
Out of Equilibrium Dynamics of Complex Systems

Leticia F. Cugliandolo

Sorbonne Universités, Université Pierre et Marie Curie
Laboratoire de Physique Théorique et Hautes Energies
Institut Universitaire de France

`leticia@lpthe.jussieu.fr`

`www.lpthe.jussieu.fr/~leticia/seminars`

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Plan of Lecture

1. Langevin equation

(derivation, time-scales)

2. Stochastic calculus

(discretisation, chain-rule, Fokker-Planck, drift-force)

3. Generating functional formalism

(Onsager-Machlup, Martin-Siggia-Rose)

4. Time-reversal symmetry

(fluctuation-dissipation theorem, fluctuation theorems)

Formalism

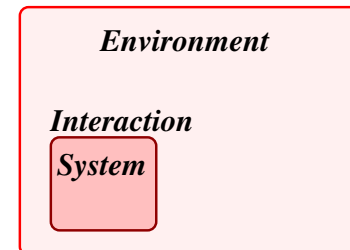
Dissipative systems

Aim

Interest in describing the **statics** and **dynamics** of a **classical or quantum physical system** coupled to a **classical or quantum environment**.

The Hamiltonian of the ensemble is

$$H = H_{syst} + H_{env} + H_{int}$$



The dynamics of all variables are given by **Newton** or **Heisenberg** rules, depending on the variables being classical or quantum.

The total energy is conserved, $E = \text{ct}$, but each contribution is not, in particular, $E_{syst} \neq \text{ct}$, and we'll take $E_{syst} \ll E_{env}$.

Reduced system

Model the environment and the interaction

E.g., an ensemble of harmonic oscillators and a linear in q_α and non-linear in x , via the function $\mathcal{V}(x)$, coupling :

$$H_{env} + H_{int} = \sum_{\alpha=1}^{\mathcal{N}} \left[\frac{p_\alpha^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} q_\alpha^2 \right] + \sum_{\alpha=1}^{\mathcal{N}} c_\alpha q_\alpha \mathcal{V}(x)$$

Equilibrium. Imagine the whole system in contact with a bath at inverse temperature β . Compute the reduced **classical** partition function or **quantum** density matrix by tracing away the bath degrees of freedom.

Dynamics. **Classically** (coupled Newton equations) and **quantum** (easier in a path-integral formalism) to get rid of the bath variables.

In all cases one can integrate out the oscillator variables as they appear only quadratically.

Reduced system

Statistics of a classical system

Imagine the coupled system in canonical equilibrium with a megabath

$$\mathcal{Z}_{syst + env} = \sum_{env, syst} e^{-\beta H}$$

Integrating out the environmental (oscillator) variables

$$\mathcal{Z}_{syst}^{red} = \sum_{syst} e^{-\beta \left(H_{syst} - \frac{1}{2} \sum_a \frac{c_a^2}{m_a \omega_a^2} [\mathcal{V}(x)]^2 \right)} \neq \mathcal{Z}_{syst} = \sum_{syst} e^{-\beta H_{syst}}$$

One possibility : assume weak interactions and drop the new term.

Trick : add $H_{counter}$ to the initial coupled Hamiltonian, and choose it in such a way to cancel the quadratic term in $\mathcal{V}(x)$ to recover

$$\mathcal{Z}_{syst}^{red} = \mathcal{Z}_{syst}$$

i.e., the partition function of the system of interest.

Reduced system

Dynamics of a classical system : general Langevin equations

The system, p, x , coupled to an **equilibrium environment** evolves according to the multiplicative noise non-Markov **Langevin equation**

Inertia

friction

$$\underbrace{m\ddot{x}(t)} + \mathcal{V}'(x(t)) \overbrace{\int_{t_0}^{\infty} dt' \gamma(t-t') \dot{x}(t') \mathcal{V}'(x(t'))} = \underbrace{-\frac{\delta V(x)}{\delta x(t)}}_{\text{deterministic force}} + \mathcal{V}'(x(t)) \underbrace{\xi(t)}_{\text{noise}}$$

deterministic force

noise

The friction kernel is $\gamma(t-t') = \Gamma(t-t')\theta(t-t')$

The **noise** has zero mean and correlation $\langle \xi(t)\xi(t') \rangle = k_B T \Gamma(t-t')$ with

T the temperature of the bath and k_B the Boltzmann constant.

Reduced system

Dynamics of a classical system : general Langevin equations

The system, p, x , coupled to an **equilibrium environment** evolves according to the multiplicative noise non-Markov **Langevin equation**

Inertia

friction

$$\underbrace{m\ddot{x}(t)} + \mathcal{V}'(x(t)) \overbrace{\int_{t_0}^{\infty} dt' \gamma(t-t') \dot{x}(t') \mathcal{V}'(x(t'))} =$$
$$\underbrace{-\frac{\delta V(x)}{\delta x(t)}} + \mathcal{V}'(x(t)) \underbrace{\xi(t)}$$

deterministic force

noise

Important Noise arises from lack of knowledge on bath ; noise can be multiplicative ; memory kernel generated ; equilibrium assumption on bath variables implies detailed balance between friction and noise

Additive white noise

Separation of time-scales & special coupling

In classical systems one usually takes a bath kernel with the **smallest relaxation time**, $t_{env} \ll t_{all}$ other time scales.

