INVARIANCE $\rightarrow$
SCALE INVARIANCE

MORE LATER

THE RESULTS ABOVE

GINZBURG CRITERION

ARE OK AS LONG AS
lhs $\lesssim$ rhs

GAUSSIAN FLUCT AROUND THE HOMOG. S.P.

$$\langle \delta\phi_{\text{coh}}^2 \rangle \sim \langle \phi \rangle^2 = \phi_{\text{SP}}^2$$

PROOF
OF $\langle \phi \rangle =$
 ϕ_{SP}

$$\delta\phi_{\text{coh}} \equiv \frac{1}{V_{\text{coh}}} \int d^d x (\phi(\vec{x}) - \phi_{\text{SP}})$$

SEE
NOTES,
EXC.

$$V_{\text{coh}} = \xi^d$$

WITH ξ CORREL. LENGTH

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \frac{1}{V_{\text{coh}}^2} \int d^d x \int d^d y \left((\phi(\vec{x}) - \phi_{\text{SP}})(\phi(\vec{y}) - \phi_{\text{SP}}) \right)$$

DEFINITION OF $C^{\text{CONN}}(\vec{x}, \vec{y})$

FOR A HOMOGENEOUS PROBLEM

TRANSL. INV.

ISOTROPY

EXPECTED $\Rightarrow$

$$C^{\text{conn}}(\vec{x}, \vec{y}) = C^{\text{conn}}(\|\vec{x} - \vec{y}\|)$$

Call $r = \|\vec{x} - \vec{y}\|$

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \frac{1}{V_{\text{coh}}} \int d^d r \, C^{\text{conn}}(r)$$

$$= \frac{1}{V_{\text{coh}}} \Omega_d \int dr \, r^{d-1} C^{\text{conn}}(r)$$

WE SHOWED IT ABOVE.

if $C^{\text{conn}}(r) \sim r^{2-d-\eta} e^{-r/\xi}$

CLOSE TO T_c

WITH $\eta=0$

WITHIN LANDAU

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \frac{\Omega_d}{\xi^d} \int dr \, r^{d-1} r^{2-d} e^{-r/\xi}$$

$$= \frac{\Omega_d}{\xi^d} \int \frac{dr}{\xi} \frac{r}{\xi} e^{-r/\xi} \xi^2$$

$$= \Omega_d \xi^{2-d} \underbrace{\int_{d/\xi}^{4/\xi} dy y e^{-y}}_{A'}$$

A NUMBER

$$\langle \delta\phi_{\text{coh}}^2 \rangle = \xi^{2-d} A$$

CLOSE TO THE CRITICAL POINT

$$\langle \phi \rangle = \phi_{\text{sp}} \simeq |T_c - T|^\beta \quad (\text{LANDAU})$$

$$\text{if } \xi \sim |T - T_c|^{-\nu} \implies |T - T_c| \sim \xi^{-1/\nu}$$

$$\implies \phi_{\text{sp}} \sim \xi^{-\beta/\nu}$$

WITHIN THE LANDAU APPROX

$$\beta = 1/2$$

$$\nu = 1/2$$

$$\langle \phi \rangle = \phi_{\text{sp}} \sim \xi^{-1}$$

BACK TO THE GINZBURG CRITERION

$$\langle \phi_{coh}^2 \rangle \stackrel{<}{\sim} \langle \phi \rangle^2 \Rightarrow$$

$$\xi^{2-d} \stackrel{<}{\sim} \xi^{-2} \Rightarrow \xi^{4-d} \stackrel{<}{\sim} 1$$

$$d \geq 4 \quad \Delta \quad d_u = 4$$

ABOVE $d=4$
 $\Rightarrow$ OK

- if $d \geq 4$ THEN $\langle \phi_{coh}^2 \rangle \stackrel{<}{\sim} \langle \phi \rangle^2$ Δ

MEAN FIELD OK

- if $d < 4 \Rightarrow$ NO! FOR $T \rightarrow T_c$

$d_u \equiv$ UPPER CRITICAL DIMENSION

STILL AT $d < 4$ ONE CAN APPROACH T_c BUT NOT GET TOO CLOSE TO IT SO THAT $\xi^{4-d} < 1$ IF ξ REMAINS FINITE. $\Rightarrow$ AWAY FROM CRITICAL REGION

$$\xi \sim |T - T_c|^{-\nu} \Rightarrow (T - T_c)^{-\nu(4-d)} < 1$$

FIRST ORDER PHASE TRANS. WITHIN LANDAU'S FIELD THEORY

- FIELD DRIVEN

$$- \int d^d x \ h(\vec{x}) \phi(\vec{x})$$

- TEMP DRIVEN

$$\int d^d x \ \phi^3(\vec{x})$$

LIKE IN $p=3$ FM MODEL

THIS TERMS CHANGE THE FORM OF $F[\phi]$
AND ALLOW FOR METASTABLE STATES, AND
JUMPS IN ORDER PARAM. AT THE 1ST ORDER
TRANSITION

APPLICATIONS

- COND-MATTER
- HIGH ENERGY ϕ HIGGS FIELD
 A_μ FIELD
- etc.

EXTENSIONS TO VECTOR VARIABLES

- VECTOR FIELDS

$$\vec{\phi}(\vec{x})$$

LATER : 2dXY MODEL $\Leftrightarrow$
 $O(2)$ FIELD TH. IN
 $d=2$

27.2 CORR FUNCT & GLOBAL SUSCEPTIBILITY

BACK TO GENERICS. $\theta \Rightarrow$ ORDER PAR. $\langle \theta \rangle$

$$C(\vec{x}, \vec{y}) = \langle \delta \theta(\vec{x}) \delta \theta(\vec{y}) \rangle \quad \text{corr fun}$$

(I will NOT WRITE "CONN" ANYMORE)

$$\sim r^{2-d-\eta} \underbrace{f(r/\xi)}$$

BEYOND
WHAT WE'VE
DONE SO FAR

ADOMALOUS EXP.

CUT-OFF FUNCT

TRANS INV.
ISOTROPY

$$\xi \sim |T - T_c|^{-\nu}$$

CORR LENGTH

$$\chi(\vec{x}, \vec{y}) \equiv \frac{\delta \langle \theta(\vec{x}) \rangle_h}{\delta h(\vec{y})} \bigg|_{h \rightarrow 0} \quad \text{LINEAR SUSCEP.}$$

$$H \rightarrow H - \int d^d x \, h(\vec{x}) \theta(\vec{x})$$

RELATION BETWEEN χ AND C

$$\chi(\vec{x}, \vec{y}) \equiv$$

$$\frac{\delta}{\delta h(\vec{y})} \frac{\int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')} \phi(\vec{x})}{\int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')}} \phi(\vec{x})$$

HAVE TO TAKE VARIATION IN NUM & DEN
AND THEN SET $h=0$

$$\underline{\text{NUM}} \rightarrow \frac{\int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')} \phi(\vec{x}) \phi(\vec{y})}{\int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')}} \phi(\vec{x}) \phi(\vec{y})$$

$$\underline{\text{DEN}} \rightarrow \frac{\int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')} \phi(\vec{x}) \int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')} \phi(\vec{y})}{\left(\int \mathcal{D}\phi e^{-\beta H + \beta \int d^d x' h(\vec{x}') \phi(\vec{x}')} \right)^2}$$

$$\Rightarrow \langle \phi(\vec{x}) \phi(\vec{y}) \rangle - \langle \phi(\vec{x}) \rangle \langle \phi(\vec{y}) \rangle$$

WITH AVERAGES AT $h=0$

$$\chi(\vec{x}, \vec{y}) = \beta C(\vec{x}, \vec{y})$$

LOCAL

FDT

RECALL, IT'S CORR.
CORR. FUNCT.

