
Advanced Statistical Physics

Lectures by **Leticia F. Cugliandolo**^{1,2}

Exercise sessions **Ada Altieri**³ & **Marco Tarzia**^{1,2}

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Lectures Monday, 08:30 - 10:30

Exercises Monday, 10:45 - 12:45

Focus

Classical and Quantum Phase transitions

Phenomenology, concepts & formalism

Critical phenomena & universality, topology, renormalization, nucleation, and disorder

Current research, *e.g.*

Non-Equilibrium Universality : From Classical to Quantum and Back, KITP UC Santa Barbara 2021

Quantum localization and glassy physics, Summer School at Cargèse 2023

Out-of-equilibrium Dynamics and Quantum Information of Many-body Systems with Long-range Interactions

KITP UC Santa Barbara 2023

All material (program, lectures & TDs) downloadable from the webpage

www.lpthe.jussieu.fr/~leticia/enseignement.html

Plan

Mathematical preliminaries (should have been done during the summer)

1. One introductory lecture
2. Six lectures on classical statistical physics
3. Six lectures on quantum statistical physics
4. Twelve TD sessions in half group
5. One homework to be done during the mid-term holidays
6. Final written exam (concepts and exercises)

Final Mark 20% homework and 80% written exam

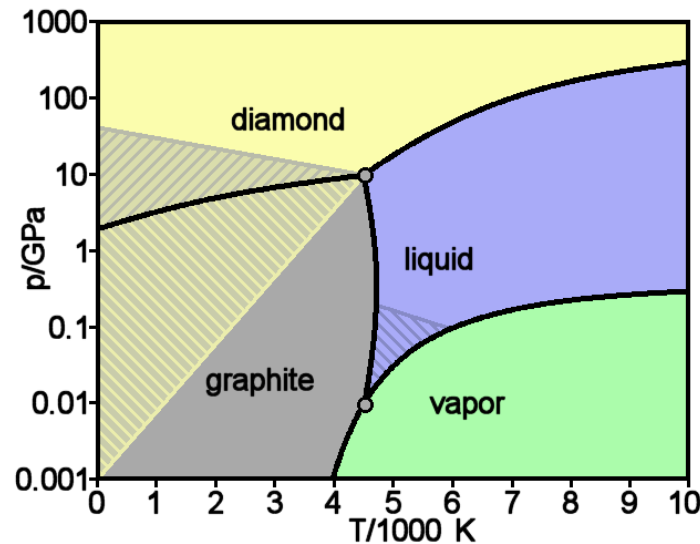
Plan

1. Interest and background, principles and formalism, e.g.
 - (In)Equivalence of ensembles for (long) short-range interactions
 - Systems' reduction (role of environments)
2. Phase transitions in the classical case
 - Important concepts (phase diagrams, order parameters, spontaneous symmetry breaking, etc.)
 - Uncommon mechanisms (e.g. topological phases, condensation)
 - Renormalization group ideas
 - Effects of quenched randomness
3. Quantum statistical physics
 - Generics : Linear response, Kubo formula, FDT, etc.
 - Quantum – classical equivalence, imaginary time representation
 - Quantum spin chains (tricks)

Phase transitions

Phase diagram

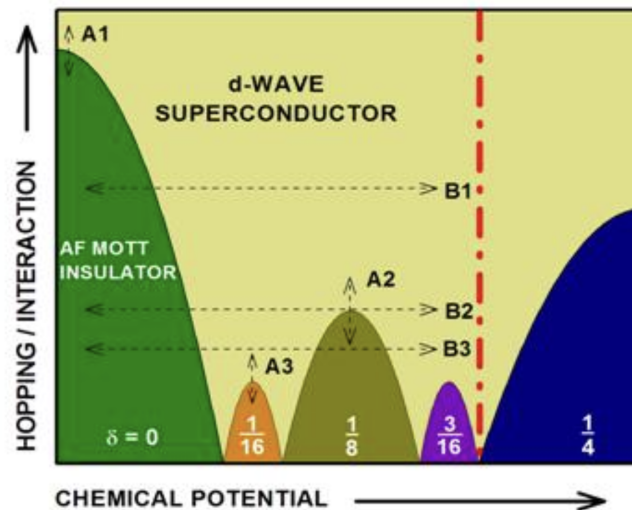
Graphite



Theoretical phase diagram of carbon, which shows the state of matter for varying temperatures and pressures. The hatched regions indicate conditions under which one phase is metastable, so that two phases can coexist.

Phase diagrams

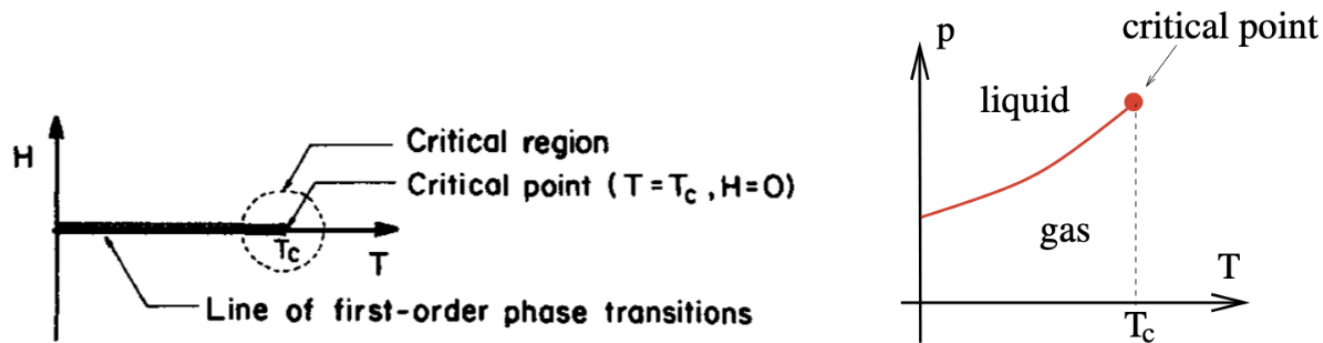
High T_c superconductors



... to propose a bosonic effective quantum Hamiltonian based on the projected $SO(5)$ model with extended interactions, which can be derived from the microscopic models of the cuprates. The global phase diagram of this model is obtained using mean-field theory and Quantum Monte Carlo simulations ...

Universality

Para-Ferro magnetism & Liquid-Gas



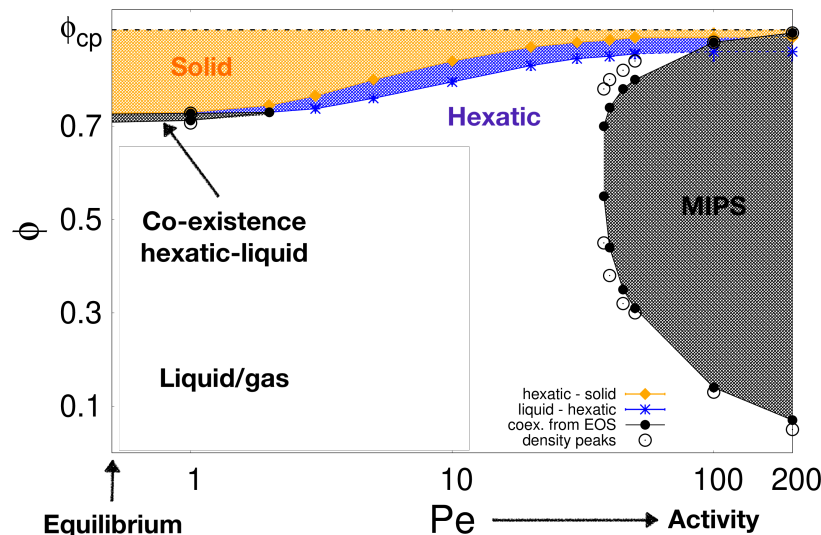
Very different variables and control parameters and, still, same critical properties

Non-Equilibrium Universality : From Classical to Quantum and Back, KITP UC Santa

Barbara 2021

Active Brownian disks

Phase diagram with **solid**, **hexatic**, **liquid**, co-existence and MIPS



1st order **hexatic-liquid** close to $Pe = 0$

KT-HNY **solid-hexatic**

- universal dislocation unbinding

Breakdown of KT-HNY **hexatic-liquid** picture

- disclination unbinding within the **liquid** phase

- **percolation** of defect clusters in the liquid

Pressure $P(\phi, Pe)$ (EoS), correlations $G_T(r)$, $G_6(r)$, distributions of ϕ_i , and $|\psi_{6i}|$
defect identification & counting

Digregorio, Levis, Suma, LFC, Gonnella & Pagonabarraga, PRL 121, 098003 (2018)

Digregorio, Levis, LFC, Gonnella & Pagonabarraga, Soft Matter 18, 566 (2022)

Long-Range Interactions

Statistical mechanics and dynamics of solvable models with long-range interactions

A. Campa (1), T. Dauxois (2), S. Ruffo (3) ((1) Complex Systems and Theoretical Physics Unit, ISS and INFN, Rome, Italy (2) Laboratoire de Physique and CNRS, ENS-Lyon, France (3) Dip. di Energetica, Univ. Firenze and INFN, Italy)

The two-body potential of systems with long-range interactions decays at large distances as $V(r) \sim 1/r^\alpha$, with $\alpha \leq d$, where d is the space dimension. Examples are: gravitational systems, two-dimensional hydrodynamics, two-dimensional elasticity, charged and dipolar systems. Although such systems can be made extensive, they are intrinsically non additive. Moreover, the space of accessible macroscopic thermodynamic parameters might be non convex. The violation of these two basic properties is at the origin of ensemble inequivalence, which implies that specific heat can be negative in the microcanonical ensemble and temperature jumps can appear at microcanonical first order phase transitions. The lack of convexity implies that ergodicity may be generically broken. We present here a comprehensive review of the recent advances on the statistical mechanics and out-of-equilibrium dynamics of systems with long-range interactions. The core of the review consists in the detailed presentation of the concept of ensemble inequivalence, as exemplified by the exact solution, in the microcanonical and canonical ensembles, of mean-field type models. Relaxation towards thermodynamic equilibrium can be extremely slow and quasi-stationary states may be present. The understanding of such unusual relaxation process is obtained by the introduction of an appropriate kinetic theory based on the Vlasov equation.

Comments: 118 pages, review paper, added references, slight change of content

Subjects: **Statistical Mechanics (cond-mat.stat-mech)**

Journal reference: Physics Reports 480 (2009), pp. 57–159

Long-Range Interactions

Notes on the Statistical Mechanics of Systems with Long-Range Interactions

David Mukamel

Thermodynamic and dynamical properties of systems with long range pairwise interactions (LRI) which decay as $1/r^{d+\sigma}$ at large distances r in d dimensions are reviewed in these Notes. Two broad classes of such systems are identified: (a) systems with a slow decay of the interactions, termed "strong" LRI, where the energy is super-extensive. These systems are characterized by unusual properties such as inequivalence of ensembles, negative specific heat, slow decay of correlations and ergodicity breaking. And (b) systems with faster decay of the interaction potential where the energy is additive, thus resulting in less dramatic effects. These interaction affect the thermodynamic behavior of systems near phase transitions, where long range correlations are naturally present. Long range correlations are often present in systems driven out of equilibrium when the dynamics involves conserved quantities. Steady state properties of driven systems are considered within the framework outlined above.

