Advanced Statistical Physics

Exam

22nd December, 2023

Surname :

Name :

Master :

Write your surname & name clearly and in CAPITAL LETTERS.

You can write in English or French, as you prefer.

No books, notes, calculator nor mobile phone allowed.

Not only the results but also the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

The answers must be written neatly within the boxes.

The problems roughly follow the order of the chapters in the Lecture Notes but are not necessarily of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you found difficult.

Conceptual questions **30pt**

1 - Explain what "ergodicity" means. Support your answer with an equation. **2pt**

Ergodicity means that after a sufficiently long time, time and ensemble averages coincide, that is, well-behaved global observables satisfy

$$\overline{O}(t) = \langle O \rangle$$

with

$$\overline{O}(t) \equiv \lim_{\tau \gg t_0} \frac{1}{\tau} \int_t^{t+\tau} dt' O(t')$$

independent of t for t sufficiently large, and

$$\langle O \rangle \equiv \frac{1}{N} \sum_{a=1}^{N} O_a$$

N being the number of elements of the ensemble.

2.a - Define the term "phase". **2pt**

A state of matter
No singularity in the free-energy for all the states in a phase
A domain of parameter space in which the system displays similar behaviour

2.b - Give an example. **1pt**

Solid; ferromagnetic.

3.a - Define the term "phase transition". **2pt**

A sharp change between states of matter.

3.b - Give an example. **1pt**

From solid to liquid; from paramagnetic to ferromagnetic.

4 - Define the order parameter in the context of phase transitions. **2pt**

A quantity that characterizes the degree of order in a system.

5 - Describe the behaviour of the order parameter across a second-order phase transition. **2pt**

Continuous.

For example,

$O(g) = 0$ on one side of the disordered side of the transition and

$O(g) \sim (|g - g_c|^\beta)$ on the ordered one with β the critical exponent

6.a - Define the spatial connected correlation function in a problem described by, say, a scalar field. **2pt**

$$C(r) \equiv \langle [\phi(\vec{r}') - \langle \phi(\vec{r}') \rangle][\phi(\vec{r}' + \vec{r}) - \langle \phi(\vec{r}' + \vec{r}) \rangle] \rangle$$

6.b - Describe the spatial connected correlation function close and at a second order phase transition. Use an equation, a plot and explain them. **3pt**

$$C(r) \sim r^{2-d-\eta} \, e^{-r/\xi(g)}$$

with $\xi(g) \sim |g - g_c|^{-\nu}$ the equilibrium correlation length which diverges when the control parameter approaches the critical point g_c with the critical exponent ν

The drawing of hte connected correlation: In a log-log plot, straight line with a cut-off away from g_c and a full straight line at g_c.

Drawing of the correlation length divergence on both sides of g_c

7.a - Describe the behaviour of the order parameter across a first-order phase transition. **1pt**

Jumps

7.b - Describe the behaviour of the correlation length across a first-order phase transition. **1pt**

It does not diverge, finite

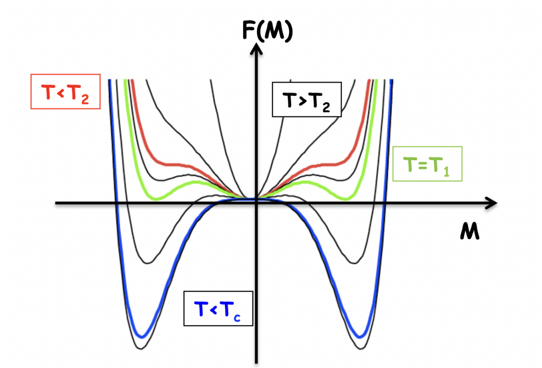


Figure 1: The curves represent the Curie-Weiss or mean-field free-energy density of a magnetic system in contact with baths at different temperatures indicated in the figure, as a function of the magnetisation density, called M here. T_S (in blue) is the name given to the temperature at which the minimum at $M = 0$ disappears.