DIVERGENCE OF $\xi \Rightarrow$ DIVERGENCE OF χ

INTEGRATED OVER FULL
VOLUME

Proof

GLOBAL OR

TOTAL SUBE

$$\int d^d x \int d^d y \underbrace{\chi(\vec{x}, \vec{y})}_{\chi(r)} = \chi$$

INTEGRATING
OVER VOLUME
TWICE

HOM / ISOT

$$= V \Omega_d \int dr r^{d-1} \chi(r)$$

FDT

$$= V \Omega_d \beta \int dr r^{d-1} C(r)$$

ONE
VOLUME
FACTOR

ONE VOLUME INT

if $C(r) \sim r^{2-d-\eta} f(r/\xi) \Rightarrow$

$$\chi = V \Omega_d \beta \int dr r^{d-1} r^{2-d-\eta} f(r/\xi)$$

$$= V \Omega_d \beta \int dr r^{1-\eta} f(r/\xi)$$

$$= V \Omega_d \beta \xi^{2-\eta} \underbrace{\int \frac{d\tau}{\xi} \left(\frac{\tau}{\xi}\right)^{1-\eta} f(\tau/\xi)}_{\text{CONST A}}$$

THIS ONE IS INTEGRATED TWICE OVER VOLUME

CONST A

$$\chi = V \Omega_d A \beta \xi^{2-\eta}$$

IF $\eta < 2 \Rightarrow$
 $\xi \rightarrow \text{DIVER.} \Rightarrow$
 $\chi \text{ DIVER.}$

POWER LAW DECAY OF CORR FOR

$\eta < 2 \Rightarrow$ DIVERGENCE OF χ
 WITH $\xi^{2-\eta}$ (TWICE INTEGRATED)

$$\text{CHECK } \chi \sim V \xi^{2-\eta} \quad \downarrow \text{ONE VOLUME } L^d \quad \left\{ \begin{array}{l} \frac{\chi}{L^d} \sim \xi^{2-\eta} \\ \sim |T-T_c|^{-\nu(2-\eta)} \end{array} \right.$$

RELATION BETWEEN EXPONENTS $\longrightarrow$

DIVERGENCE
 CHANGED WRT
 $\chi \sim |T-T_c|^{-\gamma}$

MORE TO COME

SHOW VALUES
HERE

EXPONENTS $\Rightarrow$ UNIVERSALITY

$$\alpha = \alpha' = 0 \quad \beta = 1/2 \quad \gamma = \gamma' = 1 \quad \delta = 3$$

$$v = 1/2 \quad \eta = 0 \quad \underline{\text{MF}}$$

$$\alpha = \alpha' = 0 \quad \beta = 1/8 \quad \gamma = \gamma' = 7/4 \quad \delta = 15$$

$$v = 1 \quad \eta = 1/4 \quad \underline{d=2 \text{ ENSAGER}}$$

$$\alpha = \alpha' = 0.11 \quad \beta = 0.325 \quad \gamma = 1.24 \quad \delta = 4.82$$

$$v = 0.63 \quad \eta = 0.032 \quad \underline{d=3 \text{ APPROX}}$$

THE SAME VALUES IN MANY MANY $\neq$ SYST
MICROSC DOESN'T MATTER

APART FROM FEATURES WE WILL SEE DO
MATTER (DIM, DIM ORDER, SYMM.)
PARAM

d

n

SYMM

SUMMARY

TO BE SHOWN WITH SLIDES

LANDAU $\phi(\vec{x})$

$F(\phi)$ BASED ON ASSUMPTIONS

2nd

1st

GINZBURG

LIMITS OF VALIDITY $d > d_u$ EXACT

$d \leq d_u$ FAR FROM T_c ok.

$$\xi \sim |T - T_c|^{-\nu}$$

$$G_{\text{CONN}}(\vec{r}) \sim r^{2-d-1} f(r/\xi)$$

$f(r/\xi)$ CUT-OFF $r \gg \xi$

$$\chi = \int d^d x \int d^d y \chi(\vec{x}, \vec{y}), \text{ FDT,}$$

$$\propto \beta \xi^{2-\eta}$$

DIVERGES WITH ξ

SET OF EXPONENTS

2.8 SCALING AND UNIVERSALITY

SHOW THAT THE SINGULAR PART OF f DEPENDS ON THE CONTROL PARAMETERS AS.

$$f_{\text{sing}} = |t|^{2-\alpha} g_f \left(\frac{h}{|t|^4} \right)$$

$$t = \frac{T - T_c}{T_c}$$

THIS IS A HOMOGENEOUS FUNCTION OF THE PARAMETERS (t, h)

THIS LEADS US TO :

$\Rightarrow$ FIND RELATIONS BETWEEN EXPONENTS

$\alpha, \beta, \gamma, \delta, \nu, \eta$ NOT ALL INDEP.

IN CURIE-WEISS MODEL

$$f(t, h) = \min_m \left(\frac{t}{2} m^2 + u m^4 - h m \right)$$

ALREADY EXPANDED
FOR $m \approx 0$

I MAY BE CHANGING NUMERICAL FACTORS:

IT IS NOT IMPORTANT

$$t \equiv \frac{T - T_c}{T_c}$$

$$u = \frac{\lambda}{4}$$

(IN PREVIOUS
NOTATION)

USE THE EQ. FOR THE ORDER PARAMETER

$$t m + 4 u m^3 - h = 0 \quad \Rightarrow \quad m_{\text{sol}}(u, t, h)$$

NOT IMPORTANT
↑
wavy line → IMPORTANT

AND WRITE $f(m; t, h) = f(m_{\text{sol}}; t, h)$

$$= f(t, h) \quad \text{FOR SING. PART}$$

①

For

$$\begin{array}{l} t < 0 \\ h = 0 \end{array}$$

$$4u m^3 + t m = 0 \Rightarrow$$

$$m^2 = \frac{-t}{4u} \Rightarrow$$

$$f(t, h) = \frac{t}{2} \frac{(-t)}{4u} + u \frac{t^2}{16u^2}$$

$$f(t, h) = -\frac{t^2}{16u}$$

PARAMETER DEPENDENCE
OF FREE-ENERGY DENSITY

②

For

$$\begin{array}{l} h \neq 0 \\ t = 0 \end{array}$$

$$4u m^3 - h = 0 \Rightarrow$$

$$m = \left(\frac{h}{4u} \right)^{1/3}$$

$$f(t, h) = u \left(\frac{h}{4u} \right)^{4/3} - h \left(\frac{h}{4u} \right)^{1/3}$$

$$f(t, h) = \frac{3}{4^{4/3}} \frac{h^{4/3}}{u^{1/3}}$$

PUTTING THESE TOGETHER,
IN CURIE WEISS:

$$f_{\text{sing}}(t, h) = |t|^2 g_f\left(\frac{h}{|t|^A}\right)$$

THIS IS AN EXAMPLE OF AN HOMOGENEOUS FUNCTION OF ITS ARGUMENTS

WITH $g_f(y=0) = -\text{CONST}$ ✓ ($h=0$)

$$g_f(y \rightarrow \infty) = y^{4/3} \quad |t| \rightarrow 0$$

$$|t|^2 \frac{h^{4/3}}{|t|^{A \cdot \frac{4}{3}}} \sim h^{4/3} \Rightarrow 2 - \frac{A \cdot 4}{3} = 0 \Rightarrow$$

$$A = \frac{3}{2}$$

SUGGEST TO WRITE THE

SINGULAR PART OF THE FREE-ENERGY DENSITY
AS A FUNCT OF CONTROL PARAM.

IN CASES WITH $\alpha \neq 0$ & EQUAL ON BOTH SIDES OF TRANS
 $\alpha = \alpha'$, e.g. BEYOND CURIE-WEISS:

$$f_{\text{sing}}(t, h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^A}\right)$$

JUSTIFICATION OF POWER $2-\alpha$ IN 1ST FACTOR $C_V \sim |t|^{-\alpha}$

RECALL $\langle H \rangle = - \frac{\partial \ln Z}{\partial \beta}$

$$C_V = \frac{\partial \langle H \rangle}{\partial T} = \frac{\partial}{\partial T} \left(- \frac{\partial \ln Z}{\partial \beta} \right) = \frac{\partial}{\partial T} \frac{\partial}{\partial \beta} (\beta F)$$

$$= \frac{\partial}{\partial T} \left[F + \beta \frac{\partial F}{\partial \beta} \right]$$

$$= \frac{\partial F}{\partial T} + \frac{\partial \beta}{\partial T} \frac{\partial F}{\partial \beta} + \underbrace{\beta \frac{\partial}{\partial T} \frac{\partial F}{\partial \beta}}_{\text{MOST SING. PART}}$$

$$C_V \sim \beta \frac{\partial}{\partial T} \frac{\partial F}{\partial \beta} \Rightarrow \text{THE MOST SING PART COMES FROM } \frac{\partial^2}{\partial T^2}$$

$$C_V \sim \frac{\partial^2 F}{\partial T^2} \sim N \frac{\partial^2 f_{\text{sing}}}{\partial T^2}$$

$$C_V \sim |t|^{-\alpha} \Leftrightarrow f_{\text{sing}} \sim |t|^{2-\alpha} g\left(\frac{h}{|t|^\lambda}\right)$$

$$\frac{\partial^2 f_{\text{sing}}}{\partial T^2} \sim |t|^{-\alpha} \quad \text{OK.}$$

THIS PART DOESN'T CONT
 $h \rightarrow 0$: CONST

WE JUSTIFIED f_{sing} BEYOND CURIE-WEISS ($\alpha \neq 0$)

CONSEQUENCES:

