Advanced Statistical Physics

Exam

January, 2022

Surname :

Name :

Master :

Write your surname & name in CAPITAL LETTERS.

Not only the results but especially the clarity and relevance of the explanations will be evaluated.

Focus on the questions asked and answer them (and not some other issue).

The answers must be written neatly within the boxes.

The problems follow the order of the chapters in the Lecture Notes but are not of increasing difficulty.

The exam is long but do not panic, if you are blocked by some problem, jump to the next one and come back later to the one you find difficult.

1. Ergodicity

Figure 1 shows the numerical evaluation of the solution of a stochastic (Langevin) equation, ruling the time ($\boldsymbol{t}$) evolution of a real variable $\boldsymbol{X}$. We do not need to specify this equation but only remark that it depends on a parameter **a**. The evolution of $\boldsymbol{X}$ for different initial conditions $\boldsymbol{X}(0)$ and different random noise realisations is the one in the panel above for $\mathbf{a} > 0$ and the one in the panel below for $\mathbf{a} = 0$.

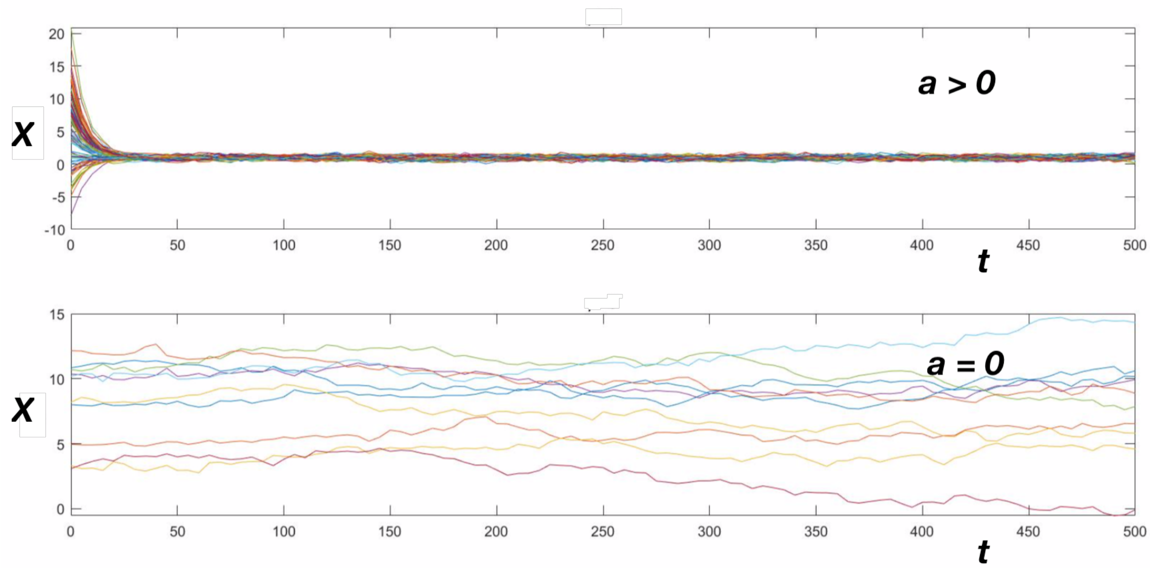

Figure 1: The solution of a stochastic equation with $\mathbf{a} > 0$ above and $\mathbf{a} = 0$ below. The different curves correspond to different initial conditions and different noise realizations.

What do you conclude about the ergodic properties of the system in the two cases? Give the condition needed to satisfy ergodicity and discuss whether it holds or not in the two cases.

2. Phase transitions.

Consider the classical Heisenberg model in three dimensions in canonical equilibrium with a bath at temperature T. This model is defined by the Hamiltonian

$$H = -J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_i \ , \tag{1}$$

where $J > 0$, $\vec{s}_i$ are placed at the vertices of a three dimensional lattice with unspecified geometry and they are vectors with three components, $\vec{s}_i = (s_i^1, s_i^2, s_i^3)$, each of them taking real values, $-\infty < s_i^a < \infty$ for $a = 1, 2, 3$, but constrained to have unit modulus, $|\vec{s}_i|^2 = (s_i^1)^2 + (s_i^2)^2 + (s_i^3)^2 = 1$. The sum in Eq. (1) runs over nearest-neighbours on the lattice and there are N spins in the system.

1 – In the absence of any phase transition consideration, which is the canonical average of $\vec{s}_i$ at all temperatures? Justify your answer with a mathematical proof.

