Statistical Physics – M2 – ICFP condensed matter

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TD 6 : Quenched randomness

The objective of this session is to better understand the different mechanisms that can trigger disorder in a given system, such as introduction of topological defects, dilution, random magnetic field, frustration and also detachment/emergence of domain walls and surface fluctuations.
For more details, see also References below [1].

Exercise 1 : diluted ferromagnet

Let us consider a spin lattice and assign to each site a certain probability p_i to be occupied or empty. If $p_i = 1 \ \forall i$, we recover the Ising model back, otherwise if $p_i < 1$ we end up with a diluted model.
We define a diluted system on a cubic lattice by the Hamiltonian :

$$H = -\sum J_{ij}\sigma_i\sigma_j + h\sum \sigma_i \tag{1}$$

where $\sigma_i = \pm 1$ and $J_{ij} = 1$ with probability p and 0 with probability $(1-p)$ respectively. For $h \neq 0$ there is no phase transition, whereas for $h = 0$ we can define the evolution in temperature $T_c(p)$ as a function of p.

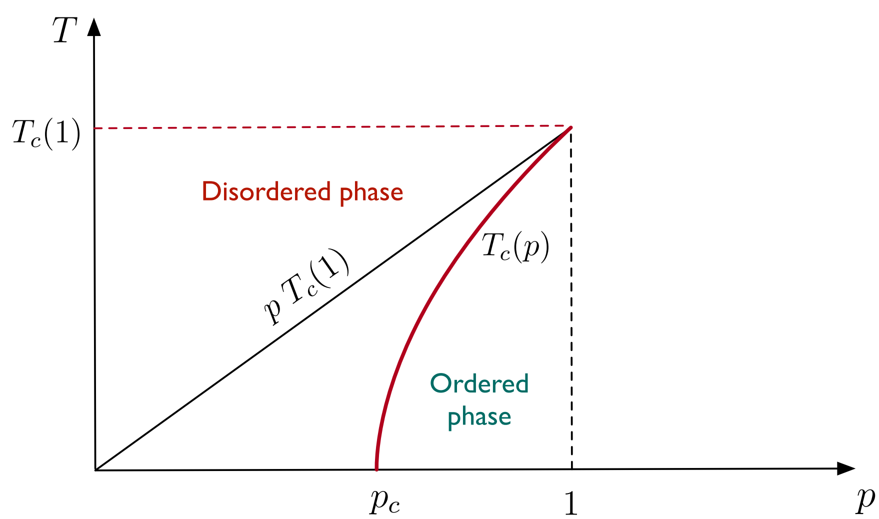


FIGURE 1 – Phase diagram of a diluted ferromagnet.

Clearly we recover : $T_c(1) = T_c^{\text{Ising}} = T_c^{\text{pure}}$. Furthermore,

1. R. B. Griffiths, Phys. Rev. Lett. **23**, 17 (1969) ;
A. J. Bray, Phys. Rev. Lett. **60**, 720 (1988) ;
M. Randeria, James P. Sethna *et al.*, Phys. Rev. Lett. **54**, 1321 (1985).

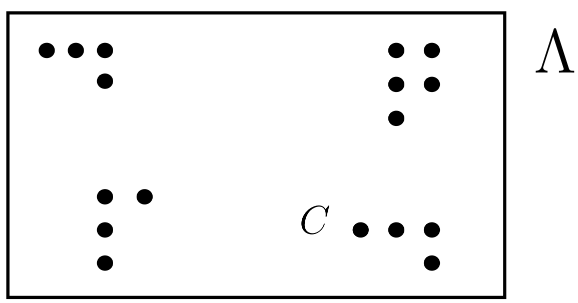
— If $p < p_c$ (probability of link percolation), $T_c(p) = 0$;
— $T_c(p) < pT_c(1)$;
— In the disordered phase with $T > T_c(p)$ and $T < T_c(1)$, Griffiths singularities can occur leading to a divergent high-temperature expansion (even without undergoing a true phase transition).

These features for the different phases are summarized in Fig. 1 below.

1) What should you expect for the probability of site occupancy p ? What are the observables that might depend on p ?

2) Considering the Hamiltonian in Eq. (1), how can you define site dilution and link dilution respectively ?

