# Advanced Statistical Physics

## Exam

January 2022

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SURNAME :
NAME :
MASTER :

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8h30 11h30

Write your answers within the BOXES.

You can write in ENGLISH or FRENCH, as you prefer.

Use a CLEAR handwriting.

No books, notes, calculator nor mobile phone allowed.

Clarity and organization of the answers are important, and will be taken into account to mark the exam.

If doubt exists as to the interpretation of any question, the candidate is urged to consult the examiners in the room and to submit with the answer paper a clear statement of any assumptions made.

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**Question 1.** Figure 1 is extracted from an experimental paper. $X$ is an observable of interest in the context of this study. In the two systems analysed, the outcomes of many measurements of $X$ is organised in the histograms shown in panels A and B. What can you say about the measurement of $X$? And about the behaviour of another (uncorrelated with $X$) observable $Y$?

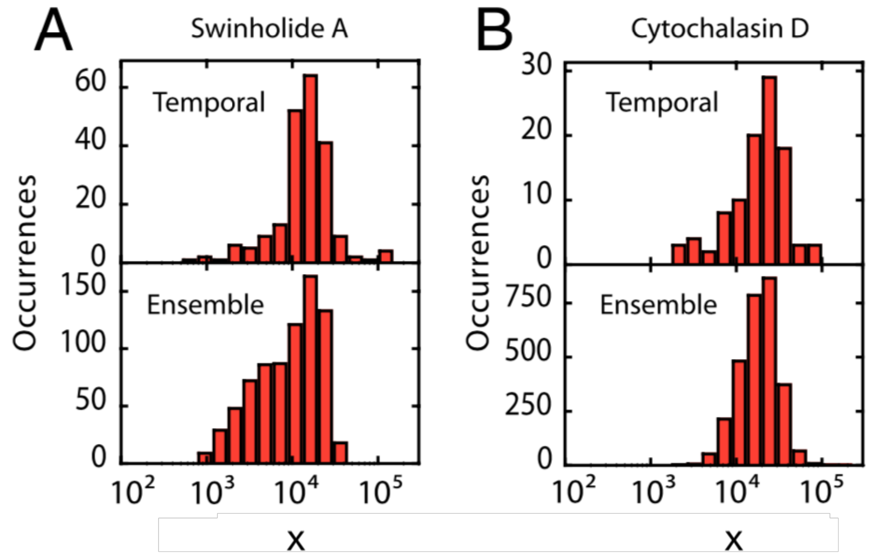


Figure 1: Histograms of the observable $X$.

**Question 2.** What is the idea behind the Ginzburg-Landau approach? Describe the assumptions made, the basic ingredients, what is this theory intended to capture, and its limits of validity (when spatial variations are taken into account). Be precise and complete, but also concise. (The next question is related, so read it before answering this one so as to avoid writing twice the same ideas.)

**Question 3.** In particular, formulate the Ginzburg-Landau approach for a first-order phase transition (with no applied field). Illustrate the mechanism with one (or two) graphical sketch(s). Write an equation for the free-energy functional, as a function of a scalar field, and define and discuss the meaning of each parameter and each term in the form proposed. What are the main characteristics/properties of first order phase transitions?

**Question 4.** Which is the mechanism driving the transition in the 2dXY model? Give the functional form of the spatial magnetic correlation function in the low temperature phase. What is the linear magnetic susceptibility in this phase? And the correlation length? What is so particular of this phase?

**Question 5.** Take a model with a second order phase transition driven by temperature $T$ and under a magnetic field $h$. What can you say about the parameter dependence of the singular part of the free-energy density at the critical point? What is it useful for?

**Question 6.** Figure 2 shows four snapshots of the $2d$ Ising Model (red spins are up, yellow spins are down) in contact with a bath at temperature $T$ and no applied field. (a) is a Monte Carlo generated configuration, in equilibrium at $T$. (b)-(c)-(d) have been obtained after successive block spin transformation and re-scaling.

(a) (b) (c) (d)

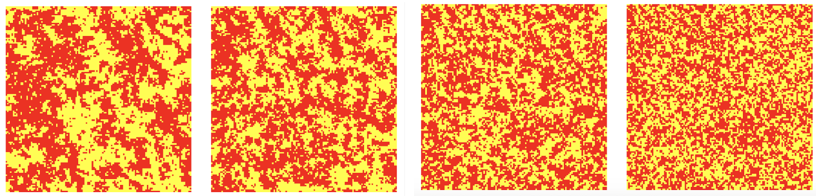

Figure 2: Snapshots of the $2d$IM.

(a) Give the generic and formal mathematical expressions of these transformations.

(b) How does the temperature $T$ compare to the critical one?

(c) Explain which would be the configurations obtained after repeated application on these transformations for initial configurations at $T < T_c$, $T = T_c$ and $T > T_c$.

**Question 7.** Take a classical Hamiltonian with an adimensional coupling constant $K = \beta J$ which under renormalisation of the degrees of freedom within blocks of linear size $\ell = ba$, with $b = 2$ and $a$ the lattice spacing, transforms as $t' = (t + 5t^2 - 3t^3)/3$ where $t = \tanh K$. (a) Find the fixed points. (b) Which is the interesting fixed point? (c) Consider the linear approximation of the recurrence close to it. (d) What is the critical exponent $\nu$? The divergence of which quantity does it characterise?

**Question 8.** Take a quantum Ising spin with Hamiltonian $\hat{\mathcal{H}} = -\Gamma \hat{\sigma}^x$. Express its quantum partition function as a classical partition function. Explain the mapping.

