# Glassy dynamics & coarsening

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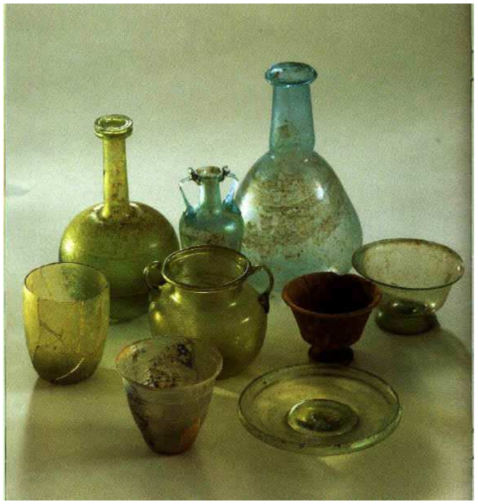
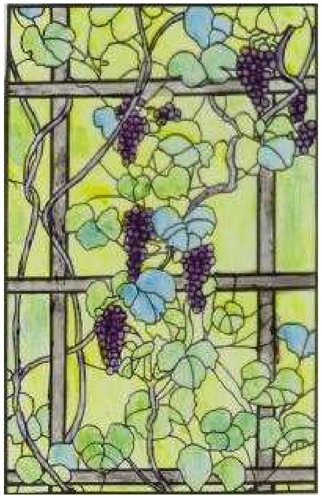
# Plan

- What is the glassy problem ? Overview.
- Fluctuations : some theoretical ideas coming from mean-field theory.
- **Let us make it simpler : back to coarsening phenomena**.

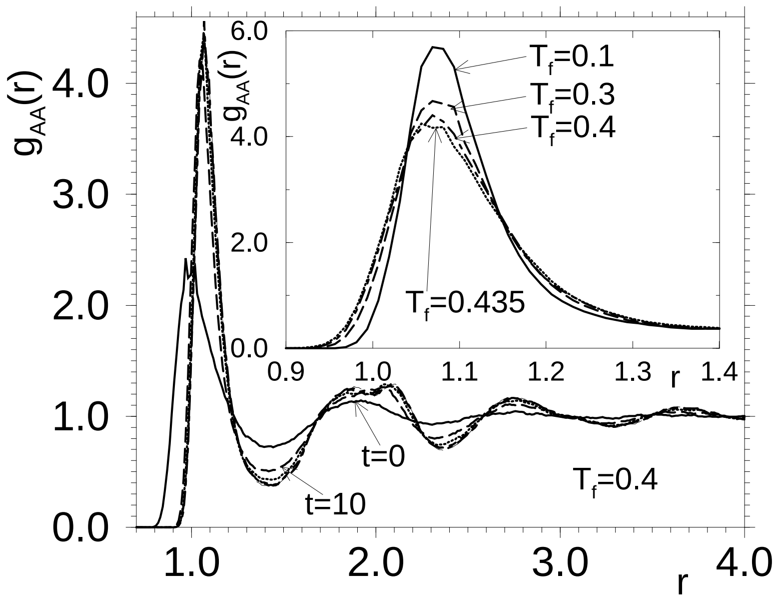


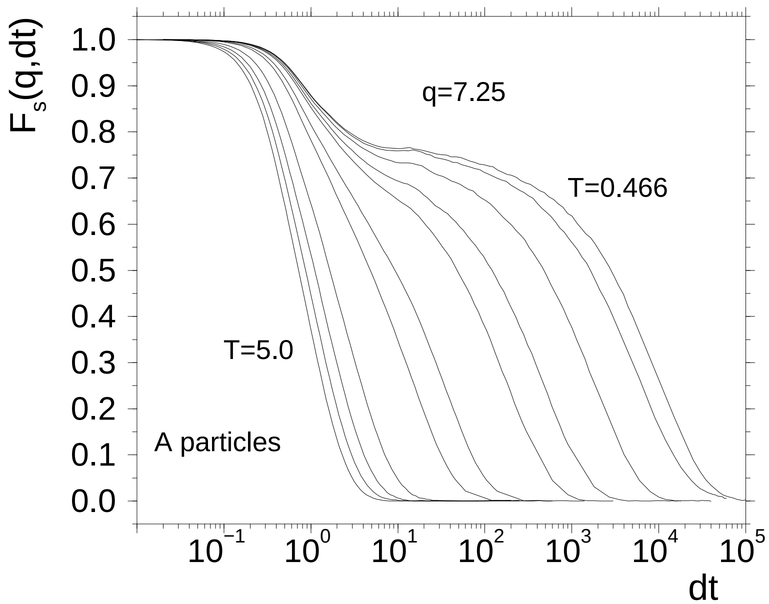


# Structural glasses

No obvious structural change

but slowing down !




LJ mixture $V_{\alpha\beta}(r) = 4\epsilon_{\alpha\beta}\left[\left(\frac{\sigma_{\alpha\beta}}{r}\right)^{12} - \left(\frac{\sigma_{\alpha\beta}}{r}\right)^{6}\right]$

$\tau_{micro} \ll \tau_{relax}$ that changes by $\approx 10$ orders of magnitude !

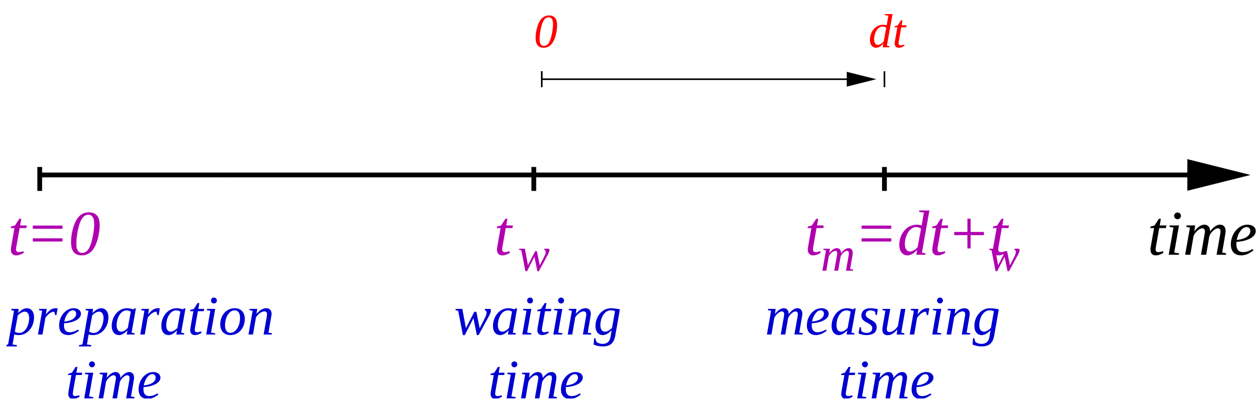
Time-scale separation & slow non-equilibrium dynamics

**Molecular dynamics, J-L Barrat & Kob, 99.**

# Random 1st order

- Upon approaching $T_g$ there is a separation of time-scales : rapid approach to a plateau, slow decay from it.
- The plateau in the correlation function can be interpreted as an order parameter, the Edwards-Anderson or non-ergodicity parameter, $q_{ea}$.
- $q_{ea}$ is finite at $T_g$
- There is no discontinuity in the linear susceptibility.
- The so-called $\alpha$-relaxation time – say, the time needed to reach $\frac{q_{ea}}{2}$ – increases when approaching $T_g$.

# Times

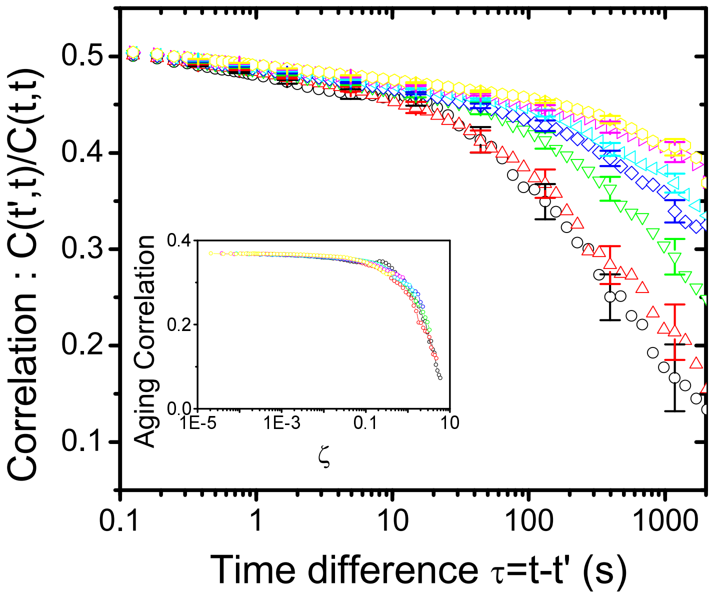


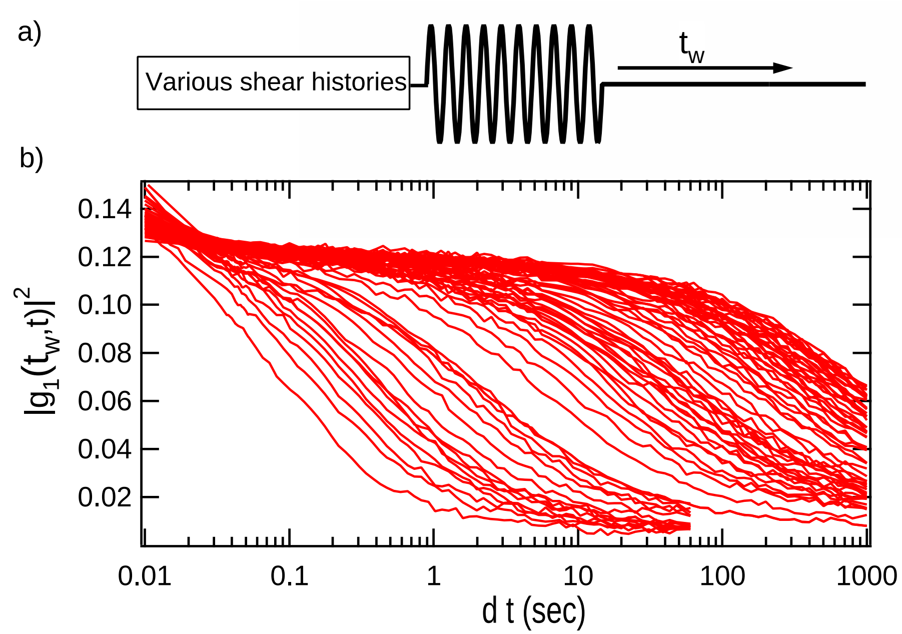

In equilibrium $t_w > t_{eq}$

Out of equilibrium $t_w < t_{eq}$

# Still lower temperature

Out of equilibrium relaxation




Spin-glass **Hérisson & Ocio, 99.**

Colloids **Viasnoff & Lequeux, 03.**

$$\tau_{micro} \ll \tau_{exp} \ll \tau_{relax}$$

The relaxation time goes beyond the experimentally accessible times

The same is observed in all other glasses.