One uses the symmetry under $\vec{s}_i \mapsto -\vec{s}_i$ to prove it.

$\langle \vec{s}_i \rangle = Z^{-1} \int ds_i^1 \int ds_i^2 \int ds_i^3 \ \delta(|\vec{s}_i|^2 - 1) \ \vec{s}_i \ e^{\beta J \sum_{\langle ij \rangle} \vec{s}_i \cdot \vec{s}_j}$

Change variables $\vec{s}_i = -\vec{\sigma}_i$ in the integral

$\langle \vec{s}_i \rangle = -Z^{-1} \int d\sigma_i^1 \int d\sigma_i^2 \int d\sigma_i^3 \ \delta(|\vec{\sigma}_i|^2 - 1) \ \vec{\sigma}_i \ e^{\beta J \sum_{\langle ij \rangle} \vec{\sigma}_i \cdot \vec{\sigma}_j} = -\langle \vec{\sigma}_i \rangle$

And then $\langle \vec{s}_i \rangle = 0$.

2 – Do you expect a finite temperature phase transition in this problem? Under which conditions on the number of spins?

Yes, if $N \to \infty$.

3 – Which would be the phases? Describe them.

A high temperature disordered paramagnetic phase. Spins pointing in all directions with no preferred order.

A low temperature ordered ferromagnetic phase. Spins pointing in one direction with thermal fluctuations distorting the perfect order (ground state).

4 – Identify an order parameter and give its mathematical expression.

$\vec{m} = N^{-1} \sum_i \langle \vec{s}_i \rangle$ is a global order parameter, the magnetization density.

The local observable $\langle \vec{s}_i \rangle$ should behave in the same way, since there is no quenched randomness in this problem and there is no reason to have heterogeneous behaviour.

5 – Which is the mechanism whereby the order parameter just defined would acquire a non vanishing value? Explain its origin in an experimental situation.

Spontaneous symmetry breaking

When the system goes through a phase transition

remanent fields which the experimentalist cannot control select one among the continuity of possible directions that the spins can take in the ground state

6 – How is this mechanism imposed mathematically?

One applies a magnetic field $\vec{h}$ in a chosen direction,

calculates the averages under this field $\langle \vec{s}_i \rangle_{\vec{h}}$ in the $N \to \infty$ limit,

and then takes the field to zero

to obtain a non-vanishing $\vec{m}$, in the direction of the field $\vec{h}$.

7 – Explain the way in which you have contoured the answer to question 1 – with the mathematical approach proposed in the answer to question 6 –.

By applying the magnetic field $\vec{h}$ in a chosen direction, one breaks the rotation symmetry.

In particular, also the one under spin inversion which was used in 1 –

Thus, the average $\langle \vec{s}_i \rangle_{\vec{h}}$ can be different from zero in the thermodynamic limit.

8 – Do you expect ergodicity breaking in this problem? Discuss similarities and differences with the Ising cases that we discussed in the Lectures.

Difficult question since we did not discuss it in the lectures. Let's see what the students write.

The pinning field selects one direction but once it's set to zero there is the zero mode related to the rotation symmetry (contrary to the discrete one of the Ising model) that they should discuss.

9 – Consider now the fully-connected model in which each spin interacts, via the same scalar product, with all other spins, $\sum_{i \neq j} \vec{s}_i \cdot \vec{s}_j$. How do you render the model well-defined in the thermodynamic limit? Justify your answer.

Let us focus on one ground state, in which all spins point in the same direction $\vec{s}_i = \vec{v}_i$, say.

Its energy is $E = -JN(N-1)$.

To make it extensive one needs to rescale $J \mapsto J/N$.

10 – Go back to the model in Eq. (1) defined on a finite three dimensional lattice which we will take to be a cubic one, with either free or periodic boundary conditions, a distinction which is not important in the infinite size limit. Establish the mean-field analysis, determine the phase diagram and sketch the behaviour of the order parameter. (Hint: you can exploit the answer to question 6 – to simplify the vectorial treatment.)

3. (In) equivalence of ensembles.

Consider a system of N spins placed on the vertices of a lattice. The potential energy is given by the sum, over all pairs of the elementary constituents, of a two-body energy $u(s_i, s_j)$.

1– Explain and illustrate, with one equation, the *extensivity* property of the energy.

The energy should be extensive, that is to say, scale with the number of spins.

This is necessary to have an interesting thermodynamic $N \to \infty$ limit allowing, e. g., for finite temperature phase transitions.