Comments: Notes of lectures given at a Summer School in Les Houches (France), 4–29 August 2008

Out of equilibrium

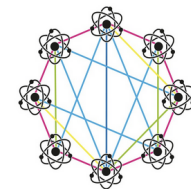


Out-of-equilibrium Dynamics and Quantum Information of Many-body Systems with Long-range Interactions

Coordinators: Nicolò Defenu, Alexey Gorshkov, Giovanna Morigi and Lea F. Santos

Scientific Advisors: Paola Cappellaro, Eugene Demler, David Mukamel, Manfred Salmhofer and Monika Schleier-Smith

The experimental progress in the field of quantum simulations has stimulated a strong theoretical interest in the quantum coherent dynamics of long-range interacting particles. The prominent collective character of these systems gives rise to novel dynamical phenomena, metastable phases of matter, and forms of dynamical criticality which do not have a counterpart in traditional quantum systems with local interactions. This program will give a broad scientific community an occasion to have systematic and organic discussions over several recently discovered non-equilibrium phenomena enabled by long-range interactions, bridging the insights coming from theory and experiment. The program will cover several aspects of long-range physics, including, but not limited to the dynamics of quantum correlations and entanglement, dynamical phase transitions, dynamically stabilized phases and metastable phases and anomalous slow dynamics and ergodicity breaking. The program will provide a framework for various research communities to discuss the core concepts and tools relevant to long-range interacting matter and will contribute to better bridging the gap between theorists and experimentalists.



DATES

Oct 16, 2023 - Dec 22, 2023

INFORMATION

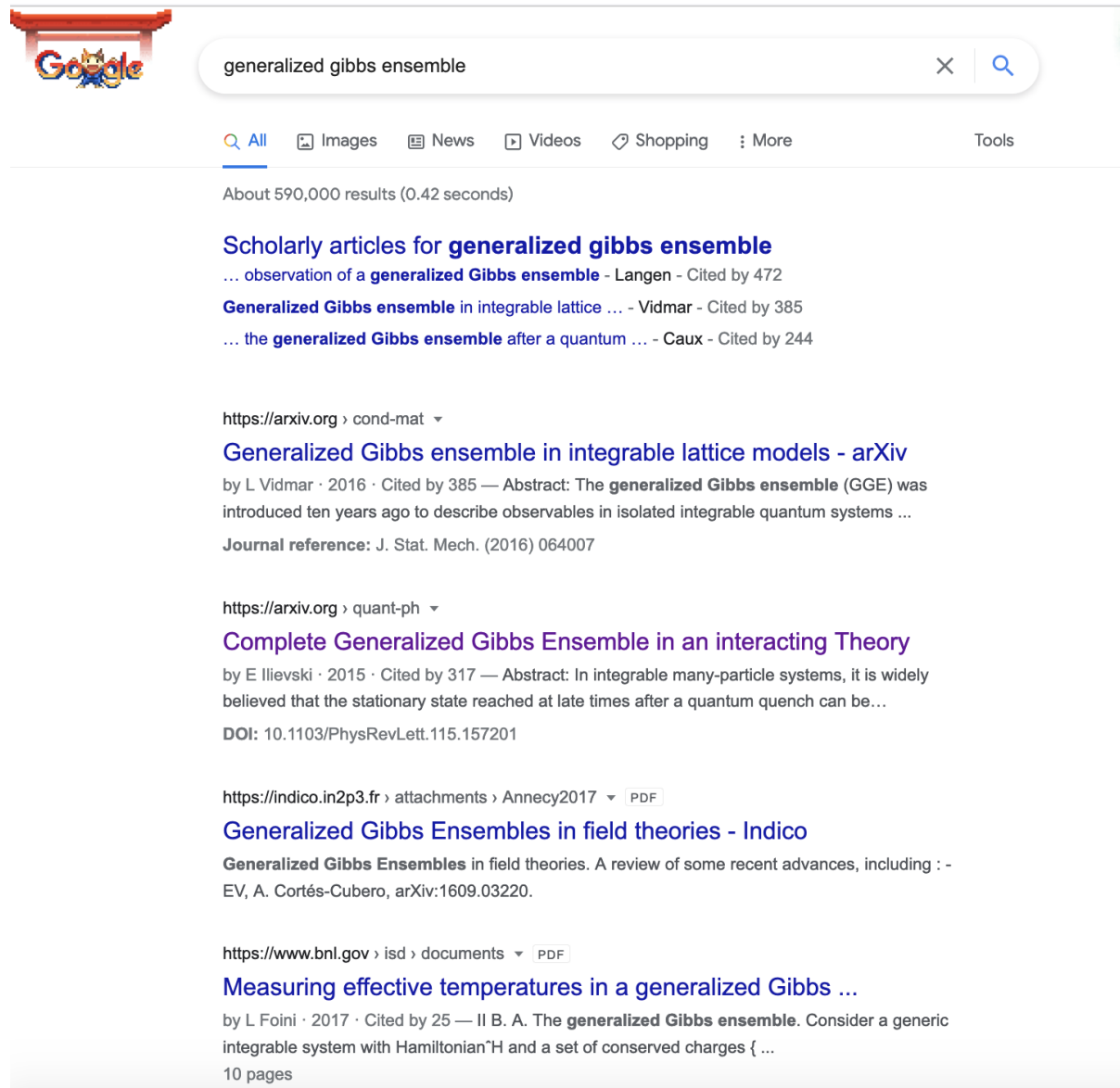
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Application deadline is:

Nov 13, 2022.

Primary deadline above

GGEs



A screenshot of a Google search results page for the query "generalized gibbs ensemble". The search bar at the top shows the query and a magnifying glass icon. Below the search bar, navigation links for "All", "Images", "News", "Videos", "Shopping", and "More" are visible, along with a "Tools" link on the right. The search results indicate "About 590,000 results (0.42 seconds)".

Scholarly articles for generalized gibbs ensemble

- ... observation of a **generalized Gibbs ensemble** - Langen - Cited by 472
- Generalized Gibbs ensemble** in integrable lattice ... - Vidmar - Cited by 385
- ... the **generalized Gibbs ensemble** after a quantum ... - Caux - Cited by 244

<https://arxiv.org> › cond-mat ▾

Generalized Gibbs ensemble in integrable lattice models - arXiv

by L Vidmar · 2016 · Cited by 385 — Abstract: The **generalized Gibbs ensemble** (GGE) was introduced ten years ago to describe observables in isolated integrable quantum systems ...

Journal reference: J. Stat. Mech. (2016) 064007

<https://arxiv.org> › quant-ph ▾

Complete Generalized Gibbs Ensemble in an interacting Theory

by E Ilievski · 2015 · Cited by 317 — Abstract: In integrable many-particle systems, it is widely believed that the stationary state reached at late times after a quantum quench can be...

DOI: 10.1103/PhysRevLett.115.157201

<https://indico.in2p3.fr> › attachments › Annecy2017 ▾ PDF

Generalized Gibbs Ensembles in field theories - Indico

Generalized Gibbs Ensembles in field theories. A review of some recent advances, including : - EV, A. Cortés-Cubero, arXiv:1609.03220.

<https://www.bnl.gov> › isd › documents ▾ PDF

Measuring effective temperatures in a generalized Gibbs ...

by L Foini · 2017 · Cited by 25 — II B. A. The **generalized Gibbs ensemble**. Consider a generic integrable system with Hamiltonian $\hat{H}$ and a set of conserved charges $\{ \dots$

10 pages

GGEs

Generalized Gibbs ensemble in integrable lattice models

Lev Vidmar, Marcos Rigol

The generalized Gibbs ensemble (GGE) was introduced ten years ago to describe observables in isolated integrable quantum systems after equilibration. Since then, the GGE has been demonstrated to be a powerful tool to predict the outcome of the relaxation dynamics of few-body observables in a variety of integrable models, a process we call generalized thermalization. This review discusses several fundamental aspects of the GGE and generalized thermalization in integrable systems. In particular, we focus on questions such as: which observables equilibrate to the GGE predictions and who should play the role of the bath; what conserved quantities can be used to construct the GGE; what are the differences between generalized thermalization in noninteracting systems and in interacting systems mappable to noninteracting ones; why is it that the GGE works when traditional ensembles of statistical mechanics fail. Despite a lot of interest in these questions in recent years, no definite answers have been given. We review results for the XX model and for the transverse field Ising model. For the latter model, we also report original results and show that the GGE describes spin-spin correlations over the entire system. This makes apparent that there is no need to trace out a part of the system in real space for equilibration to occur and for the GGE to apply. In the past, a spectral decomposition of the weights of various statistical ensembles revealed that generalized eigenstate thermalization occurs in the XX model (hard-core bosons). Namely, eigenstates of the Hamiltonian with similar distributions of conserved quantities have similar expectation values of few-spin observables. Here we show that generalized eigenstate thermalization also occurs in the transverse field Ising model.

Comments: Review article for special issue of JSTAT on non-equilibrium dynamics in integrable systems

Subjects: **Statistical Mechanics (cond-mat.stat-mech)**; Quantum Gases (cond-mat.quant-gas); Strongly Correlated Electrons (cond-mat.str-el); Quantum Physics (quant-ph)

Journal reference: J. Stat. Mech. (2016) 064007

Baths

Reservoir-controlled quantum materials

Responsable : Cristiano Ciuti (PR Université de Paris)

Enseignants : Cristiano Ciuti (PR Université de Paris)

nombre d'ECTS : 3

Langue d'enseignement : Anglais

Evaluation : examen écrit (questions sur le cours, tutoriels et articles donnés à l'avance)

Description :

In recent years, new concepts related to the coupling of quantum many-body systems to tailored reservoirs are providing elegant ways to modify the electronic, optical and mechanical properties of quantum materials in condensed matter, atomic, optical and quantum circuit platforms. Moreover, reservoir-based approaches are emerging in the context of machine learning and quantum information. This M2 course and its tutorials will provide in a pedagogical way the fundamental concepts for the reservoir-control of quantum systems and show the most recent developments, applications and experimental realizations. The interdisciplinary and self-contained character of this class should resonate with students belonging to the tracks of Condensed Matter, Quantum Physics, and Theoretical Physics.

