Artificial spin-ice & vertex models

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Why this topic ?

- Materials of possible technological importance
- that pose challenging

problems in experimental physics,

questions of fundamental interest,

and need(ed) the development of theoretical physics/mathematics tools.

Nice interplay between **theory & experiment**

Artificial spin-ice

Metamaterials: designed in the laboratory.

Artificial spin-ice

Metamaterials

Arrays of nano/micro-scale magnets

single domain magnetic islands

placed at the edges of a tiling or

the edges of a planar graph

Parameters specified by design

Artificial spin-ice

Metamaterials

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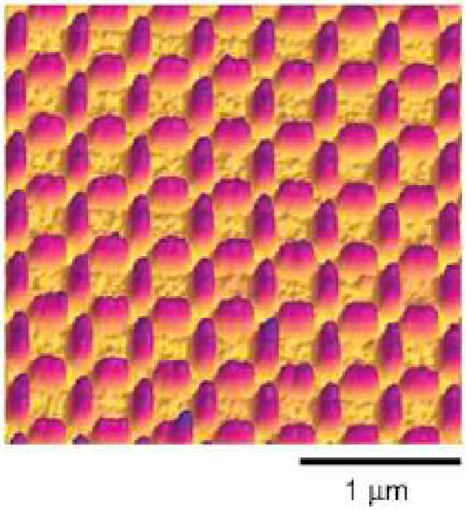
Parameters specified by design

Image: atomic foce microscopy



Square lattice

Wang et al 06

Artificial spin-ice

Metamaterials

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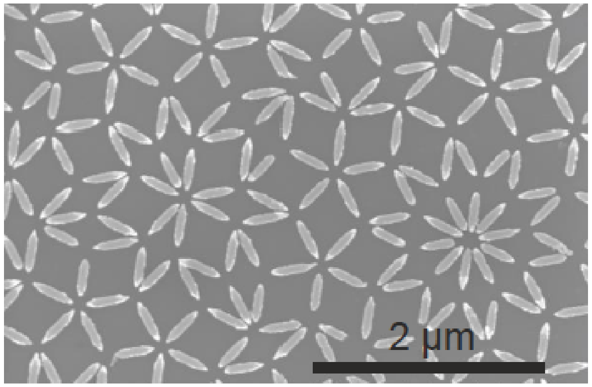


Penrose tiling

Marrows et al 14

Artificial spin-ice

Metamaterials

Arrays of micro/nano-scale magnets

single domain magnetic islands

placed at the edges of a tiling

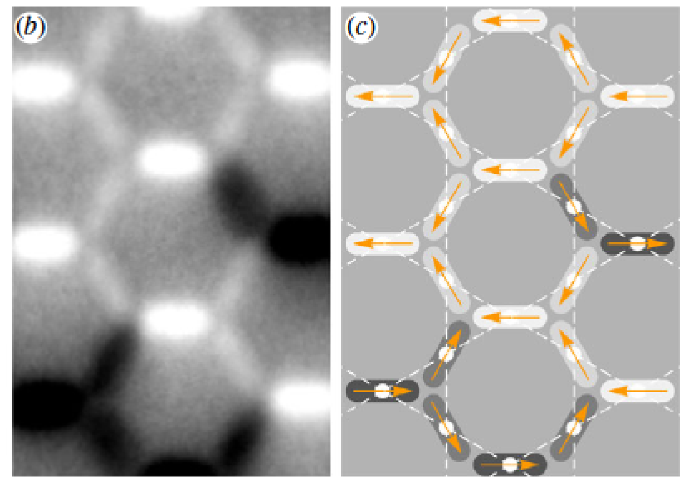
the edges of a planar graph

Easy axis magnets ⇒ Ising spins

Construction ⇒ along the edges

Photoelectron emission microscopy

(More about fabrication later)



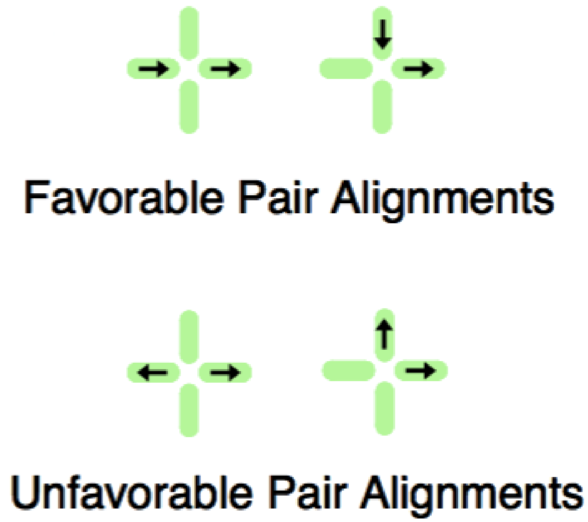
Honeycomb lattice

Hügli et al 15

Artificial spin-ice

Dipolar interactions

$$E_{12} \propto \vec{s}_1 \cdot \vec{s}_2 - 3\,\frac{(\vec{s}_1 \cdot \vec{r}_{12})(\vec{s}_2 \cdot \vec{r}_{12})}{r_{12}^2}$$



$E < 0$ favorable

$E > 0$ unfavorable

The islands meet at each vertex ; local dipolar interactions are frustrated ; that is to say, they cannot be satisfied simultaneously.

It is not possible to find a configuration of the spins that join at a vertex that minimises all pair contributions to the total energy.

Artificial spin-ice

A simpler modelling

Metamaterials

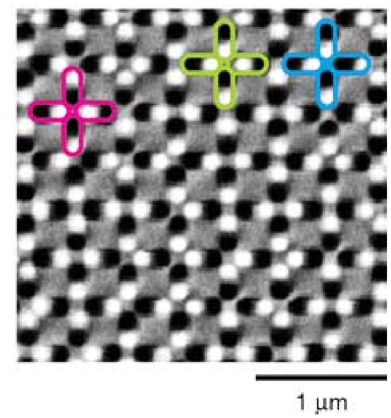
Arrays of nanoscale Ising magnets

single domain magnetic islands

placed at the edges of a tiling or

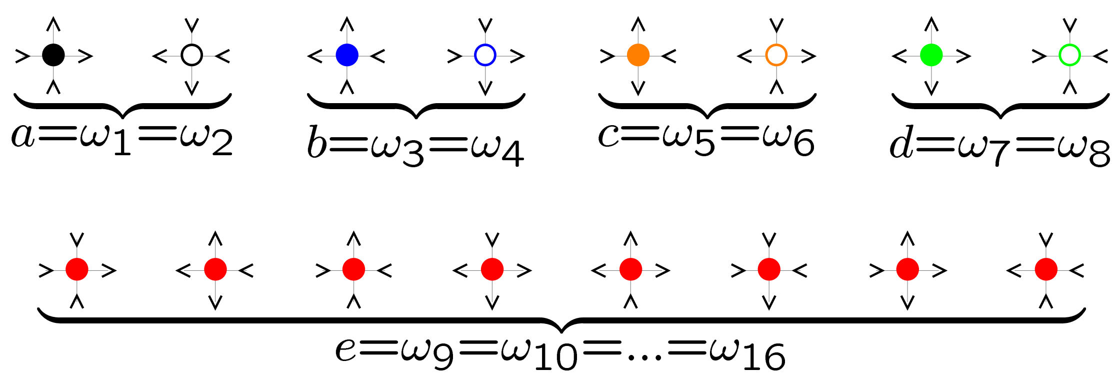
the edges of a square lattice

Parameters specified by design



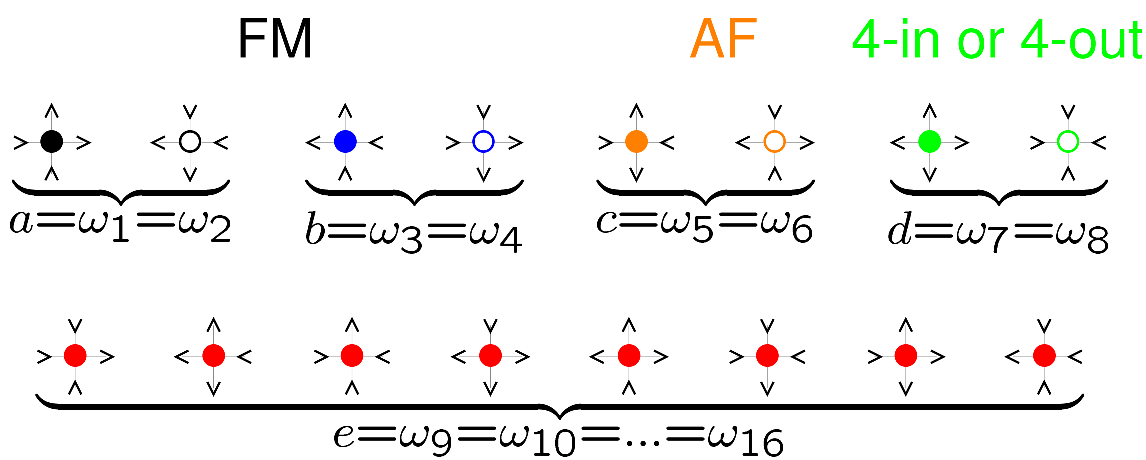
Magnetic force microscopy

Local approx: $2d$ vertex model with experimentally relevant parameters



The $2d$ 16 vertex model

with 3-in 1-out vertices: non-integrable system



3-in 1-out or 3-out 1-in

(Un-normalized) statistical weight of a vertex $\omega_k = e^{-\beta \epsilon_k}$

In the models $\boxed{a, b, c, d, e}$ are free parameters (usually, c is the scale)

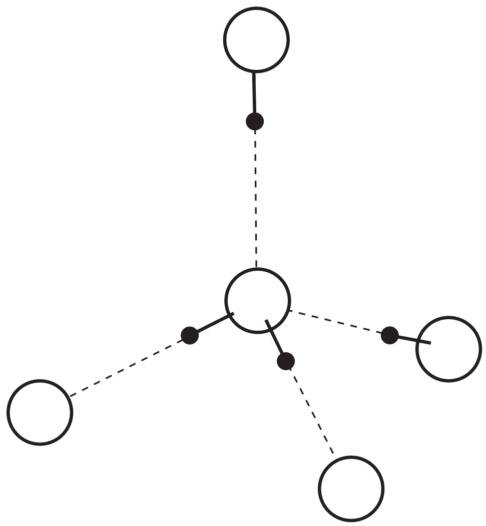
In the experiments ϵ_k depend on the sample and,

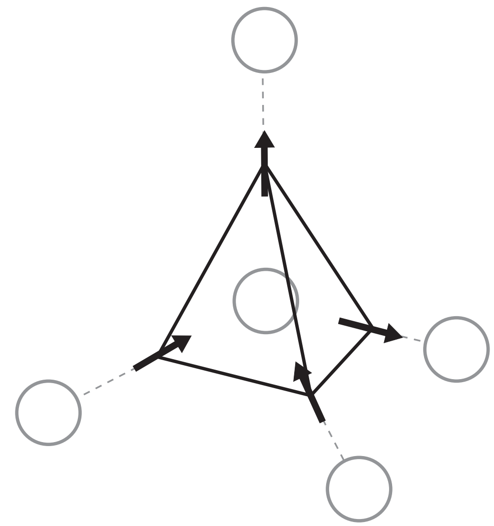
from the (planar) local energy approximation, $\boxed{\epsilon_{AF} < \epsilon_{FM} < \epsilon_{3-1} < \epsilon_{4-0}}$

The energies ϵ_k could be tuned differently by adding fields, vertical off-sets, etc.

Natural ices

Single cell unit - tetrahedron - in water-ice and spin-ice





Water-ice : coordination four lattice. **Bernal & Fowler 33** rules, two H near and two far away from each O.

Spin-ice : four (Ising) spins on each tetrahedron forced to point along the axes that join the centres of two neighbouring units (Ising anisotropy). Local interactions imply the two-in two-out ice rule ;

e.g. Dy_2 Ti_2 O_7 **Harris, Bramwell, McMorrow, Zeiske & Godfrey 97**

Artificial spin-ice

Metamaterials

Arrays of nanoscale magnets

single domain magnetic islands

placed at the edges of a tiling or

the edges of a planar graph

Ising spins along the links

Local dipolar interactions are geometrically frustrated

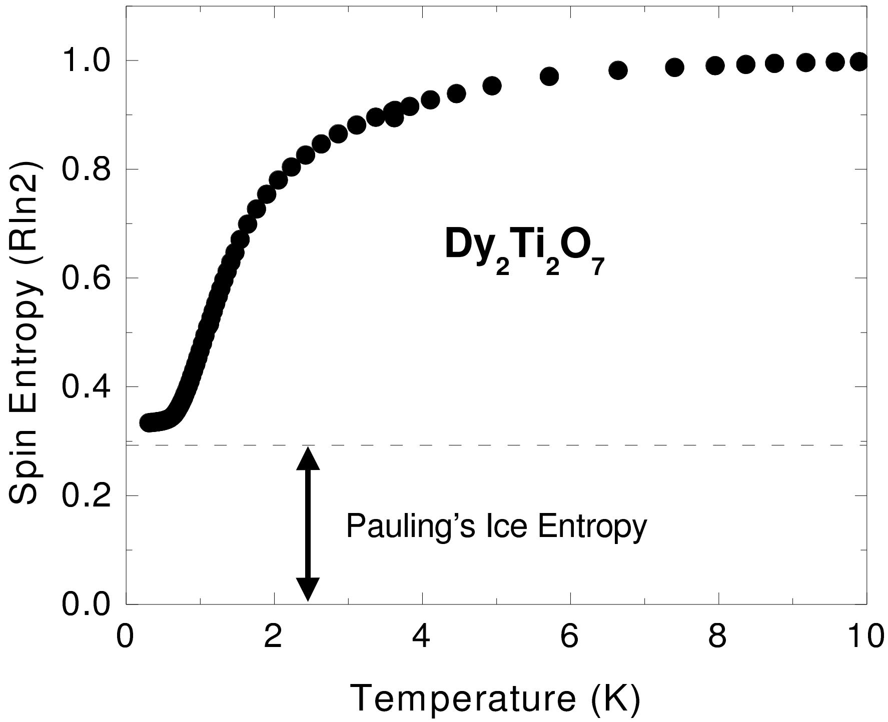
no quenched disorder

It is not possible to find a configuration of the spins that join at a vertex that minimises all pair contributions to the total energy.