Given a spin configuration, the total potential energy is a function of N and the other parameters in the problem which we do not write explicitly. $U(N) = \sum_{i \neq j} u(s_i, s_j)$ and, for it to be extensive, it should be proportional to N with proportionality constant $\overline{u}$ of order 1.

$U(N) = N\overline{u}$.

2 – Discuss qualitatively the conditions under which this potential is *additive*. Illustrate this property with one equation. You can support your argument with a sketch (drawing).

One needs short range interactions to be able to drop the contributions to the energy of the interface between any two macroscopic subsystems.

$U = U_1 + U_2 + U_{12} \sim U_1 + U_2$

since $U_{12} = o(N)$ while $U_1 = O(N)$ and $U_2 = O(N)$

Space for a drawing with the system partition in two subsystems 1 and 2.

3 – Explain why the violation of these properties may affect the equivalence of ensembles. Focus on micro-canonical and canonical measures and expand your answer with a mathematical argument.

To derive the canonical measure from the microcanonical one we use the extensivity and additivity of the energy.

They can/should develop the derivation of the canonical from the microcanonical. I will not write it here.

4. 2d XY model

1 – Write down the Hamiltonian of the $2d$ XY model and define all variables and parameters.

Hamiltonian $H = -J\sum_{\langle ij\rangle} \vec{s}_i \cdot s_j$ (if no applied field).

$J > 0$ FM coupling,

$\vec{s}_i = (s_i^1, s_i^2)$ two component vectors on each site i of the lattice.

$\sum_{\langle ij\rangle}$ sum over nearest neighbours on the lattice.

2 – Which kind of transition do you expect in this model?

Topological.

3 – Which are the excitations that drive the phase transition? You can illustrate them with drawings and give the mathematical expression for the corresponding spin configurations.

Vortices - drawings

4 – Give a qualitative argument to estimate the critical temperature and express its parameter dependence.

Peierls argument, E vs S, estimate T_{KT}

5 – What is the nature of the low temperature phase?

Critical

6 – Give the expression of the correlation function.

Power law decay at $T < T_{KT}$.

Exponential decay at $T > T_{KT}$.

7 – How does the correlation length behave at the transition? How can one prove? (Do not give the mathematical details just describe the method).

Exponential divergence.

RG via mapping to Villain's model.

5. Finite size scaling

The measurement of a linear susceptibility in a model with different linear system sizes yields the curves in panel (a) of Fig. 2. Explain what has been done in panel (b) in the same figure. To which universality class does this model belong to?

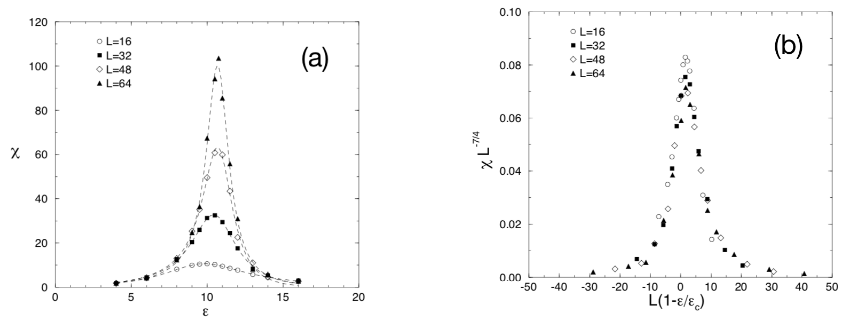

Figure 2: The susceptibility.

Critical behaviour as a function of the control parameter ϵ

Finite size Scaling $\chi L^{\gamma/\nu} = \tilde{\chi}(\epsilon L^{1/\nu})$

$2d$ Ising model exponents, since $\nu = 1$ and $\gamma = 7/4$.

X. Quantum Which are the assumptions leading to the quantum fluctuation dissipation theorem? Recall its expression.

Linear response and canonical equilibrium.

Equation

X. Quantum Figure 3 shows the time evolution of the probability distribution of atoms in an optical trap – which can be thought of a confining harmonic potential.

Are the atoms in thermal equilibrium? Discuss.

They are not.

The pdf depends on time.

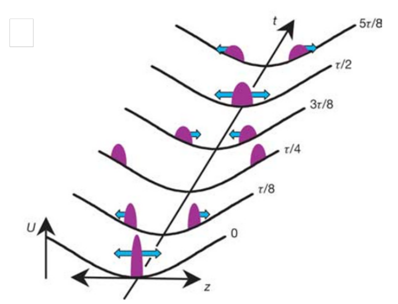

Figure 3: The probability distribution of the atom positions in an optical trap (harmonic potential) in the course of time.