3) We aim to show that upon increasing the system size increases with $T < T_c(1)$ – but not too small – the imaginary axis in the complex plane of h contains a singularity at $h = 0$. However, the spontaneous magnetization does not display any jump (indeed the magnetization could be C^∞ in h at $h = 0$).
Let us consider site dilution, *i.e.* j is occupied only if $\tau_j = 1$, otherwise the site is empty. Then we call :
— $C \equiv$ arbitrary configuration of occupied sites in the lattice region Λ ;
— $|C| \equiv$ number of sites belonging to C ;
— $P_{C,\Lambda} \equiv$ probability of occurrence of C, namely probability of obtaining C by site percolation ;
— $M_\Lambda \equiv$ average magnetization per site in Λ ;
— $M_C \equiv$ average magnetization per site in C.



Given the definitions above, what is the average magnetization per site in Λ (see Fig. below) ?

4) If we introduce the weight $z = e^{-2\beta h}$, how would you express the average magnetization restricted to C in terms of the free energy f_C ? (Suggestion : use the thermodynamic relation between the order parameter and the free energy.)

5) Optional
What can you claim about the average magnetization in the domain ? Is it bounded ? Using the expressions for M_Λ and $\eta_a(\Lambda) \equiv \frac{1}{|\Lambda|} \sum_{C:\zeta_\alpha(C)=\zeta_a} P_{C,\Lambda}\, m_a(C)$ used before, analyze M_Λ as a function of z.

Exercise 2 : Ising model in a random magnetic field

In the following we shall consider a simple Ising model to show the equivalence – close to the critical point – between a spin system in d-dimensions in a random magnetic field and a spin system in $(d-2)$-dimensions without field.

1) How would you define the free energy of the model with Lagrangian $\mathcal{L}(x) = -\frac{1}{2}\phi(x)\Delta\phi(x)+V(\phi(x))$ and averaged over the random field ? For the computation, assume the field to be Gaussian distributed.

2) If we take advantage of a tree-like diagram approximation, what would the expression for $F[h]$ look like ?

3) If you consider again the tree-like approximation, how should you write the corresponding expression for the correlation $\langle\phi(x)\phi(0)\rangle_h$ and for $\langle\phi(x)\phi(0)\rangle$?

4) Comment the result about the last integral form. How would you solve the integral for the correlation function ?
Suggestion : use the variable change $h(x) = h^{'}(x) + \tilde{h}(x)$.

5) Optional point. Now we introduce the superspace defined by d dimensions x_i, which commute, and 2 dimensions θ, $\bar{\theta}$, which are Grassmann variables. We introduce then a **super-field** $\Phi(x,\theta,\bar{\theta}) = \phi(x) + \bar{\theta}c(x) + \bar{c}(x)\theta + \bar{\theta}\theta\lambda(x)$.

• Verify that the Lagragian operator satisfies the following relation

$$\int \tilde{L}(\phi, c, \bar{c}, \lambda)\; d^dx = \int \mathcal{S}_{\text{SUSY}}(\Phi)\; d^dx d\theta d\bar{\theta}\;, \tag{2}$$

with the action $\mathcal{S}_{\text{SUSY}}(\Phi) = -\frac{1}{2}\Phi\Delta_{\text{SS}}\Phi + V(\Phi)$ and the corresponding Laplacian $\Delta_{\text{SS}} = \Delta + \frac{\partial^2}{\partial\theta\partial\bar{\theta}}$.

• Check that the super-symmetric transformations are simply rotations in the aforementioned superspace leaving the metrics $x^2 + \theta\bar{\theta}$ invariant.

6) Optional – Imry and Ma argument.
Let us consider an Ising ferromagnet in a random magnetic field in d dimensions. It is defined by the following Hamiltonian

$$H = -\sum_{i,j} J_{ij}\sigma_i\sigma_j + \sum_i h_i\sigma_i \tag{3}$$

with $P(h_i) \sim e^{-\frac{h_i^2}{2\epsilon}}$ and $h_i\bar{h}_j = h^2\delta_{ij}$.
We assume that the system without external field develops a spontaneous magnetization. • Explain on what basis the long-range order should be stable with respect to the formation of domain walls.