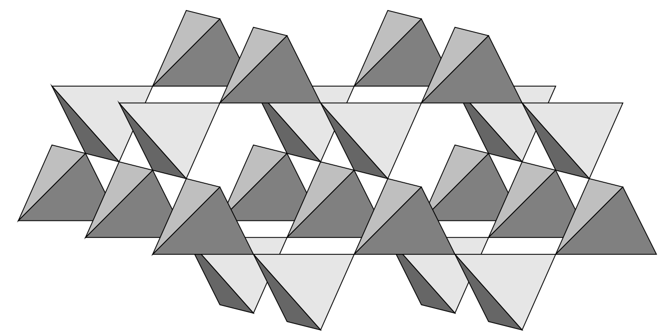
Macroscopic degeneracy of the ground state and metastable states

Natural spin-ice entropy

$$\Delta S = \int_{T_1}^{T_2} dT' \; \frac{C(T')}{T'}$$



Pyrochlore



$Dy_2\ Ti_2\ O_7$

Ising Dy spins lie along the axes

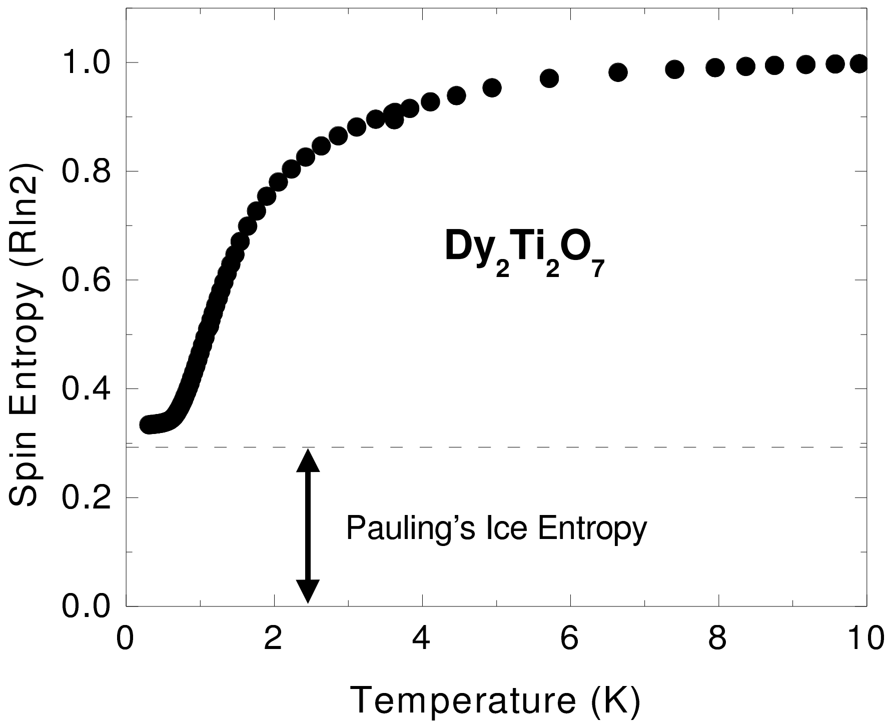
that join the nn cell units

Ramírez, Hayashi, Cava, Siddharthan & Shastry 99.

Very similar to **Giauque & Stout 33** for water ice.

Natural spin-ice entropy

$$\Delta S = \int_{T_1}^{T_2} dT' \, \frac{C(T')}{T'}$$



N tetrahedra

4 nn cell units each

$4N/2 = 2N$ links

All 2-in 2-out equivalent

$\Omega_0 \simeq 2^{2N} \left(\frac{6}{16}\right)^N$

$s_0/k_B \simeq \ln \frac{3}{2} \simeq 0.405$

Pauling 35

$s_0/k_B = \frac{3}{2} \ln \frac{4}{3} \simeq 0.431$

Transfer matrix **Lieb 76**

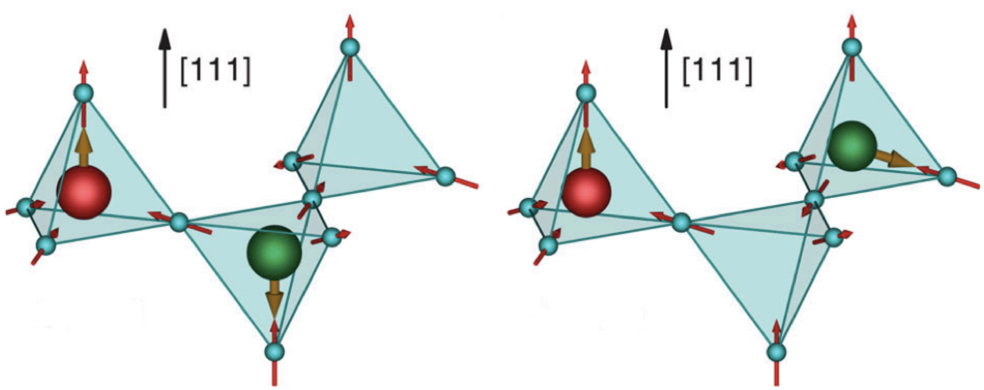
Ramírez, Hayashi, Cava, Siddharthan & Shastry 99.

Very similar to **Giauque & Stout 33** for water ice.

Natural ices

Properties and mapping

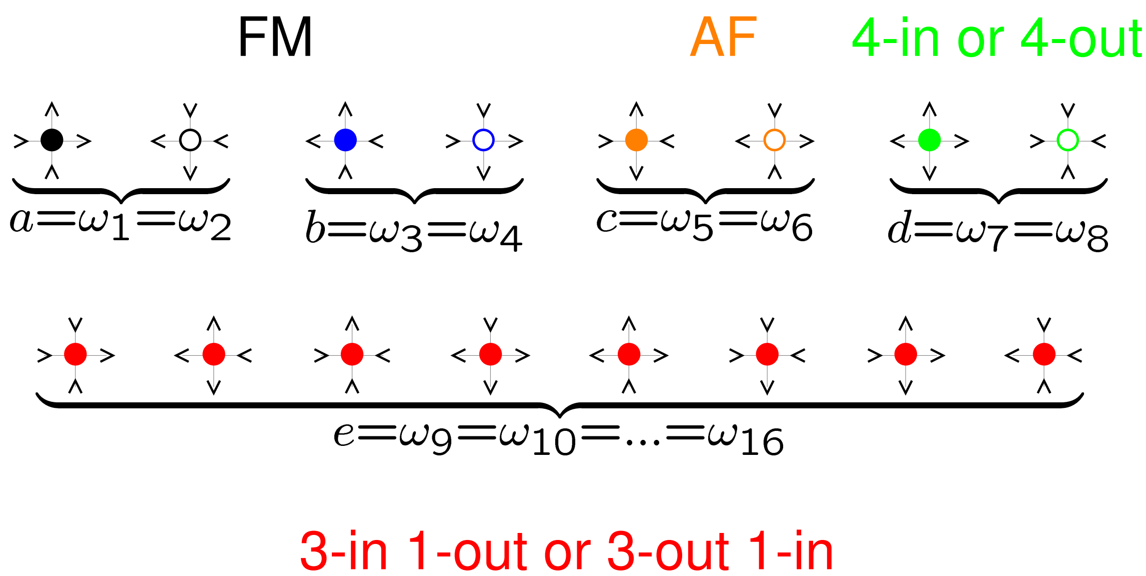
- Ice-rule breaking vertices – excitations above the ground state.
- The ± 1 Ising spins map onto a pair of “emergent” magnetic charges.
- The two-in two-out rule $\equiv$ vanishing magnetic charge in the unit cell.
- Excitation have non-vanishing charge, effective magnetic monopoles.
- The defects/charges migrate on the lattice.



Castelnovo, Moessner & Sondhi 08

The $2d$ 16 vertex model

with 3-in 1-out vertices: non-integrable system



(Un-normalized) statistical weight of a vertex $\omega_k = e^{-\beta\epsilon_k}$

In the models $\boxed{a, b, c, d, e}$ are free parameters (usually, c is the scale)

In the experiments ϵ_k depend on the sample

from the (planar) local energy approximation, $\epsilon_{AF} < \epsilon_{FM} < \epsilon_{3-1} < \epsilon_{4-0}$

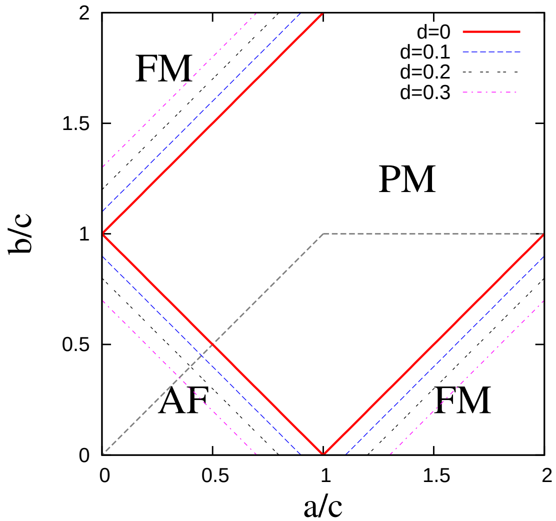
The energies ϵ_k could be tuned differently by adding fields, vertical off-sets, etc.

Static properties

What did we know ?

- **6 and 8 vertex models.**

Integrable systems techniques (transfer matrix + Bethe Ansatz), mappings to many physical (e.g. quantum spin chains) and mathematical problems.



Phase diagram

critical exponents

ground state entropy

boundary conditions effects

etc.

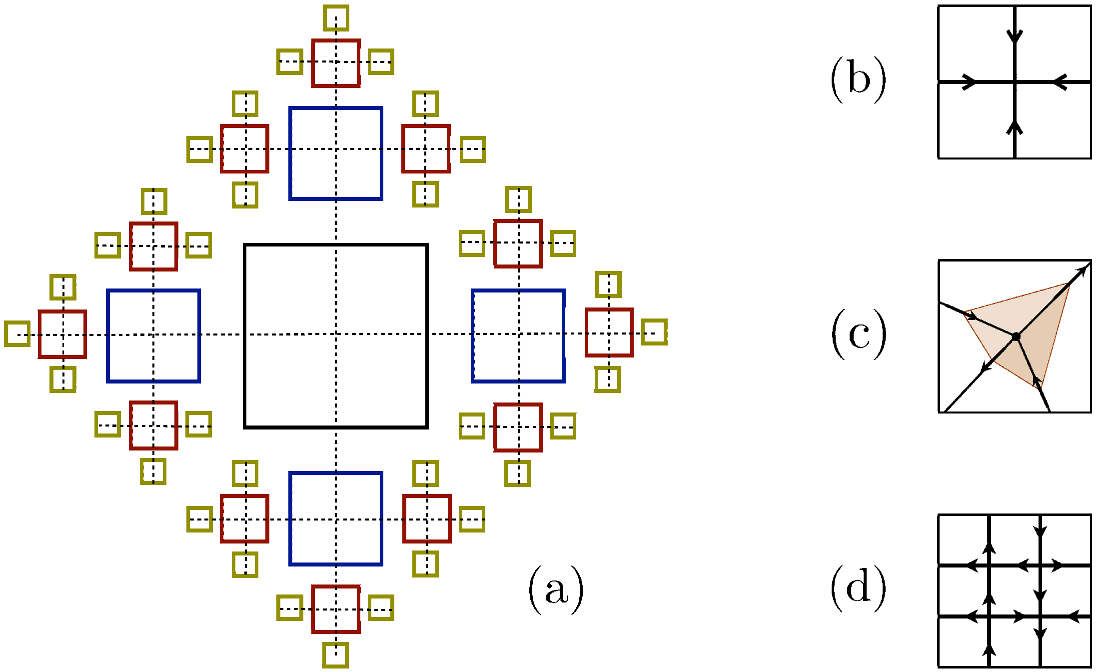
Lieb 67 ; Baxter *Exactly solved models in statistical mechanics* **82**

- **16 vertex model.**

Integrability is lost. Not much interest so far.

Equilibrium analytic

Bethe-Peierls or cavity method



Join an L-rooted tree from the left ; an U-rooted tree from above ;

an R-rooted tree from the right and a D-rooted tree from below.

Foini, Levis, Tarzia & LFC 12

Is it a powerful technique ?

in, e.g., the 6 vertex model

With a tree in which the unit is a **vertex** we find the PM, FM, and AF phases.

$s_{PM} = \ln[(a+b+c)/(2c)]$

Pauling's entropy $s_{PM} = \ln 3/2 \sim 0.405$ at the spin-ice point $a = b = c$.

Location and 1st order transition between the PM and FM phases. ✔

Location ✔ but *1st order* PM-AF transition. ✘

no fluctuations in the frozen FM phase. ✔

no fluctuations in the AF phase. ✘

With a **four site plaquette** as a unit we find the PM, FM, and AF phases.

A more complicated expression for $s_{PM}(a,b,c)$ that yields

$s_{PM} \simeq 0.418$ closer to Lieb's entropy $s_{PM} \simeq 0.431$ at the spin-ice point.

Location and 1st order transition between the PM and FM phases. ✔

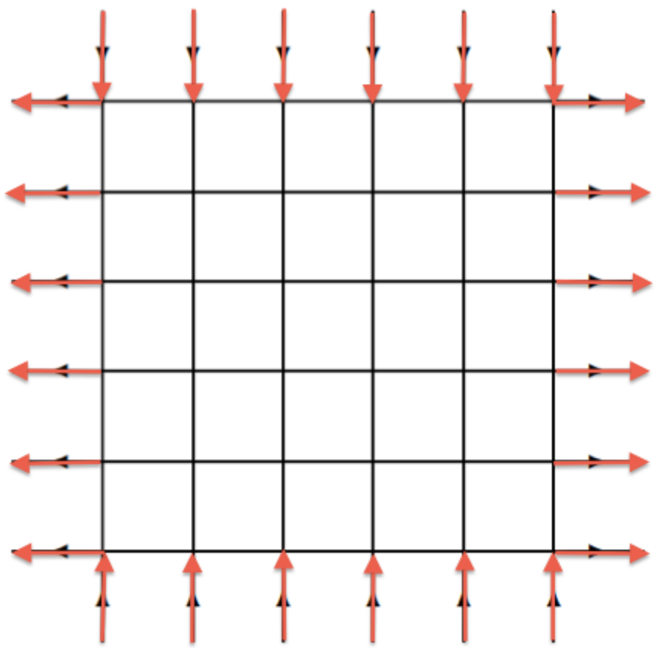
Location ✔ but *2nd order* (should be BKT) PM-AF transition. ✘

fluctuations in the AF phase and frozen FM phase. ✔

Six-vertex model

with domain-wall boundary conditions

Strongly constrained model: non-trivial effect of boundary conditions.