THE MAGN DENSITY

$$m = \frac{\partial f}{\partial h} = |t|^{2-\alpha} g'_f\left(\frac{h}{|t|^\Delta}\right) \frac{1}{|t|^\Delta}$$
$$= |t|^{2-\alpha-\Delta} g_m\left(\frac{h}{|t|^\Delta}\right)$$

WE KNOW $m \sim |t|^\beta$ FOR $\frac{h}{|t|^\Delta} \rightarrow 0$

$$\Rightarrow \boxed{2-\alpha-\Delta=\beta} \quad \& \quad g_m(y) \sim 1 \quad y \rightarrow 0$$

WE ALSO KNOW $m \sim h^{1/8}$ FOR $\frac{h}{|t|^\Delta} \rightarrow \infty$

$$\Rightarrow g_m(y) \sim y^{1/8} \quad y \rightarrow \infty$$

$$\boxed{2-\alpha-\Delta-\Delta/8=0}$$

FROM NO ZEP.
ON $|t|$ LEFT

PUTTING THE TWO TOGETHER

$$\boxed{\Delta = \beta \delta} \quad (1)$$

$$2 - \alpha - \beta \delta = \beta \Rightarrow$$

$$\boxed{\beta = \frac{2 - \alpha}{1 + \delta}} \quad (2)$$

IN THE MF CASE

$$\beta = 1/2 \quad \alpha = 0 \quad \delta = 3 : \quad \frac{1}{2} = \frac{2}{4} \quad \checkmark$$

SUSCEPTIBILITY

$$\chi = \frac{\partial^2 f}{\partial h^2} \Big|_{h \rightarrow 0} = \frac{\partial m}{\partial h} \Big|_{h \rightarrow 0}$$

BOTH m AND h
ARE "INTENSIVE"
NOT SCALING WITH
 L^d

$$= |t|^{2-\alpha} \left[g_m'' \left(\frac{h}{|t|^\Delta} \right) \frac{1}{|t|^\Delta} \frac{1}{|t|^\Delta} \right]$$

$$= |t|^{2-\alpha-2\Delta} g_m''(0) \quad h \rightarrow 0$$

SHOULD BE EQUAL TO $|t|^{-\gamma}$

$$\gamma = \alpha + 2\Delta - 2$$

$$\gamma = \alpha + 2\beta\delta - 2 \Rightarrow (3)$$

PROOF OF WIDOM'S

$$\text{FROM (2)} \quad \beta = \frac{2-\alpha}{1+\delta} \Rightarrow \alpha = 2 - \beta(1+\delta)$$

$$\text{IN (3)} \quad \gamma = \cancel{2} - \beta - \cancel{\beta\delta} + 2\beta\delta - \cancel{2} = \beta(\delta - 1)$$

WIDOM'S
IDENTITY

$$\delta - 1 = \gamma/\beta \quad (4)$$

$$\text{IN CURIE-WEISS} \quad \delta=3, \gamma=1, \beta=1/2 \Rightarrow 3-1 = 1/1/2 \quad \checkmark$$

FINALLY, ANOTHER ONE WITH A NAME:

REPLACE WIDOM'S δ IN (3)

$$\gamma = \alpha + 2\beta \left(1 + \gamma/\beta\right) - 2$$

$$\cancel{\gamma} = \alpha + 2\beta + \cancel{\gamma} - 2$$

$$2 = \alpha + 2\beta + \gamma$$

RUSHBROOKE'S
IDENTITY

$$\alpha + 2\beta + \gamma = 2$$

IN MF CURIE-WEISS $\alpha = 0$ $\beta = 1/2$ $\gamma = 1$

$$0 + 1 + 1 = 2 \quad \text{ok.} \quad \checkmark$$

RUSHBAOKE'S

WIDOM'S

$$\alpha + 2\beta\delta + \gamma = 2$$

$$\delta - 1 = \gamma/\beta$$

THESE ARE, FOR THE MOMENT, INDEP OF d

FROM 6 EXPONENTS NOW WE ARE LEFT WITH 4
INDEP. ONES ONLY. BECAUSE WE HAVE TWO RELATIONS

SPATIAL INFO IN C AND ξ START USING ξ

ASSUME SCALING OF CORRELATION LENGTH

$$\xi(t, h) \sim |t|^{-\nu} g_{\xi}\left(\frac{h}{|t|^{\Delta}}\right)$$

$$g_{\xi}(y \rightarrow 0) = \text{ct.} \quad g_{\xi}(y \gg 1) \sim y^{-\nu/\Delta}$$

$$\hookrightarrow \xi \sim h^{-\nu/\Delta}$$

$$-\beta f_{\text{sing}}(t, h) \sim L^{-d} \underbrace{\ln \mathcal{Z}}_{\Delta \text{D.M.}}$$

$$\sim L^{-d} \left(\frac{L}{\xi}\right)^d \quad \begin{array}{l} \text{UNION OF} \\ (L/\xi)^d \text{ INDEP} \\ \text{PIECES} \end{array}$$

$$\sim \xi^{-d}$$

$$\sim |t|^{+d\nu} \left[g_{\xi} \left(\frac{h}{|t|^{\Delta}} \right) \right]^{-d}$$

REPLACING
 ξ BEHAVIOUR

THIS HAS TO BE EQUAL
TO THE PREVIOUS FORM WE USED

$$f_{\text{sing}}(t, h) \sim |t|^{2-\alpha} g_f \left(\frac{h}{|t|^{\Delta}} \right)$$

$\Rightarrow$

$$2 - \alpha = d\nu$$

JOSEPHSON'S
IDENTITY

A RELATION WITH d IN IT. CAUES

HYPERSCALING

STILL HAVE TO RELATE γ TO THE OTHER EXPONENTS
WE KNOW FROM FDT AND DIVERG OF GLOBAL χ :

$$C(r) \sim r^{2-d-\gamma} \Rightarrow \frac{\chi}{L^d} \sim \xi^{2-\gamma}$$

$$\frac{\chi}{L^d} \sim |t|^{-\gamma} \sim \xi^{2-\gamma} \sim |t|^{-\nu(2-\gamma)} \Rightarrow$$

$$\gamma = \nu(2-\gamma)$$

WE SAW IT ALREADY
WHEN WE CALCULATED χ

WE PROVED FOUR RELATIONS BETWEEN
EXPONENTS

IN CONCLUSION, THERE ARE ONLY TWO
INDEPENDENT EXPONENTS.

ALL OTHER ONES CAN BE WRITTEN IN
TERMS OF THEM.

eg. ν AND η

IT WAS BELIEVED THAT $\alpha = \alpha'$ $\gamma = \gamma'$ ON TWO
SIDES OF TRANS. BUT THIS IS NOT NECESSARILY
TRUE $\Rightarrow$ APPLY ABOVE ON EACH SIDE.

HOMOGENEOUS FUNCTIONS

ONE VARIABLE

TAKE A FUNCTION OF ONE VARIABLE $f(x)$

IT IS HOMOGENEOUS OF DEGREE p IF
UNDER SCALING OF x BY A FACTOR b

$$x \rightarrow bx$$

THE FUNCTION GETS MULT. BY b^p :

$$f(bx) = b^p f(x)$$

b IS THE SCALING FACTOR

p IS THE DEGREE OF HOMOGENEITY

NB I COULD CALL $b \rightarrow b^p$ AND DO
 $x \rightarrow b^p x$, $f(b^p x) = b^p f(x)$ EQUIV.

MANY VARIABLES x_i

THEY ARE THE CONTROL
PARAM OF THE TRANSITION
e.g. H & R in our

EXAMPLE

$$b^{\frac{P_i}{P_n}} g(x_1, x_2, \dots, x_n) =$$

$$= g(b^{\frac{P_1}{P_n}} x_1, b^{\frac{P_2}{P_n}} x_2, \dots, b^{\frac{P_n}{P_n}} x_n)$$

$$\forall b \in \mathbb{R}$$

TAKE $x_n = b^{-\frac{P_n}{P_n}} \Rightarrow$ WITH ARGUMENT $= 1$

$$b = x_n^{-1/P_n}$$

THIS IS A CHOICE OF b
THE x 'S ARE THE "INDEP.
VARIABLES."

$$g(x_1, x_2, \dots, x_n) =$$

$$x_n^{\frac{P_i}{P_n}} g\left(\underbrace{x_n^{-\frac{P_1}{P_n}} x_1, x_n^{-\frac{P_2}{P_n}} x_2, \dots, 1}_{\text{ONE LESS INDEP VARIABLE}}$$

ONE LESS INDEP VARIABLE

& we appear as

$$\frac{x_i}{x_n^{P_i/P_n}}$$

case

$$\frac{P_i}{P_n} = \Delta_i \Rightarrow$$

$$\frac{x_i}{x_n^{\Delta_i}}$$

→ THE ARGUMENTS OF g
 $n-1$ SCALING VARIABLES
 g SCALING FUNCTION.