# Spin-glasses

- Spin-glasses have a traditional 2nd order phase transition with standard exponents and so on (note the difference with structural glasses).
- There is no consensus on the nature of the equilibrium low-temperature phase :

  ‘many state’ scenario (replicas) *vs* a ‘disguised ferromagnet’ (droplets).
- Still, the non-equilibrium relaxation of spin-glasses is pretty similar to the one of structural glasses.

Relatively recent viewpoint :

**Deal with all glasses simultaneously.**

# Questions

1. Can one characterize the global/bulk dynamics ?

(**Mean-field/large dimensional models**)

2. What about fluctuations ? Local/mesoscopic dynamics

Idea : accept the glass without explaining how and why it appears

and describe its dynamics in detail.

(cfr. **phonons in solids**...)

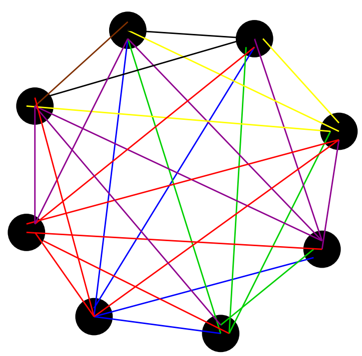
3. What happens in simpler situations ? Coarsening dynamics.

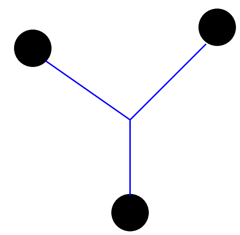
# 1. Global/bulk behaviour

# Models : statics & dynamics

**Exact treatment**

- Spins with **quenched random interactions** on a complete graph.





***p=2*** ***p=3, the unit***

- Particles in interaction on an infinitely dimensional space.

**Approximate treatment**

- Finite $d$ : self-consistent resummation (mode-coupling, etc.)

Large $N$ and/or large $d$ limit.

# $p$ spin disordered models

**Random first order transition**

$$H = -\sum_{i_1 i_2 \ldots i_p} J_{i_1 i_2 \ldots i_p} s_{i_1} s_{i_2} \ldots s_{i_p}$$

with $P(J_{i_1 i_2 \ldots i_p}) = e^{-p!\, J^2_{i_1 i_2 \ldots i_p}/(2N^{p-1} J^2)}$.

Ising, $s_i = \pm 1$, or spherical, $\sum_{i=1}^{N} s_i^2 = N$, spins.

Sum over all $p$-uplets (**mean-field**).

$p = 2$ spin-glass ; $p \geq 3$ structural glass.

- Statics : replica trick $[\ln Z_J] = \lim_{n \to 0} \frac{[Z^n] - 1}{n}$.
- Metastable states : Thouless-Anderson-Palmer approach.

**Rugged free-energy landscape**

**cfr. G. Parisi's talks.**

# Dynamic equations

**Stochastic dynamics**

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t) C(t,t_w) = \int dt' \; [\Sigma(t,t') C(t',t_w) + D(t,t') R(t_w,t')] + 2T R(t',t) \;,$$

$$(\partial_t - z_t) R(t,t_w) = \int dt' \, \Sigma(t,t') R(t',t_w) + \delta(t-t_w) \;,$$

where the self-energy and vertex are functions of $C$ and $R$ :

$$D(t,t_w) = \frac{p}{2} C^{p-1}(t,t_w) \;, \quad \Sigma(t,t_w) = \frac{p(p-1)}{2} C^{p-2}(t,t_w) \, R(t,t_w) \;.$$

and the Lagrange multiplier $z_t$ is fixed by $C(t,t) = 1$.

# Dynamics of $p$ spin models

**Analytical results**

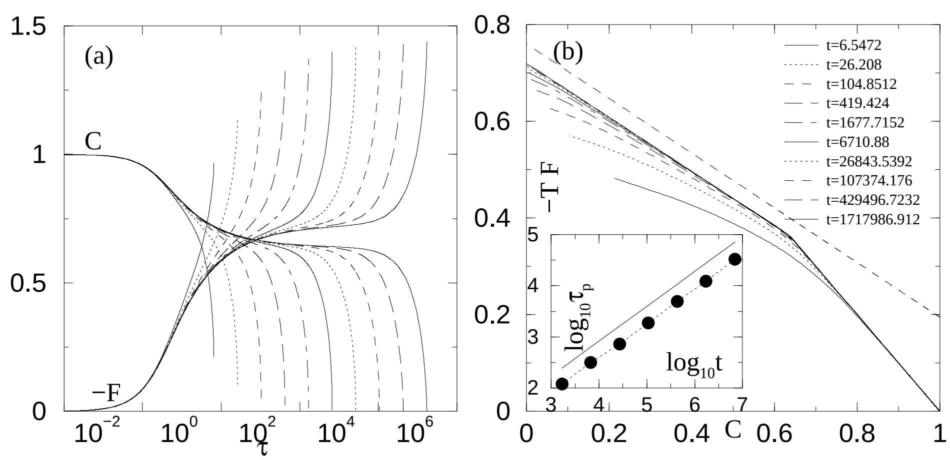
- separation of time-scales $C(t,t_w) = C_{st}(t-t_w) + C_{ag}(t,t_w)$

$$\chi(t,t_w) = \chi_{st}(t-t_w) + \chi_{ag}(t,t_w)$$

- Highly non-trivial relation between $\chi$ and $C$ : violations of FDT.
- (Approx) Time-reparametrization invariance $t \to h(t)$ in aging.

**LFC & Kurchan, 93.**

**(Smart) numerical results**




**Kim & Latz, 00.**

# The self-correlation

From now on we focus on the correlation functions. Their decay can be separated into scales; *e.g.* in the examples above there is a clear difference between the decay above and below the plateau.

**Scaling**

One can prove that within a correlation scale any two-time monotonic correlation scales as

$$C(t, t_w) = f_c\left(\frac{h(t)}{h(t_w)}\right)$$

with $h(t)$ a monotonic function of time. *This statement can be made mathematically precise*.

- Verified in the stationary regime : $h(t) = e^{-t/\tau}$ and $f_c : 1 \to q_{ea}$.
- Coarsening $h(t) = \mathcal{R}(T, t)$, the typical domain radius, $f_c : q_{ea} \to 0$.

**LFC & Kurchan, 94**

# Partial conclusions

**Pros**

- $p$**-spin disorder models** capture (though in an exaggerated manner) the phenomenology of super-cooled liquids and glasses :

  both thermodynamics and dynamics in and out of equilibrium.

- Their free-energy landscape (studied analytically with the TAP approach) allows to interpret the above results.

- An analytic solution to their non-equilibrium asymptotic dynamics exists.

  It allows one to compute analytically several but not all properties !

- Predictions of new phenomena **(fdt & $T_{\text{eff}}$, rheology, etc.)**

# Partial conclusions

**Pros - continued**

- **Is quenched disorder important ?** Approximate ways of studying models of particles in interaction lead to the same equations and predictions :

  - ⋆ real replica tricks for the statics of the equilibrium glass

    **Kirkpatrick, Thirumalai & Wolynes, 80s ; Mézard & Parisi, late 90s.**

  - ⋆ resummation techniques to study the dynamics (MCT)

    **Franz & Hertz, 96 ; Bouchaud, LFC, Kurchan, Mézard, 96 ; Latz, 00.**

# Partial conclusions

**Cons**

- The mean-field character of the theory. In structural glasses :
  - ⋆ The dynamic phase transition at $T_d$ ($\approx T_g$) does not exist !
  - ⋆ and the static one at $T_s$ ($\approx T_K$) we do not really know.
  - ⋆ Metastable states in finite $d$ systems do not have infinite life-time :

**Challenge** : Dynamics at finite times, $t(N)$.

How to go below the threshold ?

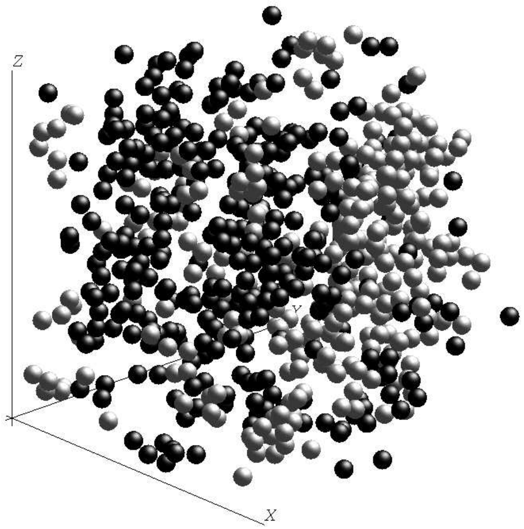
**Beyond**

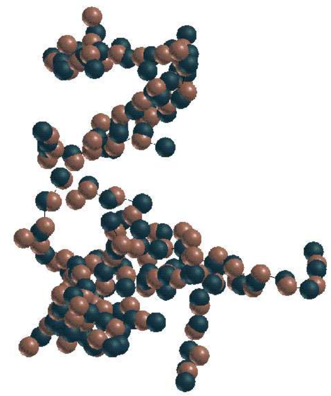
- Can one still use this approach as a guideline to explain dynamic heterogeneities ?

# 2. Fluctuations

# Heterogeneities

**Super-cooled Lennard-Jones binary mixture**



Clustering of fast and slow particles over a period $\Delta t$.

Strings of fast particles with increasing length for decreasing $T$.