Baths

Home > The Journal of Chemical Physics > Volume 148, Issue 11 > 10.1063/1.4991779



< PREV

NEXT >

No Access . Submitted: 23 June 2017 . Accepted: 21 February 2018 . Published Online: 15 March 2018

Theories of quantum dissipation and nonlinear coupling bath descriptors

J. Chem. Phys. **148**, 114103 (2018); <https://doi.org/10.1063/1.4991779>

Rui-Xue Xu, Yang Liu, Hou-Dao Zhang, and YiJing Yan^{a)}

View Affiliations



PDF

ABSTRACT

FULL TEXT

FIGURES

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METRICS

TOPICS

- Invariance properties
- Emission spectroscopy
- Nonequilibrium statistical mechanics
- Quantum dissipation
- Complex systems theory
- Stochastic processes
- Entropy
- Optical properties
- Oscillators
- Wick's theorem

ABSTRACT

The quest of an exact and nonperturbative treatment of quantum dissipation in nonlinear coupling environments remains in general an intractable task. In this work, we address the key issues toward the solutions to the lowest nonlinear environment, a harmonic bath coupled both linearly and quadratically with an arbitrary system. To determine the bath coupling descriptors, we propose a physical mapping scheme, together with the prescription reference invariance requirement. We then adopt a recently developed dissipaton equation of motion theory [R. X. Xu *et al.*, Chin. J. Chem. Phys. **30**, 395 (2017)], with the underlying statistical quasi-particle (“dissipaton”) algebra being extended to the quadratic bath coupling. We report the numerical results on a two-level system dynamics and absorption and emission line shapes.

SCIENTIFIC REPORTS



OPEN

Decoherence and control of a qubit in spin baths: an exact master equation study

Jun Jing^{1,2} & Lian-Ao Wu^{2,3}

Received: 8 August 2017

Accepted: 8 January 2018

Published online: 24 January 2018

In spin-based nanosystems for quantum information processing, electron spin qubits are subject to decoherence due to their interactions with nuclear spin environments. In this paper, we present an exact master equation for a central spin-1/2 system in time-dependent external fields and coupled to a spin-half bath in terms of hyperfine interaction. The master equation provides a unified description for free and controlled dynamics of the central spin and is formally independent of the details and size of spin environments. Different from the previous approaches, the master equation remains exact even in the presence of external control fields. Using the parameters for realistic nanosystems with nonzero nuclear spins, such as GaAs, we investigate the Overhauser's effect on the decoherence dynamics of the central spin under different distributions of bath-spin frequencies and system-bath coupling strengths. Furthermore, we apply the leakage elimination operator, in a nonperturbative manner, to this system to suppress the decoherence induced by hyperfine interaction.

Baths

Bath-induced Zeno localization in driven many-body quantum systems

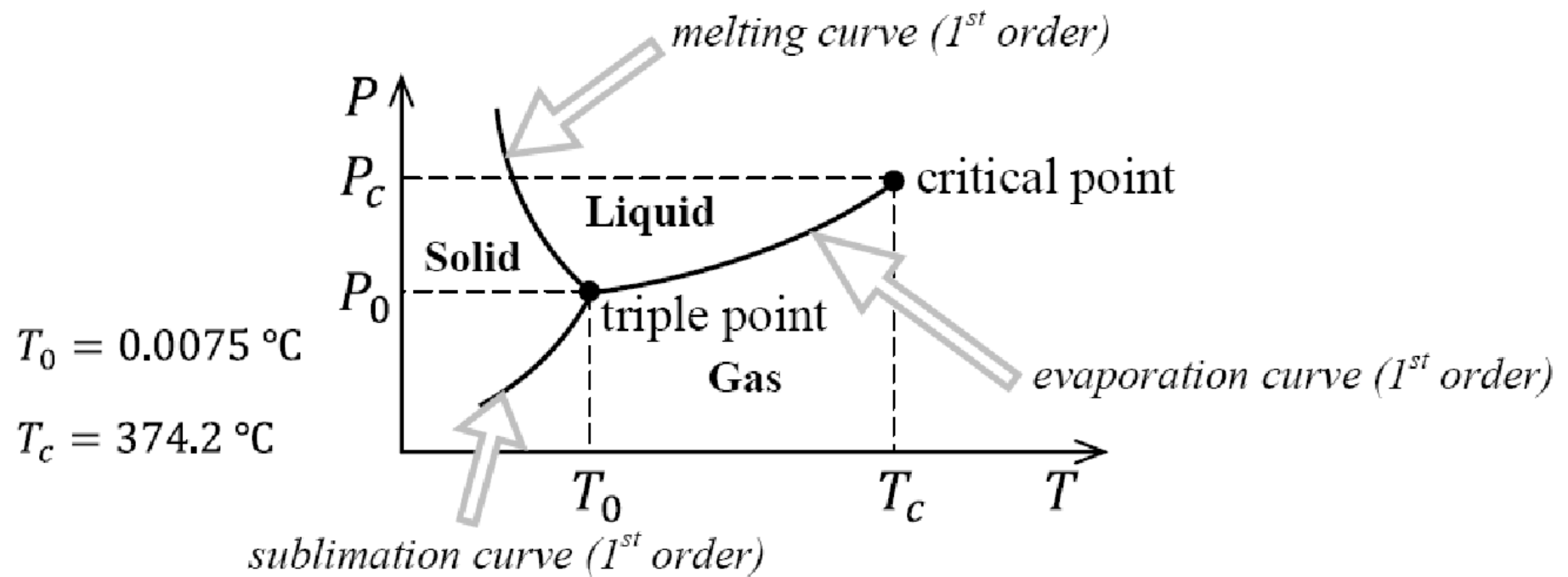
Markus Holzmann ¹, Thibaud Maimbourg ², Denis M. Basko ¹, Alberto Rosso ² [Details](#)

1 LPM2C - Laboratoire de physique et modélisation des milieux condensés

2 LPTMS - Laboratoire de Physique Théorique et Modèles Statistiques

Abstract : We study a quantum interacting spin system subject to an external drive and coupled to a thermal bath of spatially localized vibrational modes, serving as a model of Dynamic Nuclear Polarization. We show that even when the many-body eigenstates of the system are ergodic, a sufficiently strong coupling to the bath may effectively localize the spins due to many-body quantum Zeno effect, as manifested by the hole-burning shape of the electron paramagnetic resonance spectrum. Our results provide an explanation of the breakdown of the thermal mixing regime experimentally observed above 4 - 5 Kelvin.

Phase transitions

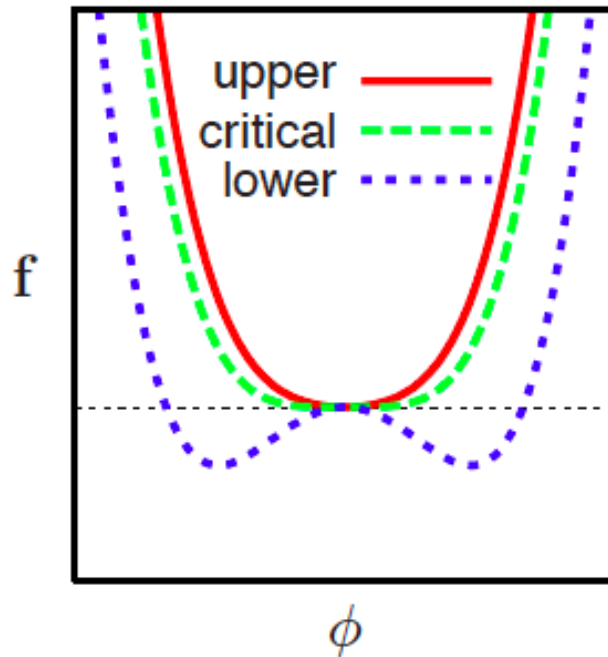


Summary 3rd Lecture

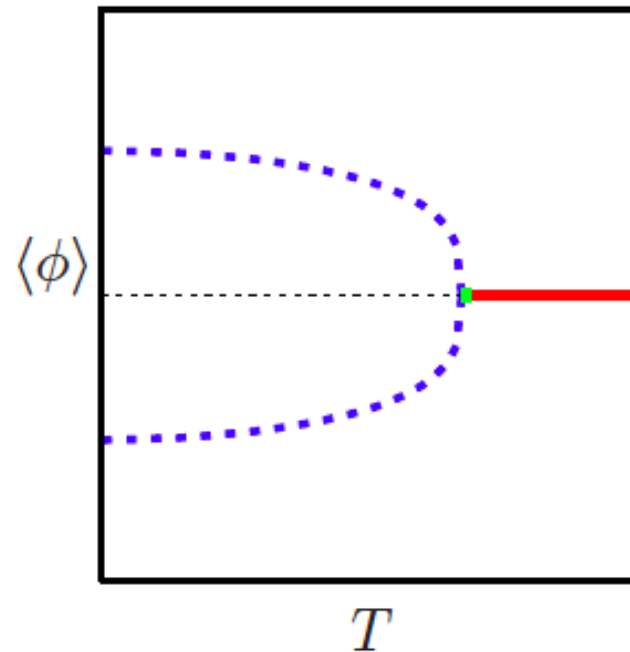
Continuous phase-transition

Bi-valued equilibrium states related by symmetry

$\mathcal{F}$



Ginzburg-Landau free-energy



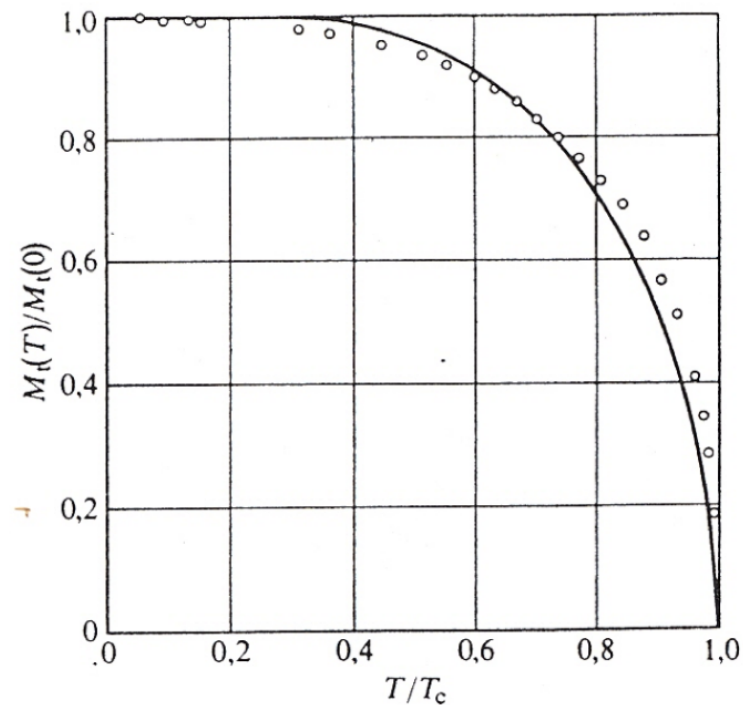
Scalar order parameter

e.g. Ising magnets at $h = 0$, $m = \langle s_i \rangle = \frac{1}{N} \sum_{k=1}^N \langle s_k \rangle$