EXAMPLE CURIE WEISS g is $f_{\text{sing}}(h, |t|)$

$$x_1 = h \quad x_2 = |t|$$

$$\frac{x_1}{x_n^{\Delta}} = \frac{h}{|t|^{\Delta}}$$

$$\frac{p_1}{p_2} = \Delta$$

THERE IS ONLY ONE Δ SINCE ONLY ONE RATIO p_1/p_2

$$f_{\text{sing}} = |t|^2 g\left(\frac{h}{|t|^{\Delta}}\right)$$

$$\Delta = \frac{3}{2}; \quad \frac{p_1}{p_n} = 2$$

WE WILL PROVE HOMOGENEITY OF f WITH RG BELOW.

WONDERS OF SCALING : $\left\{ \begin{array}{l} \text{SAME EXPONENTS } \Delta \\ \text{SAME SCALING FUNCTIONS} \end{array} \right.$

WE SAW ABOVE THAT THE FREE-ENERGY DENSITY, SUSC., MAGN., ETC ARE HOMOGENEOUS FUNCT OF THE CONTROL PARAM.

PARTICULAR CASE :

IF THE RESULT REMAINS UNCHANGED UNDER
SCALING

$$x \rightarrow bx$$

$$f(bx) = f(x)$$

$p=0$ IN THE ONE-VARIABLE
CASE ABOVE

THE FREE-ENERGY AND SO ON DON'T HAVE $p=0$
BECAUSE THERE'S A FACTOR IN FRONT OF SCALING FCT.

IN SOME PLACES SCALE INV IS ASSOC. TO NO
PRE-FACTOR. NOT IN CONT. PHASE TRANSITIONS

WHAT HAPPENS WITH LENGTH DEPEND.
OBSERV. AT T_c ?

$$\xi(T) \sim |T - T_c|^{-\nu} \quad h=0$$

$$\Rightarrow \xi(bT) \sim |bT - bT_c|^{-\nu} = b^{-\nu} |T - T_c|^{-\nu}$$

$$\xi(bT) \sim b^{-\nu} \xi(T)$$

HOMOGENEITY OF SPATIAL CORR NEAR T_c

SCALE INVARIANCE USUAL WAY OF REFERRING TO.

$L \rightarrow \infty$, JUST ξ AS ONLY SCALING LENGTH.
BUT, AT CRIT POINT $\xi \rightarrow \infty \Rightarrow$ NONE LEFT !

WHAT HAPPENS WITH $C(\vec{x}, \vec{y})$?

$$C(\vec{x}, \vec{y}) = C(|\vec{x} - \vec{y}|) = C(r)$$

$$C(r) \sim r^{2-d-\eta}$$

$$C(br) \sim b^{2-d-\eta} C(r) \quad \text{BECAUSE IT'S A POWER LAW}$$

WE CAN ALSO WRITE IT AS:

$$b^p C(\vec{x}, \vec{y}) = C(b\vec{x}, b\vec{y})$$

$$p = 2-d-\eta$$

WE ZOOM IN & OUT & STATISTICALLY WE SEE THE SAME PICTURES

NOT ONLY AT CRIT POINT OF 2ND ORDER PHASE TRANS. BUT ALSO FRACTALS, ETC.

FINITE SIZE SCALING

$a \ll \xi \ll L$ IN REAL LIFE

SCALING $\frac{\xi}{a}, \frac{L}{\xi}$ ONLY RELEVANT

TYPICAL SITUATION IN EXPERIMENTS

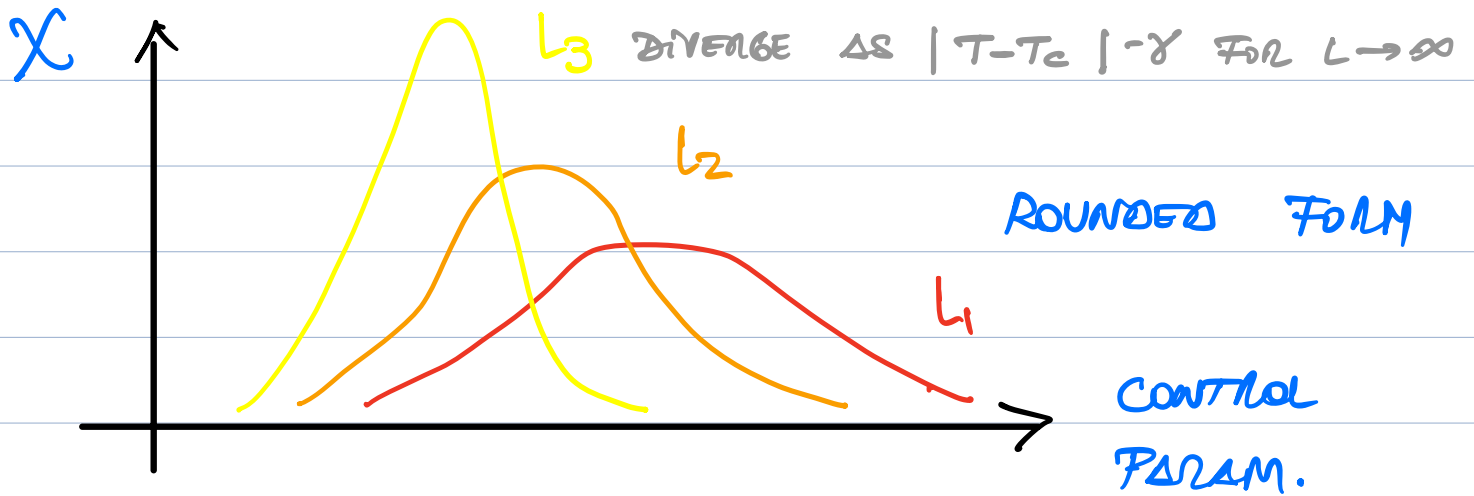
$\frac{\xi}{a} \rightarrow \infty$ $\frac{L}{\xi}$ FINITE

① QUESTION : WHAT HAPPENS WITH eg $\chi(t)$, A QUANTITY THAT WOULD DIVERGE AT T_c IF $L \rightarrow \infty$

?

OBSERVATIONS FROM EXPS & NUM SIMUL.

THIS IS THE LOCAL ONE, OR AN INTENSIVE ONE THAT SHOULD



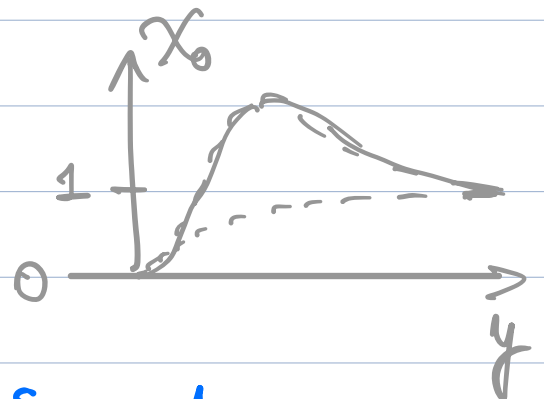
- MAXIMUM MOVES TO THE "LEFT" FOR INCREASING L
- WIDTH OF CURVE DIMINISHES WITH $L \uparrow$
- HEIGHT OF MAX $\uparrow$ WITH $L \uparrow$

ALL $\sim$ POWER LAWS

HYPOTHESIS

$$\chi \sim \xi^{\gamma/2} \chi_0\left(\frac{L}{\xi}\right)$$

$$\chi_0(y) \sim \begin{cases} \text{cst} & y \gg 1 \\ y^{\gamma/2} & y \rightarrow 0 \end{cases}$$



ENSURING

$$\begin{cases} \xi^{\gamma/2} & \text{FOR } L \gg \xi ; L \rightarrow \infty \\ L^{\gamma/2} & \text{FOR } \xi \gg L \end{cases}$$

IN TERMS OF $|t|$ AGAIN, USE $\xi \sim |t|^{-\nu}$
TO REWRITE χ AS A FCT OF $|t|$ AND L

ONE CAN GUESS IT

$$\chi(|t|, L) \rightarrow L^{\gamma/2} \tilde{\chi}(L^{1/2} |t|)$$

$$L^{\gamma/2} \tilde{\chi}(L^{1/2} |t|)$$

WHERE WAS ξ NOW THERE'S L (IN FRONT) AND ARGUMENT
CHANGE IS QUITE OBVIOUS

HOW TO MAKE A CHECK OF THIS?

PLOT $L^{-\gamma/2} \chi_L$ vs. $L^{1/2} |t|$ FOR DIFF L .

DO THE DATA POINTS FALL ON TOP OF EACH OTHER?

