Advanced Statistical Physics

TD4: Real space renormalization group

October 2021

The purpose of this set of problems is to learn how to implement the real space renormalization group ideas on simple physical systems. We will study first the percolation transition on a triangular lattice, then the Ising model on d-dimensional hyper-cubic lattices within the Migdal-Kadanoff approximation.

I. REAL-SPACE RENORMALIZATION GROUP APPROACH TO THE PERCOLATION TRANSITION

In statistical physics and mathematics, the percolation transition describes a geometric type of phase transition occurring when nodes (or links) are added to a given network. The probability of each node being occupied is independent of the other sites' occupancy, and is equal to p. Neighboring occupied sites form clusters. For an infinite size system, there is a critical value of p, called *percolation threshold* p_c, such that for $p < p_c$ the system does not have any clusters that span the entire system length, and for $p > p_c$ it does.

(A) One-dimensional chain: exact solution

1. Consider a $1d$ chain with sites occupied with probability p or empty with probability $1-p$. Define n_s to be the cluster number – the number of clusters containing s sites per lattice site. Find it in terms of p and s.
2. What is S, the average size of a finite cluster?
3. Calculate the correlation function $g(r)$ – the probability that a site, that is a distance r away from an occupied site, belongs to the same cluster. Rewrite it in terms of the correlation length as $g(r) = e^{-r/\xi}$.
4. What is the percolation threshold in this problem? What happens to S and ξ at p_c? The fact that the divergence of ξ at the percolation threshold can be described in general by a simple power law, $\xi \propto (p - p_c)^{-\nu}$.

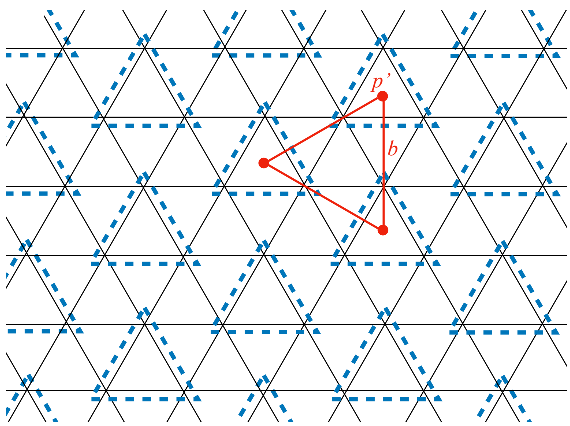
(B) One-dimensional chain: RG approach

Now we will use this exactly solvable model to demonstrate how real-space renormalization works. The idea is to replace a group of sites by a coarse-grained super-site, whose linear dimension is b, with $1 < b < \xi$. The super-site is said to be occupied if there is a cluster of the original sites that spans the length of the cell. Away from the critical point, the probability p' of the super-sites will be different from p, while at the critical point $p' = p = p_c$ since ξ is infinite. The new lattice will have a new lattice constant b, and a new correlation length ξ', measured in the units of b. Since this is an exact transformation, one has that $b\xi'(p') = \xi(p)$.

1. Show that

$$\frac{1}{\nu} = \frac{\log \frac{\mathrm{d}p'}{\mathrm{d}p}\Big|_{p_c}}{\log b}\,.$$

   This expression allows us to estimate the critical exponent ν via the renormalization group method.
2. Group the sites in our one-dimensional chain into cells of b sites. Find p'.
3. Values of p that stay constant under renormalization are called *fixed points*. What are the fixed points in this problem?
4. Calculate ν and compare it to the earlier result.



Two-dimensional triangular lattice. Sites belonging to the same triangular plaquette (dashed blue) are grouped together.

(C) Two-dimensional triangular lattice: RG approach

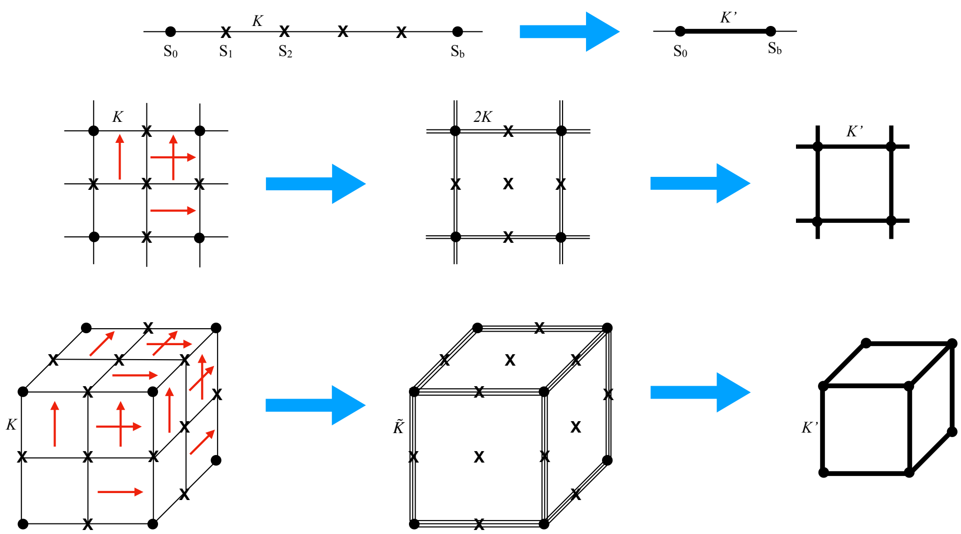
1. Consider now a two-dimensional triangular lattice. The lattice is divided in triangular plaquettes, as shown in the figure. Group together the three sites belonging to each plaquette and find p'.
2. What are the fixed points of the recursion relation? Which of them is stable and which of them is unstable?
3. Show that the lattice made by the triangular plaquettes is still a triangular lattice (but rotated by $\pi/6$ with respect to the original lattice) with a renormalized lattice constant. Find b and ν.