Continuous phase-transition

Bi-valued equilibrium states related by symmetry

Aimantation et phénomène magnétocalorique du nickel



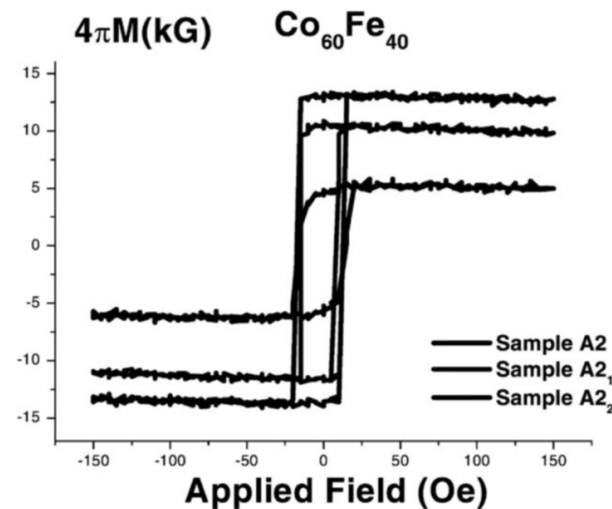
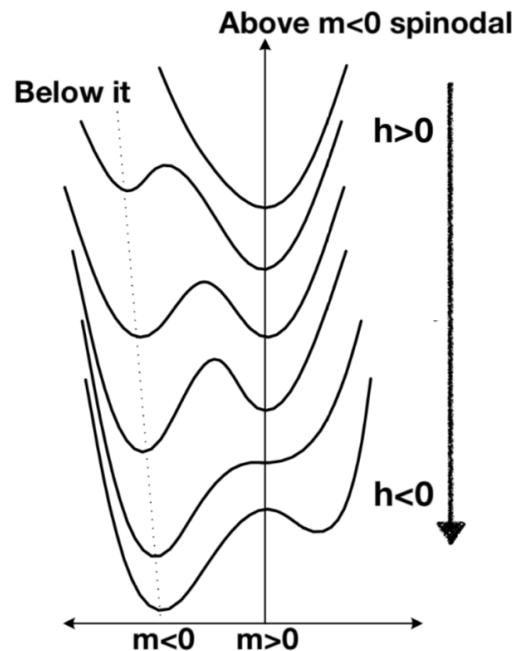
Nickel data vs. mean-field $m(T/T_c)$

Weiss & Forrer, Ann. Phys. 5, 153 (1926)

Discontinuous phase transition

Hysteresis loops

Growth Rate Effects in Soft CoFe Films



M. Vopsaroiu, K. O'Grady, M. T. Georgieva, P. J. Grundy, and M. J. Thwaites,

IEEE Transactions on magnetics 41, 3253 (2005)

Curie-Weiss Mean-Field

Exponents for a continuous ($\mathbb{Z}_2$ spont symm broken) transition

$$t = (T - T_c)/T_c$$

	exponent	definition	conditions	mean-field
Specific heat	α	$C_v \propto t ^{-\alpha}$	$t \rightarrow 0, \quad h = 0$	0
Order parameter	β	$m \propto (-t)^\beta$	$t \rightarrow 0^-, \quad h = 0$	1/2
Susceptibility	γ	$\chi \propto t ^{-\gamma}$	$t \rightarrow 0, \quad h = 0$	1
Critical isotherm	δ	$h \propto m ^\delta \text{sign}(m)$	$h \rightarrow 0, \quad t = 0$	3

Measurements

Magnetisation³ vs. temperature

Nuclear magnetic resonance in MnF₂ near a critical point

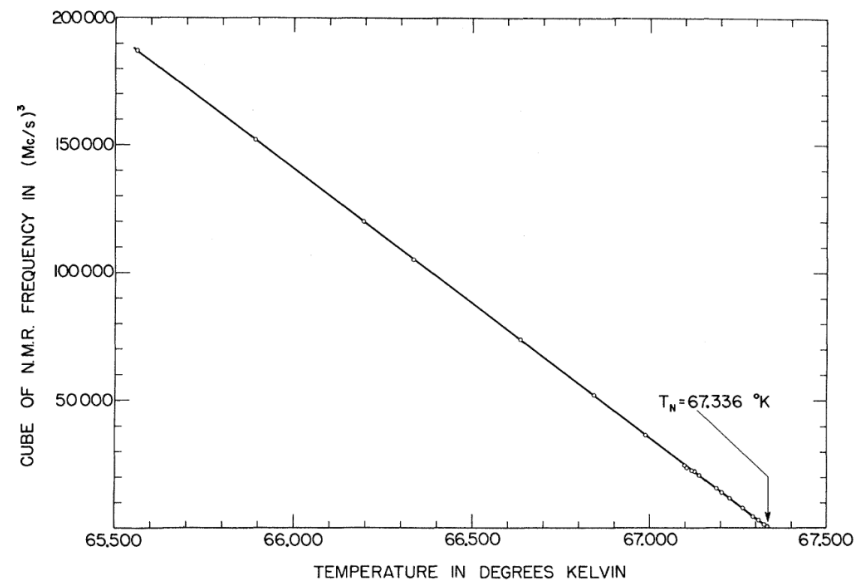


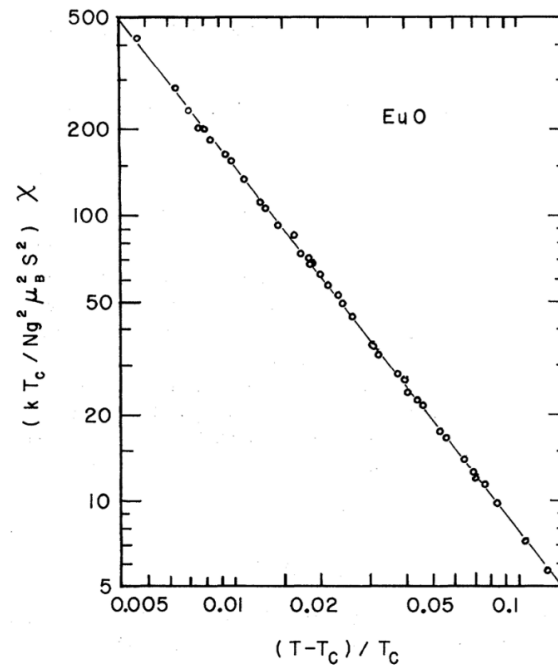
FIG. 1. Temperature dependence of the cube of the F^{19} nuclear resonance frequency for the first 1.8 degrees below T_N . The points lie on the straight line shown to within the experimental uncertainty of about 5 millidegrees.

Mean-Field yields $\beta = 1/2$

P. Heller and G. B. Benedek, Phys. Rev. Lett. 8, 428 (1962)

Measurements

Magnetic susceptibility $\chi \sim (T_c - T)^{-1.3}$

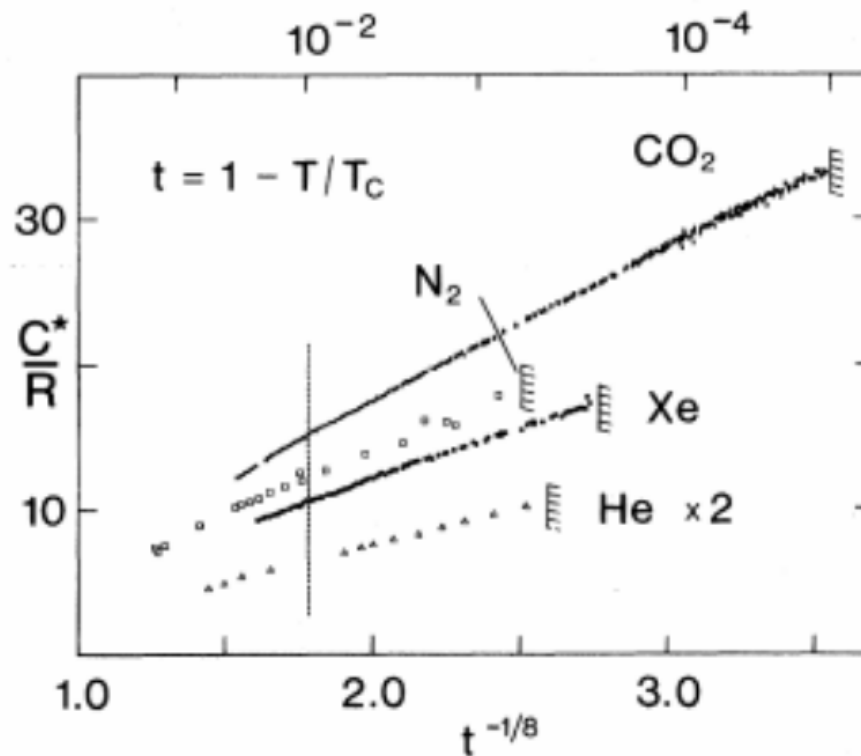


Mean-Field yields $\gamma = 1$

Measurements

Heat capacity vs reduced temperature^{-1/8}

Precision Measurement of the Specific Heat of CO₂ Near the Critical Point

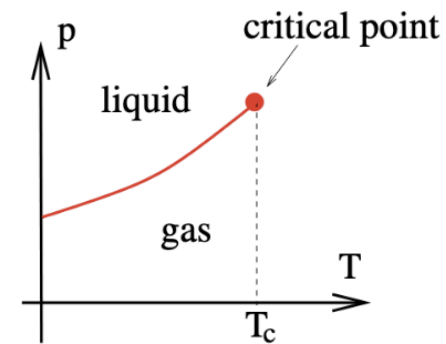
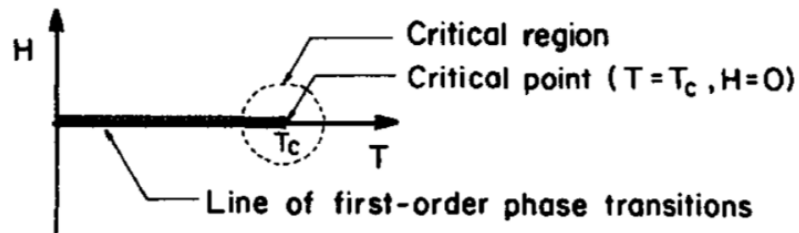