8.a - Describe the phase transition that is represented in Fig. 1. **2pt**

At T_1, first order phase transition from $M = 0$ at $T > T_1$ to $M \neq 0$ at $T < T_1$, with two possible signs.

8.b - What could happen between T_2 and T_S? Explain. **2pt**

Hysteresis between $M \neq 0$ and $M = 0$.

9 - Do you know a system in which the correlation length diverges in a full phase? **1pt**

2d XY model

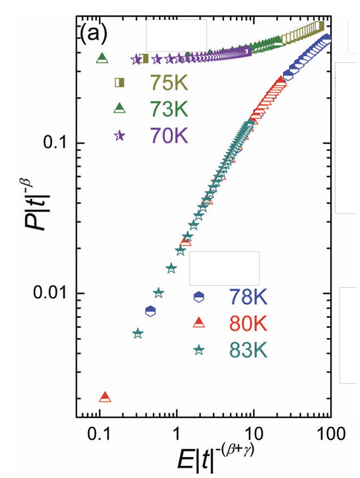


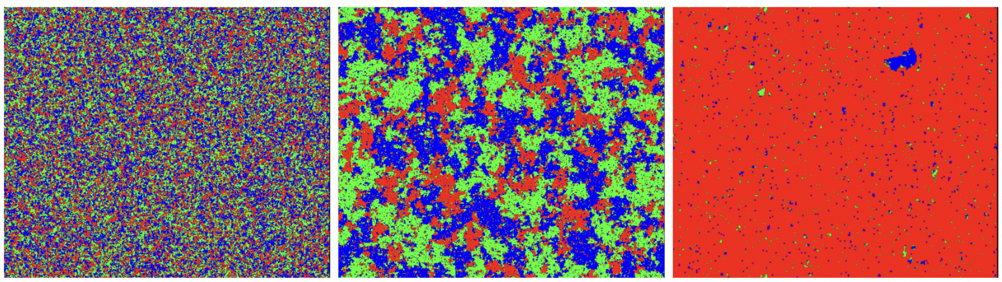
Figure 2: Image extracted from Ding & Zhong, *A theoretical strategy for pressure-driven ferroelectric transition associated with critical behavior and magnetoelectric coupling in organic multiferroics*, Phys. Chem. Chem. Phys. 22, 19120 (2020). P is the polarization, E is the modulus of an applied electric field, and the data points have been measured at different temperatures indicated in the plot.

10 - Explain what has been done in Fig. 2. Describe what are t, β and γ, used in the vertical and horizontal axes. **2pt**

A scaling plot with the purpose of identifying P_c, $t = P - P_c$ possibly divided by P_c, and the critical exponents β (of the order parameter) and γ (of the susceptibility). Note that the title of the paper is ... *for pressure driven...* so the tuning parameter to go through the transition is the pressure P

11 - Do the answer to the questions above apply to classical and quantum phase transitions or is there a difference when dealing with the quantum ones? Justify your answer. **2pt**

Yes. Classical $d+1$ dimensional - quantum d dimensional mapping, with the corresponding association of parameters



T_1 T_2 T_3

Figure 3: These are three configurations of the bidimensional Potts model with three colors. This model has variables placed on the vertices of a lattice, which can be coloured in three ways: blue, red and green. The interactions act on first neighbours on the lattice and favour that neighbouring sites be occupied by variables with the same colour. The configurations have been drawn from equilibrium at different temperatures T_1, T_2 and T_3.

12 - What do you conclude from the images, especially about the values of the three temperatures T_1, T_2 and T_3? **2pt**

Above, at and below T_c. T_3 is not zero since there are finite T fluctuations in the last figure. There has been spontaneous symmetry breaking towards the red state in this case.