Global parameters in D phase

$f^{D}_{\mathrm{dwBC}} > f^{D}_{\mathrm{pBC}}$

Korepin & P. Zinn-Justin 00

Macroscopic phase separation

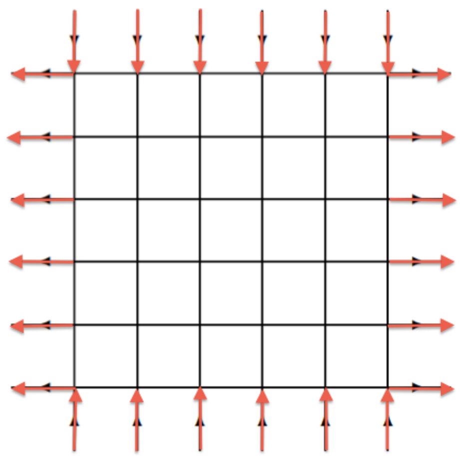
external frozen region

internal thermal region

interface : arctic curve

Six-vertex model

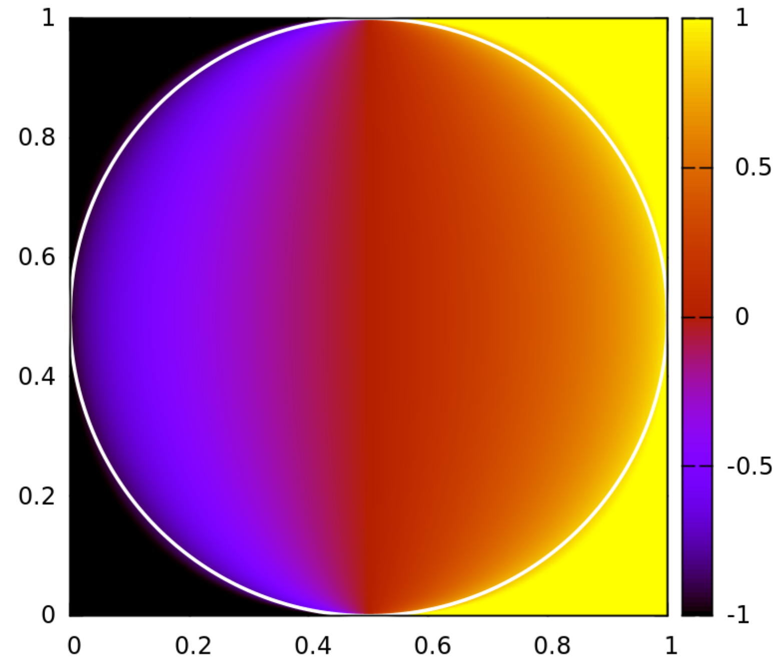
with domain-wall boundary conditions



$$\Delta \equiv (a^2 + b^2 - c^2)/(2ab)$$

$= 0$ Disordered phase

$a = b$ free-fermion case



Color code : Bethe-Peierls polarization of horizontal arrows

White curve : analytic arctic circle.

Elkies et al. 92-96 ; Jokusch et al. 98 ; Colomo & Pronko 08-14 ; Sportiello

LFC, Gonnella & Pelizzola 14

Six-vertex model

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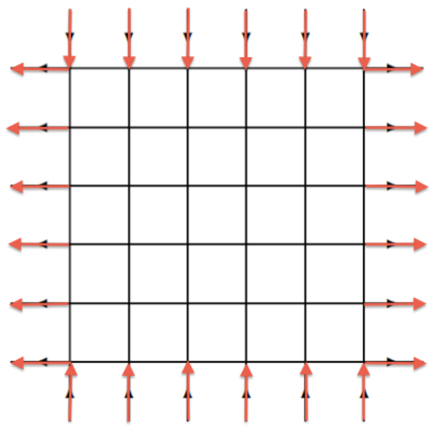
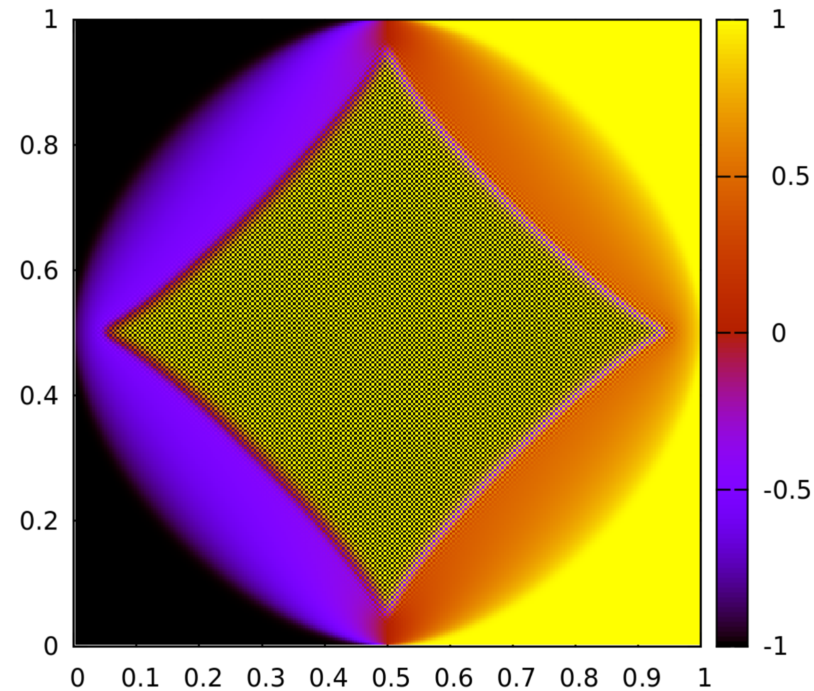




$$\Delta \equiv (a^2 + b^2 - c^2)/(2ab)$$

< -1 Anti-ferromagnetic

$a = b = 1$

$c = 2.5$

Color code : BP polarization of horizontal arrows

Double phase separation: frozen – thermal disordered - AF

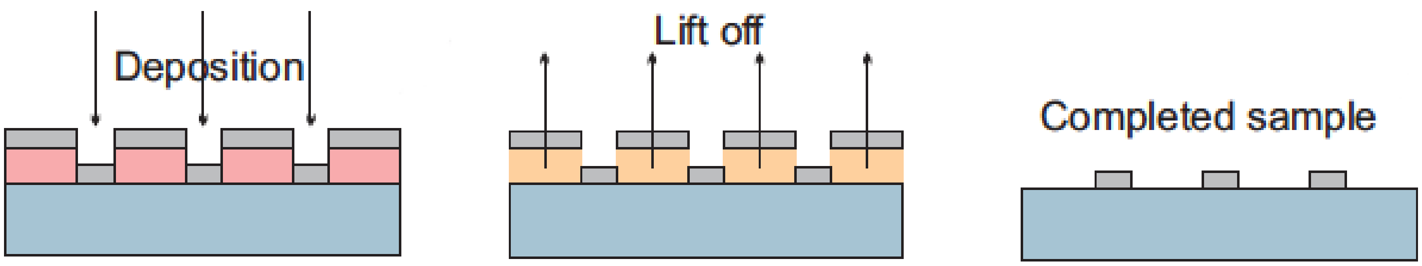
Colomo, Pronko & P. Zinn-Justin 10 ; Sportiello btw frozen and thermal

LFC, Gonnella & Pelizzola 14

Artificial spin-ice

Back to experiments: as-grown samples

- Lattice is written with electron beam lithography.
- Magnetic material is gradually poured as in the sketch.



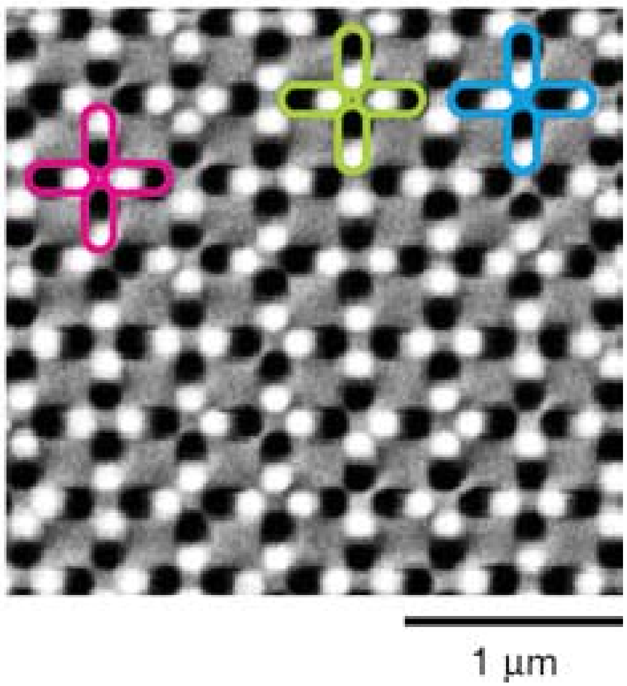
- Thermal fluctuations let the magnets flip until some size is reached.
- Configurations are henceforth blocked.

(Magnetic field annealing is also used.)

Note the similarity with granular matter & effective measure ideas.

Vertex density

How many of each kind ?



Signalled in this image

Pink : an AF vertex

Yellow : a 3-in 1-out vertex

Blue : a FM vertex

No 4-in nor 4-out vertices

Magnetic force microscopy

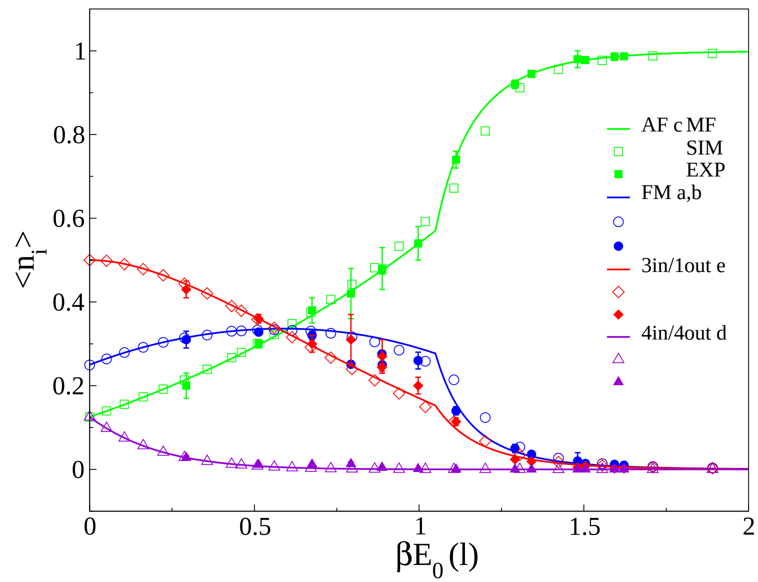
Wang et al 06

Does one sample the Gibbs-Boltzmann distribution function at β ?

Another measure ?

Vertex density

Across the PM-AF transition – numerical, analytic and exp. data



Disordered AF

PM - AF transition

AF vertices

FM vertices

3-in 1-out & 3-out 1-in e-vert.

4-in & 4-out d-vertices

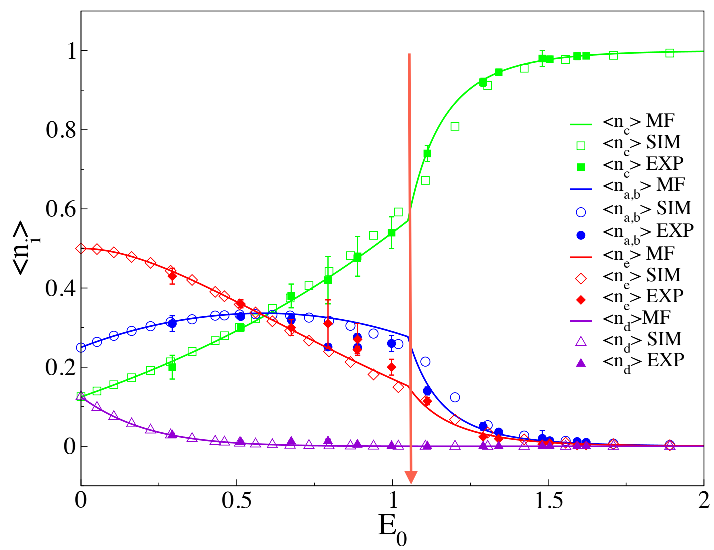
Each set of vertical points, $\beta E_0(\ell)$ value, corresponds to a different sample (varying lattice spacing ℓ or the compound). $\epsilon_{a,b,d,e} = \overline{f}(\ell) = f(\epsilon_c)$

Agreement seems perfect but experience from glassy physics...

Levis, LFC, Foini & Tarzia 13 ; experimental data courtesy of **Morgan** *et al.* **12**

Vertex density

Across the PM-AF transition – numerical, analytic and exp. data



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Levis, LFC, Foini & Tarzia 13 ; experimental data courtesy of **Morgan** *et al.* **12**

Quench dynamics

A simpler setting

- Take an initial condition in equilibrium at, e.g. $\boxed{T_0 \to \infty}$ that corresponds to $a_0 = b_0 = c_0 = d_0 = e_0 = 1$.
- Evolve it with a set of parameters a, b, c, d, e in the phases PM, FM or AF: an infinitely rapid quench at $t = 0$.
- Concretely, we use **stochastic dynamics**:

  with single spin flip updates with the usual heat-bath rule,

  and a continuous time MC algorithm (to reach long time scales).