Mean-Field yields $\alpha = 0$

J. Lipa, C. Edwards, and M. Buckingham, Phys. Rev. Lett. 25, 1086 (1970)

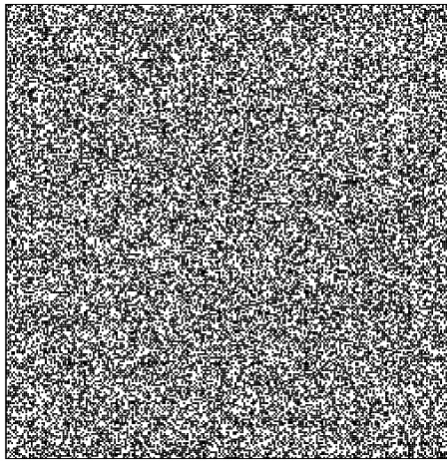
Universality

Para-Ferro magnetism & Liquid-Gas

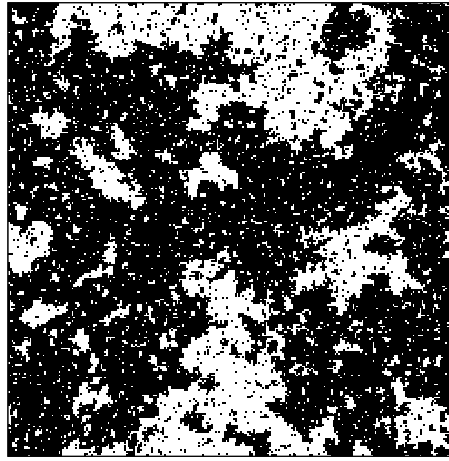


Typical configurations

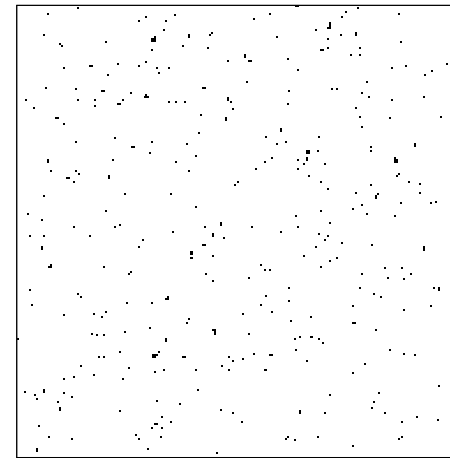
Up & down spins in a $2d$ Ising model



$$T \rightarrow \infty$$



$$T = T_c$$



$$0 < T < T_c$$

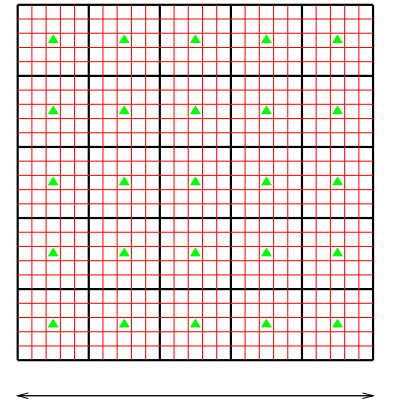
Real space viewpoint

Ginzburg-Landau

Continuous scalar statistical field theory

Coarse-grain the spin

$$\phi(\mathbf{r}) = V_{\mathbf{r}}^{-1} \sum_{i \in V_{\mathbf{r}}} s_i.$$



The partition function is $\mathcal{Z} = \int \mathcal{D}\phi e^{-\beta \mathcal{F}(\phi)}$ with

$$\mathcal{F}(\phi) = \int d^d r \left\{ \frac{1}{2} [\nabla \phi(\mathbf{r})]^2 + \frac{T-T_c}{2} \phi^2(\mathbf{r}) + \frac{\lambda}{4} \phi^4(\mathbf{r}) \right\}$$

Elastic + potential energy with the latter inspired by the results for the fully-connected model (entropy around $\phi \sim 0$ and symmetry arguments)

Uniform saddle point in the $V \rightarrow \infty$ limit : $\phi_{sp}(\mathbf{r}) = \langle \phi(\mathbf{r}) \rangle = \phi_0$

The free-energy density is $\lim_{V \rightarrow \infty} f_V(\beta, J, g) = \lim_{V \rightarrow \infty} V^{-1} \mathcal{F}(\phi_0)$

Ginzburg-Landau

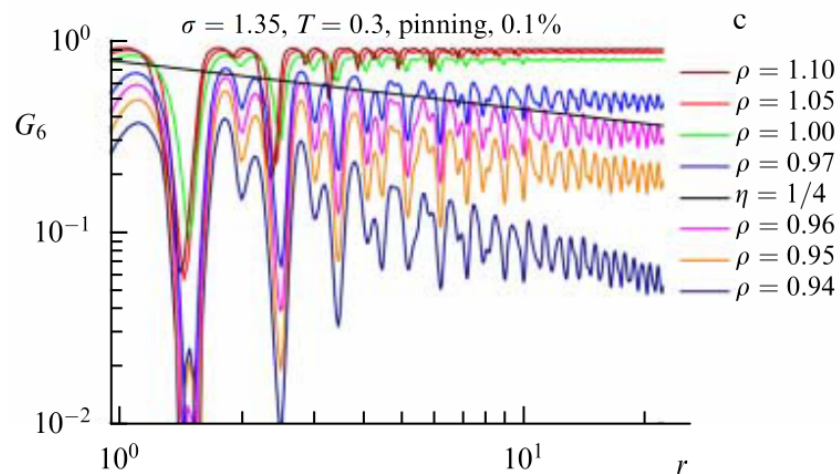
Exponents from the saddle-point analysis vs. finite d ones

	exponent	definition	conditions	mean-field
Specific heat	α	$C_v \propto t ^{-\alpha}$	$t \rightarrow 0, \quad h = 0$	0
Order parameter	β	$m \propto (-t)^\beta$	$t \rightarrow 0^-, \quad h = 0$	1/2
Susceptibility	γ	$\chi \propto t ^{-\gamma}$	$t \rightarrow 0, \quad h = 0$	1
Critical isotherm	δ	$h \propto m ^\delta \text{sign}(m)$	$h \rightarrow 0, \quad t = 0$	3
Correlation length	ν	$\xi \propto t ^{-\nu}$	$t \rightarrow 0, \quad h = 0$	1/2
Correlation function	η	$G(\mathbf{r}) \propto \mathbf{r} ^{-d+2-\eta}$	$r = 0, \quad h = 0$	0

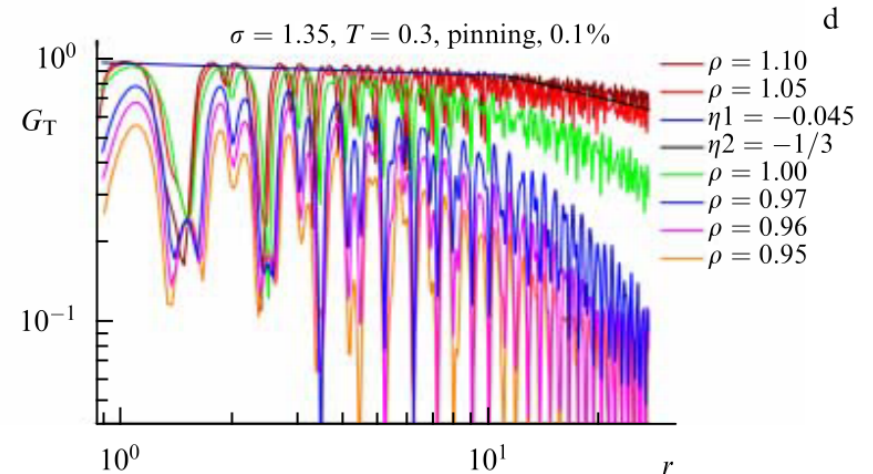
d	β	$\alpha = \alpha'$	$\gamma = \gamma'$	δ	ν	η	
2	1/8	0	7/4	15	1	1/4	exact
3	0.325	0.11	1.24	4.82	0.63	0.032	approx
MF	1/2	0	1	3	1/2	0	exact

Correlation functions

Melting: from solid to liquid in $2d$



straight lines



curved

Translational order correlation functions G_T : exponential decay $e^{-r/\xi}$

Orientalional order correlation functions G_6 : power law decay $r^{2-d-\eta}$ - critical

Correlation functions

Atomic gases in $2d$ - Stochastic Gross-Pitaevskii equation

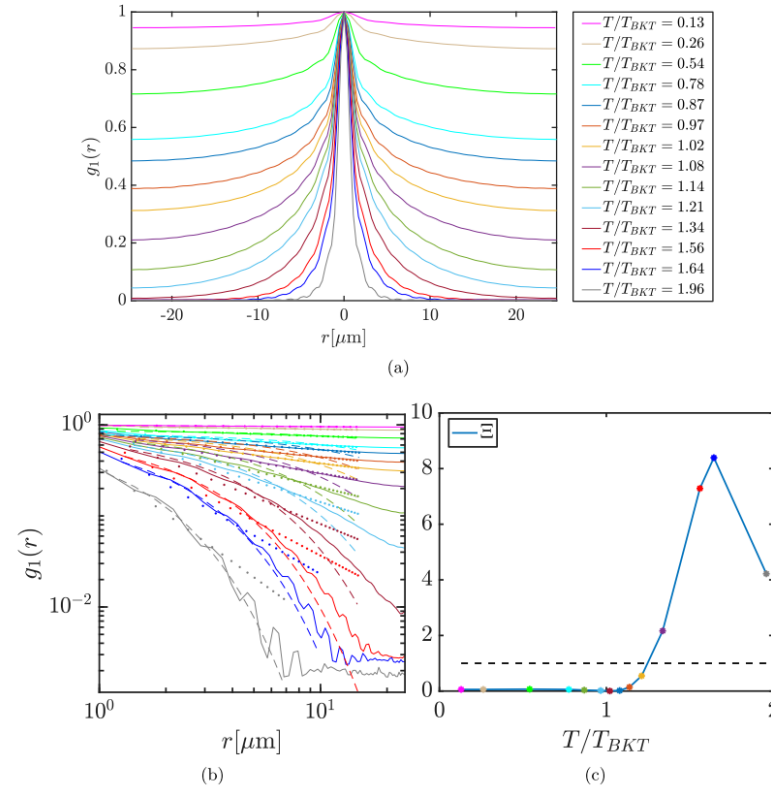
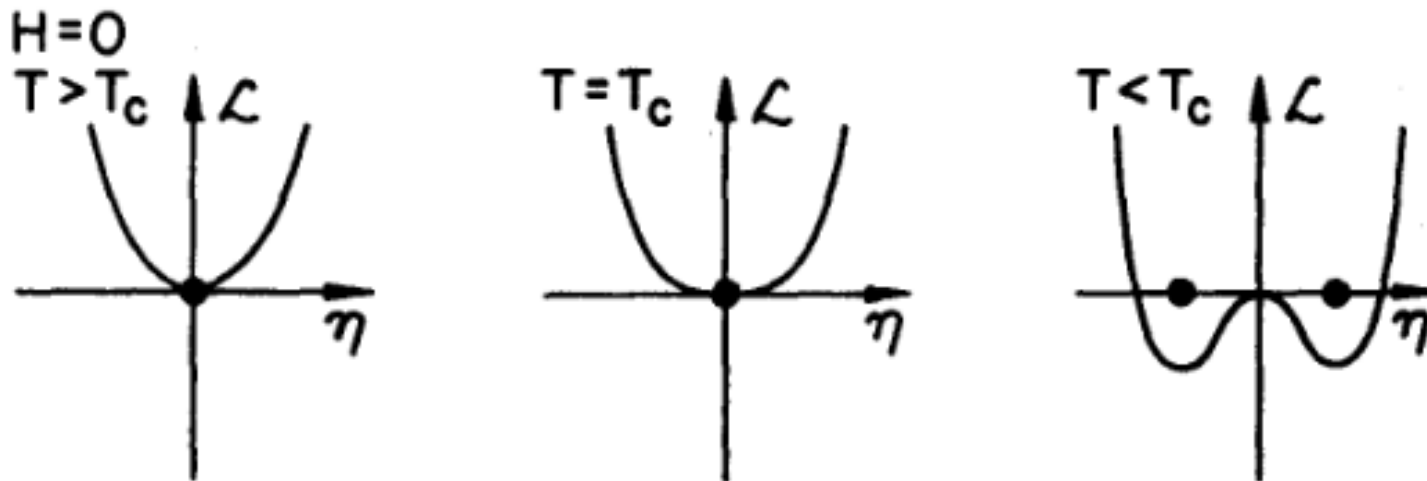


Figure 4.7: (a) Equilibrium profiles for the first-order correlation function $g_1(r)$ for different temperatures. (b) Correlation function profiles as in the previous figure, in logarithmic scale. The fitted algebraic (dotted lines) and exponential (dashed lines) functions are defined as in (4.56). (c) Ratio Ξ between the mean squared errors for the fits in (b), as defined in (4.57). The dashed line expresses the value of Ξ for which the two fits apply equally well to the data. The resulting critical point presents a shift of about ~ 0.25 with respect to the theoretical relation for T_{BKT} , equation (4.41).