Problem: The antiferromagnetic Ising model **40pt**

Take an Ising antiferromagnetic model defined by the Hamiltonian

$$\mathcal{H}_{\rm AF}[\{s_i\}] = \frac{J}{2} \sum_{\langle ij \rangle} s_i s_j - H \sum_i s_i \tag{1}$$

with $i = 1, \ldots, N$ classical Ising spins, $s_i = \pm 1$. The coupling constant is $J > 0$ and H is an external field. The sum runs over nearest neighbours on a d dimensional hypercubic lattice with coordination z and the factor $1/2$ ensures that the contribution of each link is counted once. We will consider the statistical properties of this model coupled to an equilibrium bath at temperature T.

Generic questions

1. Identify the symmetries of this Hamiltonian. Consider $H = 0$ and $H \neq 0$. **1pt**

Global spin reversal $s_i \mapsto -s_i$ for $H = 0$. Explicitly broken for $H \neq 0$.

2. Which ground states do you expect for $H = 0$? Draw it or them for $d = 1$. **1pt**

Antiferromagnetic ordering for $H = 0$, alternating up-down spins.

3. Is the ground state unique or are there many? Which would be their degeneracy in the latter case? **2pt**

For $H = 0$ two related by symmetry.
For $H \neq 0$ unique ground state.

4. Which kind of equilibrium phase do you expect at very high temperature? **1pt**

Paramagnetic, disordered.

5. You will discuss below whether you expect a finite temperature phase transition. Focus on $H = 0$.

   (a) Under which condition on the system size? **1pt**

   $N \to \infty$.

   (b) Between which phases? **1pt**

   from a PM (high T) to an AF (low T)

   (c) In which space dimensions? **2pt**

   In $d \geq 2$
   In $d = 1$ no finite T order

   (d) If you exclude the existence of a finite temperature phase transition in some particular value of d, explain why. **1pt**

   In $d = 1$ the same Peierls argument as for the FM chain

6. Take the case $H = 0$. Use symmetry arguments (based on your answer to question 1.) to evaluate the equilibrium global magnetization $m = N^{-1} \sum_{i=1} \langle s_i \rangle$ and the equilibrium local magnetization $m_i = \langle s_i \rangle$. **1pt**

$m = m_i = 0$ because of the up-down symmetry.

7. Which is the mechanism whereby the system acquires the low temperature ordering that you have identified in previous questions? **1pt**

Spontaneous symmetry breaking, due to an infinitesimal staggered pinning field, between the two equivalent equilibrium states

8. Which order parameter would you propose? **1pt**

Staggered magnetization over two sub-lattices A and B, with $m_A = -m_B$.

The mean-field approximation

We will now study this model in its mean-field approximation.

9. In order to capture the low temperature phase, can you treat all spins in the same way? **1pt**

no, separate them in two sublattices.

10. Use now the approach of Weiss, in which the interaction between neighboring spins is replaced by an interaction with an averaged value which will play the role of an order parameter and contributes to the local effective field. Derive the mean-field approximation of the Hamiltonian $\mathcal{H}_{\rm AF}$ justifying the steps followed and the approximations made. **4pt**

We call A and B the two sub-lattices. We write a spin in each of these sub-lattices as its average plus its fluctuations

$$s_j = m_A + (s_j - m_A) = m_A + \delta s_j \qquad\qquad s_k = m_B + (s_k - m_B) = m_B + \delta s_k \tag{2}$$

with $\delta s_j \ll m_A$ and $\delta s_k \ll m_B$. The quadratic part of the Hamiltonian then reads