The bath is approximated by the white form $\Gamma(t - t') = 2\gamma_0\delta(t - t')$

Moreover, one assumes the **coupling is bi-linear**, $H_{int} = \sum_a c_a q_a x$.

The Langevin equation becomes

$$m\ddot{x}(t) + \gamma_0\dot{x}(t) = -\frac{\delta V(x)}{\delta x(t)} + \xi(t)$$

with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2k_B T \gamma_0 \delta(t - t')$.

Separation of time-scales

Velocities and coordinates

For $t \gg \tau_v = m/\gamma$ one expects the velocities to equilibrate to the

Maxwell distribution $P(\{\vec{v}\}) = \prod_i P(\vec{v}_i) \propto \prod_i e^{-\beta m v_i^2 / 2}$

In this limit, one can drop $m\dot{v}_i^a$ and work with the

overdamped equation $\gamma \dot{r}_i^a = -\frac{\delta V(\{\vec{r}_i\})}{\delta r_i^a} + \xi_i^a.$

The positions can have highly **non-trivial dynamics**, see **examples**.

Message : be very careful when trying to prove equilibration.

Different variables could behave very differently.

Stochastic calculus

Two ways of writing the multiplicative noise equation

The physical eq. that comes from integrating away the bath (oscillators)

$$(\mathcal{V}'[x(t)])^2 d_t x(t) = F[x(t)] + \mathcal{V}'[x(t)] \xi(t)$$

and the equation usually found in the mathematics literature

$$d_t x(t) = f[x(t)] + g[x(t)] \xi(t)$$

are equivalent after identification

$$g[x(t)] = \frac{1}{\mathcal{V}'[x(t)]}$$

$$f[x(t)] = \frac{1}{(\mathcal{V}'[x(t)])^2} F(x(t)) = (g[x(t)])^2 F[x(t)]$$

Stochastic calculus

Discretization prescriptions

$$d_t x(t) = f[x(t)] + g[x(t)] \xi(t)$$

with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(t') \rangle = 2D \delta(t - t')$ means

$$x(t_{k+1}) = x(t_k) + f[\bar{x}(t_k)] dt + g[\bar{x}(t_k)] \xi(t_k) dt$$

with

$$\bar{x}(t_k) = \alpha x(t_{k+1}) + (1 - \alpha)x(t_k)$$

and $0 \leq \alpha \leq 1$. Particular cases are $\alpha = 0$ Itô; $\alpha = 1/2$ Stratonovich.

Stratonovich 67, Gardiner 96, Øksendal 00, van Kampen 07

Stochastic calculus

Orders of magnitude & different stochastic processes

$\xi_k \equiv \xi(t_k) = \mathcal{O}(dt^{-1/2})$ because of the Dirac-delta correlations of a **white bath**

$dx \equiv x(t_{k+1}) - x(t_k) = \mathcal{O}(dt^{1/2})$ Variable increment

What is the difference between the two terms in the right-hand-side of the Langevin eq. when they are evaluated using different discretisation schemes, that is to say different α ?

$f[\bar{x}_\alpha(t_k)] - f[\bar{x}_{\bar{\alpha}}(t_k)] = \mathcal{O}(dt^{1/2})$ vanishes for $dt \rightarrow 0$

$g[\bar{x}_\alpha(t_k)]\xi(t_k) - g[\bar{x}_{\bar{\alpha}}(t_k)]\xi(t_k) = \mathcal{O}(dt^0)$ **remains finite** for $dt \rightarrow 0$

For multiplicative noise processes the discretisation matters:
different α yields different stochastic processes.

Stochastic calculus

Discretization prescriptions

$$d_t x(t) = f[x(t)] + g[x(t)] \xi(t)$$

with $\langle \xi(t) \rangle = 0$ and $\langle \xi(t) \xi(t') \rangle = 2D \delta(t - t')$ means

$$x(t_{k+1}) = x(t_k) + f[\bar{x}(t_k)] dt + g[\bar{x}(t_k)] \xi(t_k) dt$$

with

$$\bar{x}(t_k) = \alpha x(t_{k+1}) + (1 - \alpha)x(t_k)$$

The **chain rule** for the time-derivative is (just from Taylor expansion)

$$d_t Y(x) = d_t x d_x Y(x) + D(1 - 2\alpha) g^2(x) d_x^2 Y(x)$$

Only for $\alpha = 1/2$ (Stratonovich) one recovers the usual expression.