Landau theory

Second order phase transition

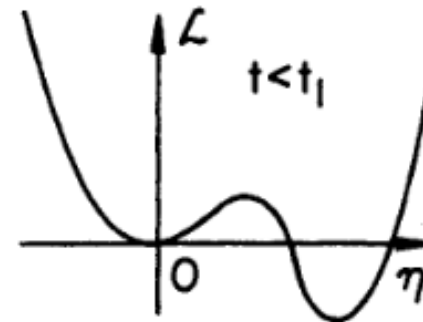
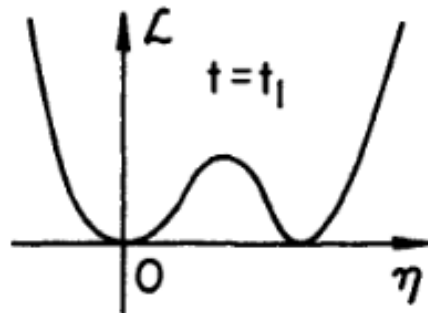
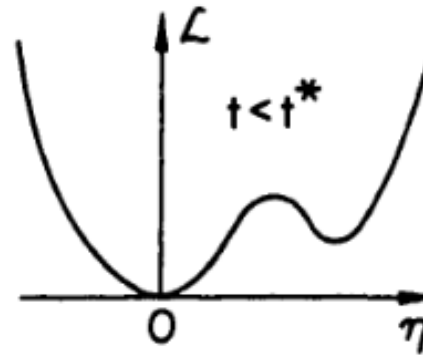
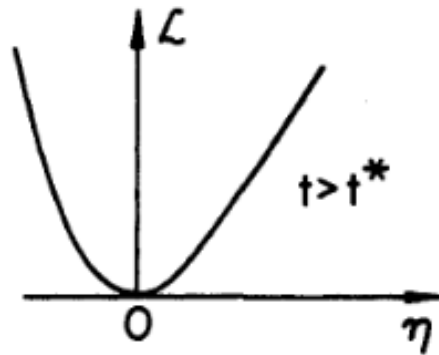


Notation : $\mathcal{L}$ is the “potential” in the Landau free-energy density

Notation : η is the “field”

Landau theory

First order phase transitions



Notation : $\mathcal{L}$ is the “potential” in the Landau free-energy density, η is the “field”, t_I is the transition

Landau theory

Important points

- The actual values of the parameters are not important
they are material/model dependent
- The dimension of the order parameter (scalar, vectorial) and the potential
decide whether the transition is second, first order or infinite order
- In 2nd order phase transitions the correlation length diverges & the linear
susceptibility as well.
- In continuous phase transitions the exponents are universal
- The strength of the Gaussian fluctuations limit the validity of the saddle-point
treatment of the Landau theory.
upper critical dimension, critical region $\xi^{4-d} < 1$ for the scalar $\lambda\phi^4$ theory

Critical Scaling

Ferromagnetic transition

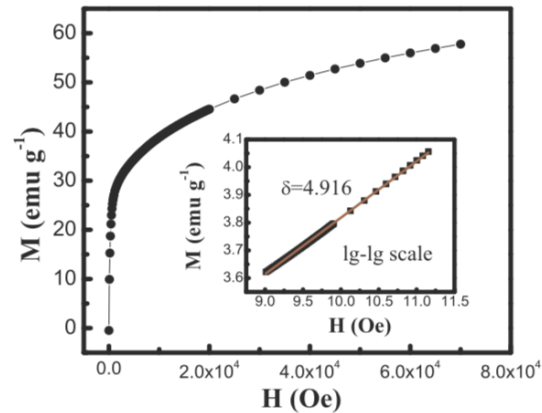


Fig. 4. Isothermal M vs H plot of $\text{Nd}_{0.55}\text{Sr}_{0.45}\text{Mn}_{0.98}\text{Ga}_{0.02}\text{O}_3$ at $T_c = 247$ K; the inset shows the same plot in log-log scale and the solid line is the linear fit following Eq. (3).

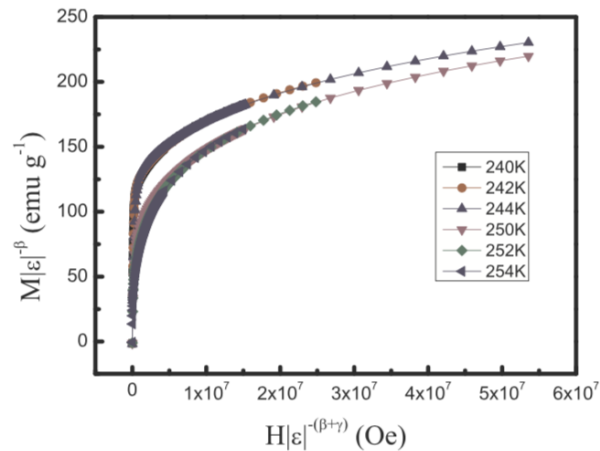


Fig. 5. Scaling plots of renormalized magnetization M vs. renormalized field H above and below T_c .

Critical Scaling

Jamming transition

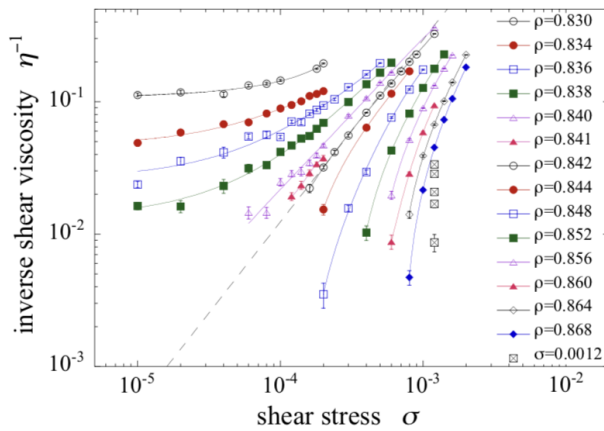


FIG. 2: (color online) Plot of inverse shear viscosity η^{-1} vs applied shear stress σ for several different values of the volume density ρ . The dashed line represents the power law dependence expected exactly at $\rho = \rho_c$ and has a slope $\beta/\Delta = 1.375$. Solid lines are guides to the eye. Points labeled $\sigma = 0.0012$ correspond to densities $\rho = 0.870, 0.872, 0.874, 0.876$, and 0.878 .

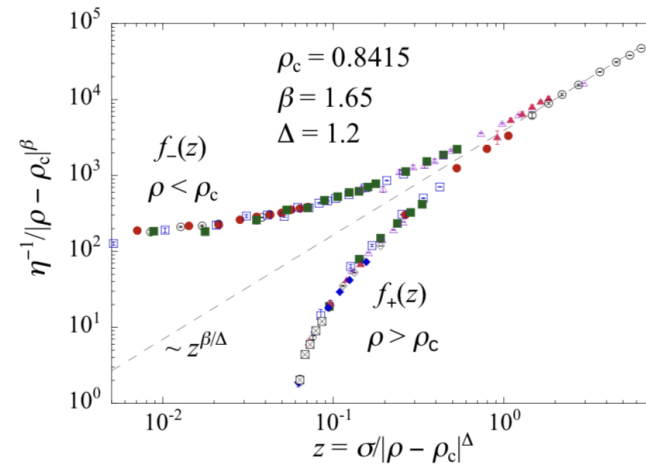


FIG. 3: (color online) Plot of scaled inverse viscosity $\eta^{-1}/|\rho - \rho_c|^{-\beta}$ vs scaled shear stress $z \equiv \sigma/|\rho - \rho_c|^{-\Delta}$ for the data of Fig. 2. We find an excellent collapse to the scaling form of Eq. (6) using values $\rho_c = 0.8415$, $\beta = 1.65$ and $\Delta = 1.2$. The dashed line represents the large z asymptotic dependence, $\sim z^{-\beta/\Delta}$. Data point symbols correspond to those used in Fig. 2.

P. Olsson and S. Teitel, *Critical Scaling of Shear Viscosity at the Jamming Transition*, Phys. Rev. Lett. 99, 178001 (2007).

Critical Scaling

Jamming transition

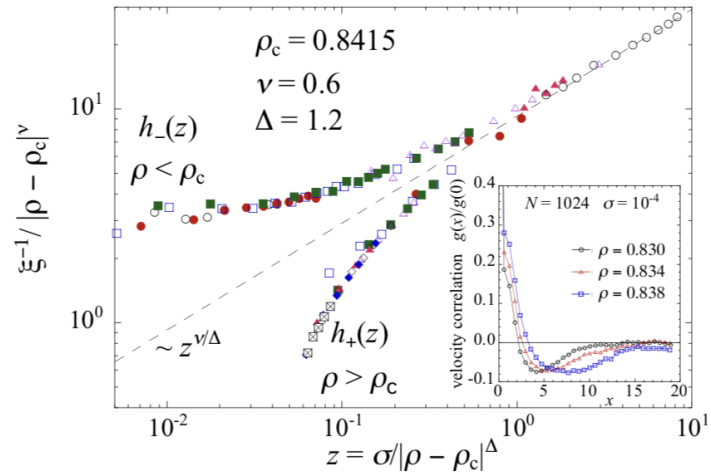


FIG. 4: (color online) Inset: Normalized transverse velocity correlation function $g(x)/g(0)$ vs longitudinal position x for $N = 1024$ particles, applied shear stress $\sigma = 10^{-4}$, and volume densities $\rho = 0.830, 0.834$ and 0.838 . The position of the minimum determines the correlation length ξ . Main figure: Plot of scaled inverse correlation length $\xi^{-1}/|\rho - \rho_c|^\nu$ vs scaled shear stress $z \equiv \sigma/|\rho - \rho_c|^\Delta$ for the data of Fig. [2]. We find a good scaling collapse using values $\rho_c = 0.8415$, $\Delta = 1.2$ (the same as in Fig. [3]) and $\nu = 0.6$. Data point symbols correspond to those used in Fig. [2].