$$\begin{aligned}
H_q(\{s_i\}, m_{A,B}) &= \frac{J}{2} \sum_{\substack{j\in A\\ k=\partial_j\in B}} (m_A + \delta s_j)(m_B + \delta s_k) + \frac{J}{2} \sum_{\substack{k\in B\\ j=\partial_k\in A}} (m_B + \delta s_k)(m_A + \delta s_j) \\
&\sim \frac{J}{2} \sum_{\substack{j\in A\\ k=\partial_j\in B}} (m_A m_B + \delta s_j m_B + m_A \delta s_k) + \frac{J}{2} \sum_{\substack{k\in B\\ j=\partial_k\in A}} (m_B m_A + \delta s_k m_A + m_B \delta s_j) \\
&= \frac{J}{2} \sum_{\substack{j\in A\\ k=\partial_j\in B}} [m_A m_B + (s_j - m_A) m_B + m_A (s_k - m_B)] \\
&\quad + \frac{J}{2} \sum_{\substack{k\in B\\ j=\partial_k\in A}} [m_B m_A + (s_k - m_B) m_A + m_B (s_j - m_A)] \\
&= \frac{J}{2} \sum_{\substack{j\in A\\ k=\partial_j\in B}} [-m_A m_B + s_j m_B + m_A s_k] + \frac{J}{2} \sum_{\substack{k\in B\\ j=\partial_k\in A}} [-m_B m_A + s_k m_A + m_B s_j]
\end{aligned} \tag{3}$$

having dropped the contributions $\mathcal{O}(\delta s^2)$. Then, putting these terms together with the ones for the coupling to the external field

$$\begin{aligned}
\mathcal{H}_{\rm AF}(\{s_i\}, m_{A,B}) &= -\frac{J}{2} z N_A m_A m_B - \frac{J}{2} z N_B m_A m_B \\
&\quad + J z m_B \sum_{j\in A} s_j - H \sum_{j\in A} s_j + J z m_A \sum_{k\in B} s_k - H \sum_{k\in B} s_k \\
&= -\frac{J}{2} z N m_A m_B + (J z m_B - H) \sum_{j\in A} s_j + (J z m_A - H) \sum_{k\in B} s_k \\
&= -\frac{J}{2} z N m_A m_B - h_A^{\rm eff} \sum_{j\in A} s_j - h_B^{\rm eff} \sum_{k\in B} s_k
\end{aligned} \tag{4}$$

11. Have you identified local fields? Which are the mathematical expressions that determine them? **2pt**

$$h_A^{\rm eff} = -Jzm_B + H \tag{5}$$
$$h_B^{\rm eff} = -Jzm_A + H \tag{6}$$

the former acting on the spins of sublattice A and the latter on the spins of sublattice B.

12. Deduce the free-energy density as a function of the order parameters. **2pt**

We write the partition sum and we calculate the sum over the now uncoupled spins on the two sublattices:

$$\begin{aligned}
\mathcal{Z} &= e^{\beta \frac{J}{2} z m_A m_B N} \sum_{s_j \in A} \sum_{s_k \in B} e^{\beta \sum_{j\in A} s_j h_A^{\rm eff}(m_B) \beta \sum_{k \in B} s_k h_B^{\rm eff}(m_B)} \\
&= e^{\beta \frac{J}{2} z m_A m_B N} \left\{ 2\cosh[\beta h_A^{\rm eff}(m_B)] \right\}^{N/2} \left\{ 2\cosh[\beta h_B^{\rm eff}(m_A)] \right\}^{N/2} \\
&= \exp\left\{ \frac{N}{2} \beta J z m_A m_B + \frac{N}{2} \ln\{2\cosh[\beta h_A^{\rm eff}(m_B)]\} + \frac{N}{2} \ln\{2\cosh[\beta h_B^{\rm eff}(m_A)]\} \right\}
\end{aligned} \tag{7}$$

The free-energy density is

$$\begin{aligned}
-\beta f(m_A, m_B) &= \frac{1}{N} \ln \mathcal{Z} \\
&= \frac{\beta J z}{2} m_A m_B + \frac{1}{2} \ln\{2\cosh[\beta h_A^{\rm eff}(m_B)]\} + \frac{1}{2} \ln\{2\cosh[\beta h_B^{\rm eff}(m_A)]\}
\end{aligned} \tag{8}$$

We check that we recover the known result for the FM case. If we set $m_A = m_B = m$,

$$-\beta f(m) = \frac{\beta J z}{2} m^2 + \ln\{2\cosh[\beta h_A^{\rm eff}(m)]\} \tag{9}$$

which is, indeed, the correct form we had for the FM case, if we change $J \mapsto -J$.