CAREFUL WITH NOTATION $\chi = \frac{\partial M}{\partial h}$ CAN BE A LOCAL QUANTITY
OR A GLOBAL ONE IF INTEGRATED OVER SPACE
THERE'S AN L^d DIFF BTW THE TWO

HOMEWORK: ON FINITE SIZE SCALING

THE RENORMALIZATION GROUP (REAL SPACE)

- LARGE CORRELATED SCALES (ξ)
- MICROSC. DETAILS SHOULD NOT MATTER

$\Rightarrow$ COLLECTIVE PHENOM.

SCALE INV.

UNIVERSALITY

VIA BLOCK SPINS

FLOW OF COUPLING CSTS.

THE 1D ISING CHAIN

$$K \equiv \beta J, \quad \beta h = 0$$

$$\mathcal{Z} = \sum_{\{S_i\}} e^{-H_K(\{S_i\})} = \sum_{\{S'_i\}} e^{-H_{K'}(\{S'_i\})}$$

SAME FCT
FORM BUT $\neq$
COUPLING
STRENGTH

BOTH SUMS ON ISING VARIABLES

$$i = 1, \dots, N$$
$$I = 1, \dots, N'$$

$$K = \beta J \quad \text{DIMENSIONAL
PARAMETER}$$

$H_k \neq H_{k'}$ SAME FUNCT FORM BUT
DIFF COUPLING CONST.

N ODD eg $N=7$
 $N-1$ EVEN eg $N-1=6 \Rightarrow \frac{N-1}{2} = 3$

$$Z = \sum_{\{S_i\}} e^{K [S_1 S_2 + S_2 S_3] + K [S_3 S_4 + S_4 S_5] + \dots}$$

THE EVEN SPINS APPEAR WITHIN THE [...] AND ONLY ONCE IN EACH

$\Rightarrow$ SUM OVER THEIR CONFS.

$$Z = \sum_{\{S_i\}} 2 \cosh K(S_1 + S_3) 2 \cosh K(S_3 + S_4) \dots$$

REWRITE EACH FACTOR IN THE FORM OF THE 1D IM
CANONICAL WEIGHT, e.g.

$$2 \cosh K(S_1 + S_3) = f(K) e^{K' S_1 S_3}$$

$$S_1 S_3 = 1$$

$$S_1 S_3 = -1$$

TWO POSSIBILITIES OUT OF

$$S_1 = \pm 1$$

$$S_3 = \pm 1$$

$\Rightarrow$

$$2 \cosh 2k = f(k) \exp k'$$

$$2 = f(k) \exp(-k')$$

TAKE THE RATIO BTW THE TWO EXPRESSIONS

$$\cosh 2k = \exp(2k') \Rightarrow$$

$$k' = \frac{1}{2} \ln \cosh 2k$$

$$f(k) = 2 \exp k' = 2 \exp \frac{1}{2} \ln \cosh 2k$$

$$f(k) = 2 (\cosh 2k)^{1/2}$$

SO WE HAVE FOUND A REWRITING SUCH THAT

$$\mathcal{Z}(k, N) = \left(f(k) \right)^{\frac{N-1}{2}} \quad \leftarrow \quad \# \text{ decim. spins}$$



SPINS

ORIGINAL COUPLING STRENGTH

NEW COUPLING

$$\times \sum_{\{S_i = \pm 1\}} e^{k' \sum_i S_i S_{i+1}}$$

SUM OVER LESS SPINS

$$\mathcal{Z}(k, N) = \left(f(k) \right)^{\frac{N-1}{2}} \mathcal{Z}(k', \frac{N+1}{2})$$

RELATION BETWEEN PARTITION
FUNCTIONS

REMAINING
SPINS

AFTER ONE STEP OF DECIMATION

YOU CAN CHECK IT WORKS FOR N EVEN & ODD

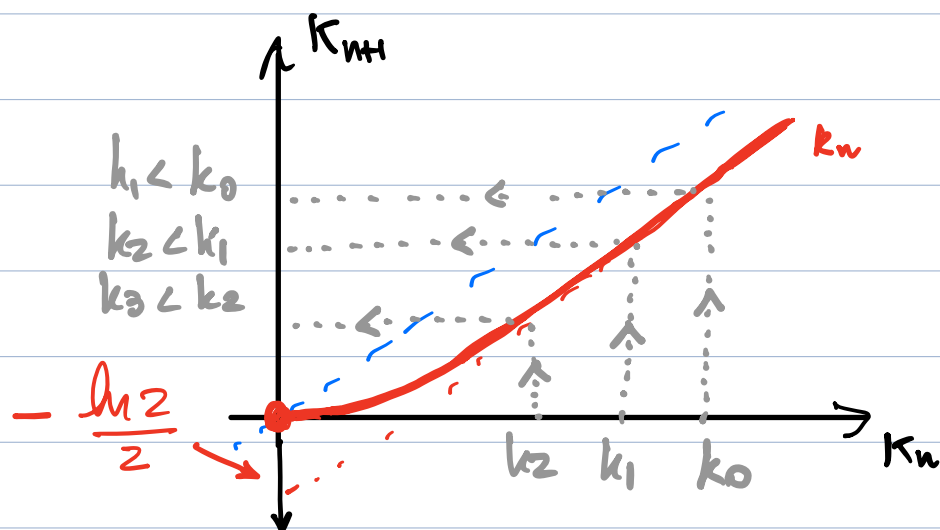
How is the evolution of the coupling constants going ?

$$k' = \frac{1}{2} \ln 2k$$

It can be thought of as a recurrence

$$k_{n+1} = \frac{1}{2} \ln 2k_n$$

with n labelling the step of the decimation.



FUNCTIONAL
FORM OF
ITERATION

$$k_n = 0 \Rightarrow k_{n+1} = 0$$

$$k_n \rightarrow \infty \Rightarrow$$

$$k_{n+1} \rightarrow \frac{1}{2} \ln \left(\frac{e^{2k_n}}{2} \right) = \frac{1}{2} (2k_n - \ln 2)$$

$$= k_n - \frac{\ln 2}{2}$$

$$\frac{d}{dk_n} \left(\frac{1}{2} \ln \cosh 2k_n \right) = \frac{1}{2} \tanh(2k_n) \neq$$

$$\xrightarrow{k_n \rightarrow 0} 0 \quad ; \quad \xrightarrow{k_n \rightarrow \infty} 1$$

TWO FIXED POINTS $k^* = 0$ ATTRACTIVE
 $k^* \rightarrow \infty$ REPULSIVE

BACK TO THE PARTITION FUNCTION, TAKE $\ln$

$\ln Z(k, N) = \sum N$ SHOULD BE $\propto N$
 THAT IS $\sum = O(1)$
 (IT'S PROP OF FREE-ENERGY)

$$\ln Z(N, k) = \frac{N-1}{2} \ln f(k) + \ln Z\left(k', \frac{N+1}{2}\right)$$

$\uparrow$ # DELETED SPINS $\uparrow$ # REMAINING SPINS

$\Rightarrow N \gg 1$

$$S(k) = \frac{1}{2} \ln f(k) + \frac{1}{2} S(k')$$

WITH $f(k)$ WRITTEN ABOVE. FROM HERE,
 $S(k')$ OBTAINED FROM $S(k)$ AND k .

$$S(k') = 2S(k) - \ln f(k)$$

ITERATE IN OPPOSITE DIRECTION. START
 FROM k' VERY SMALL $\Rightarrow$ GET $k > k'$
 (HIGH T) (LOW T)

APPROACH $T \rightarrow 0$ LIKE THIS & CONSTRUCT
 $S(k)$. EXCELLENT AGREEMENT W/ EXACT
 VALUES. SEE TABLE.

NOTE: WE HAVEN'T SEEN THE EFFECT OF $a \rightarrow 2a$ IN
 THIS CALC. BUT, IN $C(\vec{x}, \vec{y})$ WE'LL DO.