Not even for additive noise the chain rule is the usual one if $\alpha \neq 1/2$

Stochastic calculus

Fokker-Planck equation

The probability of x at time t_k

$$P(y, t_k) = \int dx_{k-1} T(y, t_k | x_{k-1}, t_{k-1}) P(x_{k-1}, t_{k-1})$$

with the condition (or transition) probability

$$\begin{aligned} T(y, t_k | x_{k-1}, t_{k-1}) &\equiv \langle \delta(y - x_{k-1} - dx) \rangle_{\xi(t_{k-1})} \\ &= \delta(y - x_{k-1} - \partial_y \{ \delta(y - x_{k-1}) \langle dx \rangle_{\xi(t_{k-1})} \} \\ &\quad + \frac{1}{2} \partial_y^2 \{ \delta(y - x_{k-1}) \langle (dx)^2 \rangle_{\xi(t_{k-1})} \} + \mathcal{O}(dx^3) \end{aligned}$$

From the Langevin equation,

$$\begin{aligned} \langle dx \rangle_{\xi(t_{k-1})} &= f[x(t_{k-1})] dt + 2D\alpha g[x(t_{k-1})] g'[x(t_{k-1})] dt \\ \langle (dx)^2 \rangle_{\xi(t_{k-1})} &= 2D g^2[x(t_{k-1})] dt \end{aligned}$$

Stochastic calculus

Fokker-Planck equations for different α

Call $y \mapsto x$, perform the integral over x_{k-1} and rearrange terms.

The Fokker-Planck equation

$$\begin{aligned}\partial_t P(x, t) = & -\partial_x [(f(x) + 2D\alpha g(x) d_x g(x)) P(x, t)] \\ & + D \partial_x^2 [g^2(x) P(x, t)]\end{aligned}$$

depends on α and g

Two processes will be statistically the same if

$$f + 2D \alpha g d_x g = f_{\text{drifted}} + 2D \bar{\alpha} g d_x g$$

Correspondence between (f, α) and $(f_{\text{drifted}}, \bar{\alpha})$

Stochastic calculus

Fokker-Planck & stationary measure

The Fokker-Planck equation

$$\begin{aligned} \partial_t P(x, t) = & -\partial_x [(f(x) + 2D\alpha g(x) d_x g(x)) P(x, t)] \\ & + D \partial_x^2 [g^2(x) P(x, t)] \end{aligned}$$

has the stationary measure

$$P_{\text{st}}(x) = Z^{-1} [g(x)]^{2(\alpha-1)} e^{\frac{1}{D} \int^x \frac{f(x')}{g^2(x')}} = Z^{-1} e^{-\frac{1}{D} U_{\text{eff}}(x)}$$

with $U_{\text{eff}}(x) = - \int^x \frac{f(x')}{g^2(x')} + 2D(1 - \alpha) \ln g(x)$

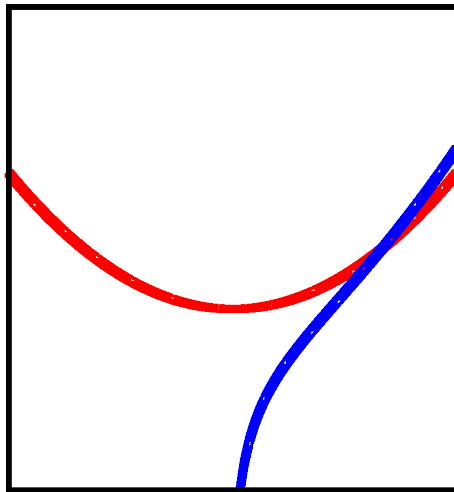
Remark : the potential $U_{\text{eff}}(x)$ depends upon α and $g(x)$

Noise induced phase transitions

Stochastic calculus

Fokker-Planck & stationary measure

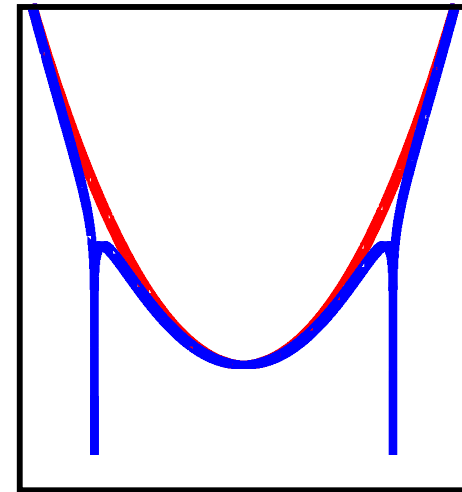
e.g. $f = -g^2 U$ and $U_{\text{eff}} = U + 2D(1 - \alpha) \ln g$