Finite size effects

Rounding of magnetization curves

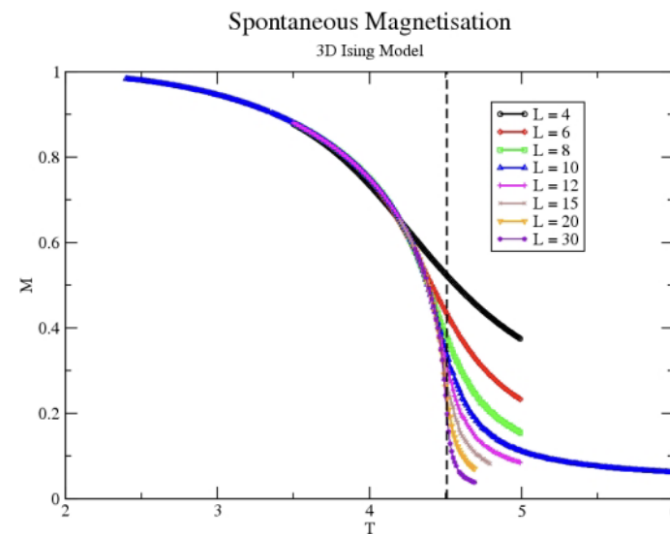
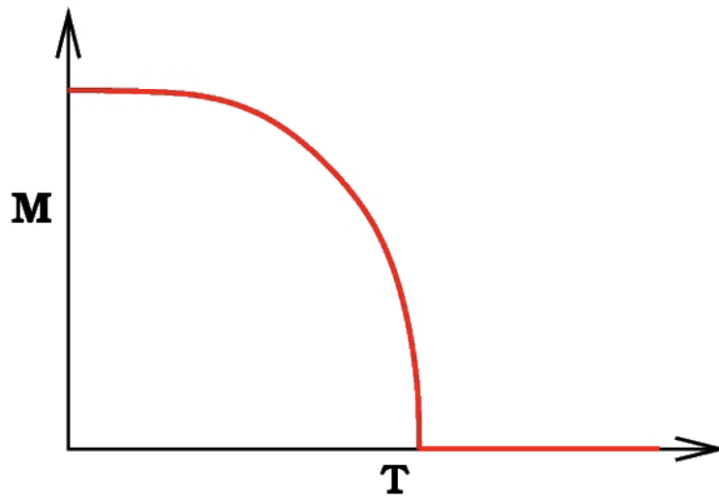
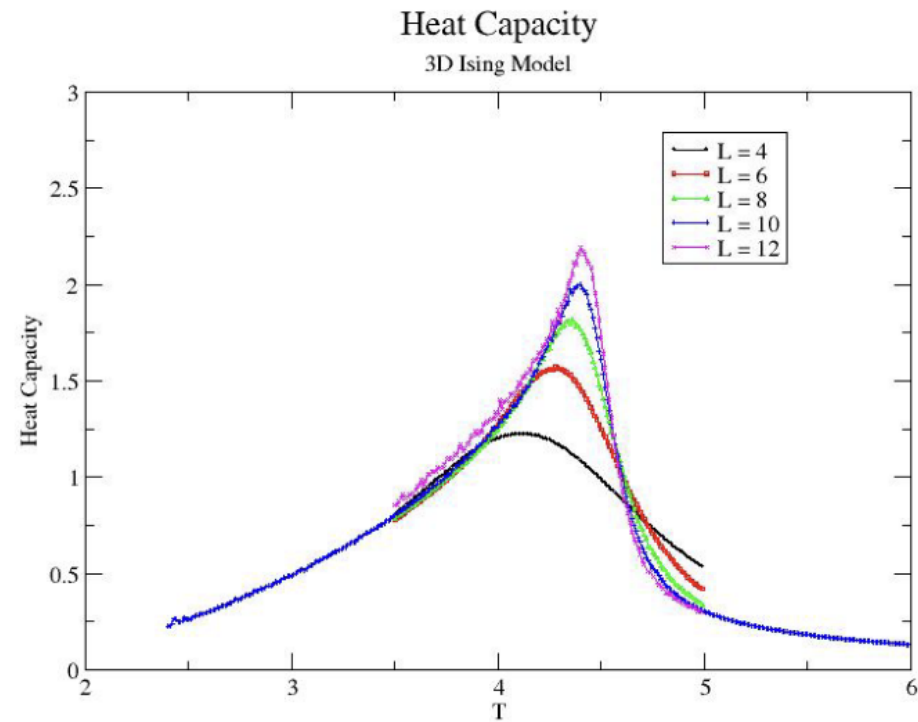


Figure: In TD limit the order parameter is 0 for $T > T_c$ but in FS the transition is smeared out.

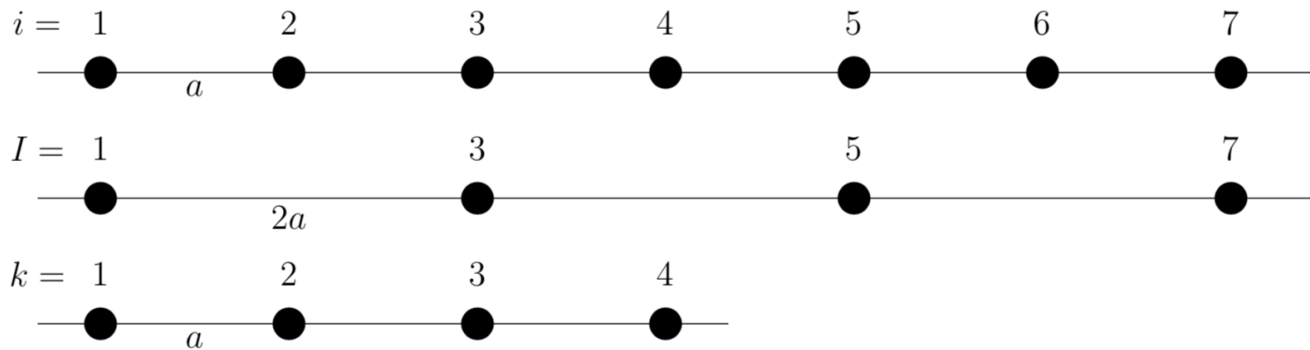
Finite size effects

Rounding of the heat capacity



Decimation

1d Ising chain $K = \beta J, \quad \zeta = \ln \mathcal{Z}$



$$K' = \frac{1}{2} \ln \cosh(2K) \rightarrow 0 \quad \text{that is} \quad (T/J)^* \rightarrow \infty$$

$$\boxed{\zeta(K') = 2\zeta(K) - \ln[2(\cosh(2K))^{1/2}]}$$

Decimation : the killing of one in every ten of a group of people as a punishment for the whole group (originally with reference to a mutinous Roman legion)

Decimation

1d Ising chain $K = \beta J$, $\zeta = \ln \mathcal{Z}$ inverse relations

$$K = \frac{1}{2} \cosh^{-1}(\exp(2K')) , \quad \zeta(K) = \frac{1}{2} \ln 2 + \frac{1}{2} K' + \frac{1}{2} \zeta(K') ,$$

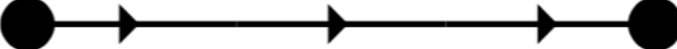
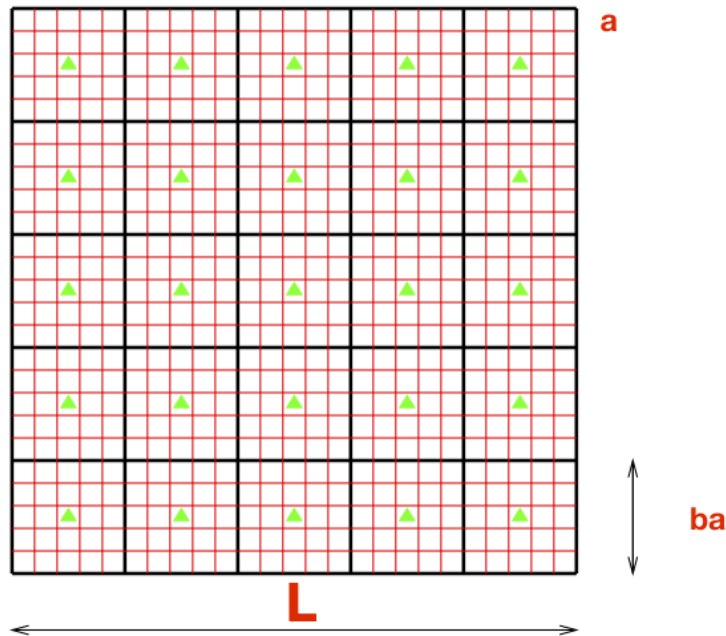
$K' = 0$  $K \rightarrow \infty$ $T \rightarrow 0$

Table I. Values of ζ for the 1D Ising model calculated from the recursion formulas Eqs. (31) and (32) of the renormalization group and from the exact formula derived from Eq. (9). ζ is related to the partition function Z through Eq. (28).

K	$\zeta(K)$	
	Renormalization group	Exact
0.01	$\ln 2$	0.693 197
0.100 334	0.698 147	0.698 172
0.327 447	0.745 814	0.745 827
0.636 247	0.883 204	0.883 210
0.972 710	1.106 299	1.106 302
1.316 710	1.386 078	1.386 080
1.662 637	1.697 968	1.697 968
2.009 049	2.026 876	2.026 877
2.355 582	2.364 536	2.364 537
2.702 146	2.706 633	2.706 634

RG

From block spins to coupling renormalization



degrees of freedom

$$b^d \mapsto 1$$

lattice spacing

$$a \mapsto ba$$

re-scaling of lengths

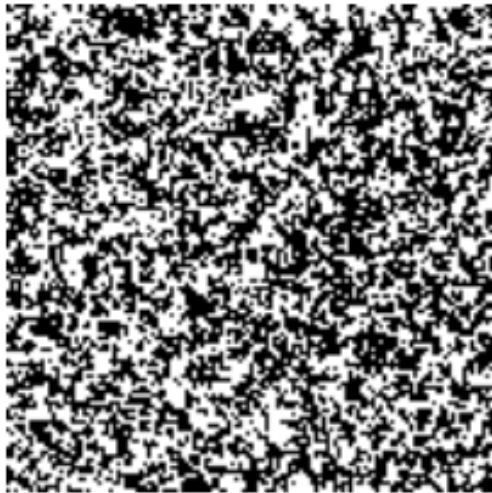
$$\vec{x} \mapsto \vec{x}/b$$

From $\mathcal{H}'_{[K']}(\{s'_I\}) = R_b \mathcal{H}_{[K]}(\{s_i\})$ to

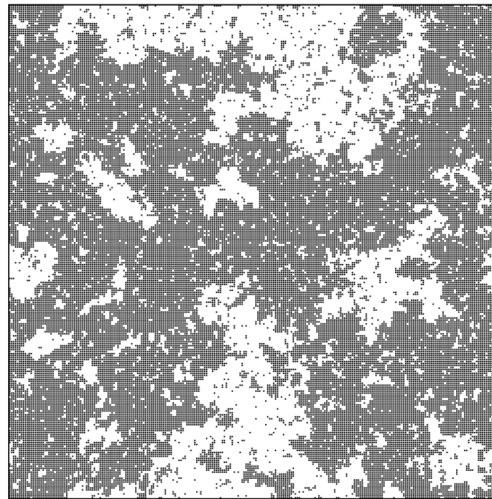
$$K'_\alpha = \sum_\beta (R_b)_{\alpha\beta} K_\beta \quad \text{close to} \quad [K_c] = [0]$$