RECAP

WE SAW SCALING AND
A VERY QUICK ANALYSIS
OF THE 1D IM $\Rightarrow$ REPEAT IT

SCALING RELATIONS

$$f_{\text{SING}}(t, h) = |t|^{2-\alpha} g_f\left(\frac{h}{|t|^4}\right)$$

$$\zeta(t, h) \sim |t|^{-\nu} g_{\zeta}\left(\frac{h}{|t|^4}\right)$$

FROM HERE WE DERIVED:

$\Rightarrow$ $\left\{ \begin{array}{l} \text{RELATIONS BETWEEN EXPONENTS} \\ \text{ONLY TWO INDEPENDENT EXPONENTS} \end{array} \right.$

FIRST ENCOUNTERED WITH RG IDEAS

1d IM - $h=0$

DECIMATION $\Rightarrow$ INTEGRATE AWAY ONE
OUT OF TWO SPINS

$$K_{n+1} = \frac{1}{2} \ln \cosh 2K_n$$

FLOW OF COUPLING CONSTANT w/ TWO
FIXED POINTS

$K^* = 0$
 $T \rightarrow 0$
STABLE

$K^* \rightarrow \infty$
 $T = 0$
UNSTABLE



CRITICAL POINT

$$\mathcal{S}(k) = \frac{1}{2} \ln f(k) + \mathcal{Z}(k')$$

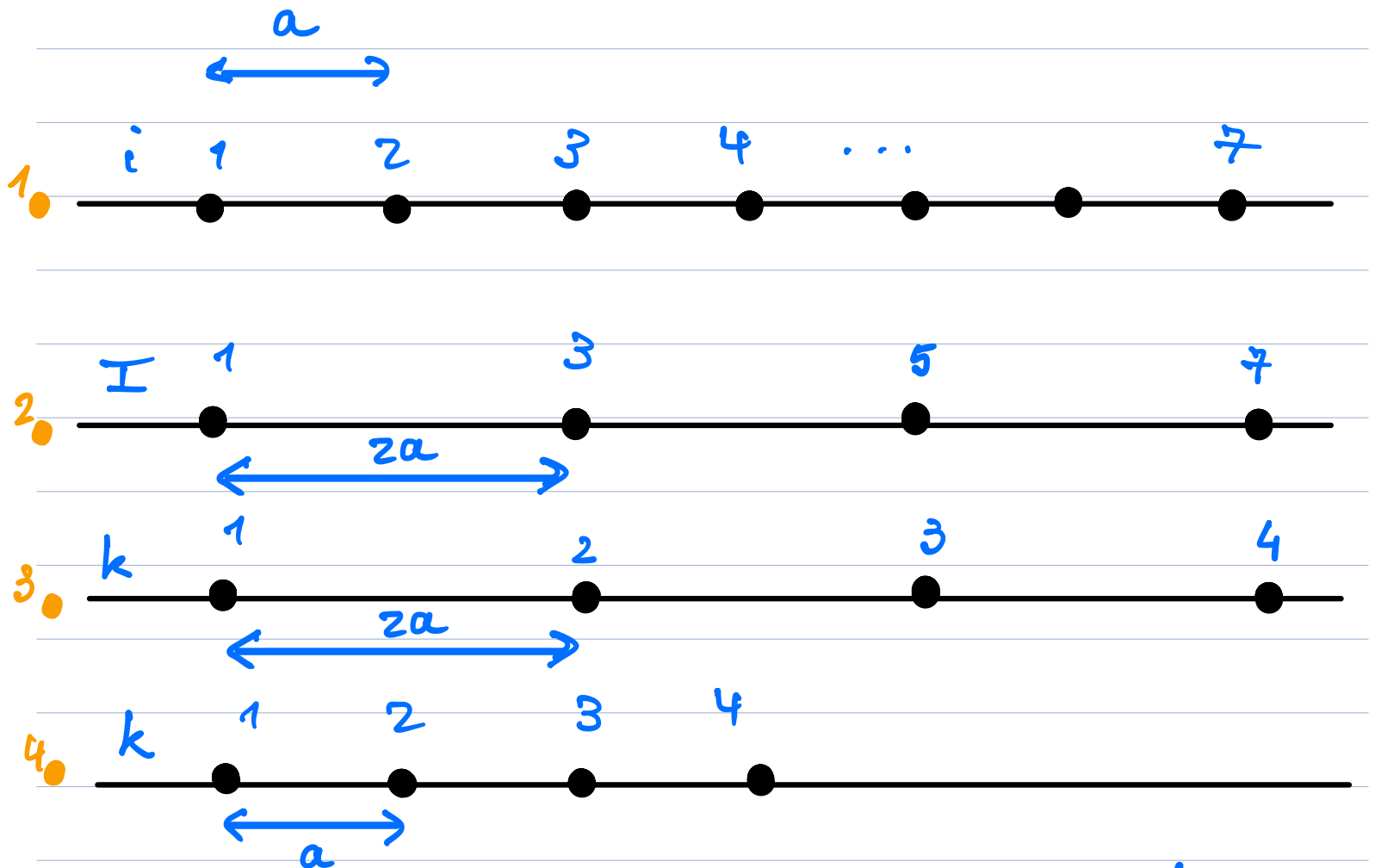
WHERE

$$\mathcal{J}(k) = N \ln \mathcal{Z}(k)$$

$$f(k) = 2 [\cosh(2K)]^{1/2}$$

(NOTE $\mathcal{J}(k) = -k_B T F(k)$)

THE SKETCH OF TRANSFORMATIONS



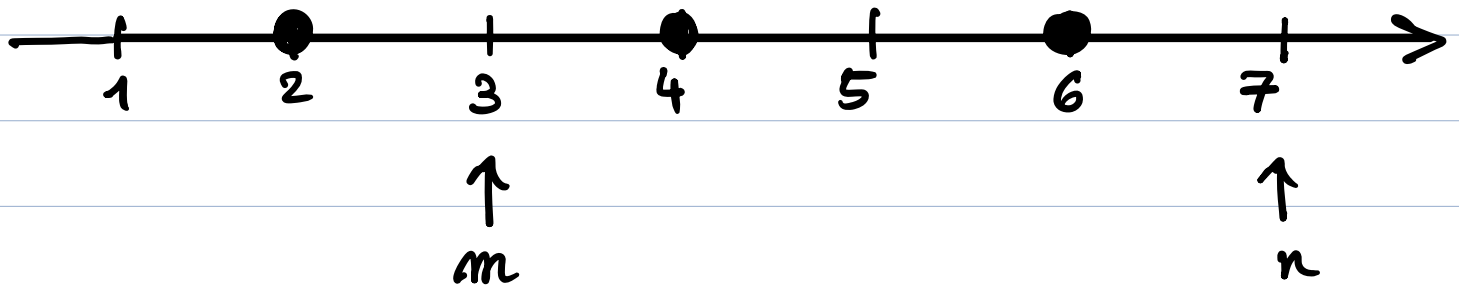
1. $i = 1, \dots, N$ SPINS DECIMATE $\frac{1-N}{2}$ SPINS:

2. $I = 1, \dots, \frac{N+1}{2}$ SPINS RELABEL REMAINING SPINS:

3. $k = \frac{I+1}{2} = 1, \dots, \frac{N+1}{2}$ SPINS SPACE RESCALING:

4. $2a \rightarrow a$

NOTE THAT IN THE LAST TRANSFORMATION
THE DISTANCES HAVE BEEN DIVIDED BY 2



● DECIMATED

m, n LABELS OF SPINS TO BE CORRELATED

$$C_{mm} = \langle S_m S_n \rangle$$

$$= \frac{1}{Z} \sum_{\{S_i = \pm 1\}} S_m S_n e^{-\beta H}$$

$$= C(\vec{x}, \vec{y})$$

$$\vec{x} = am \quad \vec{y} = an$$

THE CORRELATION LENGTH

FROM $C(\vec{x}, \vec{y}) \sim e^{-|\vec{x}-\vec{y}|/\xi(k)}$

$[\xi(k)] = L$ LENGTH UNITS

DEPENDS ON DIMENS. PARAM k

$$C(\vec{x}, \vec{y}) = \frac{1}{Z(N, k)} \sum_{\{S_i = \pm 1\}} S_m S_n e^{+k \sum_{i=1}^N S_i S_{i+1}}$$


 SAY THEY ARE ODD

- WE USE THE RELATION BETWEEN PARTITION SUMS BEFORE & AFTER DECIMATION
- WE DECIMATE THE EVEN LABELED SPINS.

$$= \frac{1}{\left(\cancel{f(k)}\right)^{\frac{N-1}{2}} Z\left(\frac{N+1}{2}, k'\right)} \sum_{\substack{\{S_I = \pm 1\} \\ I \text{ (ODD)}}} S_m S_n \left(\cancel{f(k)}\right)^{\frac{N-1}{2}} e^{k' \sum_{I \text{ (ODD)}} S_I S_{I+1}}$$

• WE RENAME $k = \frac{I+1}{2} = 1, \dots, \frac{N+1}{2}$

$$= \frac{1}{Z\left(\frac{N+1}{2}, k'\right)} \sum_{\left\{ \begin{array}{l} S_u = \pm 1 \\ k=1, \dots, \frac{N+1}{2} \end{array} \right\}} S_{\frac{m+1}{2}} S_{\frac{n+1}{2}} e^{k' \sum_{k=1}^{\frac{N+1}{2}} S_u S_{u+1}}$$

THE CORREL FCT CAN BE COMPUTED IN THE TWO WAYS & ONE SHOULD GET THE SAME RESULT IT'S JUST TWO WAYS OF CALCULATING

• IN THE SYST WITH (k, N, a)

• IN THE SYST WITH $(k', \frac{N+1}{2}, 2a)$

$$C(\vec{x}, \vec{y}) \Big|_{N, k, a} = C(\vec{x}, \vec{y}) \Big|_{\frac{N+1}{2}, k', 2a}$$

SHOULD HAVE THE SAME FUNCTIONAL FORM

CLOSE TO THE CRIT POINT EXP. DECAY
WITH A CORRELATION LENGTH ξ

$$[\xi] = L \quad \text{LENGTH DIMENSION}$$

← DIMENSIONAL ↓

$$\overline{\xi}(k) a = \overline{\xi}(k') 2a$$

SAY IT'S $\xi = 10 \text{ cm} = \overline{\xi}(k) 1 \text{ cm} = \overline{\xi}(k') 2 \text{ cm}$

$$\Rightarrow \overline{\xi}(k') = \overline{\xi}(k) / 2$$

$$\overline{\xi}(k') = \frac{\overline{\xi}(k)}{2}$$

HOW DOES ONE GET $\overline{\xi}(k)$ FOR $k \rightarrow \infty$?