$$x^2 + 2D(1 - \alpha) \ln x$$

$$g(x) = x$$

$$U(x) = x^2$$



$$x^2 + 2D(1 - \alpha) \ln(1 - x^2)$$

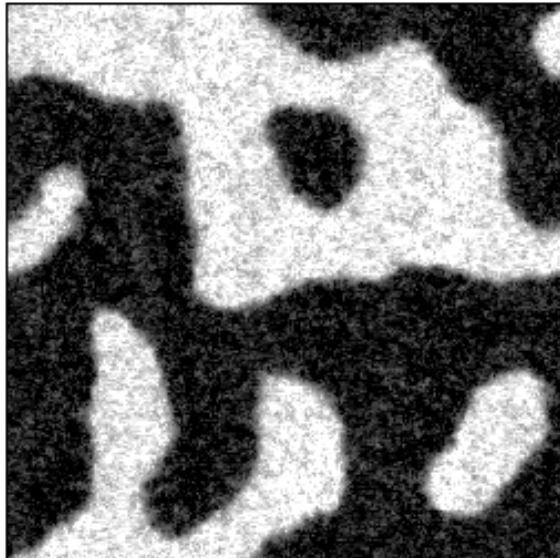
$$g(x) = (1 - x^2)$$

$$U(x) = x^2$$

Multiplicative noise

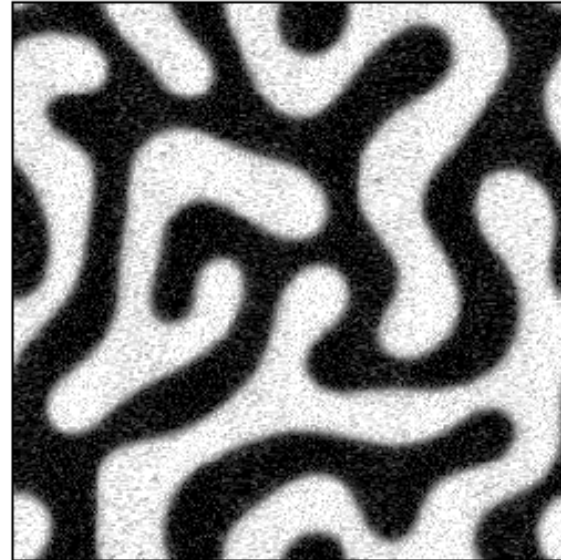
Noise induced coarsening for $\partial_t \phi = \nabla^2 \phi - \mu \phi - g \phi^3 - \phi \xi$

NCOP



$$\mathcal{R}(t, T) \simeq t^{1/2}$$

COP



$$\mathcal{R}(t, T) \simeq t^{1/3}$$

Stochastic calculus

Drift

The **Gibbs-Boltzmann equilibrium**

$$P_{\text{GB}}(x) = Z^{-1} e^{-\beta U(x)}$$

is approached if (recall the physical writing of the equation)

$$f(x) \mapsto \underbrace{-g^2(x) d_x U(x)}_{\text{Potential}} + \underbrace{2D(1 - \alpha)g(x) d_x g(x)}_{\text{drift}}$$

Potential

drift

The drift is also needed for the Stratonovich mid-point scheme

Important choice: if one wants the dynamics to approach thermal equilibrium independently of α and g the drift term has to be added.

Stochastic calculus

Fokker-Planck & stationary measure

The Fokker-Planck equation

$$\begin{aligned}\partial_t P(x, t) = & -\partial_x [(f(x) + 2D\alpha g(x) d_x g(x)) P(x, t)] \\ & + D \partial_x^2 [g^2(x) P(x, t)]\end{aligned}$$

for the drifted force $f(x) \mapsto -g^2(x) d_x U(x) + 2D(1 - \alpha)g(x) d_x g(x)$
becomes

$$\begin{aligned}\partial_t P(x, t) = & -\partial_x [(-g^2(x) d_x U(x) + 2Dg(x) d_x g(x)) P(x, t)] \\ & + D \partial_x^2 [g^2(x) P(x, t)]\end{aligned}$$

with the expected Gibbs-Boltzmann measure stationary measure

$$P_{\text{st}}(x) = Z^{-1} e^{-\frac{1}{D} U(x)}$$

independently of $g(x)$ and α

Methods

Dynamic generating functional

- Glassy models with and without disorder:

The "order parameter" is a composite object depending on two-times.

It's handy to use functional methods to write a dynamic generating functional as a path-integral

Onsager-Machlup & Martin-Siggia-Rose-Janssen-deDominicis formalisms

Similar to Feynman path-integral

The construction will follow **LFC & Lecomte**, "Rules of calculus in the path integral representation of white noise Langevin equations: the Onsager-Machlup approach", **arXiv :1704.03501**, **J. Phys. A 50, 345001 (2017)** where special care of discretisation effects was taken.