13. Derive the mean-field self consistency equations **1pt**

$$m_A = \tanh(-\beta J z m_B + \beta H) \tag{10}$$
$$m_B = \tanh(-\beta J z m_A + \beta H) \tag{11}$$

14. Consider $H \to 0$ and recall that $J > 0$. Do you find a phase transition? At which critical temperature and of which order? **2pt**

Using $m_A = -m_B$, the equations reduce to a single one

$$m_A = \tanh(\beta J z m_A) \tag{12}$$

the same equation as for the FM model. $\beta_c J z = 1$ and second order.

15. Is there a total magnetisation in this problem? **1pt**

No. But each sublattice acquires one by spontaneous symmetry breaking.

16. Can you identify a single quantity, which we will call ϕ, that is different from zero in the low temperature phase? Give its expression. **1pt**

Yes, $\phi \equiv m_A - m_B$.

17. Give the definition and calculate the linear susceptibility, defined as the variation of the order parameter that you identified with respect to an infinitesimal field $h > 0$ globally coupled to the spins, $\mathcal{H}_{\mathrm{AF}} \mapsto \mathcal{H}_{\mathrm{AF}} - h\sum_i s_i$, for $H = 0$ and $T \sim T_c$. What do you remark in the final expression? **2pt**

$$\chi = \left.\frac{\partial m_A^h}{\partial h}\right|_{h=0} \tag{13}$$

We note that $m_A^h \neq -m_B^h$ under h. We proceed as usual, we take the mean-field eqs. (10)-(11) and we expand them close to $m_A^h \sim 0$ and $m_B^h \sim 0$:

$$m_A^h \sim -\beta J z m_B^h + \beta h \tag{14}$$

$$m_B^h \sim -\beta J z m_A^h + \beta h \tag{15}$$

We replace the second equation in the first one to get a closed expression of m_A^h

$$m_A^h \sim -\beta J z(-\beta J z m_A^h + \beta h) + \beta h = (\beta J z)^2 m_A^h + (-\beta J z + 1)\beta h \tag{16}$$

which implies

$$m_A^h \sim \frac{1 - \beta J z}{1 - (\beta J z)^2}\,\beta h = \frac{1}{1 + (\beta J z)}\,\beta h \tag{17}$$

Then, at $T \sim T_c = Jz$,

$$\chi = \frac{1}{T + T_c} \qquad k_B = 1 \tag{18}$$

Note the sign, it's $T + T_c$ in the denominator.

18. If you were to guess, which would be, in your opinion, the low temperature phase of a frustrated model with ferromagnetic interactions between nearest neighbours on the lattice and antiferromagnetic interactions falling off with distance as a power law? **2pt**

Modulated, FM at short distances and reversed FM at longer distances. The length of the regions will depend on the parameters, especially the relative strength of the FM and AF couplings.

Ginzburg-Landau theory

19. Recall the functional form of the scalar field Ginzburg-Landau free-energy density of the Ising ferromagnetic model. **2pt**

$$F[\phi] = \int d^d x \left[\frac{c}{2} (\vec{\nabla}\phi)^2 + a\phi^2 + \lambda\phi^4 - H\phi \right] \tag{19}$$

with $a \propto (T - T_c)$, $\lambda > 0$.

20. From the results found in the previous section, especially in question 16, write the Ginzburg-Landau free-energy density for the anti-ferromagnetic model. **1pt**

The same as above, with the staggered order parameter ϕ defined in question 16.

The antiferromagnetic chain

Consider now the one dimensional case,

$$H_{\rm AF}[\{s_i\}] = J \sum_{i=1} s_i s_{i+1} - H \sum_i s_i \tag{20}$$

with $J > 0$, $H > 0$, and periodic boundary conditions.