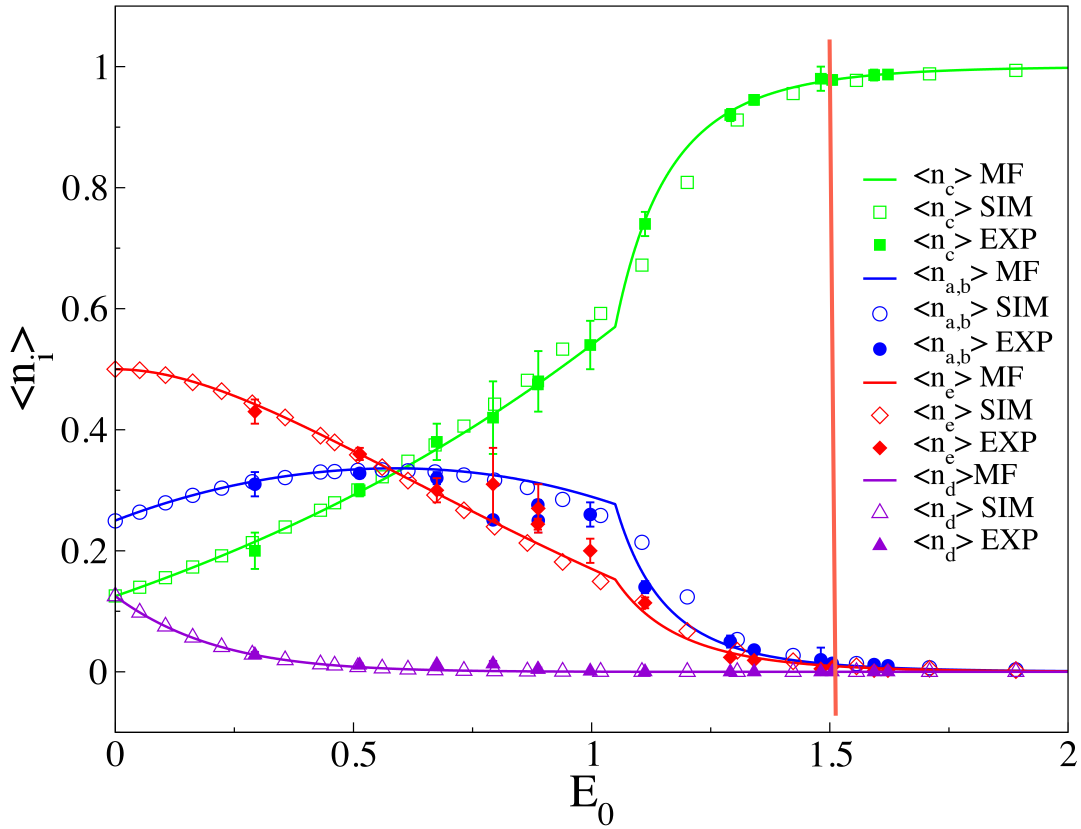
  Relevant dynamics experimentally (contrary to loop updates used to study equilibrium in the 8 vertex model)

Levis & LFC 11, 13

Dynamics in AF phase

Density of defects

$c = 1$, $a = b = 0.1$ and $d = e^2 = 10^{-10}$



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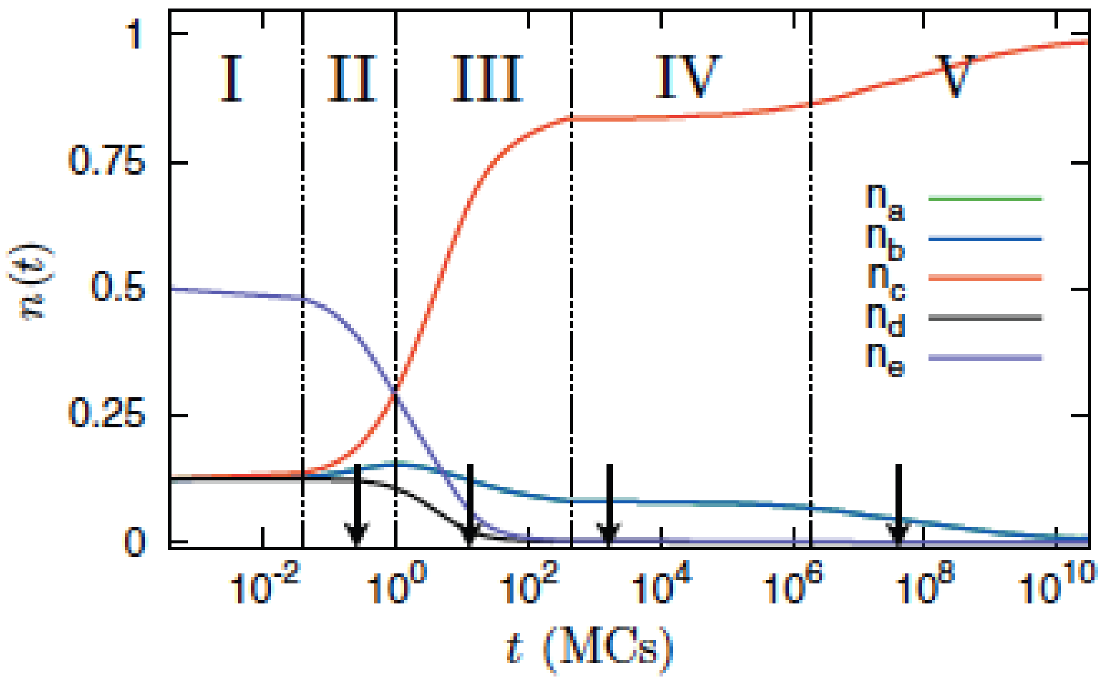


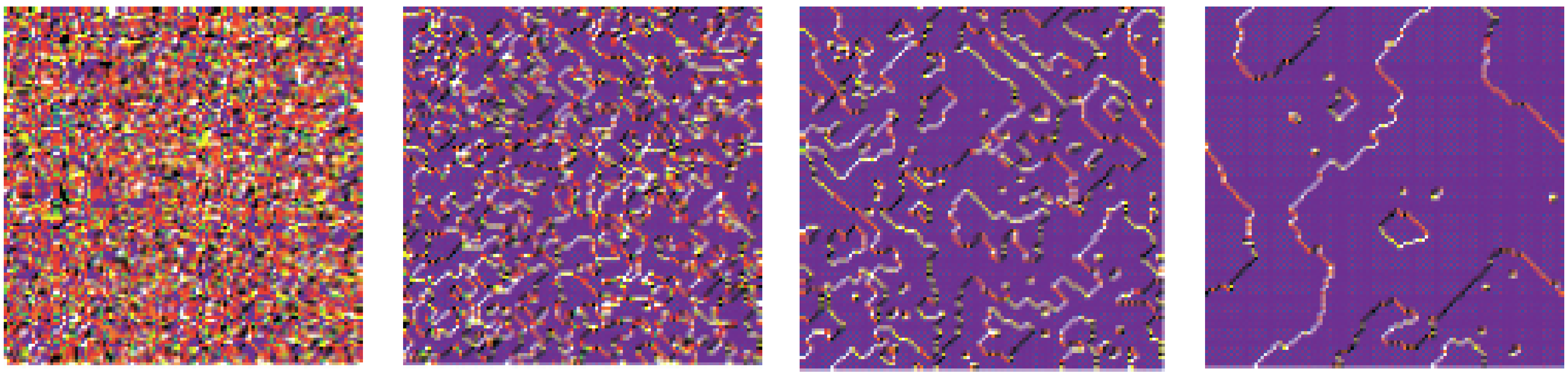
Isotropic growth of AF order with $L(t) \simeq t^{1/2}$

Dynamics in AF phase

Snapshots – coarsening

Color code. Violet background: AF order of two kinds ;

Initial state coarsening states



Isotropic growth of AF order for this choice of parameters

$\boxed{c \gg a = b}$ AF vertices are energetically preferred ;

there is no anisotropy imposed, $L^{\rm AF}(t) \simeq t^{1/2}$

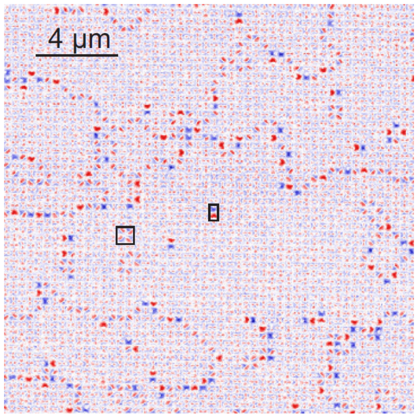
Dynamics in AF phase

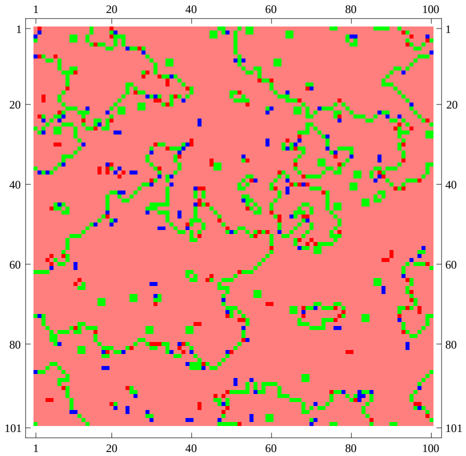
Snapshots – experiment vs. numerics

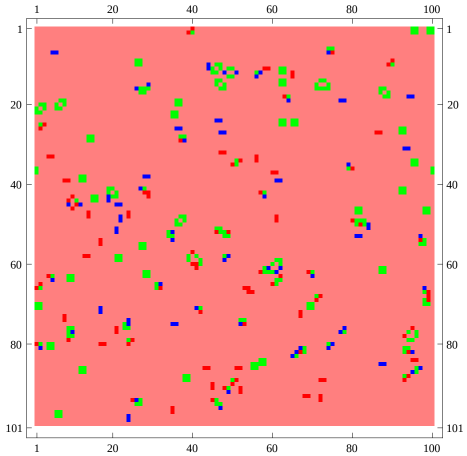
Color code. Orange background : AF order of two kinds ;

green FM vertices, red-blue defects.

Magnetic force microscopy   coarsening state   equilibrium state





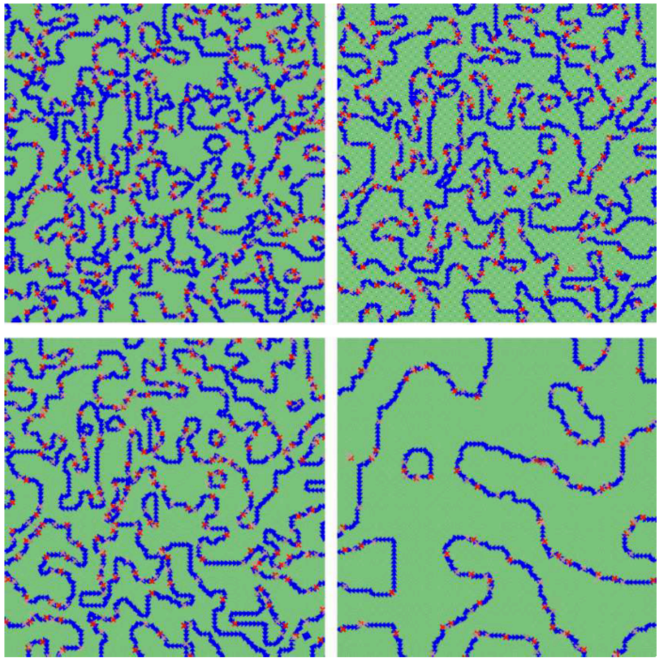


Interfaces between the two staggered AF orders

A statistical and geometric analysis of domain walls & defects should be done to conclude.

Dynamics in AF phase

Snapshots – other modelling



Budrikis et al 12

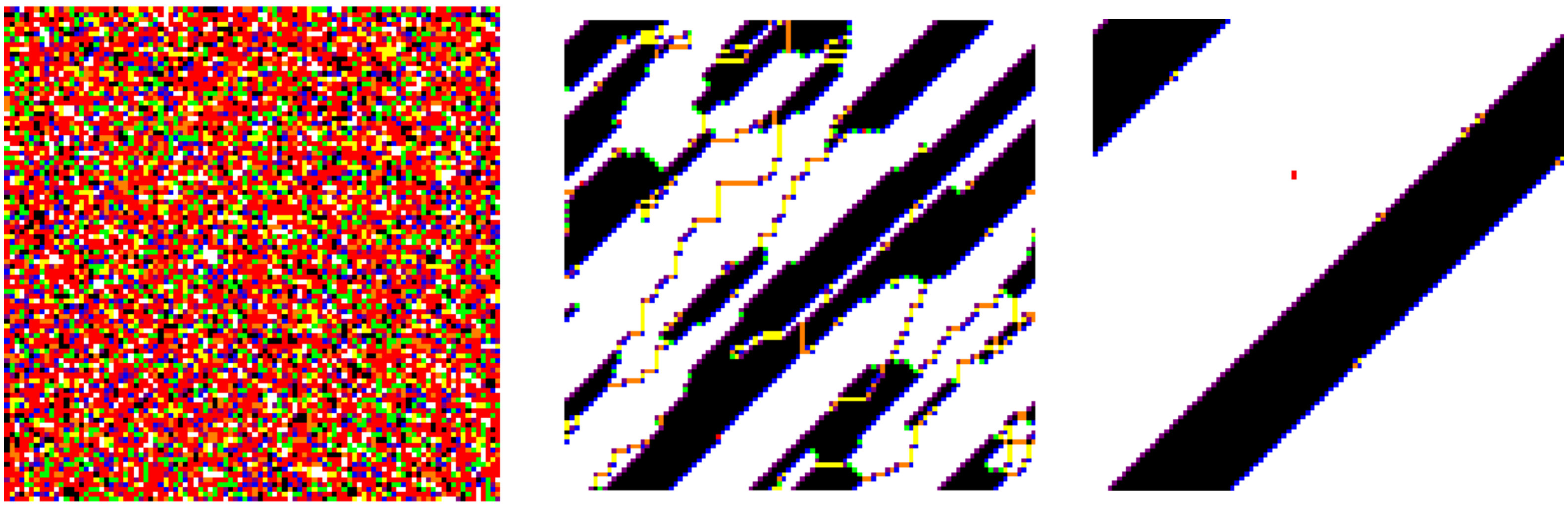
Mostame et al. 14 field quenches

Cepas & Canals 12, Cepas 14 multiple time-scales

Wysin et al. 13 dynamics in Heisenberg 2d square ice

Dynamics in the FM phase

Snapshots



Growth of stripes

Quench to a large a value : black & white vertices energetically favored.

Interesting coarsening process, $L_{\perp}^{\rm FM}(t)$ and $L_{\parallel}^{\rm FM}(t)$

Summary

Classical geometrically frustrated magnetism

spin-ice in two dimensions

$2d$ vertex models

Problems with analytic, numeric and experimental interest

Summary

Classical frustrated magnetism ; spin-ice in two dimensions.

— $2d$ **vertex models**: problems with analytic, numeric and experimental interest.

Cfr. artificial spin-ice

- Beyond integrable systems' methods to describe the **static** properties.
  — Some results of the Bethe-Peierls approximation are exact, others are at least very accurate.