Generating functional

Onsager-Machlup representation

Langevin equation in generic discretization scheme $x_k = \mathbf{R}(x_k, x_{k-1}, \xi_{k-1})$

Definition of the infinitesimal transition probability

$$T(x_k, t_k | x_{k-1}, t_{k-1}) = \int d\xi_{k-1} P_{\text{noise}}(\xi_{k-1}) \delta(x_k - \mathbf{R}(x_k, x_{k-1}, \xi_{k-1}))$$

In order to integrate over $d\xi_{k-1}$ we have to transform the δ in which ξ_{k-1} appears within \mathbf{R} to $J^{-1} \delta(\xi_{k-1} - \dots)$. The Jacobian J is needed

Generalisation of $|f'(f^{-1}(a))| \delta(f(z) - a) = \delta(z - f^{-1}(a))$

$$J = \det_{kk'} \left[\frac{\delta\{[x_k - \mathbf{R}[x_k, x_{k-1}, \xi_{k-1}; \alpha]]\}}{\delta\xi_{k'}} \right]$$

$$T(x_k, t_k | x_{k-1}, t_{k-1}) = \int d\xi_{k-1} P_{\text{noise}}(\xi_{k-1}) J^{-1} \delta(\xi_{k-1} - \tilde{\mathbf{R}}(x_k, x_{k-1}))$$

One can forget the modulus since only one solution is expected

Generating functional

Onsager-Machlup representation

The transition probability now reads

$$T(x_k, t_k | x_{k-1}, t_{k-1}) = \sqrt{\frac{1}{4\pi k_B T dt}} \frac{1}{|g(\bar{x}_{k-1})|} e^{S_{\text{OM}}[x_k, x_{k-1}; \alpha]}$$

For $d_t x(t) = f(x(t)) + g(x(t))\xi(t)$, the **Onsager-Machlup** action is

$$S_{\text{OM}}[x_k, x_{k-1}; \alpha] \equiv \ln P_i(x_{-\mathcal{T}}) - \frac{dt}{4k_B T} \left[\frac{1}{g^2(\bar{x}_{k-1})} \left(\frac{(x_k - x_{k-1})}{dt} - f(\bar{x}_{k-1}) + \underbrace{2D\alpha g(\bar{x}_{k-1})g'(\bar{x}_{k-1})}_{\text{From the integration over the noise}} \right)^2 \underbrace{-\alpha f'(x_{k-1})}_{\text{Jacobian}} \right]$$

From the integration over the noise

Jacobian

Generating functional

Onsager-Machlup representation, continuous time notation

The transition probability now reads

$$T(x_k, t_k | x_{k-1}, t_{k-1}) = \sqrt{\frac{1}{4\pi k_B T dt}} \frac{1}{|g(\bar{x}_{k-1})|} e^{S_{\text{OM}}[\{x\}; \alpha]}$$

For $d_t x(t) = f(x(t)) + g(x(t))\xi(t)$, the **Onsager-Machlup** action is

$$S_{\text{OM}}[\{x\}; \alpha] \equiv \ln P_i(x_{-\tau})$$

$$-\frac{dt}{4k_B T} \left[\frac{1}{g_t^2} \underbrace{\left(d_t x_t - f_t + 2D\alpha g_t d_x g_t \right)^2}_{\text{From the integration over the noise}} \underbrace{-\alpha d_x f_t}_{\text{Jacobian}} \right]$$

From the integration over the noise

Jacobian

Generating functional

From Onsager-Machlup to Martin-Siggia-Rose

Use the Hubbard-Stratonovich (Gaussian integral) trick to go from the exponential of a square to the one of a linear term

$$e^{-\frac{1}{2}y^2} = \int d\hat{x} e^{\pm i\hat{x}y + \frac{1}{2}(i\hat{x})^2}$$

that with the parameters in the action and a convenient choice of sign

$$e^{-\frac{1}{2(2k_B T)}y^2} = \int d\hat{x} e^{-i\hat{x}y + \frac{1}{2}(2k_B T)(i\hat{x})^2}$$

In our case $y_t = (d_t x_t - f_t + 2D\alpha g_t d_x g_t)/g_t$

Further change of variables $\hat{x}_t \mapsto g_t \hat{x}_t$

Generating functional

MSR path-integral representation

The “joint probability distribution of the trajectory” reads

$$P(\{x\}; \alpha) = \int \mathcal{D}\{\hat{x}\} \prod_{k=0}^{N-1} (4\pi k_B T dt)^{-1/2} e^{S_{\text{MSR}}[\{x\}, \{\hat{x}\}; \alpha]}$$

and the **Martin-Siggia-Rose-Janssen 79** action is

$$S_{\text{MSR}}[\{x\}, \{\hat{x}\}; \alpha] \equiv \ln P_i(x_{-\tau}) + \int \left[-i\hat{x}_t \underbrace{(\underbrace{d_t x_t}_{\text{proportional to } \gamma_0 \text{ (not written)}} - f_t + \underbrace{2D\alpha g_t d_x g_t}_{\text{proportional to } \gamma_0 \text{ (not written)}})}_{\text{proportional to } \gamma_0 \text{ (not written)}} + \underbrace{D(i\hat{x}_t)^2 g_t^2}_{\text{Jacobian}} - \alpha d_x f_t \right]$$