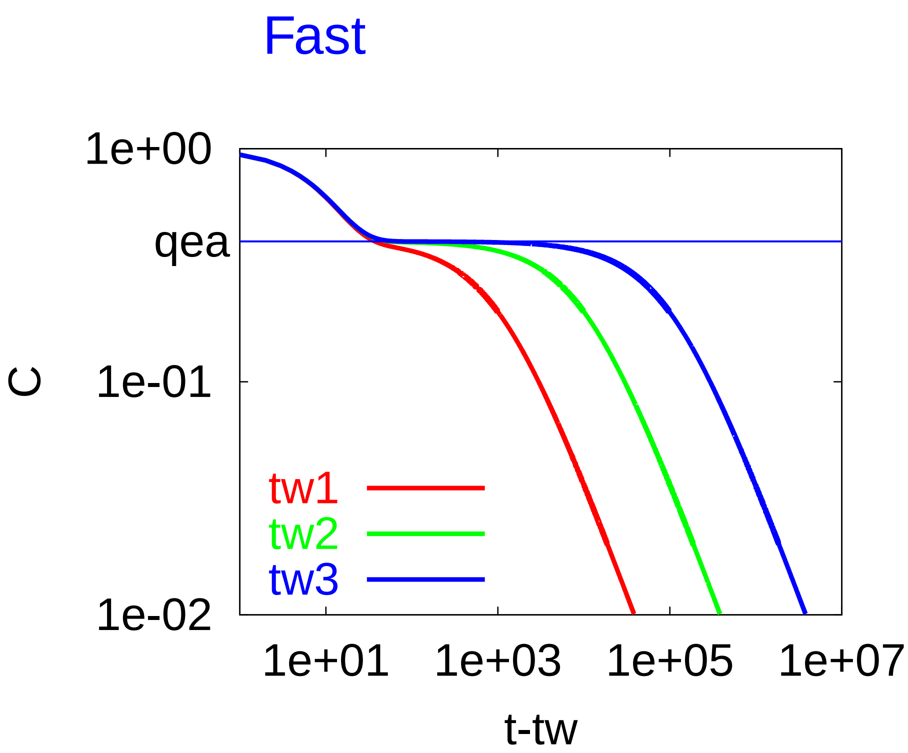
**Molecular dynamics S. Glotzer et al, 90s**

$p$-spin – mode-coupling theory : **divergent correlation length** at $T_d$.

**Wolynes** *et al*, **late 80s ; Franz & Parisi, 95 ; Biroli** *et al*, **03-06.**

# Separation of time-scales

In the long $t_w$ limit




Slow

$$C_{ag}(t, t_w) \approx f_c\left(\frac{h(t)}{h(t_w)}\right)$$

$$\partial_t C_{ag}(t, t_w) \ll C_{ag}(t, t_w)$$

log-log scale !

Eqs. for the slow relaxation $C_{ag} \equiv C < q_{ea}$ :

**Approx. asymptotic time-reparametization invariance** $t \to h(t)$

# Time-reparametrization

Example : the equation $(\partial_t - z_t)R(t,t_w) = \int dt' \; \Sigma(t,t')R(t',t_w)$

- Take $t - t_w \gg t_w$ use $z_t \to z_\infty$, drop $\partial_t R$ & separate the fast contributions to the integral :

$$\tilde{z}_\infty R_{ag}(t,t_w) \sim \int_{t_w}^{t} dt' \; D'[C_{ag}(t,t')] \, R_{ag}(t,t') \, R_{ag}(t',t_w) \; . \qquad (1)$$

- The transformation

$$t \to h_t \equiv h(t) \; , \qquad \begin{cases} C_{ag}(t,t_w) \to C_{ag}(h_t,h_{t_w}) \; , \\ R_{ag}(t,t_w) \to \frac{dh_{t_w}}{dt_w} \, R_{ag}(h_t,h_{t_w}) \; . \end{cases}$$

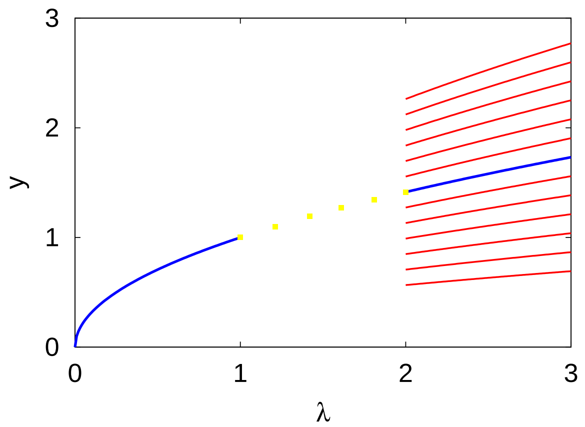
with $h_t$ positive and monotonic leaves eq. (1) **invariant** :

$$\tilde{z}_\infty \, R_{ag}(h_t,h_{t_w}) \sim \int_{h_w}^{h_t} dh_{t'} \; D'[C_{ag}(h_t,h_{t'})] R_{ag}(h_t,h_{t'}) \, R_{ag}(h_{t'},h_{t_w}) \; .$$

# Time reparametrization

A nuisance

Similar to the matching problem in non-linear diff. eqs.




$$\frac{dy}{d\lambda} = g[y(\lambda)]$$

Many asymptotic solutions

but only one is selected by the behaviour at small $\lambda$.

# Dynamic glassy problem

$$C_{ag}(t,t_w) \sim f_c\left(\frac{h(t)}{h(t_w)}\right)$$
$$\chi_{ag}(t,t_w) \sim f_\chi\left(\frac{h(t)}{h(t_w)}\right)$$

One can compute analytically $f_c$ and $f_\chi$ [ also $\chi_{ag}(C_{ag})$]

but not the 'clock' $h(t)$!

$h(t)$ is determined by matching to the stationary regime

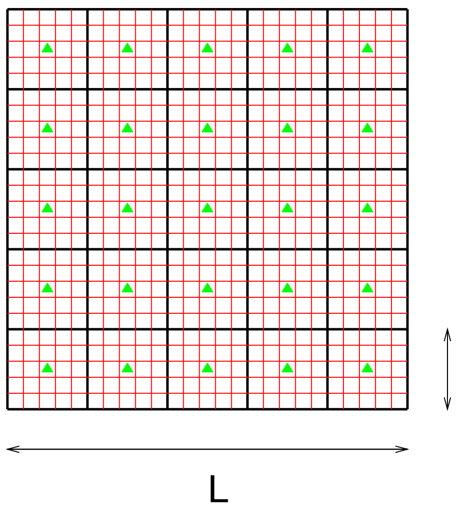
(short $t - t_w$ behaviour).

# Fluctuations

Idea : use the time-reparametrization symmetry

**to characterize the fluctuations**

**Coarse-graining**




$$C_r(t,t_w) \equiv \frac{1}{V_r}\sum_{i\in V_r}\phi_i(t)\phi_i(t_w)\ ,$$

$$\chi_r(t,t_w) \equiv \frac{1}{V_r}\sum_{i\in V_r}\int_{t_w}^{t} dt'\ \left.\frac{\delta\phi_i(t)}{\delta h_i(t')}\right|_{h=0}\ .$$

$$V_r = \ell^d\ , \qquad\qquad a \ll \ell \ll L\ .$$

# Leading fluctuations

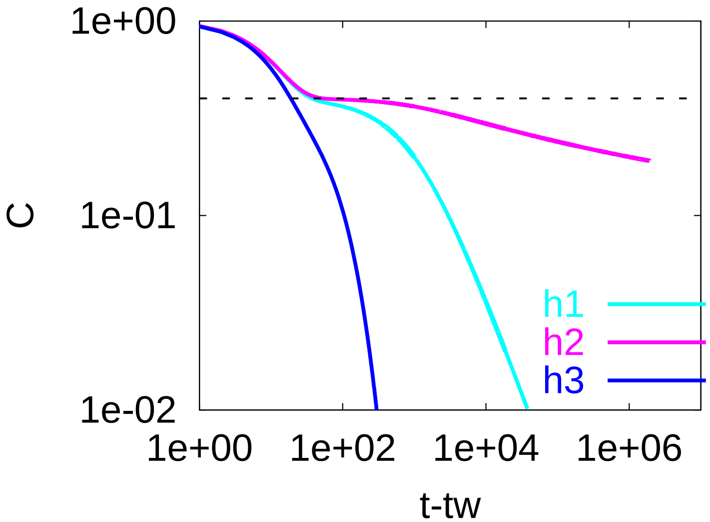
Symmetry $t \to h(t)$

. Global time-reparametrization invariance $\Rightarrow$

$$C_r^{ag}(t,t_w) \approx f_c\left(\frac{h_r(t)}{h_r(t_w)}\right)$$

.

Ex. $h_{r_1} = \frac{t}{t_0}$, $h_{r_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{r_3} = e^{\ln^a\left(\frac{t}{t_0}\right)}$ on different regions




Same $t_w$, slower and faster decays

**Castillo, Chamon, LFC,**

**Iguain, Kennett, 02, 03.**

# Turn it useful : $\sigma$ model

Easy fluctuations $t \to h_r(t) \quad \Leftrightarrow \quad$ divergent correlation length $\xi(t, t_w)$.

- Ideally : derive the action $S[h_r(t)]$.

Quasi-mean-field models, **Chamon, LFC, Franz, in progress.**

- In practice : propose the action $S[h_r(t)]$ and derive

*e.g.* $\rho[C_r^{ag}; t, t_w, \ell]$ ; $\rho[R_r^{ag}; t, t_w, \ell]$ ; $\rho[C_r^{ag}, R_r^{ag}; t, t_w, \ell]$

in the long-times limit, with $\ell/\xi \to 0$.

that can be checked numerically & experimentally.

**Kinetically const., Chamon, Charbonneau, LFC, Reichman, Sellitto, 04**

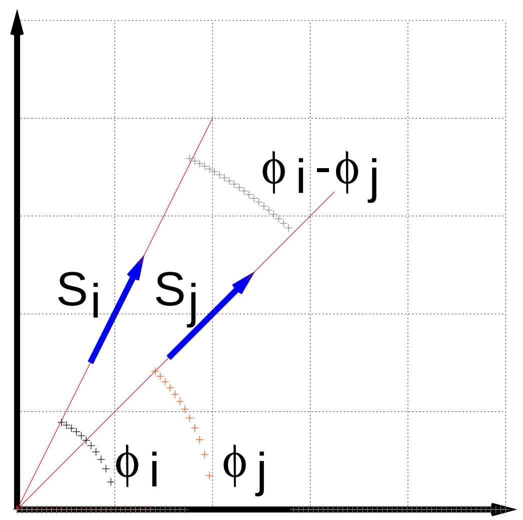
**Spin-glass, Jaubert, Chamon, LFC, Picco, 06**

**Lennard-Jones, Castillo & Parsaeian, 06**

# Parenthesis : an analogy

# The Heisenberg ferromagnet

**An analogy : a nuisance turned into a model**




Landau free-energy

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda \left[ m^2(\vec{r}) - m_0^2 \right]^2 \right\} .$$

Invariant under the global rotation $m^a(\vec{r}) = R^{ab} m^b(\vec{r})$.