Analytic challenge

- Slow coarsening (or near critical in the disordered phase) **dynamics**.

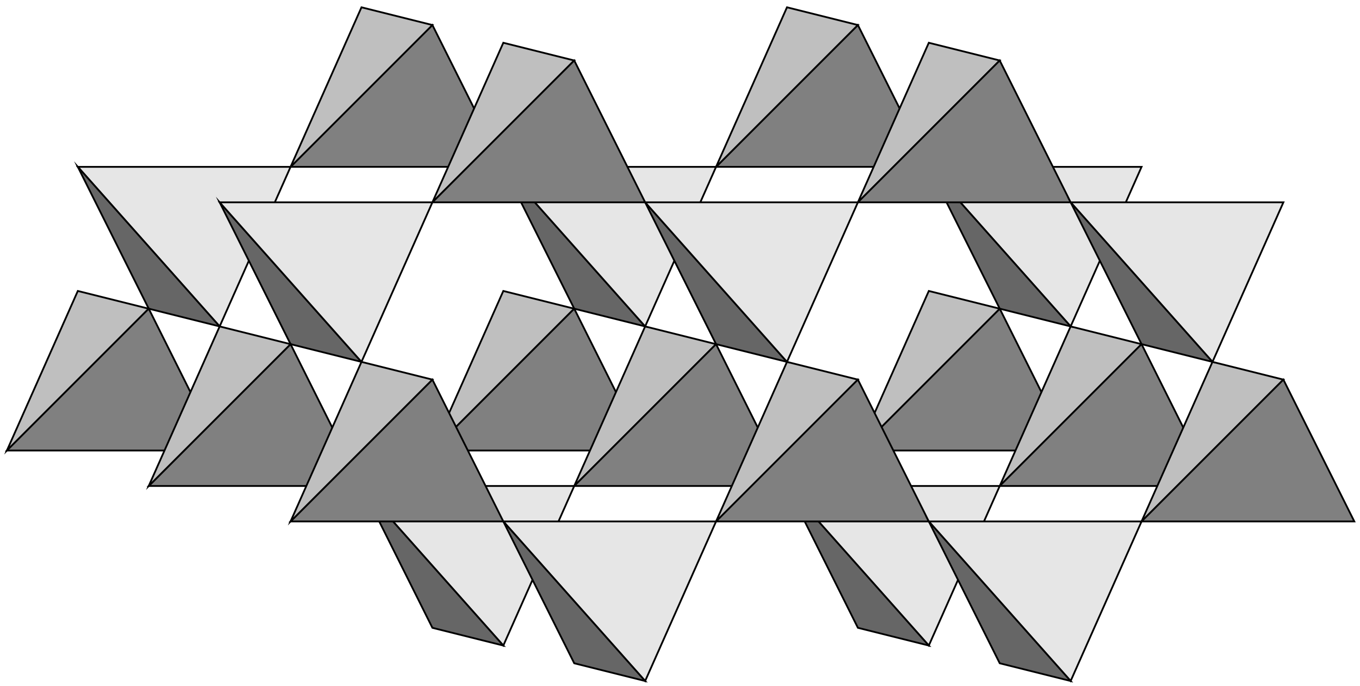
$$L_{\parallel}^{\rm FM}(t) \simeq L_{\perp}^{\rm FM}(t) \simeq L^{\rm AF}(t) \simeq t^{1/2}$$

Analytically ?

- Experiments : dynamics block, **non-equilibrium measures** ?
- Useful manipulation of **defects** (ice-breaking rule vertices).

Natural spin-ice

$3d$: the pyrochlore lattice

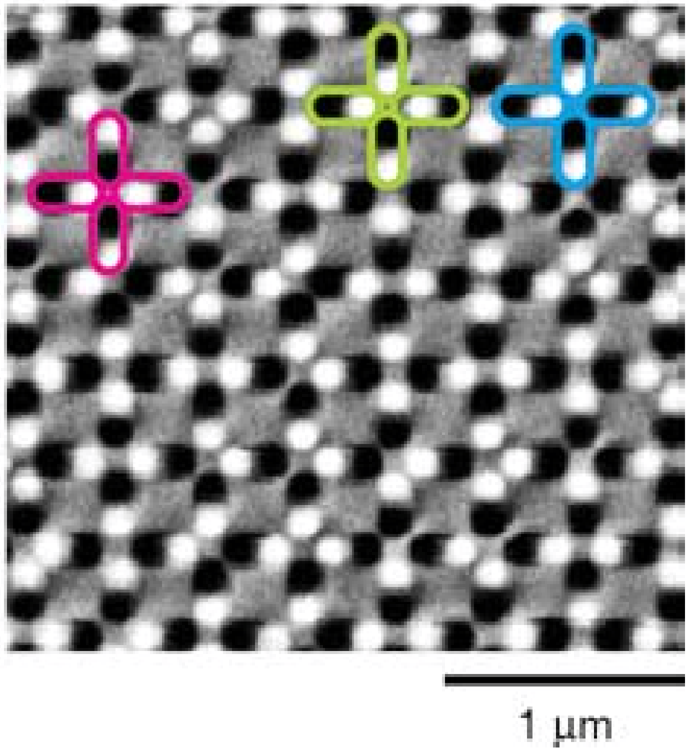


Coordination four lattice of corner linked tetahedra. The rare earth ions occupy the vertices of the tetrahedra ; **e.g. Dy$_2$ Ti$_2$ O$_7$**

Harris, Bramwell, McMorrow, Zeiske & Godfrey 97

Artificial spin-ice

Bidimensional square lattice of elongated magnets



Bidimensional square lattice

Dipoles on the edges

16 possible vertices

Experimental conditions in this fig. :

vertices w/ two-in & two-out arrows

with staggered AF order

are much more numerous

AF 3in-1out FM

Wang *et al* **06, Nisoli** *et al* **10, Morgan** *et al* **12**

Square lattice artificial spin-ice

Local energy approximation $\Rightarrow 2d$ 16 vertex model

Just the interactions between dipoles attached to a vertex are added.

Dipole-dipole interactions. Dipoles are modeled as two opposite charges. Each vertex is made of 8 charges, 4 close to the center, 2 away from it. The energy of a vertex is the electrostatic energy of the eight charge configuration. With a convenient normalization, dependence on the lattice spacing ℓ :

$$\epsilon_{AF} = \epsilon_5 = \epsilon_6 = (-2\sqrt{2} + 1)/\ell \qquad \epsilon_{FM} = \epsilon_1 = \cdots = \epsilon_4 = -1/\ell$$

$$\epsilon_e = \epsilon_9 = \ldots \epsilon_{16} = 0 \qquad \epsilon_d = \epsilon_7 = \epsilon_8 = (4\sqrt{2} + 2)/\ell$$

$$\boxed{\epsilon_{AF} < \epsilon_{FM} < \epsilon_e < \epsilon_d}$$

Nisoli *et al* 10

Energy could be tuned differently by adding fields, vertical off-sets, etc.

Static properties

What did we do ?

- Equilibrium simulations with finite-size scaling analysis.
  - Continuous time Monte Carlo.

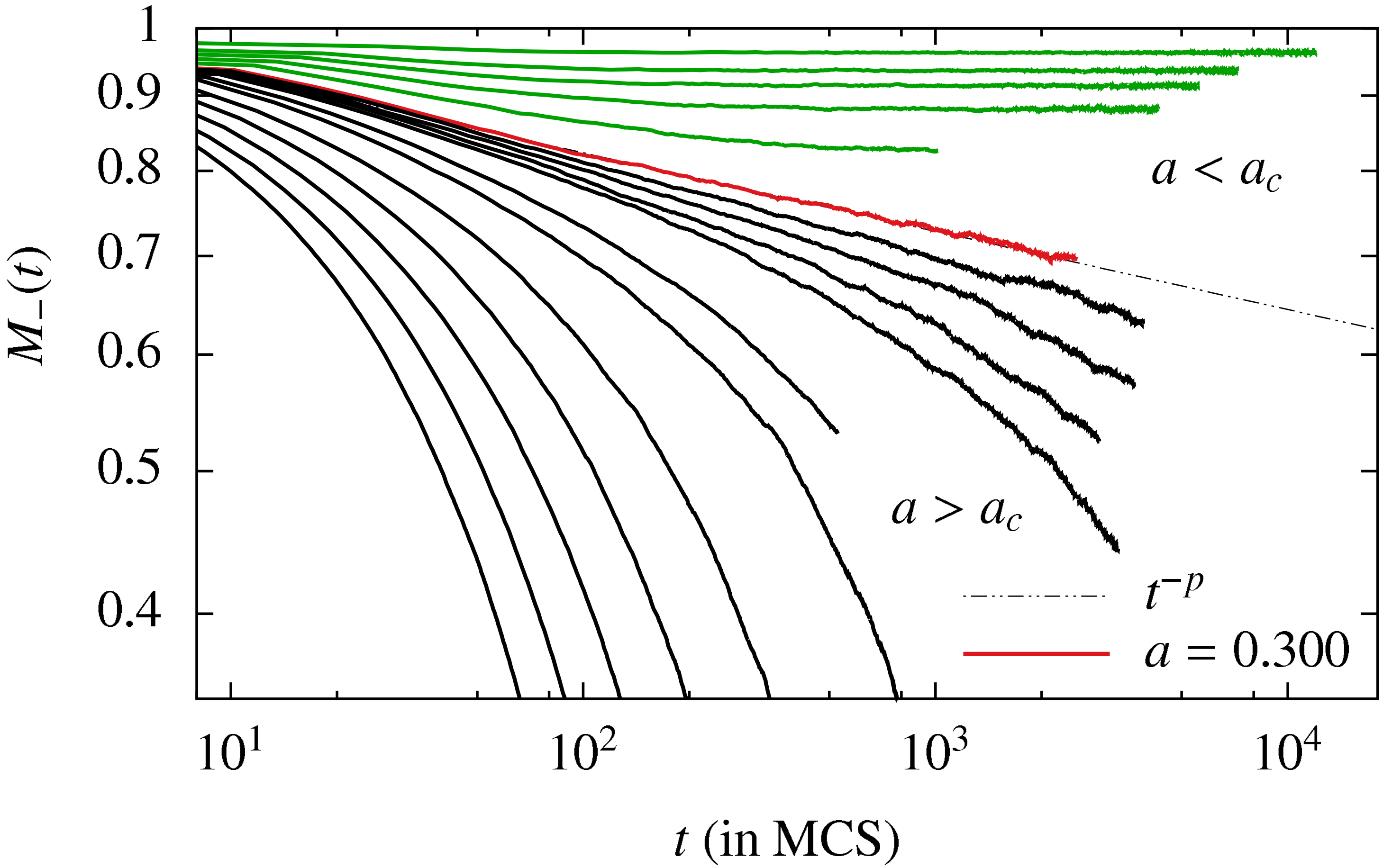
    e.g. focus on the **AF-PM transition** ; cfr. experimental data.

    AF order parameter : $$M_- = \frac{1}{2}\left(\langle|m_-^x|\rangle + \langle|m_-^y|\rangle\right)$$

    with $m_-^{x,y}$ the staggered magnetization along the x and y axes.
  - Finite-time relaxation $$M_-(t) \simeq t^{-\beta/(\nu z_c)}$$
- Cavity Bethe-Peierls mean-field approximation.
  - The model is defined on a tree of single vertices or 4-site plaquettes

Finite time relaxation

Magnetization across the PM-AF transition



$a_c = e^{-\beta_c e_1} \simeq 0.3$ with $e_1 = 0.45$ $\Rightarrow$ $\beta_c = 2.67 \pm 0.02$

Equilibrium analytic

Bethe-Peierls or cavity method

Write a (matrix) recurrence relation to compute the probability that the cavity site be occupied by each one of the six vertices.

Find the solutions as a function of the weigths ω_α.

Obtain the free-energy density.

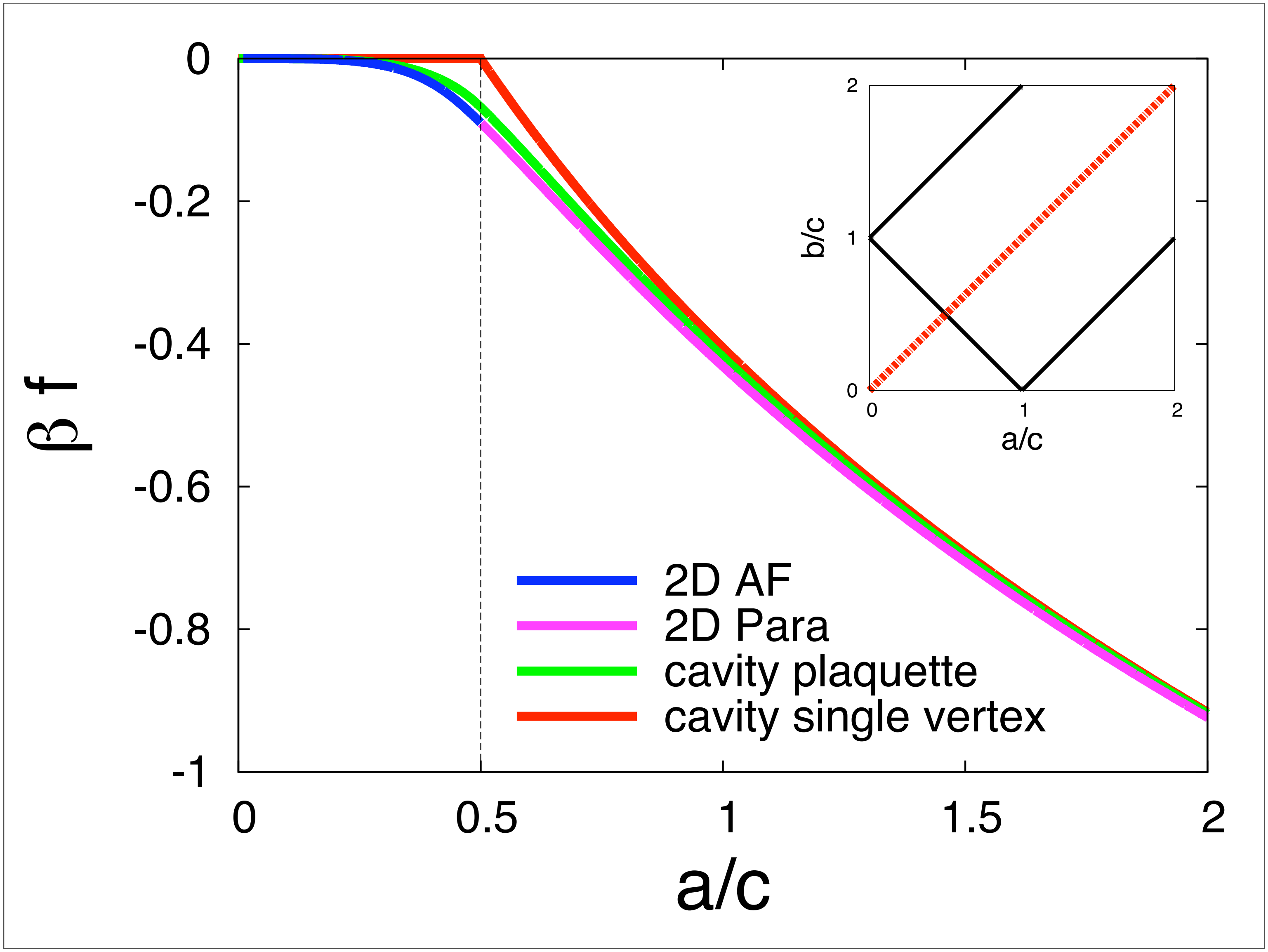
Look for transition lines.