Observable averages can now be calculated as

$$\langle A(x_t, i\hat{x}_{t'}) \rangle = \int \mathcal{D}\{x\} \mathcal{D}\{\hat{x}\} P_{\text{MSR}}(\{x\}, \{\hat{x}\}; \alpha) A(x_t, i\hat{x}_{t'})$$

Generating functional

Path-integral representation

For the drifted force $f_t = -g^2 d_x V_t + 2D(1 - \alpha)g_t d_x g_t$

$$\begin{aligned} S_{\text{MSR}}(\{x\}, \{i\hat{x}\}; \lambda, \alpha) &\equiv \ln P_i(x_{-\mathcal{T}}, \lambda_{-\mathcal{T}}) \\ &+ \int \left[-i\hat{x}_t (d_t x_t + g_t^2 d_x V_t - 2D(1 - 2\alpha)g_t d_x g_t) \right. \\ &\quad \left. + D(i\hat{x}_t)^2 g_t^2 - \alpha d_x f_t \right] \end{aligned}$$

Remark: The action depends on α and g .

Observable averages can now be calculated as

$$\langle A(x_t, i\hat{x}_{t'}) \rangle = \int \mathcal{D}\{x\} \mathcal{D}\{\hat{x}\} P_{\text{MSR}}(\{x\}, \{i\hat{x}\}; \alpha, \lambda) A(x_t, i\hat{x}_{t'})$$

and **do not depend** on α

Stochastic calculus

Path-integral representation for additive noise

For $g = 1$ and the force $f_t = -d_x V_t$ the action is

$$S_{\text{MSR}}[\{x\}, \{i\hat{x}\}; \alpha] \equiv \ln P_i(x_{-\tau}) \\ + \int [-i\hat{x}_t(d_t x_t + d_x V_t) + D(i\hat{x}_t)^2 + \alpha d_x^2 V_t]$$

Observable averages can now be calculated as

$$\langle A(x_t, i\hat{x}_{t'}) \rangle = \int \mathcal{D}\{x\} \mathcal{D}\{\hat{x}\} P_{\text{MSR}}(\{x\}, \{i\hat{x}\}; \lambda, \alpha) A(x_t, i\hat{x}_{t'})$$

and do not depend on α

Methods

Dynamical symmetry & exact results

The functional path-integral formalism allows one to obtain exact identities (fluctuation-dissipation theorem, fluctuation theorems) as consequences of a dynamic symmetry and its symmetry breaking.

Details in :

“Symmetries of generating functionals of Langevin processes with colored multiplicative noise” **Aron, Biroli & LFC, J. Stat. Mech. P11018 (2010)** ; “Dynamical symmetries of Markov processes with multiplicative white noise”, **Aron, Barci, LFC, González Arenas & Lozano, J. Stat. Mech. 053207 (2016)**

Possible (though not easy) to extend to quantum system.

“(Non) equilibrium dynamics : a (broken) symmetry of the Keldysh generating functional”
Aron, Biroli & LFC, arXiv:1705.10800

Symmetry

Transformations in the path-integral representation

Let us define

$$d_t^{(\alpha)} x_t \equiv dx_t - 2D(1 - 2\alpha)g_t dx g_t$$

and group two terms in the action due to the coupling to the bath

$$S_{\text{diss}}[x, i\hat{x}] = \int -i\hat{x}_t [d_t^{(\alpha)} x_t - Di\hat{x}_t g_t^2]$$

This expression suggests to use the generalized transformation on the time-dependent variables $\{x_t, i\hat{x}_t\}$

$$\mathcal{T}_c = \begin{cases} x_t & \mapsto x_{-t} , \\ i\hat{x}_t & \mapsto i\hat{x}_{-t} + D^{-1}g_{-t}^{-2}d_t^{(\alpha)} x_{-t} , \end{cases}$$

and $\alpha \mapsto 1 - \alpha$

Remember $D = \beta^{-1} = k_B T$

Symmetry

Transformations in the path-integral representation

For initial conditions drawn from $P_i(x) = Z^{-1}e^{-\beta V(x)}$ and

$f(x) = -g^2(x)d_x V(x) + 2D(1 - \alpha)g(x)d_x g(x)$ one proves

$$S_{\text{det+jac}}[\mathcal{T}_c i\hat{x}, \mathcal{T}_c x; \mathcal{T}_c \alpha] = S_{\text{det+jac}}[i\hat{x}, x; \alpha]$$

that implies

$$P[\mathcal{T}_c i\hat{x}, \mathcal{T}_c x; \mathcal{T}_c \alpha] = P[i\hat{x}, x; \alpha]$$

Note that we have to use the non-trivial chain rule.