21. Call $K = \beta J$ and $\mathrm{H} = \beta H$, write the partition function, and identify the transfer matrix. **1pt**

$$\mathbb{T} = \begin{pmatrix} e^{-K+\mathrm{H}} & e^{K} \\ e^{K} & e^{-K-\mathrm{H}} \end{pmatrix} \tag{21}$$

which has the same form as for the FM case, apart from the sign in front of K.

22. Is there an equivalent quantum model that represents this classical chain? **1pt**

Write $\mathbb{T}$ in terms of Pauli matrices, $\mathbb{T} = a\mathbb{I} + b\hat{\sigma}^x + c\hat{\sigma}^z$ and find a, b, c as we did in the lectures. No need to include $\hat{\sigma}^y$ since all real in $\mathbb{T}$.

Problem: The hierarchical Berker lattice

10pt

The Berker diamond lattice is a hierarchical lattice that can be constructed iteratively as explained below and illustrated in the figure: Starting from two vertices and a single edge, one recursively replace each edge by $2c$ edges, inserting c vertices in between, where $c \geq 1$ is an integer parameter. The first steps of this procedure are represented on the figure for $c = 2$. To mimic Euclidean d-dimensional lattices, c is taken equal to 2^{d-1}.

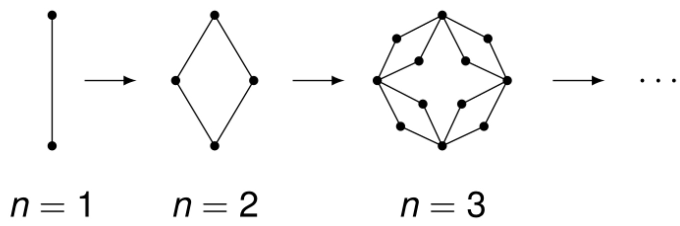


Figure 4: Iterative construction of the hierarchical diamond lattice with $c = 2$. We will also call it a graph.

We place the Ising spins on the vertices of the graph obtained after a certain number of steps of this recursive procedure, and consider the Hamiltonian

$$H = -\frac{J}{2} \sum_{\langle ij \rangle} s_i s_j \tag{22}$$

where the sum is over the $\langle ij \rangle$ edges of the graph and $J > 0$. We will study this model by the decimation method, eliminating the spins in the reverse order of their introduction.

1. How many edges and how many vertices are present in the graph after n steps? **1 pt**

Edges, case $c = 2$. $n = 1 : e_1 = 1, \quad n = 2 : e_2 = 4, \quad n = 3 : e_3 = 16.$
Rule $e_{n+1} = 2c\, e_n$ for $n = 1, \ldots$ and $e_1 = 1$. Then

$$e_n = (2c)^{n-1}$$

For $c = 2$ becomes $e_n = 4^{n-1}$ and one can check the first values above.

Vertices, case $c = 2$. $n = 1 : v_1 = 2, \quad n = 2 : v_2 = 4, \quad n = 3 : v_3 = 12.$
Recursion $v_{n+1} = v_n + c e_n$ for $n = 1, \ldots$ with $e_1 = 1$ and $v_1 = 2$. Then

$$\begin{aligned} v_{n+1} &= v_1 + c \sum_{k=1}^{n} e_k = 2 + c \sum_{k=1}^{n} (2c)^{k-1} = 2 + c \sum_{k=0}^{n-1} (2c)^k \\ &= 2 + c\, \frac{1 - (2c)^n}{1 - (2c)} \end{aligned}$$

where we used $\sum_{k=0}^{n} x^k = \dfrac{x^{n+1} - 1}{x - 1}$. For $c = 2$ the result becomes $v_n = 2 + 2/3\,(4^{n-1} - 1)$

2. Setting, as usual, $t = \tanh(\beta J)$, find the recursion relation giving the new dimensionless coupling constant t' as a function of the coupling constant t at the previous step of the decimation procedure. **2 pt**