Typical configurations

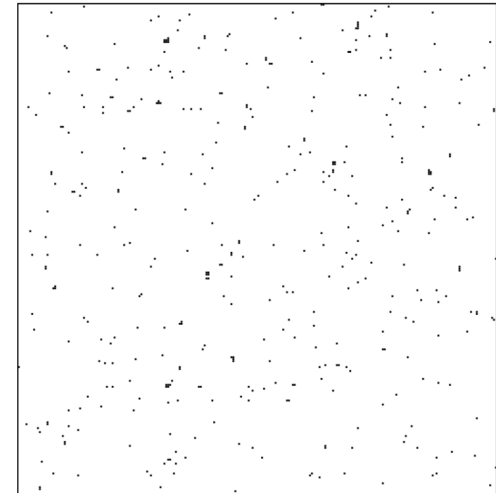
Up & down spins in a $2d$ Ising model



$$T = 2T_c$$



$$T = T_c$$



$$0 < T < T_c$$

Real space viewpoint

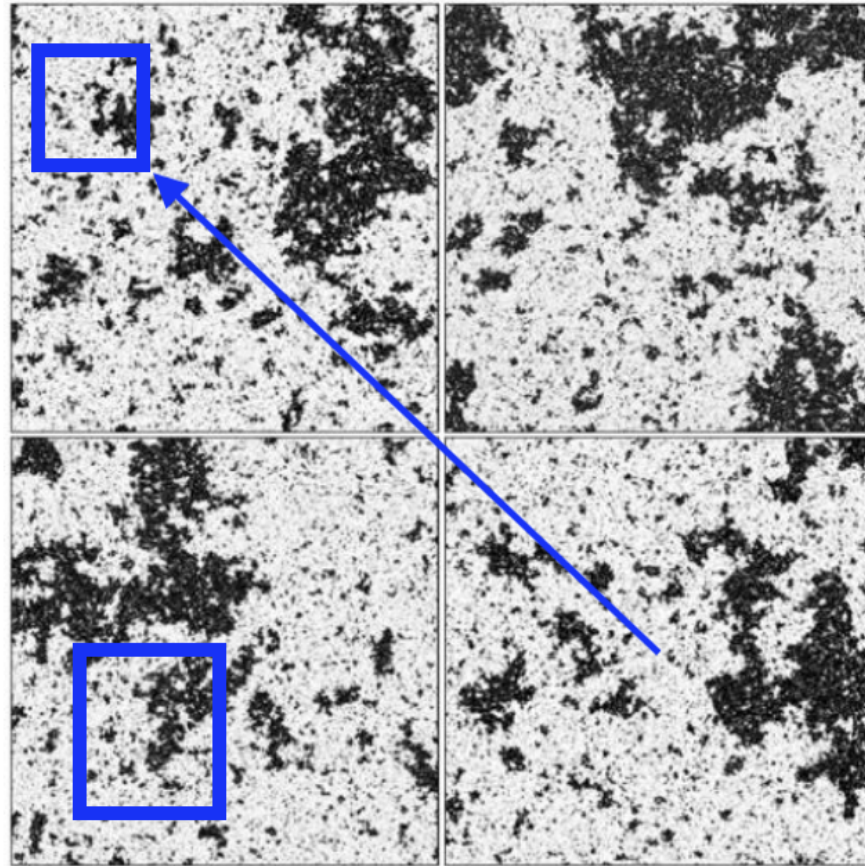
Zoom out by observing at larger scales

move away from criticality if $T \neq T_c$

$[K_c]$ repulsive fixed points

Scale invariance

2d Ising model at T_c



D. Ashton, *Scale invariance in the critical Ising model*,

<https://www.youtube.com/watch?v=fi-g2ET97W8>

RG

finite d Ising model

Two adimensional parameters in $\mathcal{H}_{[K]}(\{s_i\})$

$$K_1 = \beta J = K$$

$$K_2 = \beta h = K$$

Composition $R(b_1 b_2) = R(b_1)R(b_2)$ and symmetries

$$K' = b^{y_K} K$$

$$H' = b^{y_H} H$$

Repulsive fixed point

$$y_K > 0$$

$$y_H > 0$$

RG

finite d Ising model

Identity of total free-energy $\implies$ homogeneity

$$F(K', H') = F(K, H) \implies b^{-d} f(b^{y_K} H, b^{y_H} H)$$

choosing $b = |K|^{-1/y_K} \implies$ scaling

$$\begin{aligned} f(K, H) &= |K|^{d/y_K} f(1, |K|^{-y_H/y_K} H) \\ &= |K|^{d/y_K} g_f \left(\frac{H}{|K|^{y_H/y_K}} \right) \end{aligned}$$

with $\Delta = y_H/y_K$ and $2 - \alpha = d/y_K$

RG

Generic

Transformation

$$K'_\alpha - K_\alpha^* = \sum_\beta R(b)_{\alpha\beta} (K_\beta - K_\beta^*)$$

in other terms
$$R(b)_{\alpha\beta} = \left. \frac{\partial K'_\alpha}{\partial K_\beta} \right|_{[K^*]}$$

To solve one needs to diagonalize the linear (vectorial) relation above

Use the

eigenvalues λ^i and left orthonormal eigenvectors u_α^i of the matrix $\mathbb{R}(b)$

$$\sum_j u_\beta^j u_\gamma^j = \delta_{\beta\gamma}$$

RG

Generic

Transformation

$$\delta K_\alpha = K'_\alpha - K_\alpha^* = \sum_\beta R(b)_{\alpha\beta} (K_\beta - K_\beta^*) = \sum_\beta R(b)_{\alpha\beta} \delta K_\beta$$

multiply by the left eigenvector u_α^i and sum over α

$$\begin{aligned} \delta \kappa'^i &\equiv \sum_\alpha u_\alpha^i \delta K'_\alpha = \underbrace{\sum_\alpha u_\alpha^i \sum_\beta R(b)_{\alpha\beta}}_{\lambda^i \sum_\beta u_\beta^i} \sum_\gamma \underbrace{\left(\sum_j u_\beta^j u_\gamma^j \right)}_{\delta_{\beta\gamma}} \delta K_\gamma \\ &= \lambda^i \sum_j \sum_\beta \underbrace{u_\beta^i u_\beta^j}_{\delta^{ij}} \sum_\gamma u_\gamma^j \delta K_\gamma \\ &= \lambda^i \sum_\gamma u_\gamma^i \delta K_\gamma = \lambda^i \delta \kappa^i \end{aligned}$$

RG

Generic

Diagonalized transformation

$$\delta \kappa'^i \equiv \sum_{\alpha} u_{\alpha}^i \delta K'_{\alpha} = \lambda^i \delta \kappa^i$$

thus

$$\lambda^i = \left. \frac{\partial \delta \kappa'^i}{\partial \delta \kappa^i} \right|_{[K^*]} = \left. \frac{\partial \kappa'^i}{\partial \kappa^i} \right|_{[K^*]}$$

and

$$\lambda^i = b^{y_i} \quad \Longrightarrow \quad y_i = \frac{d \ln \lambda^i}{db}$$

RG

Generic

Parameters

- $y_i > 0$ – *relevant*, with the corresponding $\kappa_i(b)$ coupling growing under coarse-graining, i.e., getting away from the fixed point
- $y_i < 0$ – *irrelevant*, with the $\kappa_i(b)$ coupling vanishing under coarse-graining, i.e., approaching the fixed point
- $y_i = 0$ – *marginal*, with the flow of the $\kappa_i(b)$ coupling determined by the higher order terms in the couplings of the RG transformation.

Concepts

The scale of observation does not modify the properties of the critical state
nor the disordered or order character of the phases

Kadanoff blocks modify the physical state by coarse-graining but,
the modern RG (Wilson) transforms the form of the model until reaching the one
that describes the critical state

At criticality and close to it the microscopic details are not important

The critical points are repulsive fixed points of the RG

Above the critical temperature, evolution to the free fixed point, $T \rightarrow \infty$

Below the critical temperature, evolution to $T = 0$

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A renormalisation transformation is a scale transformation that leaves the
partition function invariant. Since the thermodynamic properties of a sys-
tem are governed by the partition function, the physics is preserved.

Concepts

The critical behaviour does not depend on the type of subcritical order but on the number n of components of the order parameter and the dimension of space d .

Renormalisation enables the classification in universality classes with the same critical properties, depending on (n, d)

Scaling laws apply to global, macroscopic properties of systems containing a large number of elementary microscopic units.

They are found in other domains, *e.g.* finance, biology, etc.

Exponents again

Comparison

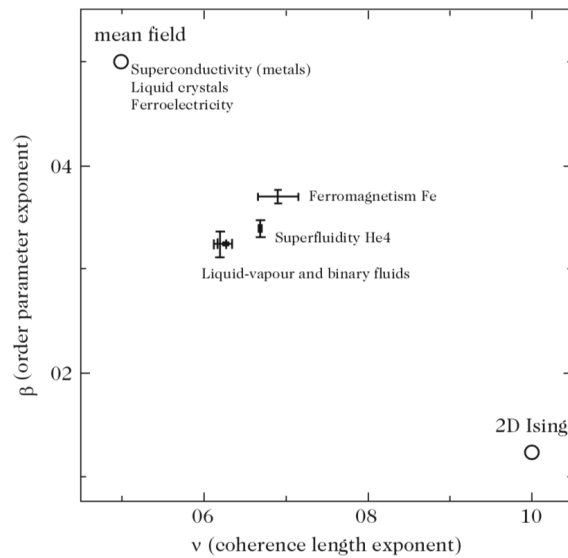


Fig. 1.22 Critical exponents β and ν : experimental results obtained for seven different families of transitions, compared to values predicted by the mean field and 2D Ising models

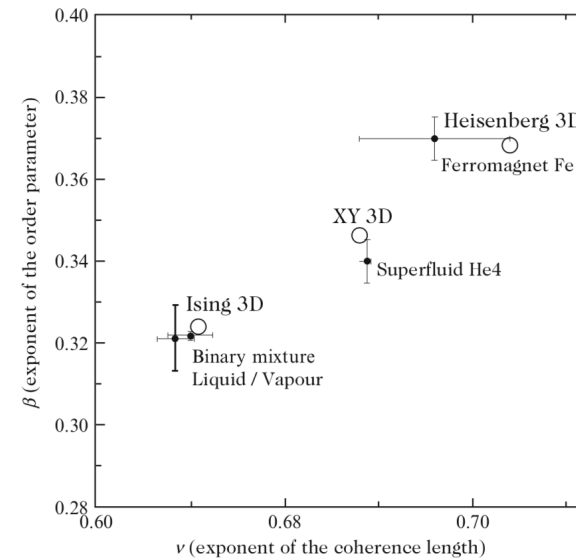


Fig. 1.23 Critical exponents measured for four families of transitions, compared to values predicted by the three corresponding models that take into account the scale invariance of the critical state

Images from

A. Lesne & M. Lagües, *Scale invariance from phase transitions to turbulence* (Springer-Verlag, 2012)