(**Global time-reparametrization invariance**)

# Explicit symmetry breaking

**A pinning field**

Landau free-energy in a field $\vec{j}$

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda \left[ m^2(\vec{r}) - m_0^2 \right]^2 + \vec{j}\,\vec{m}(\vec{r}) \right\} .$$

No longer invariant under the global rotation

A particular direction is selected by the field : $\vec{m}(\vec{r}) = m_0 \hat{j}$.

**[$\partial_t C$ and $\partial_t R$ select the 'clock' $h(t)$**

**but they become less and less important as the system ages &**

**$t - t_w$ increases !]**

# Statics of the Heisenberg ferro

Ground state : $\vec{m}(\vec{r}) = m_0\hat{\jmath}$ for all $\vec{r}$.

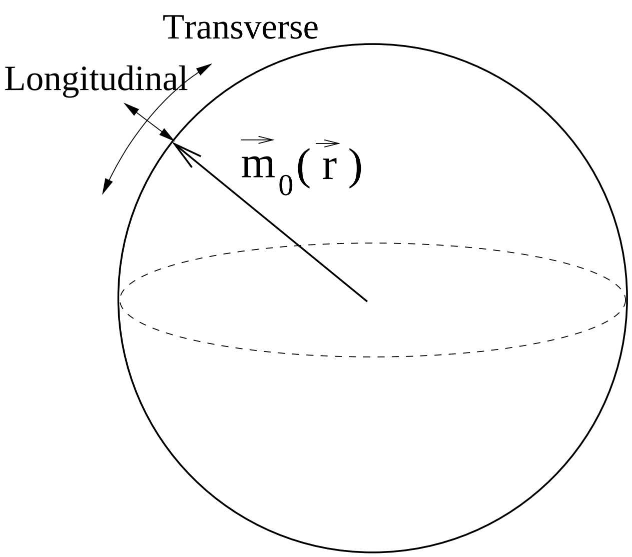
Fluctuations : $\vec{m}(\vec{r}) = \vec{m}_0 + \delta\vec{m}(\vec{r})$.

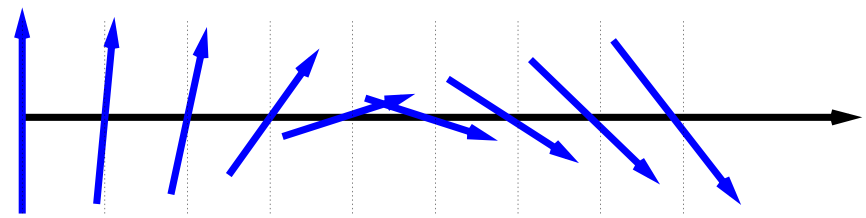
Longitudinal ($//\jmath$) hard &

transverse ($\perp\ \jmath$) (easy) fluctuations.

Spin-waves

Low energy excitation



[Time-reparametrization waves]

# End of the parenthesis

# Partial conclusions

- The proposal is that **time-reparametrization invariance** – developing asymptotically – is the symmetry responsible for a **divergent dynamic correlation length**.

(think of the similarity with **general relativity** !)

- This symmetry can be used to construct a $\sigma$-model.

(think of the similarity with **O(N), Heisenberg or xy models**.)

- This proposal has a predictive power and it **can be put to the test**, numerically and experimentally.

**Reviews : Chamon & LFC, 04, 07.**

# $\sigma$-model

Slow decay in terms of $h_r(t) \equiv e^{-\varphi_r(t)}$

$$C_r^{ag}(t,t_w) \approx f_c\left(\frac{h_r(t)}{h_r(t_w)}\right) = f_c\left(e^{-\int_{t_w}^{t} dt' \partial_{t'}\varphi_r(t')}\right)$$

The simplest

(i) global time-reparametrization invariant ;

(ii) local in space ;

(iii) positive definite ($\partial_t h_r(t) > 0 \Rightarrow \partial_t \varphi_r(t) > 0$) ;

(iv) invariant under $\varphi_r(t) \to \varphi_r(t) + \Phi(r)$ as $C_r^{ag}$ effective action is

$$\mathcal{A} = K \int d^d r \int dt \, \frac{[\nabla \partial_t \varphi_r(t)]^2}{\partial_t \varphi_r(t)}$$

Similar to a Gaussian surface after a redefinition of time and field.

# Simpler problems ?

Which is the 'simplest' problem with out of equilibrium dynamics where one can analyse fluctuations ?

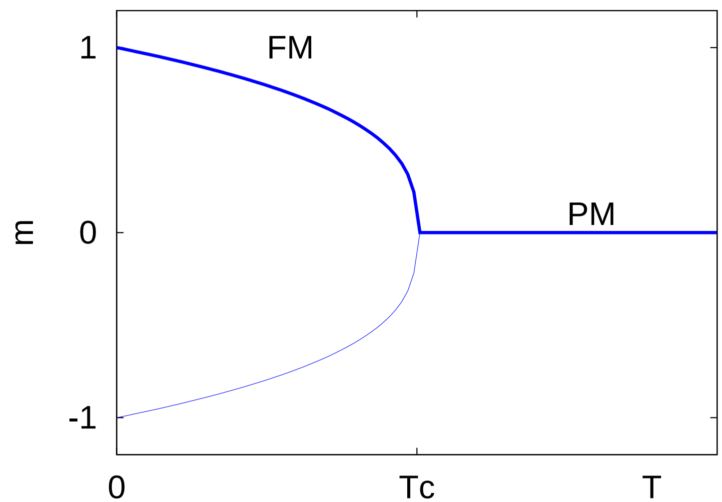
Two natural candidates, that are obviously related :

- Plain domain growth phenomena.
- Elastic lines (Edwards-Wilkinson model).

# 3. Coarsening

# Ferromagnets in equilibrium

A ferromagnetic system in contact with a heat bath at temperature $T$ under no applied field ($h = 0$) acquires a magnetization density $m$ below a critical temperature $T_c$ :




$m$ is the order parameter.

**Curie-Weiss mean-field theory (1907), Ginzburg-Landau theory (1937, 1950), Wilson renormalization group (1971).**

# The standard Ising model

$$H = -J \sum_{\langle ij \rangle} s_i s_j \ ,$$ **Ising, 1925**

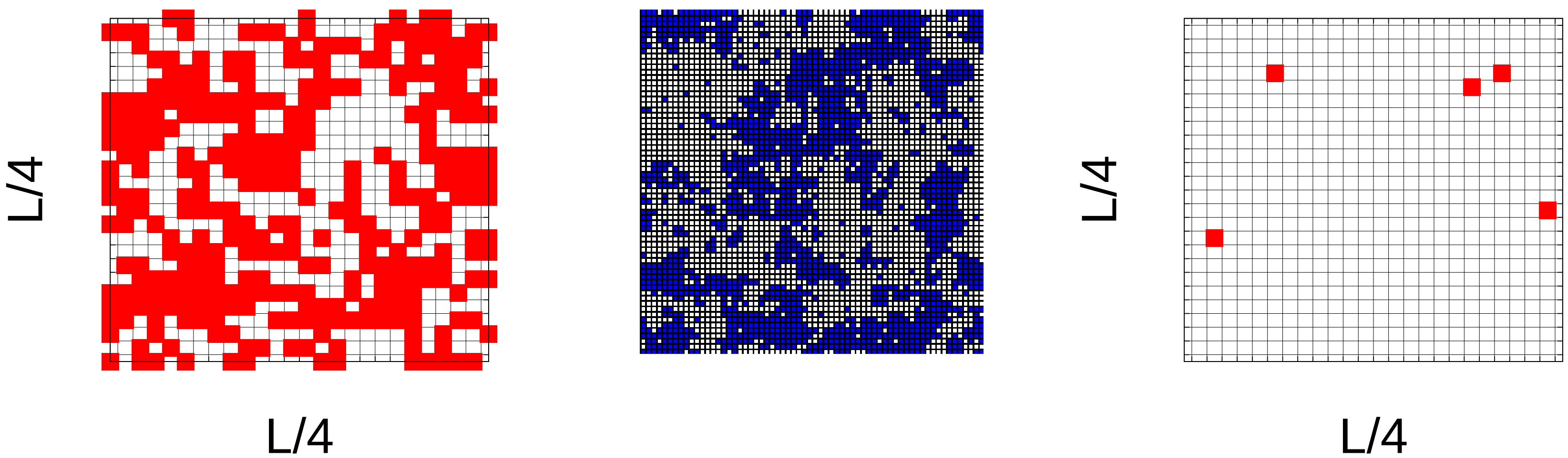
- The spins $s_i$ take bimodal values, $s_i = \pm 1$.
- The sum $\sum_{\langle ij \rangle}$ runs over nearest neighbours on a $d$ dimensional, typically hypercubic, lattice.
- $J > 0$ is the coupling strength.