We focus on the partition sum over the "internal" spins s_i with $i = 1, \ldots, c$, to two "external" ones that we call s_I and $s_{I'}$

$$\begin{aligned} Z(s_I, s_{I'}) &= \sum_{\{s_i=\pm1\}} e^{\beta J \sum_{i=1}^{c} s_i(s_I+s_{I'})} = \{2\cosh[\beta J(s_I + s_{I'})]\}^c \\ &= e^{\ln\{2\cosh[\beta J(s_I+s_{I'})]\}^c} = e^{c\ln\{2\cosh[\beta J(s_I+s_{I'})]\}} \\ &= e^{c\ln 2}\, e^{c\ln[\cosh(\beta J(s_I+s_{I'})]} \end{aligned}$$

The first factor is a constant. The second one should be rewritten in the form of the original Hamiltonian. One notes

$$s_I + s_{I'} = \left\{ \begin{array}{ll} 2 & s_I = s_{I'} = 1 \\ 0 & s_I \neq s_{I'} \\ -2 & s_I = s_{I'} = -1 \end{array} \right. \quad \Rightarrow \quad \cosh[(\beta J(s_I + s_{I'})] = \left\{ \begin{array}{ll} \cosh(2\beta J) & s_I s_{I'} = 1 \\ 1 & s_I s_{I'} = -1 \end{array} \right.$$

Then,

$$Z(s_I, s_{I'}) = e^{c\ln 2}\, e^{c\frac{\left(1+s_I s_{I'}\right)}{2}\ln[\cosh(\beta J)]} = e^{c\ln 2}\, e^{(c/2)\ln[\cosh(\beta J)]}\, e^{(c/2)\ln[\cosh(\beta J)]\, s_I s_{I'}}$$

Calling $K = \beta J$ and K' the new adimensional coupling

$$K' = (c/2)\ln[\cosh K] \quad \Rightarrow \quad e^{(2/c)K'} = \frac{1}{2}\left(e^{2K} + e^{-2K}\right)$$

We transform this relation by adding or subtracting one on both sides:

$$e^{(1/c)K'}\left(e^{(1/c)K'} \pm e^{-(1/c)K'}\right) = \frac{1}{2}\left(e^{K} \pm e^{-K}\right)^2$$

Dividing one by the other

$$\tanh[(1/c)K'] = (\tanh K)^2 \quad \Rightarrow \quad \tanh K' = \tanh\{c\tanh^{-1}\left[(\tanh K)^2\right]\}$$

Using now $t = \tanh K$ and $t' = \tanh K'$

$$t' = \tanh\{c\tanh^{-1} t^2\}$$

3. Show that for $c = 1$ one recovers the recursion relation of the one-dimensional Ising model. **2pt**

We take $c = 1$

$$t' = \tanh\{\tanh^{-1} t^2\} = t^2$$

which the relation we already know for the $d = 1$ Ising chain

Note that $dt'/dt = 2t > 0$ for all $t > 0$

4. What are the values of t corresponding to the high and low temperature limits? Check that these correspond to fixed points of the renormalization transformation. **2pt**

$t = \tanh K = \tanh(\beta J)$

Low temperature $T \to 0$ means $\beta \to \infty$, that is $\beta J \to \infty$, $\tanh(\beta J) \to 1$ and $t \to 1$

High temperature $T \to \infty$ means $\beta \to 0$, that is $\beta J \to 0$, $\tanh(\beta J) \to 0$ and $t \to 0$

Fixed points?

Low T, $t \to 1$, is $1 = \tanh[c \tanh^{-1} 1]$?
Yes, since $\tanh^{-1} 1 = \infty$, and then the result in the left-hand-side follows.

High T, $t \to 0$, is $0 = \tanh[c \tanh^{-1} 0]$?
Yes, since $\tanh^{-1} 0 = 0$ and then the result in the left-hand-side follows.