Moreover, the transformation leaves the integral measure invariant (no Jacobian) and the interval of integration as well.

Symmetry

Consequences of the transformation: FDT

From this result we can prove exact equilibrium relations such as the **fluctuation-dissipation theorem** linking the (causal) linear response to a field that changes the force as $f_t \mapsto f_t + h_t$

$$R(t, t') = \left. \frac{\delta \langle x(t) \rangle}{\delta h(t')} \right|_{h=0} \propto \theta(t - t')$$

and the correlation function in a model independent way :

$$R(t, t') - R(-t, -t') = \beta \partial_{t'} \langle x(-t)x(-t') \rangle$$

that for a stationary problem (in equilibrium) becomes

$$R(t - t') - R(t' - t) = \beta \partial_{t'} C(t' - t) = \beta \partial_{t'} C(t - t')$$

Broken symmetry

Relation under "any" transformation

$$\frac{P(\mathcal{T}_c x, \mathcal{T}_c \hat{x}; \mathcal{T}_c \alpha, \bar{\lambda})}{P(x, \hat{x}; \alpha, \lambda)} = e^{\Delta S(x, \hat{x}; \alpha, \lambda)}$$

with ΔS the variation of the full action (and measure)

$\mathcal{T}_c x$ and $\mathcal{T}_c \hat{x}$ are transformed trajectories,

$\bar{\lambda}$ the transformed parameter in the potential,

$\mathcal{T}_c \alpha$ a different discretisation parameter ;

and from here obtain relations between observables by averaging this relation : equilibrium fluctuation dissipation ($\Delta S = 0$), or out of equilibrium theorems ($\Delta S \neq 0$).
e.g., **Jarzynsky 97, Crooks 00 & many others**

Broken Symmetry

Consequences of the transformation: Fluctuation-theorems

For initial conditions drawn from $P_i(x) = Z^{-1}e^{-\beta V(x)}$ and

$$f(x, \lambda_t) = -g^2(x)\partial_x V(x, \lambda_t) + 2D(1 - \alpha)g(x)d_x g(x)$$

one

proves

$$\frac{P[\mathcal{T}_c i\hat{x}, \mathcal{T}_c x; \mathcal{T}_c \alpha, \bar{\lambda}_t = \lambda_{-t}]}{P[i\hat{x}, x; \alpha, \lambda_t]} = e^{\beta W - \beta \Delta F}$$

with

$$W = \int dt d_t \lambda_t \partial_\lambda V(x, \lambda)$$
$$\Delta F = \ln Z(\lambda_{\mathcal{T}}) - \ln Z(\lambda_{-\mathcal{T}})$$

the work, and free-energy difference between initial and fictitious final states.

Exact out of equilibrium relations such as the **Jarzynski relation** follow

$$\langle e^{-\beta W} \rangle = e^{-\beta \Delta F}$$

Coloured noise

Langevin equation & generating functional

The generic **Langevin equation** for a particle in $1d$ is

$$m\ddot{x}(t) + \mathcal{V}'[x(t)] \int_{-\mathcal{T}}^t dt' \Gamma(t-t') \mathcal{V}'[x(t')] \dot{x}(t') = F(t) + \xi(t) M'[x(t)]$$

with the coloured noise

$$\langle \xi(t) \xi(t') \rangle = k_B T \Gamma(t-t')$$

The dynamic generating functional is a path-integral

$$\mathcal{Z}_{dyn}[\eta] = \int dx_{-\mathcal{T}} d\dot{x}_{-\mathcal{T}} \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S[x, \hat{x}; \eta]}$$

with $i\hat{x}(t)$ the 'response' variable.

$x_{-\mathcal{T}}$ and $\dot{x}_{-\mathcal{T}}$ are the initial conditions at time $-\mathcal{T}$.

Martin-Siggia-Rose-Jenssen-deDominicis formalism

**Why care about
multiplicative noise ?**

Magnetisation precession

Bloch equation

Evolution of the time-dependent *3d* magnetisation density per unit volume, $\mathbf{M} = (M_x, M_y, M_z)$, with constant modulus $M_s = |\mathbf{M}|$

$$\mathbf{d}_t \mathbf{M} = -\bar{\mu} \mathbf{M} \wedge \mathbf{H}_{\text{eff}}$$

$\bar{\mu} \equiv \gamma \mu_0$ is the product of $\gamma = \mu_B g / \hbar$, the gyromagnetic ratio, and μ_0 , the vacuum permeability constant (μ_B Bohr's magneton and g Lande's g -factor)