One finds

$$\frac{T_c}{J} \begin{cases} = 0 \ , & d = 1 \quad \text{exact (\textbf{Ising, 1925}),} \\ \sim 2.27 \ , & d = 2 \quad \text{exact (\textbf{Onsager, 1944}),} \\ \sim 4.5 \ , & d = 3 \quad \text{num. (\textbf{D. P. Landau, 1976}).} \end{cases}$$

# Equilibrium configurations

$2d$ slices of a $3d$ Ising model




$T \to \infty$ — Random configuration — $s_i = \pm 1$

$T \sim T_c$ — Structures of all sizes — Self-similarity

$T < T_c$ — Essentially ordered — Thermal fluct. ($m < 1$)

# Ginzburg-Landau

Coarse-graining $\Rightarrow$ the local magnetization density

$$\phi(\vec{x}) = \frac{1}{\ell^d} \sum_{i \in V_{\vec{x},\ell}} s_i \; , \qquad Z = \sum_{\phi} e^{-\beta F(\phi)} \; .$$

Symmetry arguments ($\phi \to -\phi$) and $\langle \, \phi \, \rangle \sim 0$ at $T \sim T_c$ suggest

$$F(\phi) = \int d^d x \left[ \underbrace{\frac{c}{2}(\nabla \phi)^2}_{\text{Energy-cost domain-wall}} + \underbrace{\frac{T - T_c}{T_c}\phi^2 + \frac{\lambda}{4}\phi^4}_{\text{Symmetric double-well}} \right]$$

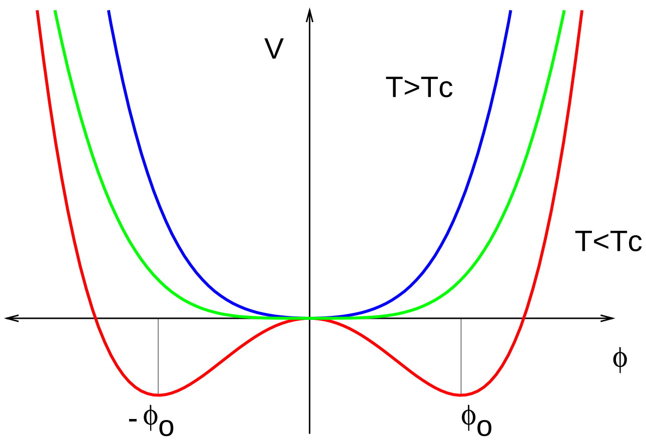
Energy-cost domain-wall

Symmetric double-well

# Ginzburg-Landau

Large volume limit

$F \approx L^d \Rightarrow$ saddle-point, mean-field or stationary phase approx.

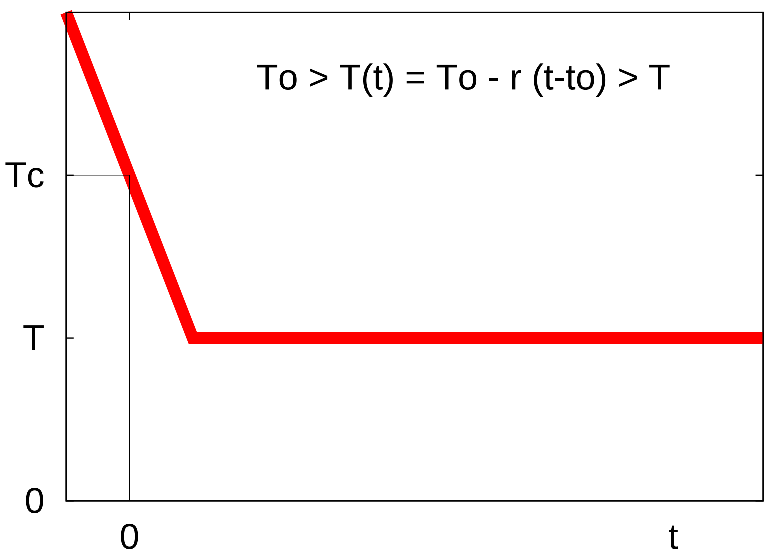


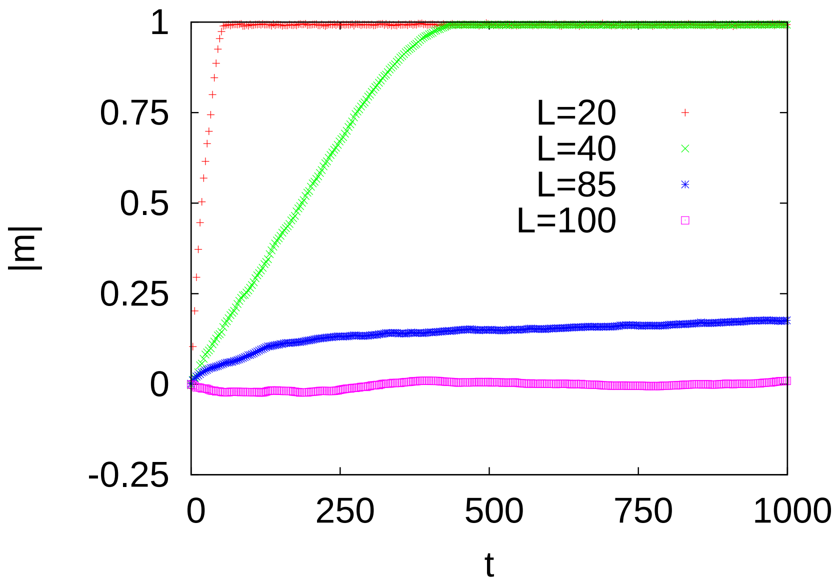

$$\langle\, \phi(\vec{x}) \,\rangle = \phi_o \propto (T_c - T)^{\frac{1}{2}} \,, \qquad \beta = \frac{1}{2} \,.$$

Essentially correct but for the critical region (*e.g.* $\beta \sim 1/3$).

# Evolution

A rapid quench



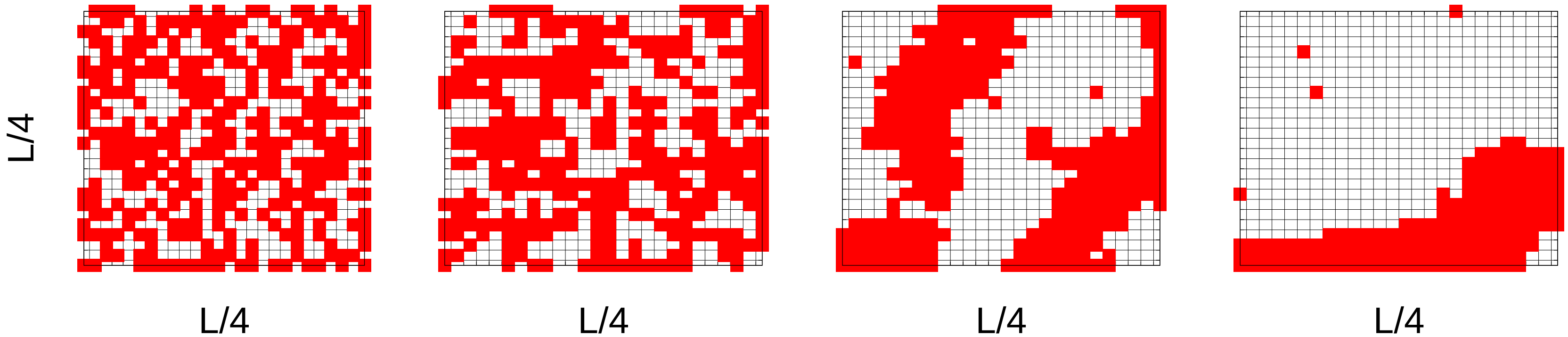

Stochastic dynamics ; Monte Carlo updates

Note : the order parameter ($m$) is not conserved.

Slow dynamics

# Domain growth

After a rapid quench




$t = 0$ MCs $t = 1$ MCs $t = 10$ MCs $t = 10^2$ MCs

# Time-dep. Ginzburg-Landau

$$\phi(\vec{x}) \to \phi(\vec{x},t) = \frac{1}{\ell^d} \sum_{i\in V_{\vec{x},\ell}} s_i(t)$$

Langevin dynamics in $F(\phi)$, model A **(Hohenberg & Halperin 1977).**

$$\begin{aligned} \gamma \frac{\partial \phi(\vec{x},t)}{\partial t} &= -\frac{\delta F(\phi)}{\delta \phi(\vec{x},t)} + \eta(\vec{x},t) \\ &= \nabla^2 \phi(\vec{x},t) + a\phi(\vec{x},t) - \lambda \phi^3(\vec{x},t) + \eta(\vec{x},t)\ , \end{aligned}$$

with $\gamma = t_0^{-1}$ and $\eta$ a Gaussian white noise,
$\langle\, \eta \,\rangle = 0$ and $\langle\, \eta(\vec{x},t)\eta(\vec{x}',t') \,\rangle = 2k_B T \gamma\, \delta(\vec{x}-\vec{x}')\delta(t-t')$.

# Scaling theory

*At late times there is a single length-scale, the typical radius of the domains* $\mathcal{R}(T,t)$*, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by* $\mathcal{R}(T,t)$*, e.g.*

$$C(r,t) \equiv \langle\, s_i(t) s_j(t)\,\rangle|_{|\vec{x}_i - \vec{x}_j| = r} \sim m_{eq}^2(T)\; f\left(\frac{r}{\mathcal{R}(T,t)}\right),$$

$$C(t,t_w) \equiv \langle\, s_i(t) s_i(t_w)\,\rangle \sim m_{eq}^2(T)\; f_c\left(\frac{\mathcal{R}(T,t)}{\mathcal{R}(T,t_w)}\right),$$

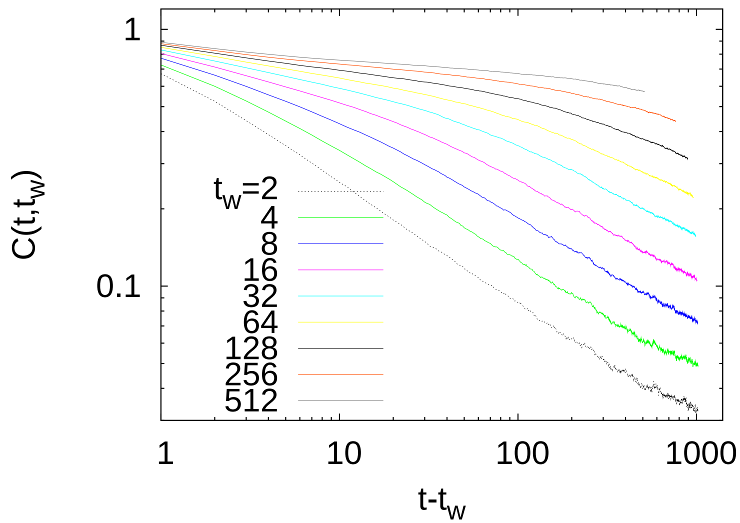
*etc.* when $r \gg \xi(T)$, $t, t_w \gg t_0$ and $C < m_{eq}^2(T)$.