**Question 9.** Which is the assumption leading to the Kubo relation? Recall its expression. Does such a relationship allow you to state that the system is in equilibrium?

**Question 10.** Figure 3 shows the expected value of $\langle \hat{x}^2(t) \rangle$ of two kinds of atoms K and Rb in a mixture as a function of time. According to the experimental data-points what can you conclude about the equilibration of the system?

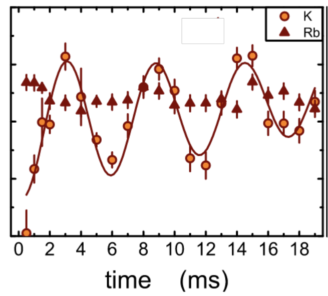


Figure 3: $\langle \hat{x}^2(t) \rangle$ as a function of time in an atomic mixture.

**Question 11.** At which temperature do quantum phase transitions occur? Which are the fluctuations driving these transitions? Which is the state of the system affected by these phase transitions?

**Question 12.** Consider an electron gas confined to a two dimensional space at temperature $T$ and under a perpendicular magnetic field $B$. This is a quantum system in equilibrium at very low temperatures. In panel **A** in Fig. 4, the resistance $R_s$ as a function of $B$ at different temperatures is plotted. The units of $R_s$ and $B$ are given between parenthesis in the axes labels and the temperatures (in Kelvin) at which the various curves are obtained are indicated within the panel, from red at 0.07 K to blue at 0.04 K. Explain what the authors have done and the implications of the plot and inset in panel **B**.

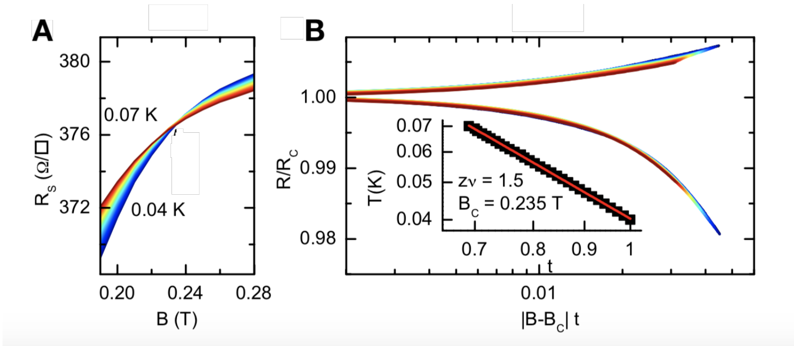


Figure 4: Resistance experimental measurement in a two dimensional electron gas.

# Problem

The Hamiltonian of the Potts model is

$$H = -J \sum_{\langle ij \rangle} \delta_{s_i s_j} \tag{1}$$

The spin variables $s_i$ can take $q$ natural values, $s_i = 1, \ldots, q$. These are also called colours in the literature and we will called them this way henceforth. The case $q = 2$ is the Ising model. The sum runs over nearest neighbours on a cubic lattice in $d$ dimensions and each pair is counted only once. The coupling is ferromagnetic $J > 0$ and $\delta_{ab}$ is the Kronecker delta. We couple the model to a heat bath at temperature $T$.

1. **Generic.**

   (a) Describe the ground states.

   (b) Which phases do you expect? Characterise them in terms of typical spin configurations.

2. **Mean-field treatment.**

   (a) Consider the fully-connected case in which each spin interacts with all the other ones in the sample and each pair is counted once in the sum. Write the Hamiltonian. How should you modify the model to let it have an interesting thermodynamic limit?

   (b) You are going to characterise order in this problem using $q$ order parameters, which you will call $x_a$ with $a = 1, \ldots, q$, and take them to be the proportion of sites taking each colour. Which condition do they satisfy?

(c) Set up the mean-field analysis of this problem writing the free-energy density as a function of the $x_a$ in the $N \gg 1$ limit. You can argue which should be the form of each contribution in the large $N$ limit without making the explicit derivation.

(d) Supposing that in the low temperature phase the symmetry is broken in favour of the first color, rewrite this free-energy density distinguishing the corresponding order parameter, say $x_1$, with respect to the other $q-1$ ones which you will take to be identical $x_2 = \ldots = x_q$.

(e) Express $x_1$ as $x_1 = (1/q)[1+(q-1)s]$, with $s$ a new parameter and rewrite the free-energy as a function of $s$.

(f) If one Taylor expands the free-energy density around $s \sim 0$ keeping up to cubic contributions in powers of $s$, using $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \mathcal{O}(x^5)$, one finds

$$\beta[f(s) - f(0)] = \frac{(q-1)}{2q}\,(q-\beta J)\,s^2 - \frac{(q-1)(q-2)}{6}\,s^3 + \mathcal{O}(s^4)$$

What do you conclude about the phase transition in the mean-field Potts model for generic $q$?

(g) Do you expect to find the same behaviour working in the micro-canonical ensemble? Explain.

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3. **Decimation of the unidimensional Potts model**

Consider the Potts model on a $d = 1$ space with periodic boundary conditions.

(a) Take three nearby sites on the line, labelled $k - 1$, $k$ and $k + 1$. Perform the partition sum over the central spin $s_k$ (decimation). Establish the recurrence relations for the parameter $e^K$ with $K = \beta J$.

(b) Which are the fixed points? Determine whether they are stable or unstable.

(c) Is there a finite temperature phase transition?

(d) Do you know an argument to show that there is no finite temperature phase transition in this problem without going through this kind of calculation?

(e) Do you expect a finite temperature phase transition in $d = 2$?