NOTICE THAT $k_{n+1} = \frac{1}{2} \ln \text{ch } 2k_n$
FOR LARGE k_n

$$\rightarrow \frac{1}{2} \ln(e^{2k_n} / 2)$$

$$= \frac{1}{2} (2k_n - \ln 2)$$

$$k_{n+1} - k_n = \frac{-\ln 2}{2} \quad \text{DECREASE PER ITERATION STEP}$$

SAY ONE STARTS FROM $k_0 = k$. HOW LONG IN n DOES IT TAKE TO GET TO $k_f \sim \Theta(1)$? CALL THAT LIMITING n ; $\bar{n}$:

$$k_f = \Theta(1) = k - \frac{\ln 2}{2} \cdot \bar{n} \Rightarrow$$

$$\bar{n} \sim \frac{2k}{\ln 2}$$

IN EACH STEP OF DECREASES BY THE SAME AMOUNT

THEN $\bar{\Sigma}(k') = \frac{\bar{\Sigma}(k)}{2}$ AFTER $\bar{n}$ STEPS

$$\Rightarrow \bar{\Sigma}(k_f) = \frac{\bar{\Sigma}(k)}{2^{\bar{n}}} = \Theta(1)$$

$$\bar{\Sigma}(k) \sim 2^{\bar{n}} = 2^{\frac{2k}{\ln 2}} \sim e^{2k}$$

EXPON. DIVERGENCE OF $\bar{\Sigma}(k)$ SPECIFIC OF $d=1$

THE LAST IDENTITY IS A CONSEQ. OF

$$2 = e^{\ln 2} \Rightarrow$$

$$2^{\frac{2k}{\ln 2}} = \left(e^{\ln 2} \right)^{\frac{2k}{\ln 2}}$$

$$= e^{\ln 2 \cdot \frac{2k}{\ln 2}} = e^{2k}$$

$$\bar{\zeta}(k) \sim e^{2k}$$

NOTE THAT THIS IMPLIES

$$\zeta(k) \sim a e^{2k}$$

THE LENGTH UNITS ARE RECOVERED. WHAT'S
IMPORTANT IS THE EXP DIVERGENCE
AS $T \rightarrow 0$

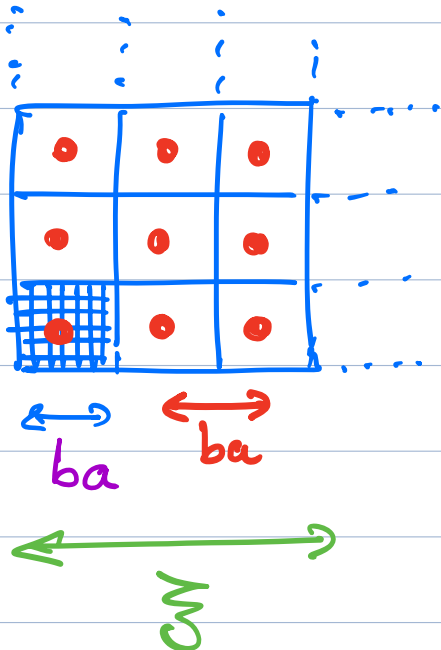
KADANOFF BLOCK SPINS & RENORMALIZATION

AS LANDAU, BUILD BLOCK SPINS

$$S_I' = b^{-x} \sum_{i \in B_I} S_i \quad [b] = 1$$

NORMALIZATION

SINCE SOME SPINS IN B_I ARE REVERSED
THE SUM IS NOT EQUAL TO $N_I = \# \text{ SPINS IN } B_I$ } BUT b^{-x}
WITH $x < d$



• S_I' ISING VARIABLES

NEW LATTICE CONST MAGNIFIED

$$a' = ba \quad (\text{NB } b > 1 \text{ A NUMBER})$$

$$\tilde{\zeta} = \bar{\zeta} \cdot a = \bar{\zeta}' a' = \bar{\zeta}' ba$$

$\bar{\zeta}$ CORREL. LENGTH IN ABSOLUTE
UNITS (eg. cm)

$$\bar{\zeta} = \bar{\zeta}' b \Rightarrow \bar{\zeta}' = \bar{\zeta} / b$$

$\bar{\zeta}'$ REDUCED ! IT'S AS SYST FURTHER
AWAY FROM CRITICALITY

SCALE TRANSF. TO GET BACK TO ORIGINAL

$$x'_\alpha = \frac{x_\alpha}{b} \quad \alpha = 1, \dots, d$$

BLOCK SPINS : ISING VARIABLES OCCUPYING
THE SITES OF A LATTICE W/ SAME LATTICE
SPACING a AFTER DISTANCE RE-SCALING ($\times 1/b$)
BUT LESS SPINS : $N' = N/b^d$

FOR THE 1D ISING CHAIN WE CALCULATED

BY DECIMATION THE NEW HAMILTONIAN

$$N \rightarrow \frac{N+1}{2}, \quad k \rightarrow k', \quad a \rightarrow a' = \frac{1}{2}a = 2a$$

IN GENERAL, WE DON'T KNOW HOW TO DO IT.

PROPOSE

$$H'(\{s'_i\}) = R_b H(\{s_i\})$$

FOR A TRANSFORMATION WITH SCALE FACTOR b

PROPERTIES

COMPOSITION

- APPLY TRANSF b & APPLY TRANSF b SHOULD BE EQUIV TO APPLYING TRANSF b^2 . IT'S MULTIPLICATIVE.

$$R_b^2 = R_b R_b \Rightarrow$$

$$R_b^n = (R_b)^n$$

n TIMES

THIS TRANSF SHOULD HAVE Δ FIXED POINT AT CRITICALITY

$$H'(\{s_i'\}) = H(\{s_i\})$$

FOR $\xi \gg r \gg ba \gg a$

WE CAN WITH A STAR THE FIXED POINT.

IN SIMPLE CASES, SAME FUNCT FORM

DESCRIBING LONG DIST PROP.

THE NEW PARAMETERS AT THE FIXED POINT $\Rightarrow$ TO BE FOUND.

$$R_b H^* = H^*$$

COUPLING CONST FLOW

NB. FORM OF HAMILT.

MAY CHANGE WRT.

ORIGINAL ONE

$$H(\{S_i\}) = - \sum_A J_A S_A$$

ALL POSSIBLE GROUPS OF SPINS

(NOTE THAT IN A GENERIC PROBLEM ONE
COULD START WITH TWO-BODY INTERACTIONS
AND GENERATE MANY-BODY ONES AFTER
DECIMATION)

$$\beta J_A = k_A \quad \Rightarrow \quad [k] = \underbrace{(k_1, k_2, \dots)}_{\text{ENSEMBLE OF PARAMETERS}}$$

$$[k'] = R_b [k]$$

TRANSFORMATION OVER FINITE LENGTH $b < +\infty$

ANALYTIC FUNCTION OF COUPLING CONSTANTS

RG TRANSFORM

- EXPAND IN TAYLOR

$$K'_\alpha = \sum_\beta R_{\alpha\beta}(b) K_\beta + \dots$$

MEASURING K_α FROM CRIT POINT

MEANING $K_\alpha - K_\alpha^c \rightarrow K_\alpha$

eg. $\frac{T-T_c}{T_c}$, βh IN FM-PM PROBLEM

$K_\alpha^c = 0 \quad \forall \alpha. \implies$ NO ZEROth ORDER TERMS IN THE TAYLOR EXPANSION

$$K' = A(b) K + B(b) H + \dots$$

$$H' = C(b) K + D(b) H + \dots$$

for $b=1$, NO TRANSF $\Rightarrow$

$$A(1)=1 \quad B(1)=0$$

$$C(1)=0 \quad D(1)=1$$

USE SYMM. TO KILL SOME TERMS

$$\text{eg. ISING } \{s_i \rightarrow -s_i\} \quad \{H \rightarrow -H\}$$
$$\{k \rightarrow k\}$$

$$\Rightarrow B(b)=0$$
$$C(b)=0$$

$$K' = A(b)K + \dots$$

... h.o.t.

$$H' = D(b)H + \dots$$

FINALLY, IMAGINE YOU APPLY TWO TRANSF
IN A ROW W/ SCALE b . SHOULD BE THE
SAME AS APPLYING 1 TRANSF WITH $b \cdot b$

$$A(b)A(b) = A(b^2)$$

$\Rightarrow$

$$D(b)D(b) = D(b^2)$$

$$A(b) = b^{\gamma_k}$$

$$D(b) = b^{\gamma_H}$$

$$\gamma_k > 0$$

$$\gamma_H > 0$$

TO MOVE AWAY FROM CRIT
w/ TRANSF. PERFORMED

IN SUMMARY

$$\begin{aligned} k' &= b^{\gamma_k} k \\ H' &= b^{\gamma_H} H \end{aligned}$$

DROPPING h.o.t.