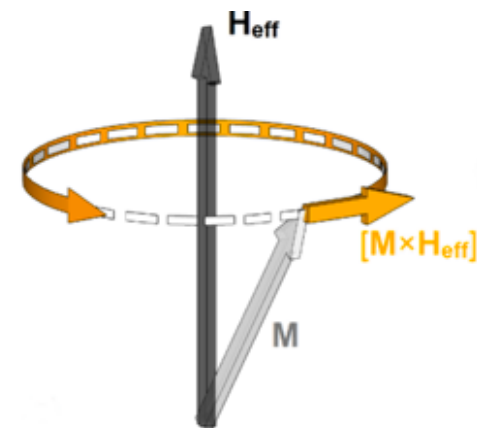
For the initial condition $\mathbf{M}(t_i) = \mathbf{M}_i$

the magnetisation **precesses** around \mathbf{H}_{eff}

with $2\mathbf{M} \cdot \mathbf{d}_t \mathbf{M} = \mathbf{d}_t |\mathbf{M}|^2 = 0$

and $\mathbf{d}_t (\mathbf{M} \cdot \mathbf{H}_{\text{eff}}) = 0$ (if $\mathbf{H}_{\text{eff}} = c\mathbf{t}$)

Bloch 32



Dissipative effects

Landau-Lifshitz & Gilbert equations

$$d_t \mathbf{M} = -\frac{\bar{\mu}}{1 + \gamma_0^2 \bar{\mu}^2} \mathbf{M} \wedge \left[\mathbf{H}_{\text{eff}} + \frac{\gamma_0 \bar{\mu}}{M_s} (\mathbf{M} \wedge \mathbf{H}_{\text{eff}}) \right]$$

Landau &
Lifshitz 35

$$d_t \mathbf{M} = -\bar{\mu} \mathbf{M} \wedge \left(\mathbf{H}_{\text{eff}} - \frac{\gamma_0}{M_s} d_t \mathbf{M} \right)$$

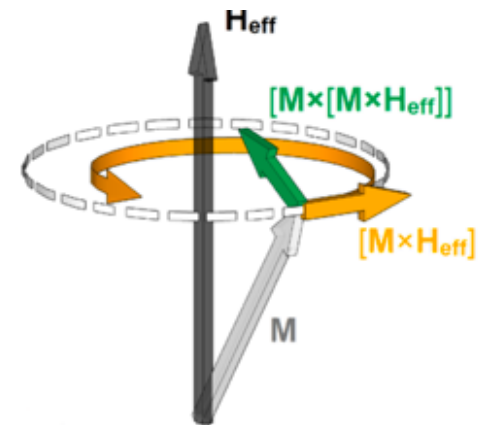
Gilbert 55

2nd terms in RHS: dissipative mechanisms slow

down the precession and push \mathbf{M} towards \mathbf{H}_{eff}

with $2\mathbf{M} \cdot d_t \mathbf{M} = d_t |\mathbf{M}|^2 = 0$

and $d_t (\mathbf{M} \cdot \mathbf{H}_{\text{eff}}) > 0$



Thermal fluctuations

À la Langevin in Gilbert's formulation

$$d_t \mathbf{M} = -\bar{\mu} \mathbf{M} \wedge \left(\mathbf{H}_{\text{eff}} + \mathbf{H} - \frac{\gamma_0}{M_s} d_t \mathbf{M} \right)$$

\mathbf{H} is a white random noise, with zero mean $\langle H_i(t) \rangle = 0$ and correlations

$$\langle H_i(t) H_j(t') \rangle = 2D \delta_{ij} \delta(t - t')$$

The (diffusion) parameter D is proportional to $k_B T$

Brown 63

The noise \mathbf{H} **multiplies** the magnetic moment \mathbf{M} and one cannot always write $2\mathbf{M} \cdot d_t \mathbf{M} = d_t M^2$ (only if the Stratonovich calculus is used)

This is the **Markov stochastic Landau-Lifshitz-Gilbert-Brown (sLLGB)** multiplicative white noise stochastic differential equation.

Subtleties of Markov multiplicative noise processes are now posed.

Thermal fluctuations

À la Langevin in Gilbert's formulation

$$d_t \mathbf{M} = -\bar{\mu} \mathbf{M} \wedge \left(\mathbf{H}_{\text{eff}} + \mathbf{H} - \frac{\gamma_0}{M_s} d_t \mathbf{M} \right) - \frac{2D(1-2\alpha)\bar{\mu}^2}{1 + \bar{\mu}^2 \gamma_0^2} \mathbf{M}$$

\mathbf{H} is a white random noise, with correlations $\langle H_i(t) H_j(t') \rangle = 2D \delta_{ij} \delta(t-t')$

The (diffusion) parameter D is proportional to $k_B T$

Brown 63

The modulus of the magnetic moment is now conserved $d_t \mathbf{M}^2 = 0$ for all α

One also proves that the dynamics approaches the asymptotic Gibbs-Boltzmann distribution

$$P_{\text{GB}}(\mathbf{M}) \propto e^{-\frac{1}{D} \mathbf{M} \cdot \mathbf{H}_{\text{eff}}}$$