This method can be applied to the 16 vertex model.

Foini, Levis, Tarzia & LFC 13

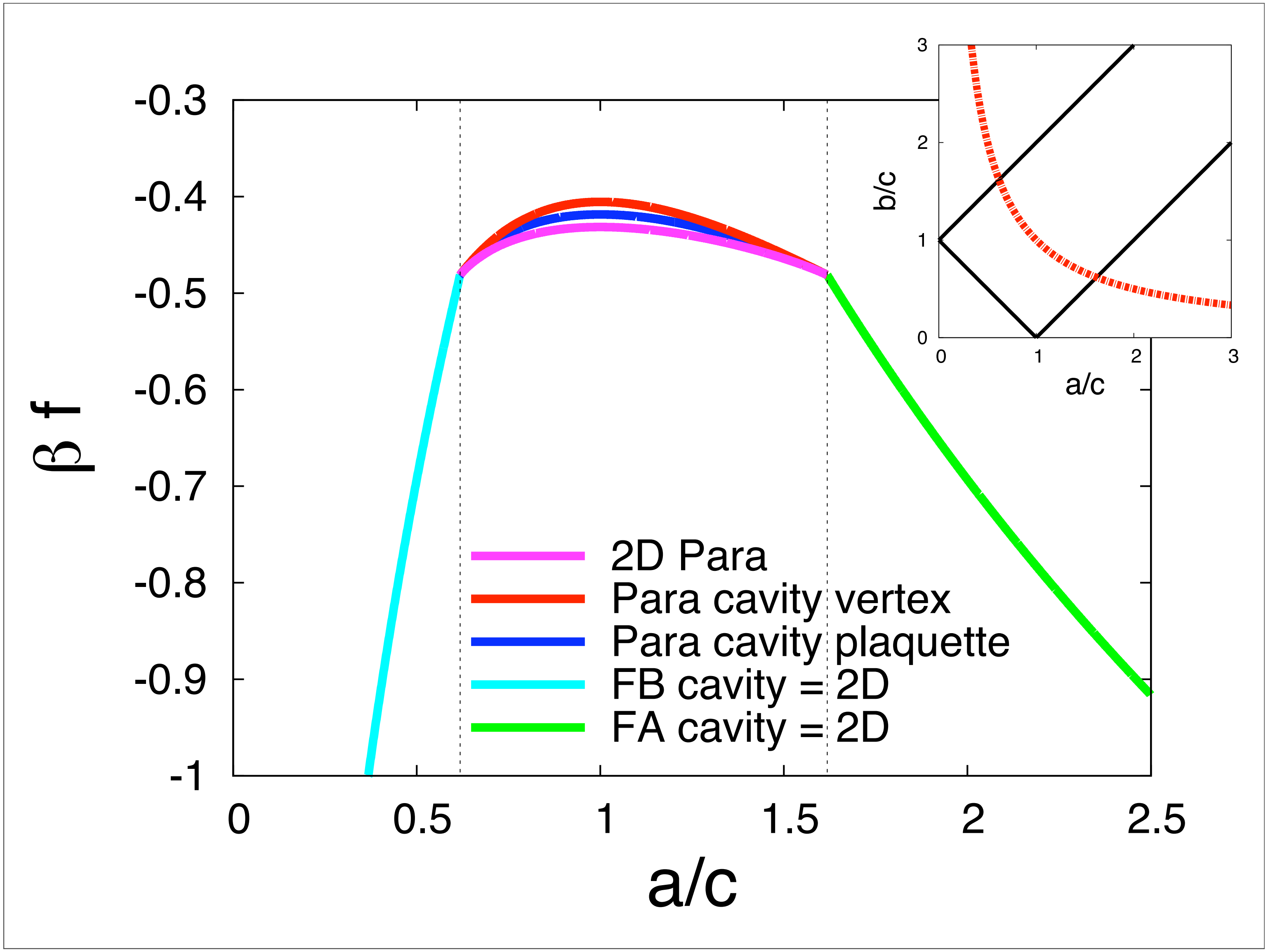
Equilibrium analytic

6 vertex : AF - D transition, cavity vs Baxter's exact solution



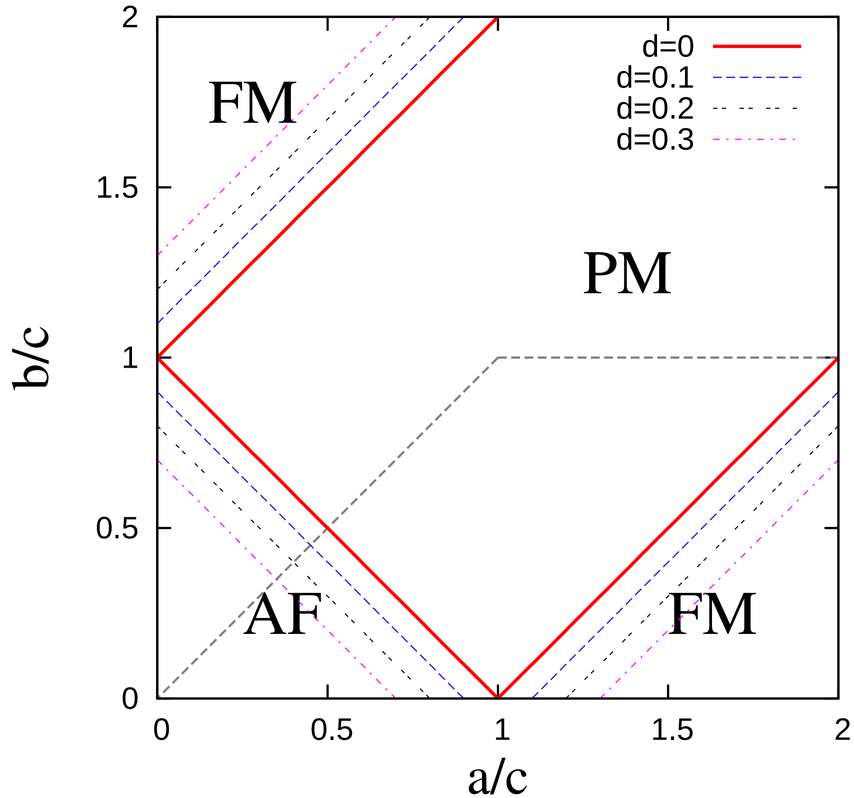
Equilibrium analytic

6 vertex : FM - D transition, cavity vs. Baxter's exact solution



The $2d$ 8 vertex model

Integrable system (transfer matrix + Bethe Ansatz)



No type e vertices.

2nd order phase transitions

$$\Delta_8 = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}$$

$\Delta_8 = \pm 1$ transition lines

With three-in one-out vertices

Integrability

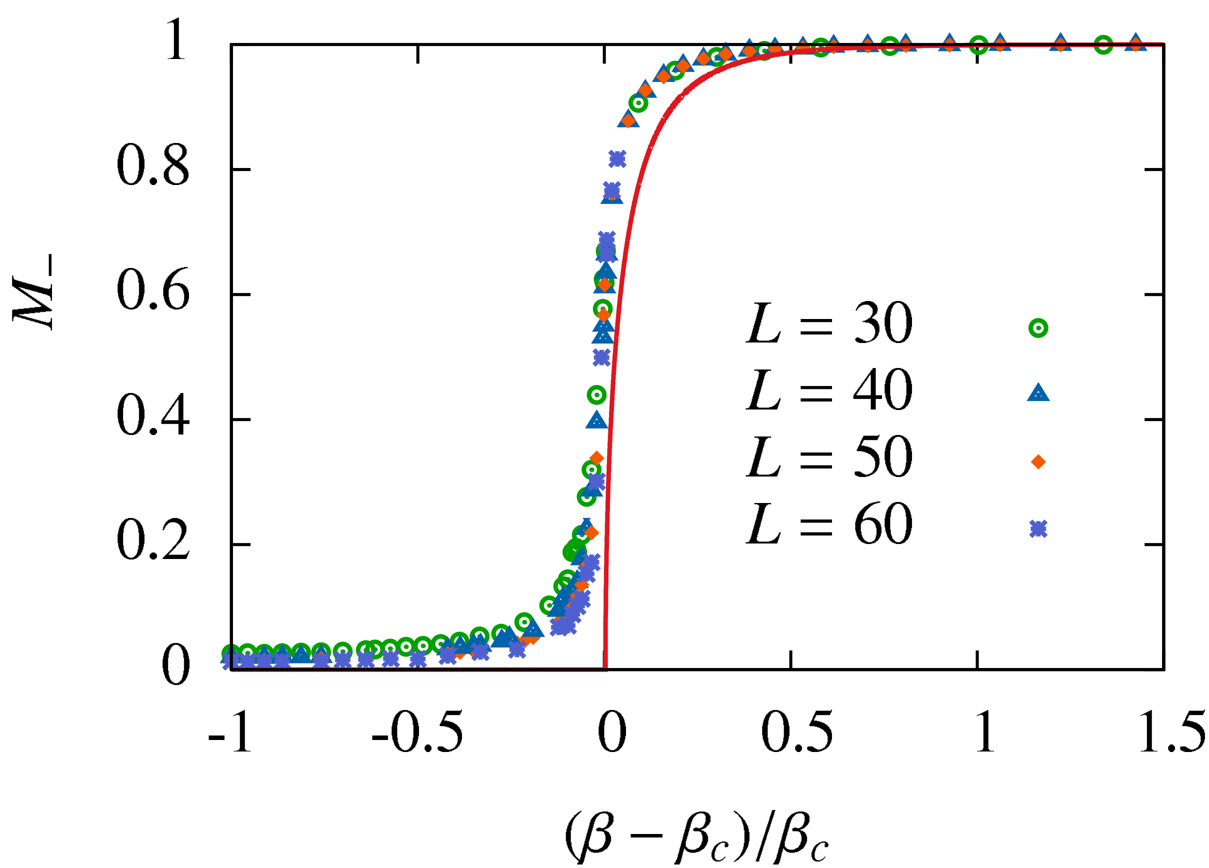
is lost.

Lieb 67 ; Baxter *Exactly solved models in statistical mechanics* **82**

Equilibrium CTMC

Magnetization across the PM-AF transition

Vertex energies set to the values explained above.

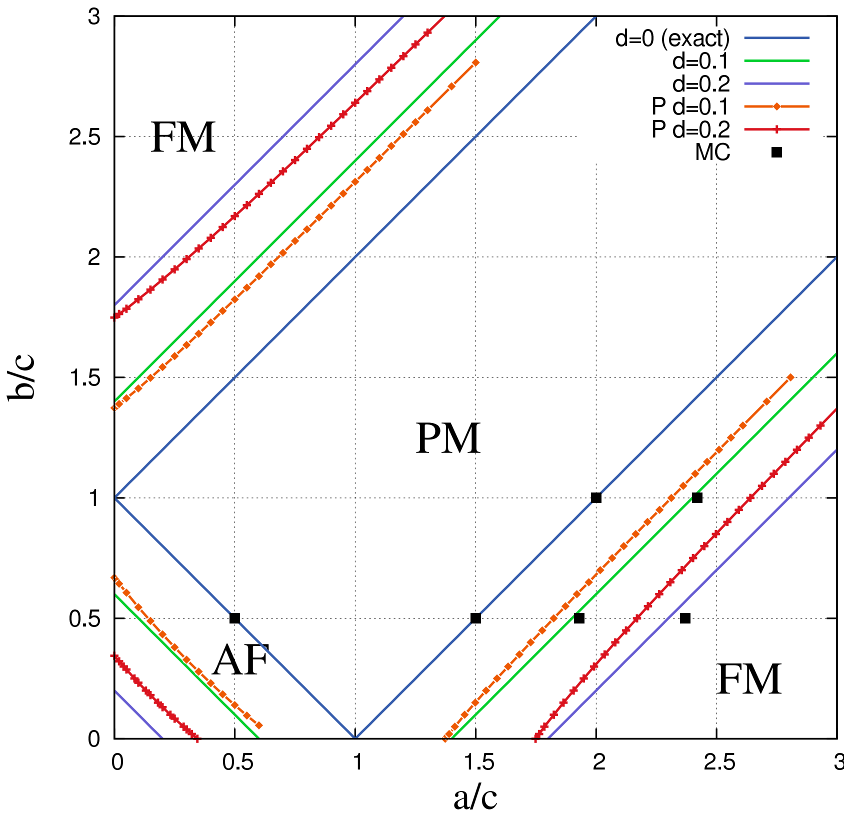


Solid red line from the Bethe-Peierls calculation.

