Advanced Statistical Physics

Exam

April 2024

Surname :

Name :

Master :

Write your surname & name clearly and in CAPITAL LETTERS.

You can write in English or French, as you prefer.

No books, notes, calculator nor mobile phone allowed.

Not only the results but also the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

The problems roughly follow the order of the chapters in the Lecture Notes but are not necessarily of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you found difficult.

A. Conceptual questions

1. Can we check ergodicity from a single sample path?
2. What is the microcanonical measure?
3. What is the canonical measure?

B. Harmonic oscillator

Take a one dimensional harmonic oscillator with mass m and spring constant k, in equilibrium with a thermal bath at temperature T.

1. Calculate the averaged position $\langle x \rangle$.
2. Calculate the averaged momentum $\langle p \rangle$.
3. Calculate the mean-square displacement $\langle (x - \langle x \rangle)^2 \rangle$.
4. Calculate the averaged kinetic energy $\langle K \rangle$.
5. Is there a phase transition in this system? Justify your answer.

C. Phase transitions

1. Explain what is a phase transition. How would you characterize it?
2. Discuss the terms first and second order phase transition. Explain the differences that justify the different names. Give an example of each.
3. Write the Landau free-energies for typical second and first order phase transitions and explain how they depend on the control parameters.
4. What has Ginzburg added to the description of the phase transition? Discuss the Ginzburg criterium and its consequences.
5. Which is the mechanism driving the transition in the 2dXY model? Give the functional form of the spatial magnetic correlation function in the low temperature phase. What is the linear magnetic susceptibility in this phase? And the correlation length? What is so particular of this phase?

D. The spin 1 model

Mean-field theory of a spin-1 ferromagnet A model of a spin-1 ferromagnet on a cubic lattice is given by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_i s_i \tag{1}$$

with a positive interaction constant J between nearest neighbors. The N spin variables s_i each can take three different values $s_i = -1, 0, 1$.

1. Qualitatively discuss the ground state in the limits $h \gg J$ and $h \ll J$.

2. Derive the mean-field free-energy for this model.
3. Derive and solve the mean-field equation for the magnetization m.
4. Find the critical temperature for the onset of ferromagnetism (in the absence of a field).
5. Expand the mean-field equation about the critical point and determine the critical exponents β (magnetization), γ (susceptibility), and δ (critical isotherm).

E. Finite size scaling

The measurement of a linear susceptibility in a model with different linear system sizes yields the curves in panel (a) of Fig. 1. Explain what has been done in panel (b) in the same figure. To which universality class does this model belong to?

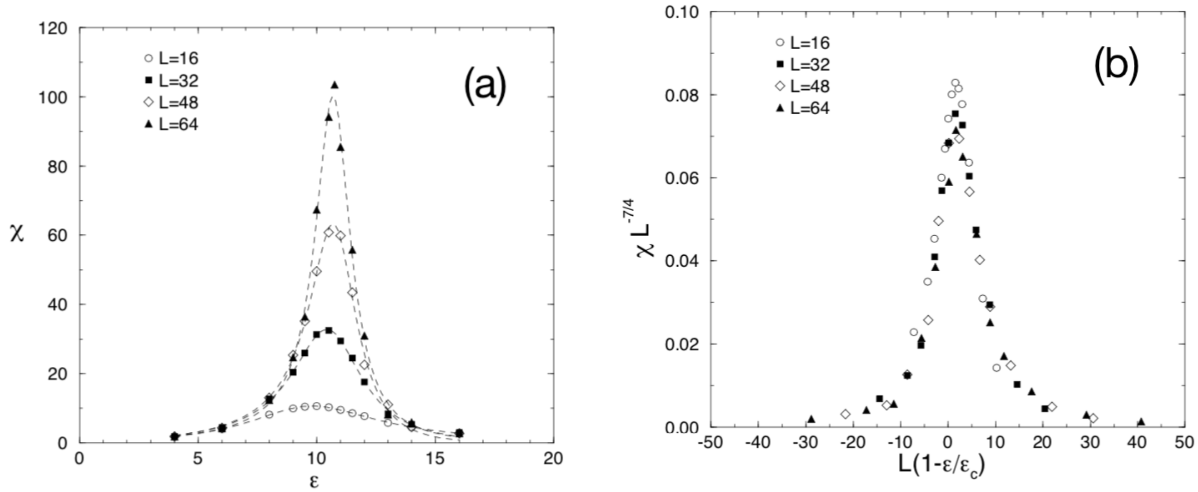

Figure 1: The susceptibility.

F. Renormalisation group.

Take an Ising chain with alternating coupling constants

$$H = -J_1 \sum_{\substack{i \text{ odd} \\ i=1}}^{L} s_i s_{i+1} - J_2 \sum_{\substack{i \text{ even} \\ i=2}}^{L} s_i s_{i+1} \ , \tag{2}$$

with $J_1 > 0$, $J_2 > 0$, $s_i = \pm 1$, and periodic boundary conditions $s_{L+1} = s_1$. The system is in equilibrium with a bath at inverse temperature β and one defines $K_1 = \beta J_1$ and $K_2 = \beta J_2$. See Fig. 2 for a sketch.

1. Write the probability weights of the group of four neighbouring spins, s_1, s_2, s_3, s_4, with s_1 and s_4 at the borders of this left segment of the chain on the one hand, and s_4, s_5, s_6, s_7, with s_4 and s_7 at the borders of this right segment of the chain on the other hand. (See Fig. 2.)
2. Write the result of the decimation of the two central spins (s_2 and s_3 on the one hand, and s_5 and s_6 on the other) in the two cases, separately.

You have the choice of working with the exponential representation of the factors (which will lead to relatively ugly expressions and we do not recommend it) or to proceed as in a TD (which will lead to more compact expressions that we will use below) following the steps:

(a) Represent the exponentials as $e^{Ks_is_j} = A(K)[1+s_is_jB(K)]$ with parameters $A(K)$ and $B(K)$ that you have to fix.

(b) Rewrite the factors $e^{-\beta H_l}$ (for the left segment) and $e^{-\beta H_r}$ (for the right segment) in this representation.

(c) Sum over the internal spin configurations.

(d) Identify the spin-dependent terms and read from them the relation between new and old parameters.

3. You must have derived

$$\tanh K'_1 = (\tanh K_1)^2 \tanh K_2$$
$$\tanh K'_2 = (\tanh K_2)^2 \tanh K_1$$

in question 2 –. Which are the fixed points of the recurrence?

4. Which one(s) is(are) stable/attractive and instable/repulsive? Give first an intuitive argument to guess the answer and then prove it (at least sketch the calculation to be done, if you do not have time to do all the calculations).

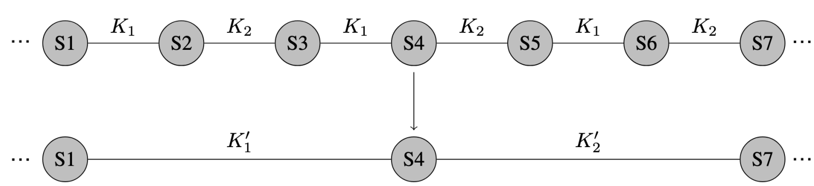

Figure 2: The Ising chain with alternating coupling constants, J_1 and J_2.