5. Study the behavior of the RG transformation around these trivial fixed points and discuss their stability for $c > 1$. **1 pt**

Use

$$\frac{d \tanh y}{dy} = \frac{1}{\cosh^2 y} \qquad\qquad \frac{d \,\mathrm{arctanh}\, y}{dy} = \frac{1}{1 - y^2}$$

then

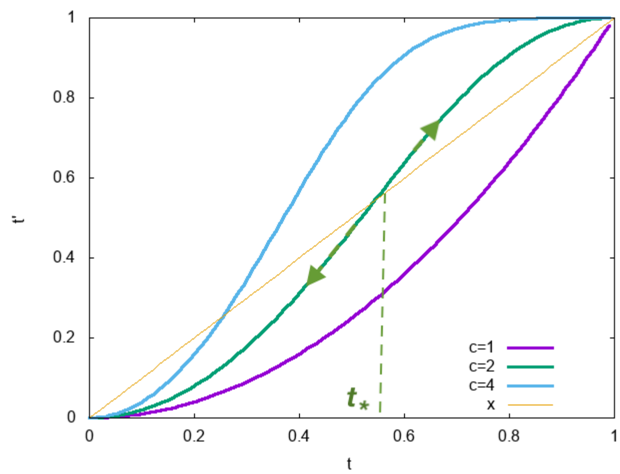
$$\frac{dt'}{dt} = \frac{1}{\cosh^2[c\,\mathrm{arctanh}\, t^2]} \frac{c}{1 - t^4} 2t$$

Now, for $c > 1$ at $t \ll 1$ that means $T \to \infty$:

$$t \ll 1 \qquad T \to \infty \qquad \frac{dt'}{dt} = 2ct + \mathcal{O}(t^5) \to 0$$

and at $t = 1$ that is $T \to 0$, the derivative is also positive and goes to zero exponentially

$$t \to 1^- \qquad T \to 0 \qquad \frac{dt'}{dt} \to \frac{2ct}{1 - t^4} \frac{1}{e^{2c\,\mathrm{arctanh}\, 1^-}} \to 0$$



For $c > 1$, there must be a crossing at a non-zero value of t, as shown in the plot.

6. The distance between the boundary sites of the lattice is equal to 2^n edges after n steps: this naturally fixes the length scale after n iterations as 2^n. For $c = 2$ the fixed point of the RG transformation is found at $t_\star \simeq 0.5437$. Compute the critical exponent ν. **1 pt**

At each iteration step the length scale doubles so the correlation length changes as

$$\xi' = \xi/2$$

Close to the non-trivial fixed point $t_\star$, $\xi \sim |K - K_\star|^{-\nu}$; then

$$|K' - K_\star|^{-\nu} = \frac{1}{2}|K - K_\star|^{-\nu} \qquad \Rightarrow \qquad \frac{1}{2} = \frac{\xi'}{\xi} = \left[\left.\frac{dK'}{dK}\right|_{K^*}\right]^{-\nu}$$

$$\nu = \frac{\ln 2}{\ln \left.\dfrac{dK'}{dK}\right|_{K_*}}$$

We can transform the derivatives

$$\left.\frac{dt'}{dt}\right|_{t_*} = \left.\frac{dt'}{dK'}\,\frac{dK'}{dK}\,\frac{dK}{dt}\right|_{K_*} \qquad \Rightarrow \qquad \left.\frac{dK'}{dK}\right|_{K_*} = \left.\frac{\dfrac{dt'}{dt}}{\dfrac{dt'}{dK'}\,\dfrac{dK}{dt}}\right|_{t_*}$$

and the factors in denominator are regular and we can forget them. Then we need to estimate

$$\nu = \frac{\ln 2}{\ln \left.\dfrac{dt'}{dt}\right|_{t_*}} = \frac{\ln 2}{\ln \left.\left\{\dfrac{2ct}{(1-t^4)\cosh^2[c\tanh^{-1}(t^2)]}\right\}\right|_{t_*}}$$

Yields $\nu \sim 1.33$.