Suggested by experiments and numerical simulations. Proved in

- Ising chain with Glauber dynamics.
- Langevin dynamics of the O($N$) model with $N \to \infty$, and the spherical ferromagnet. **Review Bray, 1994.**

# Self correlation

$$C(t, t_w) = N^{-1} \sum_{i=1}^{N} s_i(t) s_i(t_w)$$

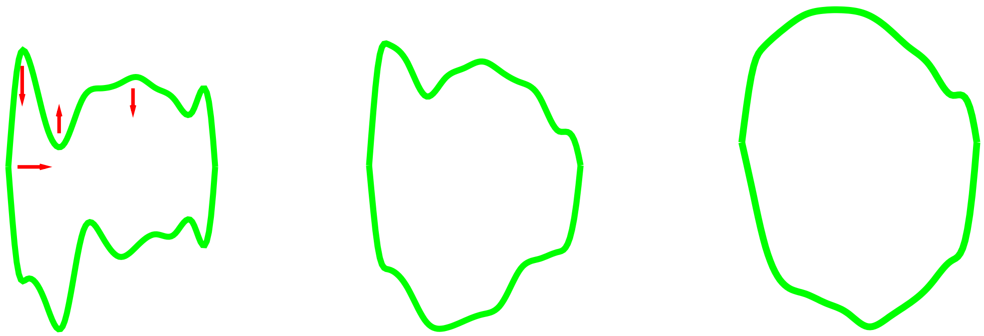



$$C(t, t_w) = C_{st}(t - t_w) + C_{ag}\left(\frac{\mathcal{R}(T,t)}{\mathcal{R}(T,t_w)}\right)$$

Equil. fluct. in domains | Domain wall motion

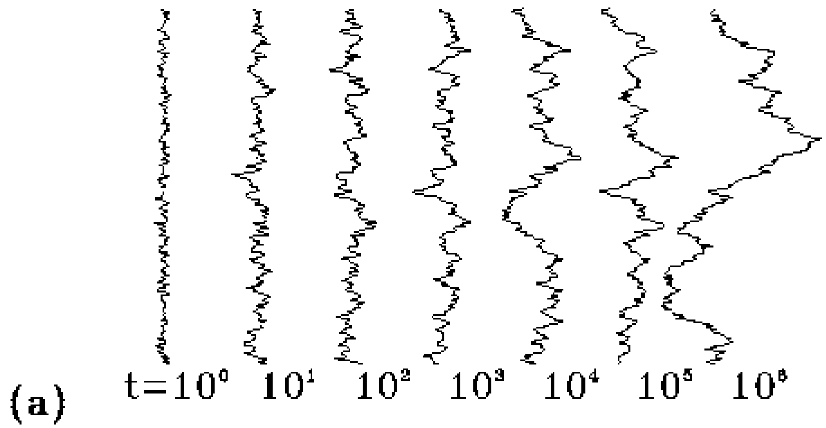
# Domain wall motion

(1) Curvature driven ($T = 0$) : $\vec{v} \propto K\hat{n}$ ; **Allen & Cahn, 79**



(2) Domain wall roughening ($T > 0$)

(3) Domain wall roughening and pinning by quenched disorder

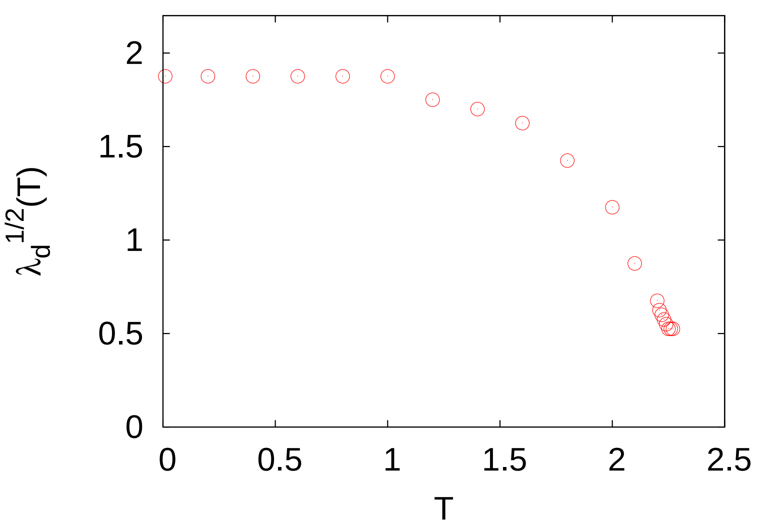



*e.g.* elastic line in random media ; **Kolton** *et al*, **05**

# MC dynamics 2dIM

The typical length-scale ⇔ a typical area

$$\mathcal{R}(T,t) \sim \sqrt{\lambda(T)\; t} \qquad \Leftrightarrow \qquad A(T,t) \sim \lambda(T)\; t$$




NB the exponent $\frac{1}{2}$ is independent of $T$ and the details of the dynamics, lattice, *etc.* as long as the order parameter is non-conserved & there is no disorder.

The $T$-dependence in $\lambda(T)$ is due to the roughening of the domain walls.

# Domain wall motion

The Allen-Cahn argument implies that at zero temperature the walls tend to become circular.

Temperature roughens the walls, opposing the previous the trend.

For this reason, $\lambda(T)$ decreases to zero at $T_c$.

**What happens locally ?**

Basic question : what does $\mathcal{R}(T,t)$ really mean ?

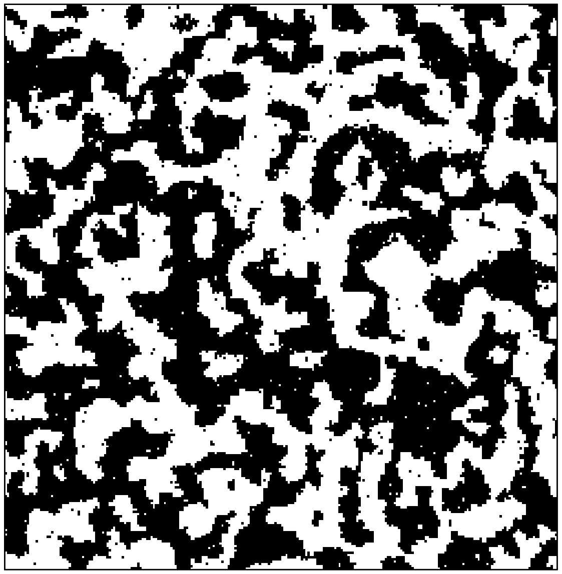
Can we compute, *e.g.* the number of domains with size $A$ at time $t$ per unit area of the system ($L^2$) ?

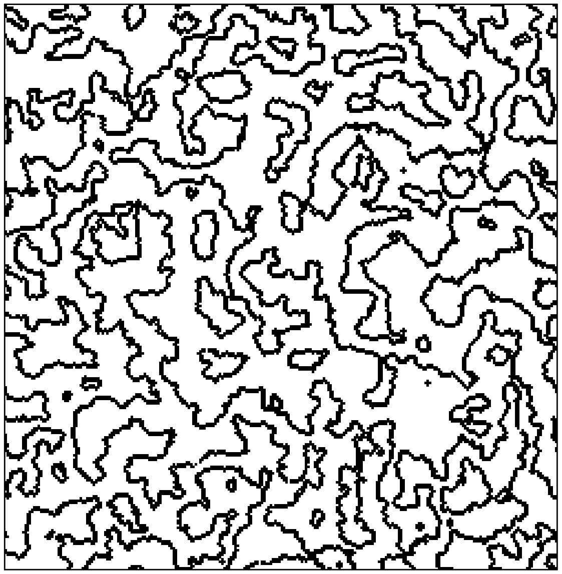
First : a short detour, domains and hulls.

# Geometry

An instantaneous configuration

$t = 32$ MCs $\qquad T = 1.5$



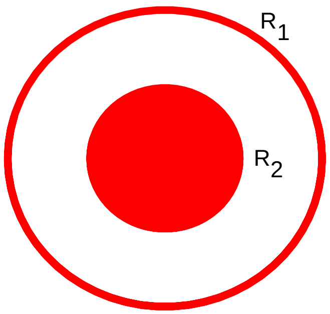


Domains Walls

So typical means...

**Arenzon, Bray, LFC, Sicilia**

# Domain & hull areas




Two hulls

$$A_1 = \pi R_1^2$$
$$A_2 = \pi R_2^2$$

Two domains

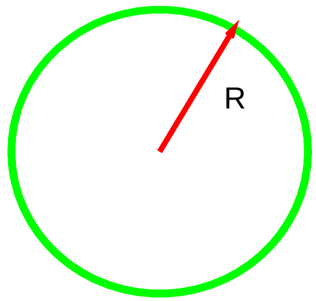
$$A_1 = \pi(R_1^2 - R_2^2)$$
$$A_2 = \pi R_2^2$$

*Hull : the interior of a domain boundary.*

- Typically hulls tend to be larger than domains ($A_1^h > A_2^h$).
- There are as many hulls as domains (two).
- Each spin belongs to one and only one domain (*e.g.* spin at the center).
- A spin can belong to more than one hull (*e.g.* spin at the center).

# Hull area $t$-dependence

Take a sphere with radius $R$, area $A = \pi R^2$ and perimeter $L = 2\pi R$.



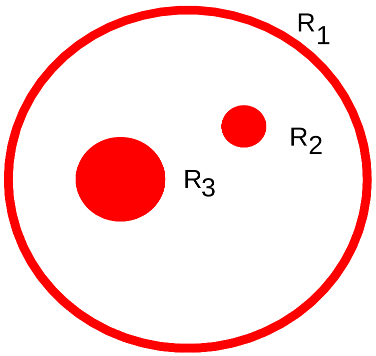

The time-variation of the *hull area*, $\frac{dA}{dt} = 2\pi R \, \frac{dR}{dt} = L\, v$ , in the case $v = -\frac{\lambda}{2\pi}\kappa$, with the curvature $\kappa = \frac{1}{R}$, is just constant

$$\frac{dA}{dt} = -\lambda$$

For a generic geometry, the velocity of the wall is proportional to the local mean curvature, and the hull area also decreases with constant velocity.