Static properties

Equilibrium phase diagram 16 vertex model

- MC simulations & cavity Bethe-Peierls method



Phase diagram

critical exponents

ground state entropy

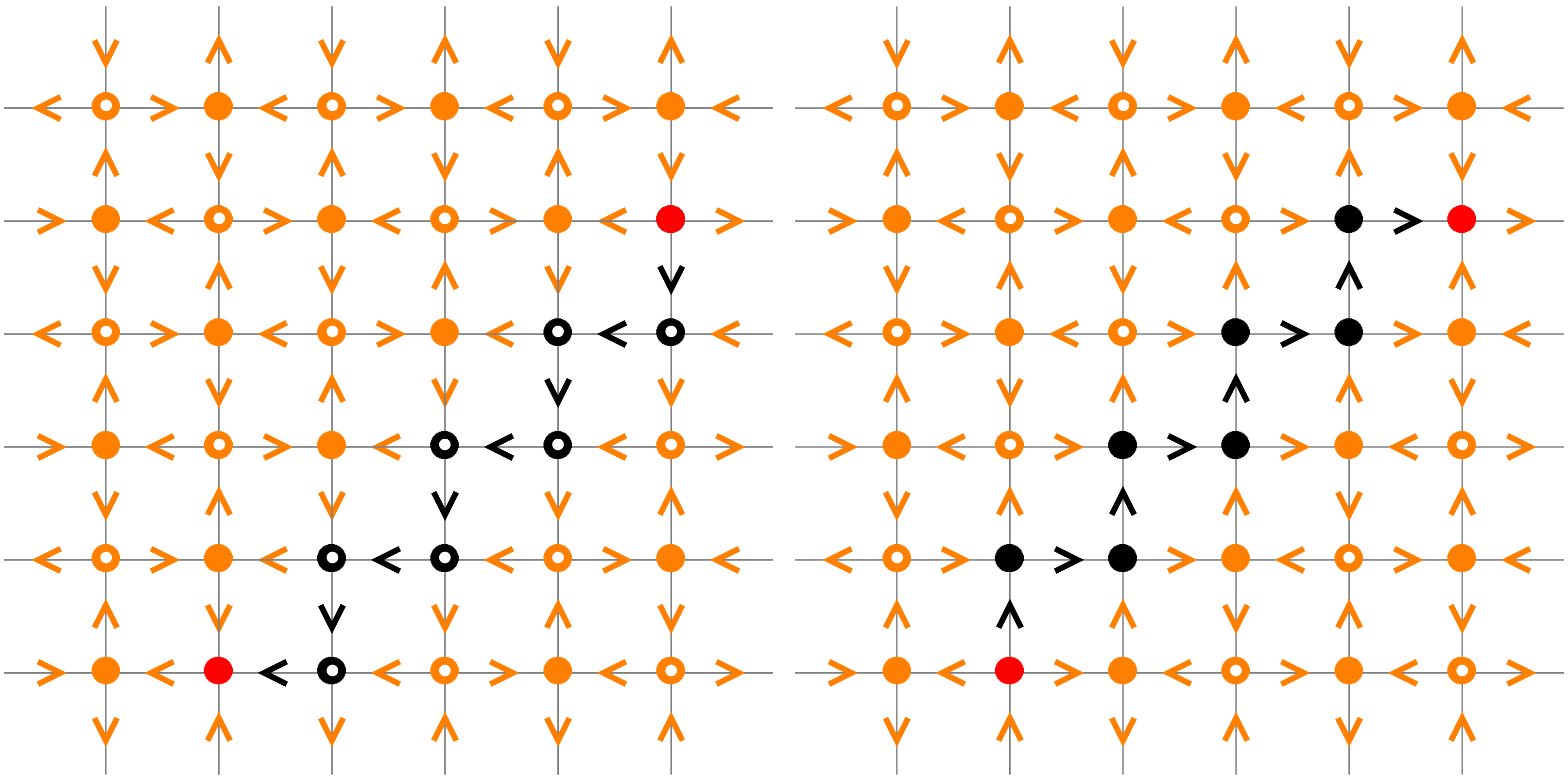
equilibrium fluctuations

etc.

Foini, Levis, Tarzia & LFC 12

Fluctuations

Sketch



The probability of such fluctuations can be estimated with the Bethe-Peierls calculation on a tree of four-site plaquettes !

Equilibrium : the tree vs 2d

16 vertex model

- The cavity method can deal with the generic vertex model.

  More complicated recursion relations, more cases to be considered, but no further difficulties.

- The transition lines do not get parallelly translated with respect to the ones of the 6-vertex model. ?

  They are all of 2nd order. ✔

  They are remarkably close to the numerical values in $2d$. ✔

  The exponents : on the tree they are mean-field, in $2d$?

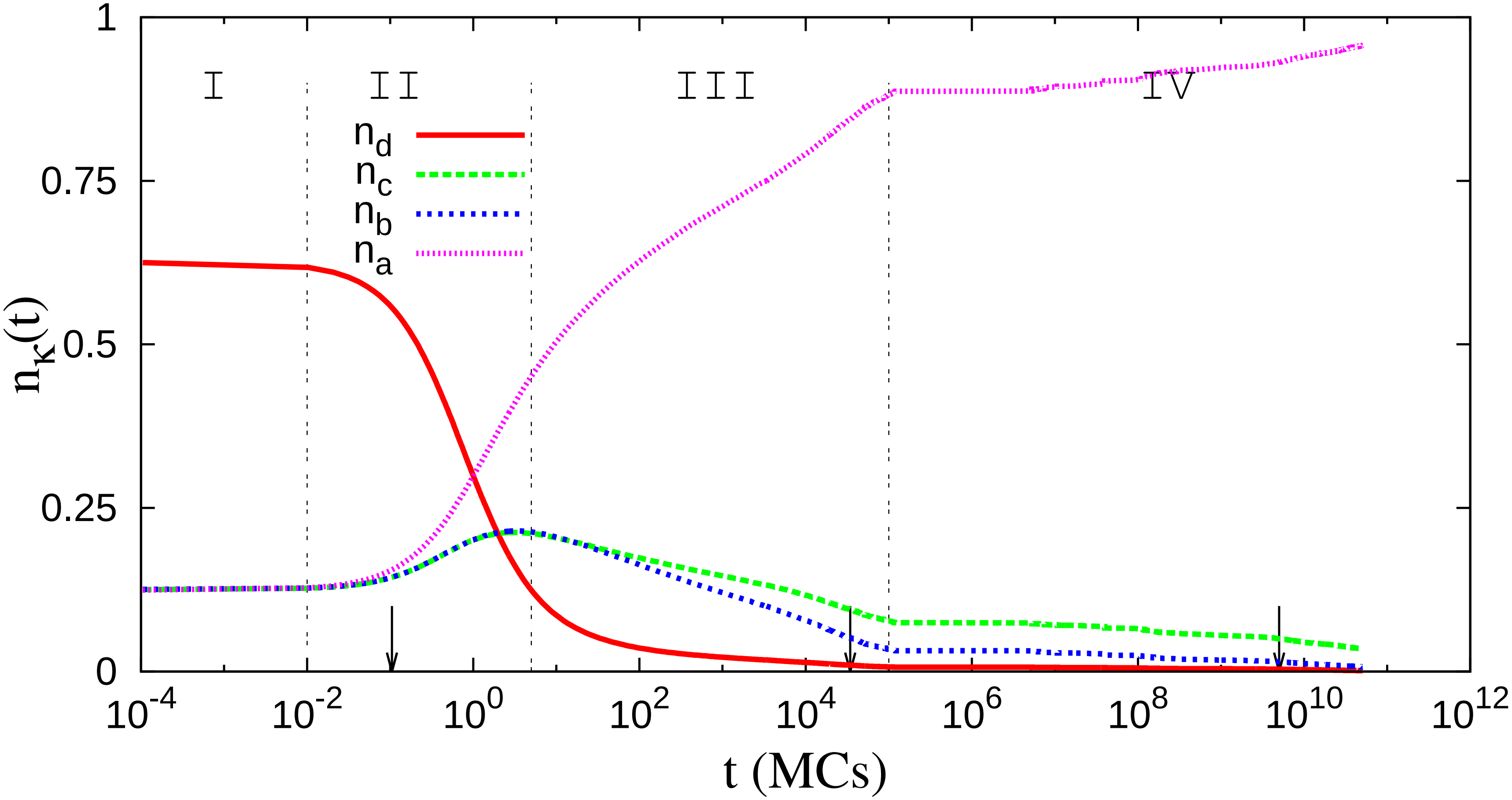
- MF expression for Δ_{16} In $2d$?

- The quantum Ising chain for the 16 vertex model should include new terms.

Foini, Levis, Tarzia & LFC 12

Dynamics in the FM phase

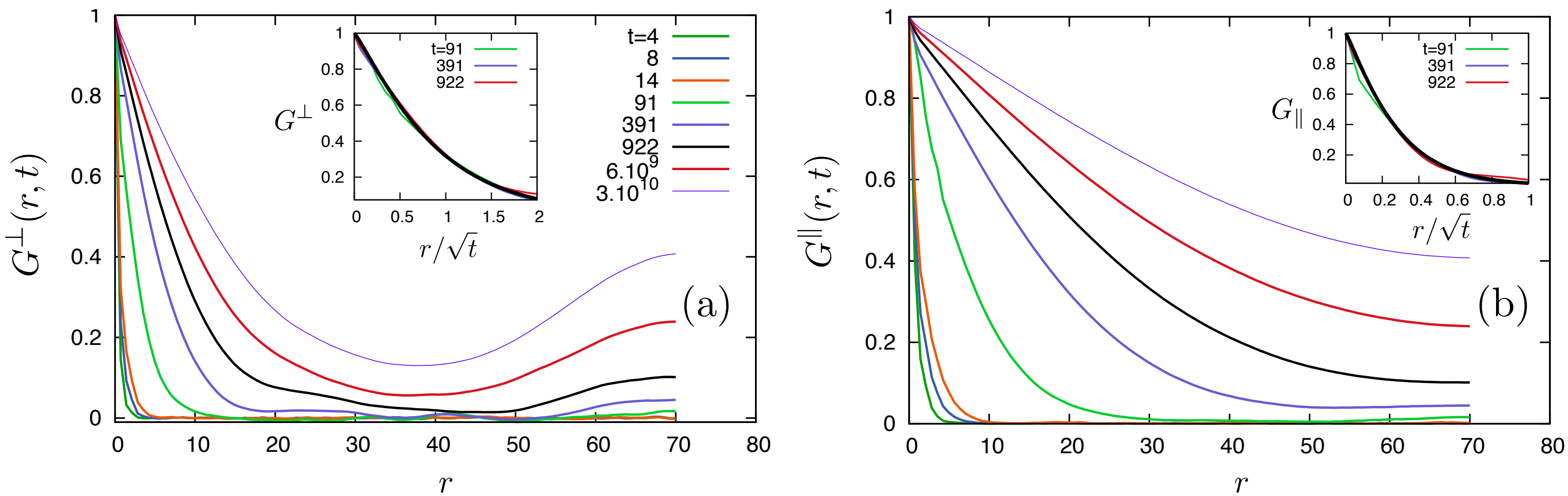
Density of defects ($d = e$ here)



Four regimes

Dynamics in the FM phase

Dynamic scaling and growing lengths



$$G^{\perp}(r,t),\ G^{\|}(r,t) \simeq F_{\|,\perp}(r/L(t))$$

Stretched exponential $F(x) = e^{-(x/w)^v}$ with $v_{\|} \simeq v_{\perp} \simeq 0.15$ and $\neq w_{\|,\perp}$

the same growing length $L_{\|}(t),\ L_{\perp}(t) \simeq t^{1/2}$

until a band crosses the sample, then a different mechanism.

Scaling theory

At late times there is a single length-scale, the typical radius of the domains $L(t,g)$*, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by* $L(t,g)$*, e.g.*

$$C(r,t) \equiv \langle\, s_i(t) s_j(t)\,\rangle|_{|\vec{x}_i - \vec{x}_j| = r} \sim \langle \phi \rangle^2_{eq}(g)\; f\left(\frac{r}{L(t,g)}\right),$$

$$C(t,t_w) \equiv \langle\, s_i(t) s_i(t_w)\,\rangle \sim \langle \phi \rangle^2_{eq}(g)\; f_c\left(\frac{L(t,g)}{L(t_w,g)}\right),$$

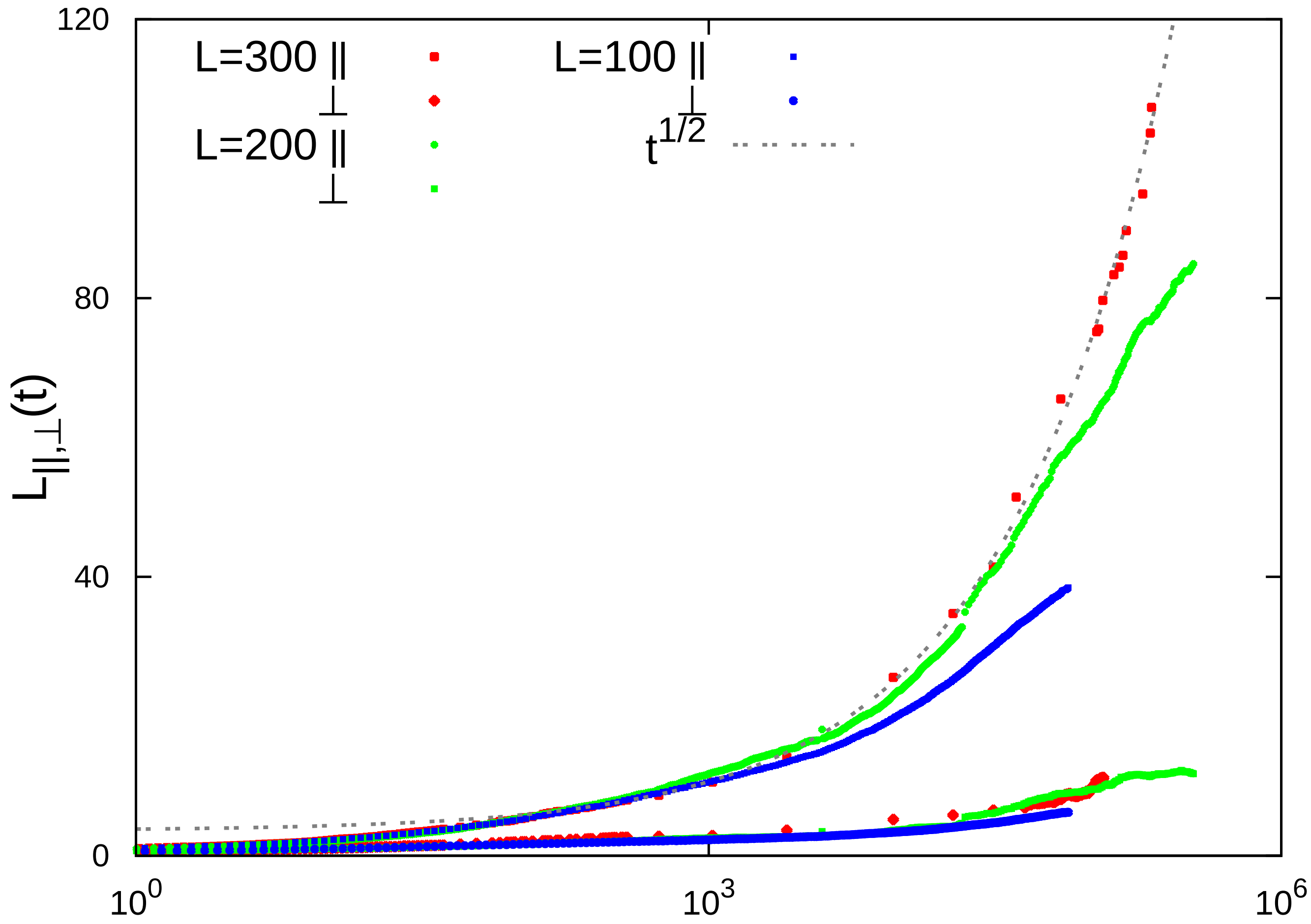
etc. when $r \gg \xi(g)$, $t, t_w \gg t_0$ and $C < \langle \phi \rangle^2_{eq}(g)$.