II. RENORMALIZATIION OF THE ISING MODEL À LA MIGDAL-KADANOFF

We consider the Ising model on a d-dimensional hyper-cubic lattice with spins $S_i = \pm 1$, described by the Hamiltonian:

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} S_i S_j \,,$$

where $\langle i,j \rangle$ denotes the pairs of nearest neighbors on the lattice. We introduce the dimensionless coupling constant $t = \tanh(K)$ with $K = \beta J > 0$.



Top: Decimation of $b-1$ spins along a one-dimensional chain. Middle: Migdal-Kadanoff procedure on a two-dimensional square lattice for $b = 2$. Bottom: Migdal-Kadanoff procedure on a three-dimensional cubic lattice for $b = 2$. Bonds are moved along the directions of the arrows. Spins marked with circles are kept and spins marked with crosses are integrated out.

(A) Decimation rule in one dimension

We start by the case $d = 1$ in which the decimation can be carried out exactly. We study the renormalization procedure illustrated in the figure by which $b-1$ spins are integrated out (crosses) and the other spins are kept (circles). The thick bonds indicate the new coupling t' obtained upon decimation.

1. Show that $e^{KS_iS_{i+1}}$ can always be written as $A + BS_iS_{i+1}$, where the constants A and B depend on K and must be determined.
2. Consider three subsequent spins on the chain, S_1, S_2, and S_3. For S_1 and S_3 being fixed, integrate out S_2 and compute:

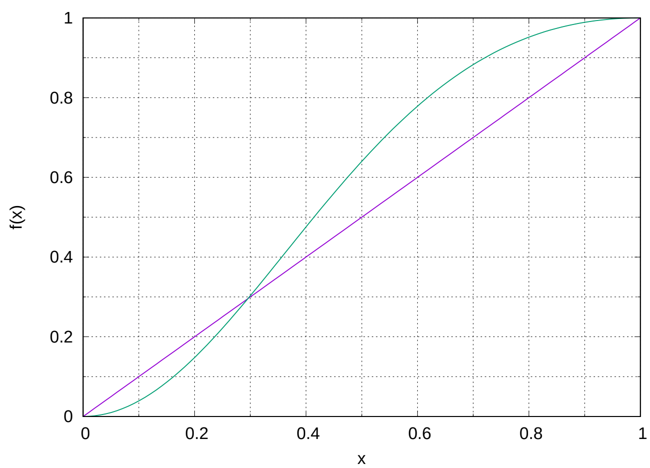
$$\sum_{S_2=\pm 1} e^{K(S_1S_2+S_2S_3)} \,.$$

3. Determine the recursion relation for the coupling constant when $b-1$ consecutive spins on the chain are integrated out.

(B) **The two-dimensonal model**

Unfortunately, a simple decimation in d larger than 1 cannot be performed exactly. To circumvent this problem approximations have to be made. We follow here an idea put forward by Migdal (1975) and Kadanoff (1976). We consider a two-step decimation procedure on a square lattice sketched in the figure. We start by the case $b = 2$. The approximation consists in *moving* the bonds that are not connected to the spins that are kept (circles). These spins are then connected by bonds of strength $2K$ instead of K.

1. Spins marked by crosses now are decimated. What is the new coupling constant t' after decimation?



Plot of the function $f(x) = 4x^2/(1+x^2)^2$ for $x \in (0,1)$.

2. Discuss the fixed points of the recursion relation (we can use here the fact that $\tanh(2x) = 2\tanh(x)/(1+\tanh^2(x))$). Using the figure above provide an approximate value for the fixed point K_c. The exact value, computed by Onsager, is $K_c^{\text{exact}} = \log(1+\sqrt{2})/2 \approx 0.44$.
3. The correlation length (measured in units of the lattice constant) of the model depends only on K. Imposing that ξ measured in units of the original lattice spacing is the same before and after the decimation, and imposing that in the vicinity of K_c the correlation length diverges as $\xi \propto (K - K_c)^{-\nu}$, compute the critical exponent ν within the Migdal-Kadanoff approximation.
4. We now generalize the procedure described above to any b. We move the bonds that are not connected to the spins that are kept along the x and y directions. The spins marked with a cross are then integrated out by one-dimensional decimation. What is the value of the new coupling constant?
5. We consider the analytic continuation limit in which an infinitesimal fraction of spins is renormalized at each step (and an infinitesimal fraction of bonds is moved). We denote $b = 1 + \epsilon$, with $\epsilon \ll 1$. Show that:

$$K' \approx K + \epsilon \left[K + f(K) \log \tanh(K) \right] ,$$

with $f(K)$ a function to be determined.

6. The exact value $K_c^{\text{exact}} = \log(1+\sqrt{2})/2 \approx 0.44$ is now a fixed point of the recursion relation. Compute the value of the critical exponent ν.

(C) **Generalizations to arbitrary dimensions**

We now consider the Ising model on the d-dimensional hyper-cubic lattice and consider the Migdal-Kadanoff decimation procedure for arbitrary b.

1. Compute N', the number of sites kept after the decimation, as a function of N and b.
2. Compute B', the number of bonds that survive after the bond moving step, as a function of N' and d.
3. Imposing that the total number of couplings is the same before and after the bond moving step, determine the value of the coupling constant $\tilde{K}$ (see figure) as a function of b and d.
4. Determine the recursion relation for any arbitrary value of b and d.
5. Discuss the stabiliy of the low-temperature ($K \to \infty$) fixed point and connect the result to the lower critical dimension of the model.