Advanced Statistical Physics

TD8: Quantum Ising model

December 2021

We consider the quantum Ising model in d dimensions made by N spins 1/2 interacting via ferromagnetic interactions in presence of a transverse field:

$$\hat{\mathcal{H}} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x \,,$$

where $\langle i,j \rangle$ denotes all pairs of nearest neighbor on a d-dimensional lattice and $\sigma_i^{x,y,z}$ are the Pauli matrices.

I. **Ground state properties**

(a) What is the energy of the ground state when $J = 0$? Is the ground state degenerate? Determine the expectation values of $\langle \sigma_x \rangle$ and $\langle \sigma_z \rangle$ in this ground state.
(b) What is the energy of the ground state when $\Gamma = 0$? Is the ground state degenerate? Determine the expectation values of $\langle \sigma_x \rangle$ and $\langle \sigma_z \rangle$ in this ground state.
(c) Let us now restore both couplings J and Γ. Plot the qualitative behavior of $\langle \sigma_x \rangle$ and $\langle \sigma_z \rangle$ as a function of Γ for $J = 1$.

II. **Mean-field approach**

(a) In the mean-field approach one replaces the instantaneous fluctuating fields by their average values. Consider a site i of the lattice and its neighbors $j = 1, \ldots, 2d$. We introduce the average magnetization per spin:

$$m = \frac{1}{N} \sum_i \langle \sigma_i^z \rangle \,.$$

Replace the operators σ_j^z by their expectation values, and write the mean field Hamiltonian as:

$$\hat{\mathcal{H}}_{\text{mf}} = -\sum_i \vec{h}_{\text{mf}} \cdot \vec{\sigma}_i \,.$$

Give the expression of the effective field $\vec{h}_{\text{mf}}$ in terms of d, J, Γ, and m.
(b) What is the energy of the ground state of $\hat{\mathcal{H}}_{\text{mf}}$? Determine the expectation values of $\langle \sigma_x \rangle$ and $\langle \sigma_z \rangle$ in the ground state and find the self-consistent equation for m.
(c) Show that there exist a critical value of Γ above which the only solution of this equation is $m = 0$.

(d) Find the solution of the self-consistent equation for m below Γ_c.

(e) We now investigate the effect of thermal fluctuations. Find the self-consistent equation for m at finite T and express it in terms of m, Γ, Γ_c, and β only.

(f) Find the equation that gives the critical value of the field $\Gamma_c(\beta)$ above which the only solution of the self-consistent equation for m is $m = 0$ as a function of Γ_c and β. Determine the values of $\Gamma_c(\beta = 0)$ and $\Gamma_c(\beta \to \infty)$ and draw the mean-field phase diagram of the model.