Suggested by experiments and numerical simulations. Proved for

- Ising chain with Glauber dynamics.
- Langevin dynamics of the $O(N)$ model with $N \to \infty$, and the spherical ferromagnet. **Review Bray 94.**

Dynamics in the FM phase

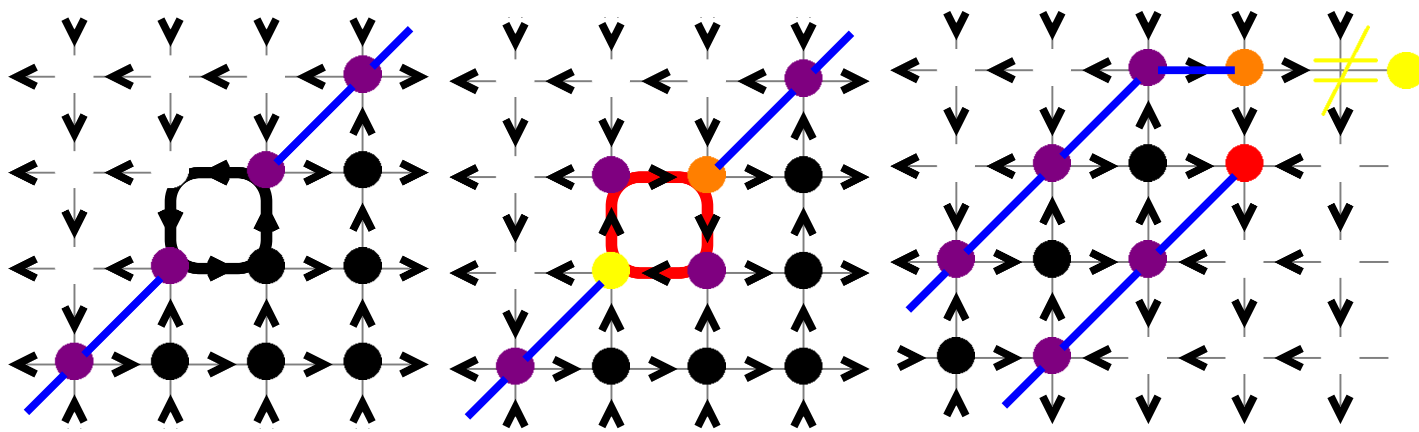
Growing lengths



Anisotropic growth of FM order with $L_{||}(t) \simeq t^{1/2}$

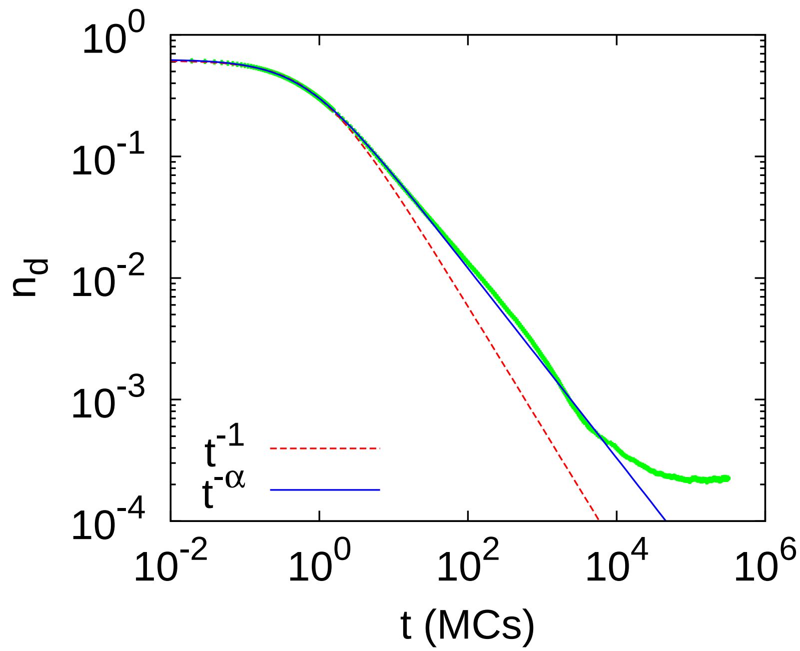
Dynamics in the FM phase

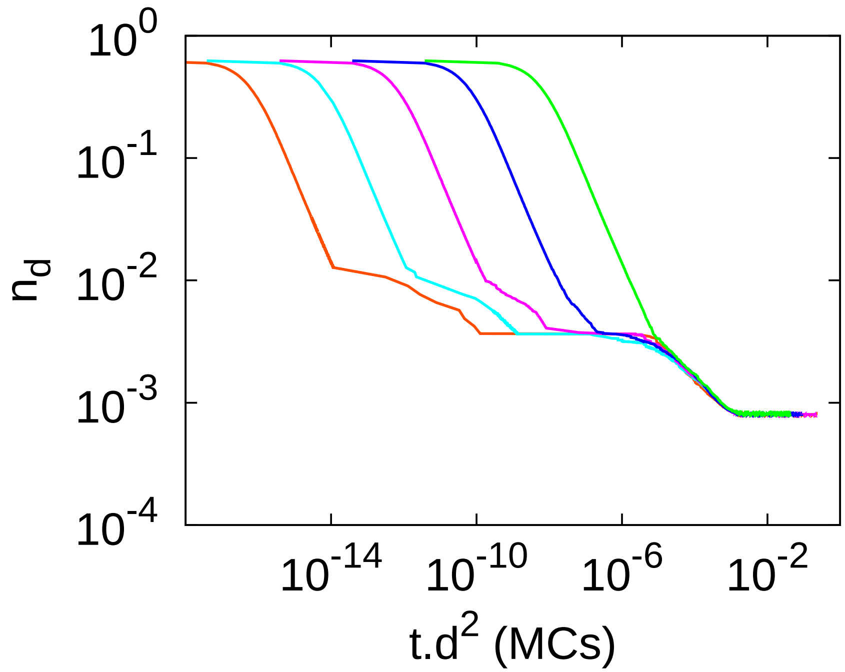
Some elementary moves



Dynamics in the D phase

Density of defects





Short-time decay $t^{-0.78}$

Different from MF approximation

to reaction - diffusion model t^{-1}.

$n_d \simeq f(td^2)$

Scaling below the plateau.

Single spin-flip dynamics

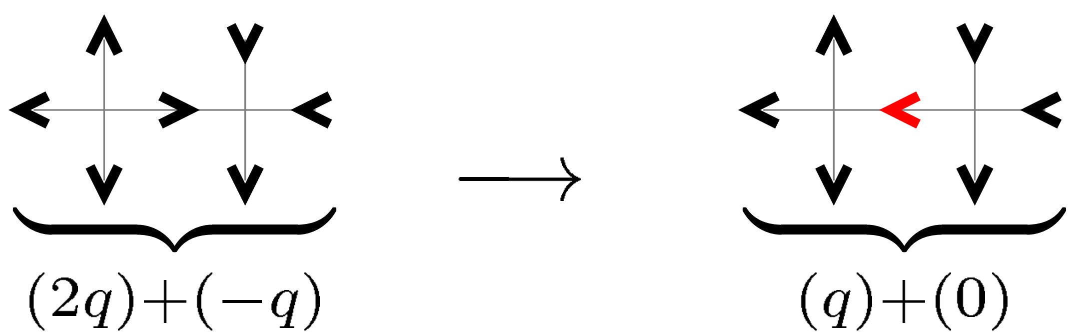
Reaction-diffusion picture in terms of the vertex charges

$$
\begin{array}{llll}
\text{Reaction} & & \Delta E & \\
(2q)_d + (-q)_e & \to (q)_e + (0)_a & \epsilon_a - \epsilon_d \propto \ln a/d & < 0 \\
(q)_e + (-q)_e & \to (0)_a + (0)_c & \epsilon_a + \epsilon_c - 2\epsilon_e \propto \ln ac/e^2 & < 0 \\
(q)_e + (q)_e & \to (2q)_d + (0)_a & \epsilon_d + \epsilon_a - 2\epsilon_e \propto \ln da/e^2 & \lessgtr 0 \\
(q)_e + (q)_e & \to (2q)_d + (0)_c & \epsilon_d + \epsilon_c - 2\epsilon_e \propto \ln dc/e^2 & \lessgtr 0
\end{array}
$$

since $\epsilon_{AF} < \epsilon_{FM} < \epsilon_e < \epsilon_d$.

e.g.,

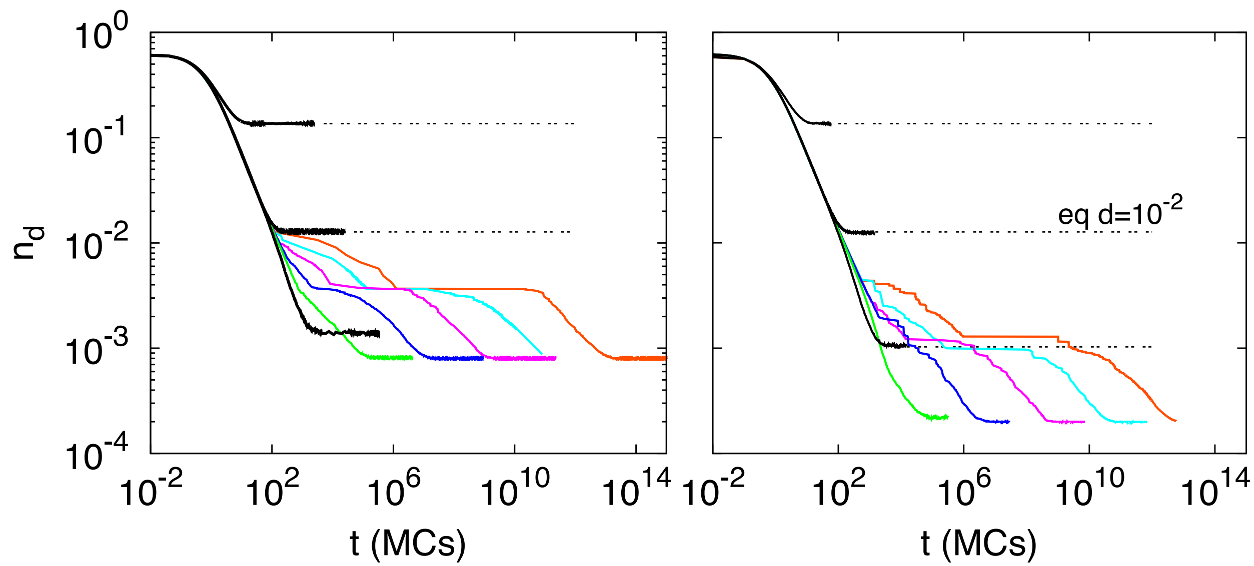
the first reaction is



Attn : “Directional diffusion" : vertices have to meet in the “good" direction.

Dynamics in the PM phase

Density of defects, $n_d = \#$**defects**$/\#$**vertices**



Relevant experimental sizes $L = 50$ $L = 100$

$a = b = c,\ d/c = e/c = 10^{-1}, 10^{-2}, \ldots, 10^{-8}$ from left to right.

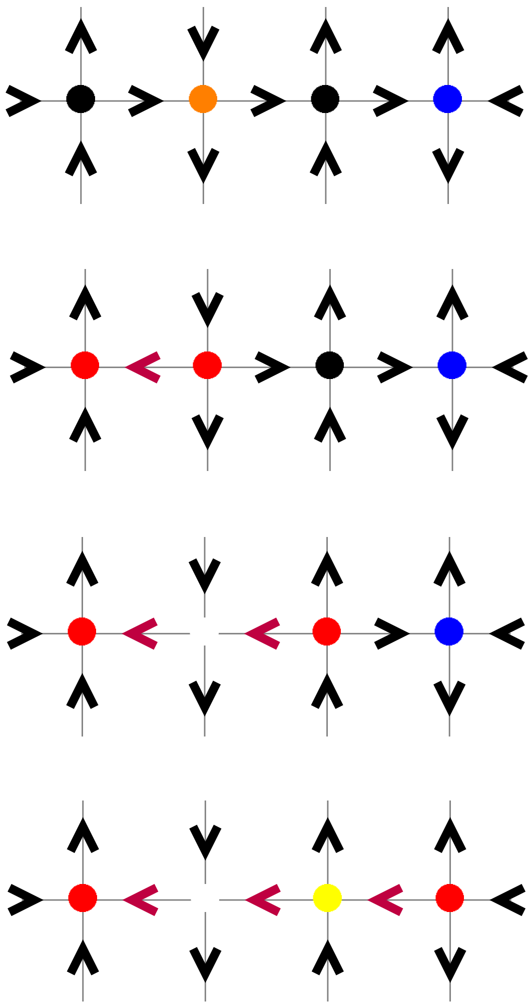
For $e = d \gtrsim 10^{-4}c$ the density of defects reaches its equilibrium value.

For $e = d \lesssim 10^{-4}c$ the density of defects gets blocked at $n_d \approx 10/L^2$.

It eventually approaches the final value $n_d \approx 2/L^2$ indep. of bc ; rough estimate for t_{eq} from reaction-diffusion arguments.

Deconfined monopoles

Ice-rule vs. ice-rule breaking vertices



Just spin-ice vertices

Two (3 in or 3 out) red defects

One lattice spacing apart

Two lattice spacings apart

NB, once created, the energy remains constant iff $a = b = c$.