NOW, IN FREE ENERGY

$$F(k', H') = F(k, H) \quad \text{AT CRITICALITY}$$

$$N' f(k', H') = N f(k, H)$$

$$f(k, H) = \left(\frac{N'}{N} \right) f(k', H')$$

$$f(k, H) = b^{-d} f(b^{y_k} k, b^{y_H} H)$$

HOMOGENEITY PROP. PROVEN!

Choose now $b = k^{-1/y_k}$

$$f(k, H) = k^{d/y_k} f\left(1, \frac{H}{k^{y_H/y_k}}\right)$$

$$\Delta \equiv \frac{y_H}{y_k}$$

$$\frac{d}{y_k} = 2 - \alpha$$

$$f(k, H) = k^{2-\alpha} g_f\left(\frac{H}{|k|^4}\right)$$

WE FOUND THE SCALING FORM.

RELATION BETWEEN PHYSICAL EXPONENTS
AND THE POWERS IN PARAM TRANSF

$$\frac{\overline{\xi}(R_b(k))}{\overline{\xi}(k)} = \frac{1}{b}$$

RECALL 1dIM
CALCULATION
& ARGUMENT

ASSUMING $\overline{\xi} \sim (k - k_c)^{-\nu}$ POWER LAW DIVERGENCE

$$\begin{aligned} \frac{1}{b} &= \left(\frac{R_b(k) - R_b(k^*)}{k - k^*} \right)^{-\nu} = \left(\frac{R_b(k) - k^*}{k - k^*} \right)^{-\nu} \\ &= \left(\left. \frac{dR_b(k)}{dk} \right|_{k=k^*} \right)^{-\nu} \end{aligned}$$

FROM HERE,

$$-\ln b = -\nu \ln \left. \frac{dR_b}{dk} \right|_{k=k^*}$$

$$\nu = \frac{\ln b}{\ln \left. \frac{dR_b}{dk} \right|_{k=k^*}}$$

FOR THE POWER LAW $R_b \sim b^{y_k} k$

$$\frac{dR_b(k)}{dk} = \frac{d}{dk} (b^{y_k} k) = b^{y_k}$$

$$\ln \frac{dR_b}{dk} = y_k \ln b$$

$\Rightarrow$

$$v = 1/y_k$$

NOTE THAT $y_k = 1/v$ $y_k = \frac{d}{2-\alpha}$

$\Rightarrow d\nu = 2-\alpha$ WHICH IS THE JOSEPHSON
IDENTITY DERIVED ABOVE

WE ALSO HAD $\Delta \equiv \frac{J_H}{J_k}$, $\frac{d}{y_k} = 2-\alpha$

WHICH IMPLIES

$$J_H = \Delta \frac{d}{2-\alpha}$$

THE TRANSF OF $\xi \Rightarrow$

POWER LAW DIVERGENCE OF ξ

AGAIN USING NOT. k FOR DIST FROM CRIT POINT

$$\bar{\xi}(k_n) = \bar{\xi}(b^{n\gamma_k} k) \quad \text{REPLACING } k_n$$

$$= b^{-n} \bar{\xi}(k) \quad \begin{array}{l} \text{USING FLOW AWAY} \\ \text{FROM CRITICALITY} \\ n\text{-TRANSFORMS} \end{array}$$

MULTIPLYING BY b^n

$$b^n \bar{\xi}(b^{n\gamma_k} k) = \bar{\xi}(k)$$

CHOOSING $b^n = \left(\frac{b}{k}\right)^{\frac{1}{\gamma_k}} = (bk^{-1})^{\frac{1}{\gamma_k}}$

ONE HAS $b^{n\gamma_k} k = (bk^{-1})^{\cancel{\gamma_k}/\gamma_k} k = b$

& REPLACING ABOVE

$$(bk^{-1})^{\frac{1}{\gamma_k}} \bar{\xi}(b) = \bar{\xi}(k)$$

ISOLATING THE k DEPENDENCE

$$\bar{\xi}(k) \sim k^{-1/\gamma_k}$$

(THE REST IS A NUMBER $b^{1/\gamma_k} \bar{\xi}(b)$)

Σ SHOWED POWER LAW DIVERGENCE

WITH

$$\nu = 1/\gamma_k$$

GENERAL

$$[k'] = R_b [k]$$

TRANSF PARAM.

$$\xi(k') = \frac{\xi(k)}{b}$$

TRANSF CORR LENGTH

FIXED POINT FROM CORR LENGTH

$$\xi^* \rightarrow 0$$

OR

$$\xi^* \rightarrow \infty$$

TRIVIAL

INTERESTING

STABLE

UNSTABLE UNDER
FLOW

ATTRACTIVE

REPULSIVE

CLOSE TO THE FIXED POINT

$$[k'] = R_b [k]$$

$\Rightarrow$

$\alpha = 1, \dots, \# \text{ PARAM.}$

$$K'_\alpha(b) - k_\alpha^* = R_{\alpha\beta}(b) (k_\beta - k_\beta^*)$$

LINEAR FORM CLOSE TO $[k^*]$

$$\delta K'_\alpha(b) = R_{\alpha\beta}(b) \delta k_\beta$$

$$R_{\alpha\beta}(b) = \left. \frac{\partial K'_\alpha(b)}{\partial k_\beta} \right|_{[k^*]} \text{ AT FIXED POINT}$$

MATRICIAL FORM

$$\delta K' = R \delta k \quad \# \text{PARAM} \times \# \text{PARAM MATRIX}$$

DIAGONALIZE WITH $UU^T = U^TU = \mathbb{1}$

$$\underbrace{U \delta k'}_{\downarrow} = \underbrace{U R U^T}_D \underbrace{U \delta k}_{\downarrow}$$

$$\delta \kappa'(b) = D(b) \delta \kappa \quad \delta \kappa_{\Delta PP4} \text{ is } \delta \kappa \text{ ROTATED BY } U$$



DIAGONAL MATRIX WITH
EIGENVALUES $\{\lambda_\alpha\}$ IN DIAGONAL

FROM TRANSF. COMPOSITION RULE
(MULTIPLICATIVE)

$$\lambda_\alpha(b) = b^{y_\alpha} \Rightarrow y_\alpha = \frac{d \ln \lambda_\alpha}{d b}$$

DERIVATIVE JUSTIFIED SINCE b CONT.
& CLOSE TO 1 ASSUMED.

THE TRANSF. OF PARAM. YIELDS $\lambda_\alpha(b)$
 $\Rightarrow y_\alpha$

$\Rightarrow$ DIVERGENCE OF
OBSERV. eg

$$\bar{\Sigma}(k) \sim k^{-1/y_k}$$

CLASSIFICATION OF PARAMETERS

- $y_\alpha > 0 \Rightarrow$ RELEVANT (INCREASES)
- $y_\alpha < 0 \Rightarrow$ IRRELEVANT (VANISHES)
- $y_\alpha = 0 \Rightarrow$ MARGINAL (h.o.t.)

COMMENTS

- RG TRANSF MOVED WITHIN A PHASE, DOESN'T GO ACROSS PHASE TRANS.
- REPULSIVE FIXED POINTS $\Rightarrow$ CRIT. POINTS ($\exists$ DECREASED UNDER RG)
 - IN DISORDERED PHASE RG TO $T \rightarrow \infty$
 - IN ORDERED PHASE RG TO $T \rightarrow 0$

THE EXAMPLE WE SOLVED 1DIM $T_c = 0$

SO THE FLOW WENT TO $T \rightarrow \infty$

FOR 2DIM $0 < T_c < \infty \Rightarrow$ SO FLOW AWAY

FROM IT IN THE TWO PHASES.