Advanced Statistical Physics

TD6 Quantum Classical Mapping

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1 The correlation length and the energy gap

Take the classical one dimensional Ising chain with ferromagnetic couplings, at temperature T and under a uniform external field h, and treat it with the transfer matrix formalism.

1. Write the spatial correlation between two spins at sites i and j, separated at distance $r = |\vec{r}_i - \vec{r}_j|$, $C(r) = \langle s_i s_j \rangle$, using the transfer matrix method.

2. Show that the correlation length ξ, which characterises the spatial decay of the two-point correlation function, $C(r) \sim e^{-r/\xi}$, is given by $\xi = [\ln(\lambda_1/\lambda_2)]^{-1}$ with λ_1 and λ_2 the eigenvalues of the transfer matrix $\mathbb{T}$. Give the explicit expression of ξ in terms of the parameters $K = \beta J$ and $H = \beta h$.

3. Which is the limit $\lim_{T\to 0^+} \lim_{H\to 0} \xi$? What does this mean?

4. *Optional* Check the scaling hypothesis using H and $t = e^{-2K}$ as the adimensional scaling parameters, with the latter replacing the linear displacement from a finite temperature critical point, in this case with an exponential divergence of the correlation length at $T \to 0$ [1].

Take now the quantum spin model $\mathcal{H} = -(\Delta/2)\hat{\sigma}^x$.

5. Find the spectrum.

6. Compare the energy gap, $e_1 - e_0$ to the inverse correlation length of the classical problem in the absence of the applied field.

2 The 2dIM and the transverse field quantum Ising chain

Take a two dimensional classical Ising model defined on a rectangular lattice made of square plaquettes, with lattice spacing a. The system has $N \times M$ sites on which the bimodal variables, $s_{n,m} = \pm 1$ with $n = 1, \ldots, N$ and $m = 1, \ldots, M$ are placed, and periodic boundary conditions on both directions are imposed. The ferromagnetic interactions are different on the two spatial directions, say J_x and J_y:

$$H(\{s_i\}) = -J_x \sum_{nm} s_{n,m} s_{n,m+1} - J_y \sum_{nm} s_{n,m} s_{n+1,m} \ . \tag{1}$$

We identified the spins by the two coordinates of the place they occupy on the lattice. The periodic boundary conditions imply $s_{n,M+1} = s_{n,1}$ and $s_{N+1,m} = s_{1,m}$.

1. Build a ladder, with two classical Ising chains attached to one another site by site. Call $\tau = na$ the direction along the chains, and $x = ma$ the one across the bonds between the two chains, that is $m = 1$ and $m = 2$. Write the partition function for this rectangular problem in terms of a transfer matrix $\mathbb{T}$. Note that one has to choose the "transfer direction" and sum over all the configurations of the "transverse" one. Think about the most convenient choice.

2. Introduce the basis $|s_1\rangle \otimes |s_2\rangle$ of the product of two spins 1/2 Hilbert space and the notation $\hat{\sigma}^z_1 = \hat{\sigma}^z \otimes \mathbb{I}$ and $\hat{\sigma}^z_2 = \mathbb{I} \otimes \hat{\sigma}^z$, and similarly for the other two Pauli cases.

3. Express the classical transfer matrix in terms of Pauli operators in this Hilbert space.

4. Show that $\mathbb{T}$ can be written as $e^{-\epsilon\hat{\mathcal{H}}}$ with $\hat{\mathcal{H}}$ the quantum Hamiltonian of a two spin systems. Determine the relation between the coefficients. Hint: use $e^{-\epsilon\hat{\mathcal{H}}} \sim e^{\epsilon b_0}\mathbb{I} - \epsilon(b_1\hat{\sigma}_x + b_2\hat{\sigma}_y + b_3\hat{\sigma}_z)$ and determine the coefficients b_0, b_1, b_2, b_3.

5. Use the results in the previous item to generalise to a system with $Na \times Ma$ size.

6. Find the quantum representation of the operator $\mathbb{T}^N$ in the form $e^{-N\epsilon\hat{\mathcal{H}}}$ and show that $\hat{\mathcal{H}}$ is the so-called quantum transverse field Ising chain, involving two non-commuting spin operators (or Pauli matrices).

7. Find the relations between the parameters in the two models.

3 *The multi-instanton gas (optional)*

In the lectures we expressed the partition function of a quantum particle in a double well potential as a path integral. We evaluated the latter in the semi-classical approximation and we found that this amounts to the saddle-point classical contribution and the quadratic fluctuations around it. The classical configurations are instantons, which take the particle from one well to the other, thus describing the tunnelling processes.

The problem here is to study the instanton gas which will be constrained by the requirement of it being dilute and that not too many instantons can be accommodated in a finite time interval.