# Domain area $t$-dependence

An example




$$A_i(t) = A_i(t_0) - \lambda(t - t_0)$$

while $\quad A_i > 0$

$$i = 1, 2, 3 \; .$$

- Hulls with initial area smaller than $\lambda t$ will have disappeared at $t$.
- Hulls with initial area larger than $\lambda t$ will have decreased by $\lambda t$.

$$\frac{dD_1}{dt} = \frac{dA_1}{dt} - \frac{dA_2}{dt} - \frac{dA_3}{dt} = -\lambda[1 - \nu(t)]$$

The outer domain area grows while its hull area shrinks.

$\nu(t)$ instantaneous number of internal walls.

# The evolution

The number of hulls (or domains) with area $A$ at time $t$ is given by

$$n(A,t) = \int dA_0 \; \delta(A - A(t, A_0)) \, n(A_0, t_0)$$

where $n(A_0, t_0)$ is the initial distribution and $A(t, A_0)$ is the area at time $t$ given that the area at the initial time $t_0$ was $A_0$.

For convenience we normalize by the area of the full system, $L^2$.

Thus, we need $A(t, A_0)$ and $n(A_0, t_0)$.

# The hull area distribution

$$d = 2$$

Each hull area : $\frac{dA}{dt} = -\lambda$ . Thus $A(t, A_0) = A_0 - \lambda(t - t_0)$ .

- Quench from an infinite temperature $\Leftrightarrow$ random initial condition,

  $s_i = \pm 1$ with $p = \frac{1}{2}$ ; close to critical percolation, $p_c$, in $d = 2$.

- Quench from equilibrium at $T_c$ : Ising cluster hulls at criticality.

The equilibrium hull area distribution at $p_c$ and $T_c$ are

$$n(A_0, t_0) \sim \frac{(2) c_h}{A_0^2} \,, \qquad c_h = \frac{1}{8\pi\sqrt{3}} \qquad\qquad (a^2 \ll A_0 \ll L^2)$$

Conformal field theory, scaling & numerical checks

**Cardy & Ziff, 03.**

# The domain area distribution

$$d = 2$$

- For the domain area $\frac{dA}{dt} = -\lambda\,[1-\nu(t)]$ where $\nu(t)$ is the number of internal walls of the chosen domain. One can approximate it in a 'mean-field' manner as

$$\nu(t) \sim \langle\,\nu(t)\,\rangle = A_h(t)\int_0^{A_h(t)} dA'\; n_h(A',t)$$

- The initial domain distribution is known at $T_c$ only :

$$n_d(A_0,t_0) \sim \frac{c_d\;a^{2(\tau-2)}}{A_0^\tau} \qquad\qquad (a^2 \ll A_0 \ll L^2)$$

with $\tau = 379/187 \sim 2.02674$ ; $c_d$ is not known analytically.

**Vanderzande & Stella, 89 ; Janke & Schakel, 05.**

# The predictions

$$n_h(A,t) \equiv \frac{(2)c_h}{(A+\lambda t)^2} \qquad\qquad n_d(A,t) \sim \frac{(2)c_d\,(\lambda_d t)^{\tau-2}}{(A+\lambda_d t)^\tau}$$

in the long time limit and for large areas such that $a^2 \ll A \ll L^2$.

Note that we derived (!) the expected scaling forms :

$$n_h(A,t) = (\lambda t)^{-2}\, f\left(\frac{A}{\lambda t}\right) \qquad\qquad n_d(A,t) = (\lambda_d t)^{-2}\, f\left(\frac{A}{\lambda_d t}\right) \ .$$

The new parameters are $c_d = c_h + O(c_h^2)$ and $\lambda_d = \lambda + O(c_h)$.

Moreover, the sum rules,

$$N_h(t) = N_d(t) \qquad\qquad \int dA\, n_d(A,t) = 1$$

relate $c_h$ to $\tau$ !

**Arenzon, Bray, LFC, Sicilia, 06-07.**

# Simulations

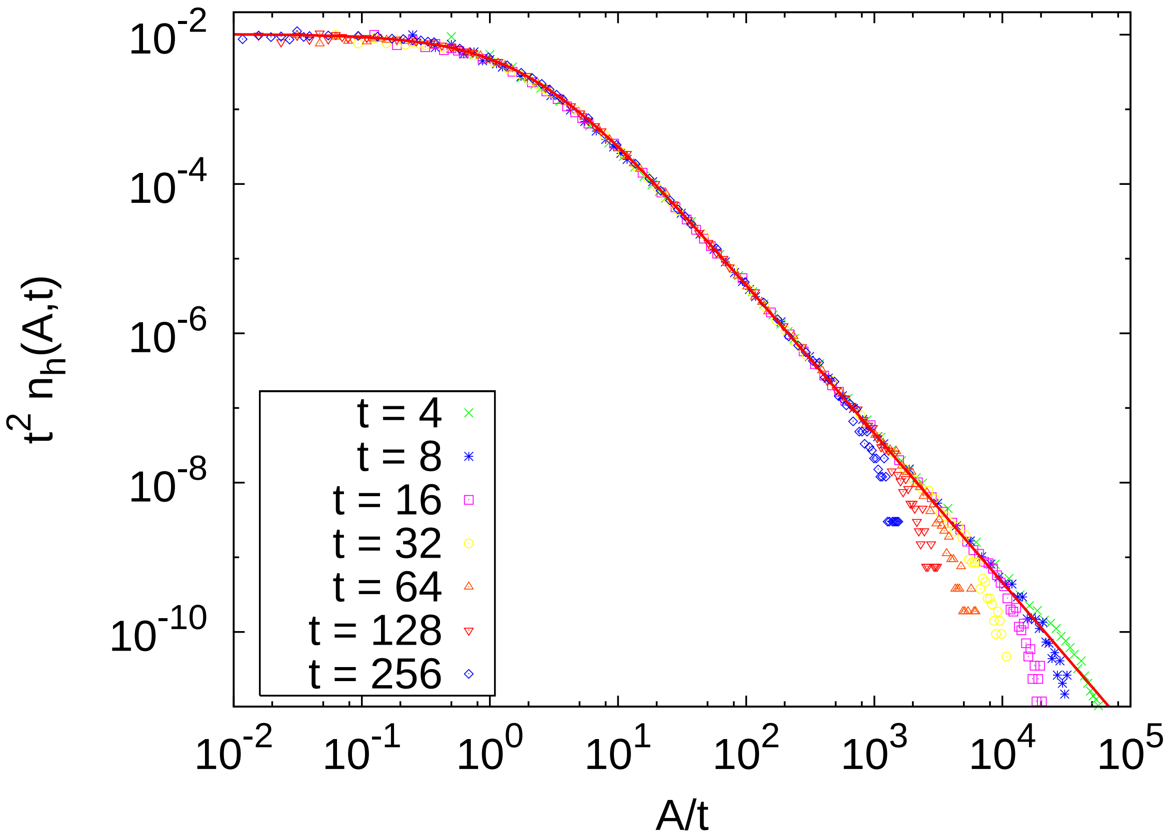
- $2d$ Ising model on a square lattice with periodic boundary conditions.
- Monte Carlo (MC) dynamics with heat-bath updates.
- $L = 10^3$, $2 \times 10^3$ samples, one time step corresponds to a MC sweep.
- Critical initial conditions generated with the Swendsen-Wang cluster algorithm to avoid critical slowing down.
- Hoshen-Kopelman algorithm to identify the domains.
- Our algorithm to identify the hulls inspired by the one used in

**R. M. Ziff, cond-mat/0510633, StatPhys22.**

# Numerical tests

Number density of (finite) *hulls* per unit area

$T = 0$ dynamics after a quench from $T \to \infty$

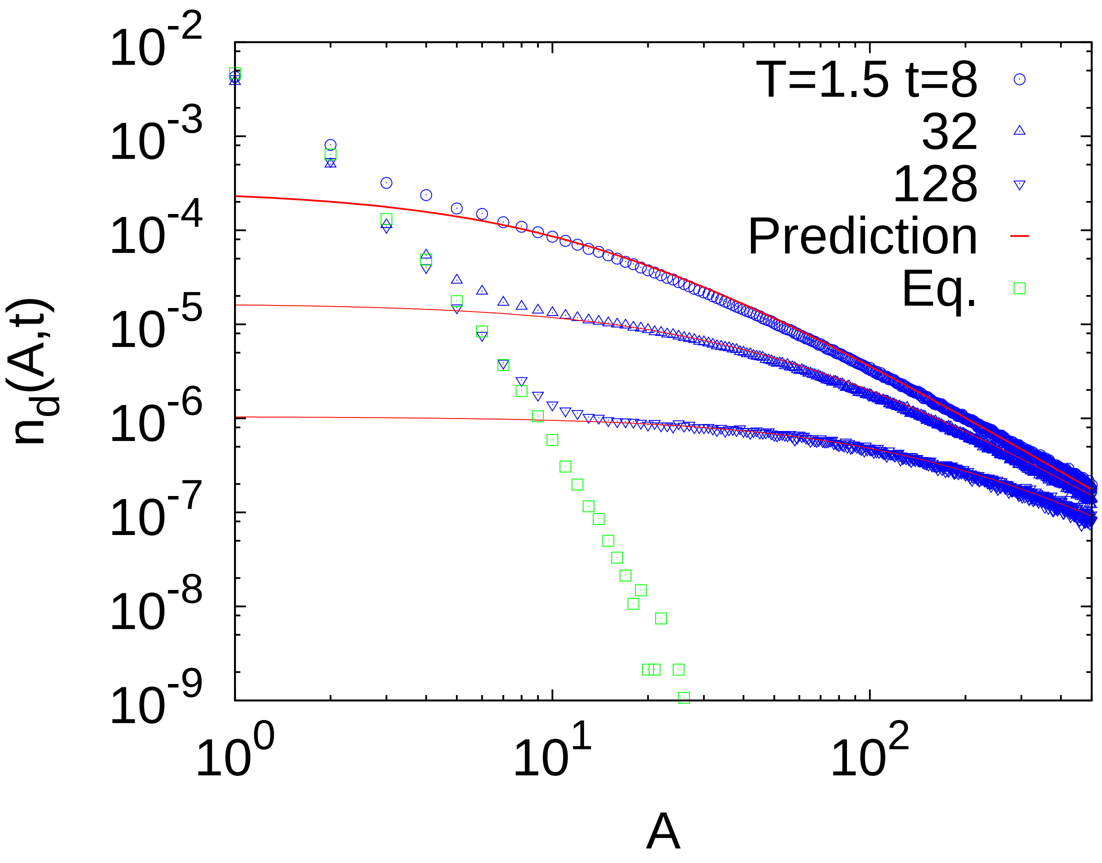



The bending is a finite size effect due to the percolating hulls.