The full transition amplitudes are

$$
\begin{aligned}
G(x_0, \tau; x_0, 0) &= \sum_{n=2k} K^n \int_0^{\tau} d\tau_1 \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \, U_n(\tau_1, \dots, \tau_n) \\
G(-x_0, \tau; x_0, 0) &= \sum_{n=2k-1} K^n \int_0^{\tau} d\tau_1 \int_0^{\tau_1} \cdots \int_0^{\tau_{n-1}} d\tau_n \, U_n(\tau_1, \dots, \tau_n)
\end{aligned}
\tag{2}
$$

with $k = 1, \dots$. In the first case n is even and in the second it is odd. U_n denotes the amplitude associated with n instantons, which take place at arbitrary times $\tau_i \in [0, \tau]$, $i = 1, \dots, n$, and $\tau_{n-1} \leq \cdots \leq \tau_1 \leq \tau$, and all these possibilities have to be added. K_n is a dimensionfull $[K_n] = [\tau]^{-n}$ constant absorbing the temporal dimension introduced

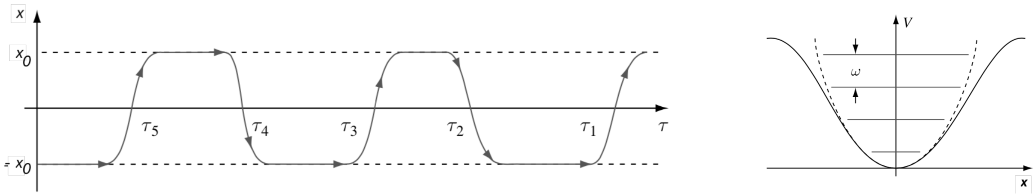

Figure 1: Left: A multi instanton configuration in the dilute limit. Right: One of the wells in the double well potential and its quadratic approximation. Figures adapted from [3].

by the time integrations. In the following, we first focus on the transition amplitude A_n, which controls the exponential dependence of the tunneling amplitude, returning later to consider the prefactor K_n.

We will consider, as in the single instanton case, that each U_n is the product of a classical and a quantum contribution, $U_n = U_n^{\rm cl}\, U_n^{\rm q}$. We will also focus on a dilute limit, as the one depicted in Fig. 1.

1. Start by studying the classical contribution, written as $U_n^{\rm cl} = e^{-S_n^{\rm cl}/\hbar}$. Estimate $S_n^{\rm cl}$ from the approximation in which each instantonic contribution is independent of the other ones and occurs in a narrow time interval.

2. Concerning the quantum contribution $U_n^{\rm q}$, we will assume that those around the rapid variation when the particle tunnels from one well to the other are negligible while the fluctuations during the long periods between transitions at τ_i and τ_{i+1}, when the particle lies in one of the two wells, are the dominating ones. To evaluate these, we approximate the well by a parabola, centred at its minimum/maximum, $V(x) \sim V(x_0) \pm V''(x_0)(x-x_0)^2/2$ and we evaluate $U(0,\tau_{i+1},0,\tau_i)$ using the path integral formalism in the semi-classical approximation. For concreteness, we take $V(x_0) = 0$.

   (a) Write U as a product and identify the two factors.

   (b) Which is the classical solution? and the corresponding action?

   (c) We go back to the original problem in real time, and we evaluate the fluctuations of the harmomic oscillator. The determinant of an operator is the product of its eigenvalues $\prod_{n=1}^{\infty} \epsilon_n$. Establish the eigenvalue equation and solve it.

   (d) Write $U^{\rm q}$ in terms of the eigenvalues of the operator.

   (e) The product for $U^{\rm q}$ looks ill-defined since it is an infinite product and in cases divergent. The idea is to regularise it by dividing by the result for the free particle, $V = 0$, and propose

   $$U^{\rm q} = \frac{U^{\rm q}}{U^{\rm q}_{\rm free}}\, U^{\rm q}_{\rm free} \tag{3}$$

   Use this trick to find a manageable expression for $U^{\rm q}$ taking advantage of the identity $\prod_{n=1}^{\infty}[1-(x/(n\pi))^2] = x/\sin x$.

   (f) Go back now to imaginary time $t \mapsto -i\tau$. Write the quantum fluctuations accumulated during the interval $[\tau_i, \tau_{i+1}]$.

(g) Evaluate now $U_n^{\rm q}$

3. Let us now return to Eqs. (2) and replace $U_n^{\rm q}$ just found. Show that G can be expressed as a factor times the series of cosh and sinh functions, using a convenient expression for $\int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 \cdots \int_0^{\tau_{n-1}} d\tau_n$

The actual density is dictated by the competition between the configurational "entropy" (favoring high density), and the "energetics", the exponential weight implied by the action (favoring low density).

References

[1] R. J. Baxter, *Exactly solved models in statistical physics* (Academic Press, 1982).

[2] L. P. Kadanoff, *Statistical Physics, statics, dynamics and renormalization* (World Scientific, 1999).

[3] B. D. Simons and A. Altland, *Condensed Matter Field Theory* (Cambridge University Press).