# Numerical tests

Number density of (finite) *domains* per unit area

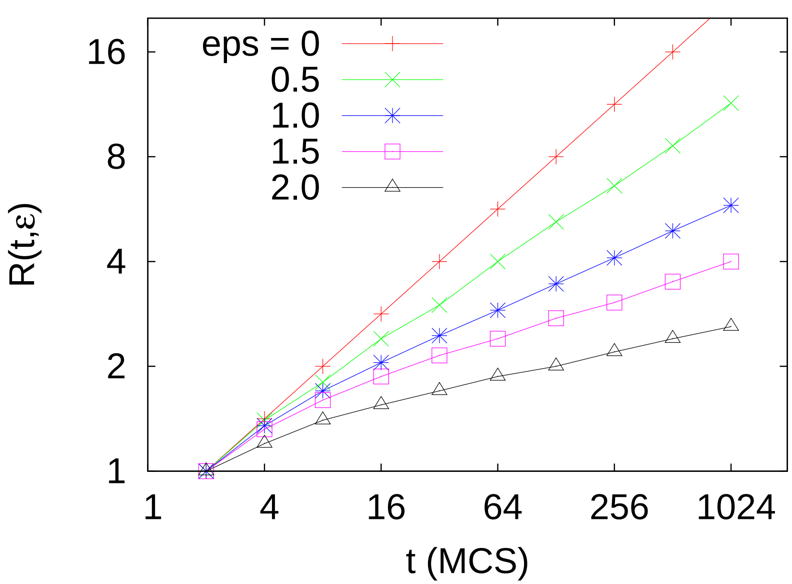
$T > 0$ dynamics after a quench from equilibrium at $T_c$

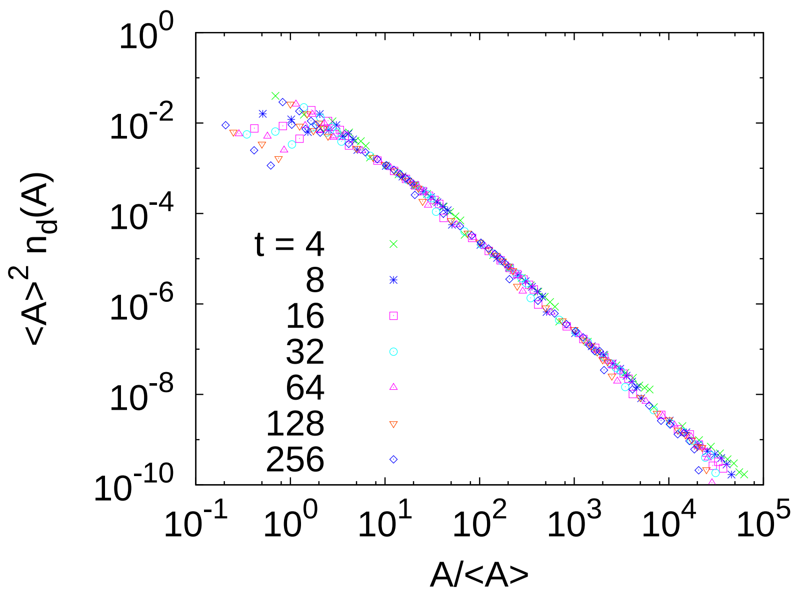


# Random ferromagnet

$$H = -\sum_{\langle ij \rangle} J_{ij} s_i s_j \ , \qquad J_{ij} \text{ uniform distributed in } [2-\epsilon, 2+\epsilon]$$

with $|\epsilon| \leq 2$. Number density of *domain* areas per unit area




$$T = 0.4 \ , \qquad \epsilon = 2 \ , \qquad \text{RIC} \ , \qquad \langle A \rangle \propto \mathcal{R}^2(T,t)$$

# Summary

- Exact results for hull area pdfs. We proved scaling !
- Approximate results for domain area. Consistent with scaling.
- The typical length-scale is not so typical after all :

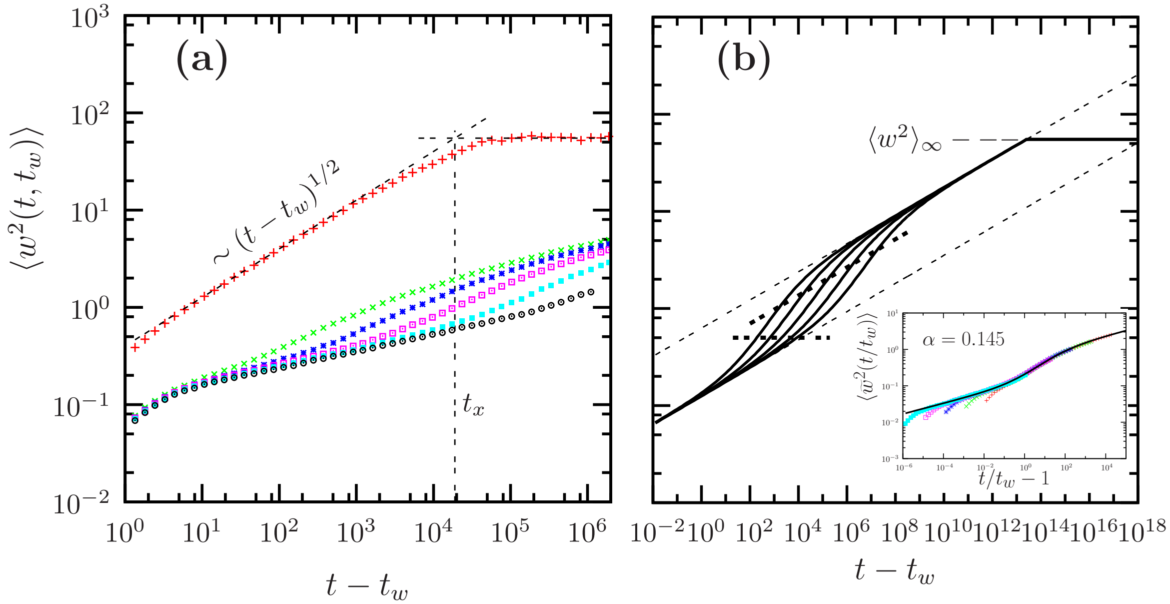
  power-law tails in $n_h$ and $n_d$.
- **What do we want to do with this ? Relate it to** $\rho(C_r; t, t_w, \ell)$.

- The large distribution of radii implies a similar one for perimeters.
- Domain walls are usually treated as elastic lines. But elastic lines also

  have an intricate behaviour – aging superimposes to the growth

  regime in the equilibrium Family-Vicsek scaling !

# Elastic lines




Aging two-time roughness $w^2(t,t_w) \equiv L^{-1}\sum_{y=1}^{L} |\delta x_y(t) - \delta x_y(t_w)|^2$ with $\delta x_y(t) \equiv x_y(t) - \overline{x}(t)$ the distance from the center of mass.

Multiplicative scaling : $\langle\, w^2(t,t_w)\,\rangle \approx R^\zeta(T,t_w)\ \mathcal{F}[\mathcal{R}(T,t)/\mathcal{R}(T,t_w)]$ with $\mathcal{R}(T,t)$ the growing length and $\zeta$ the growth exponent.

Fluctuations : $\langle w^2\rangle p(w^2) = \overline{\Phi}\,(w^2/\langle w^2\rangle; \langle \widetilde{w}^2\rangle, \langle w^2\rangle/\langle w^2\rangle_\infty)$ .

**Bustingorry, Iguain, LFC, Chamon, Domínguez, 06**

# Summary of the talks

- Phenomenology of non-equilibrium relaxation in glassy systems.
- A successful mean-field approach including statics and dynamics.

  A proposal to describe the origin of fluctuations (with predictive power).
- What can we learn from simpler problems ?

  coarsening and roughening of elastic lines.

# Future work

- Experiments :

  ⋆ video microscopy in colloidal suspensions **Makse** *et al*.

  ⋆ atomic force measurements in polymer glasses **Israeloff** *et al*.

- Numeric :

  Molecular dynamics of Lennard-Jones mixtures, soft-sphere models,

  *etc.* **Fabricius** *et al.***, Castillo** *et al.***, Loi** *et al*.

- Analysis : **Linear responses** are crucial.

  The global linear response of coarsening problems (at the observed

  times) is very anomalous (infinite effective temperature).

  Fluctuations also seem to be different.

**A better understanding ?**

# The linear response

The relaxation of the linear response

$$\chi(t, t_w) = \chi_{st}(t - t_w) + \chi_{ag}(t, t_w)$$

serves to distinguish different families (‘universality classes’) of models.

The difference arises in the aging part

$$\lim_{t_w \to \infty,\ C(t,t_w)\ \text{fixed}} \chi_{ag}(t, t_w)$$

- Structural glasses and spin-glasses : It remains finite.
- Coarsening : It vanishes.

Proved in mean-field models and other simple cases ;

supported by numerical simulations and arguments.

# Hull area $t$-dependence

A spherical hull in $d = 3$

Take a sphere with radius $R$, volume $V = \frac{4}{3}\pi R^3$ and surface $A = 4\pi R^2$.

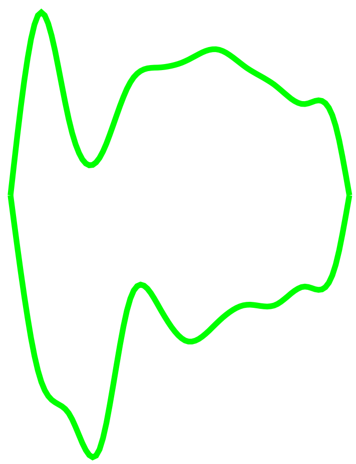
The time variation of the *hull* volume, $\frac{dV}{dt} = 4\pi R^2 \ \frac{dR}{dt}$, in the case $v = -\frac{\lambda}{2\pi}\kappa$, with $\kappa$ the *mean* curvature, *is not* constant :

$$\frac{dV}{dt} = -2R \propto -V^{1/3} \ .$$

Guess : $\frac{dV}{dt} \sim -V^{1/3}$ for generic geometries.

# Hull area $t$-dependence

A generic hull in $d = 2$



with radius $R$, area $A$ and perimeter $L$.

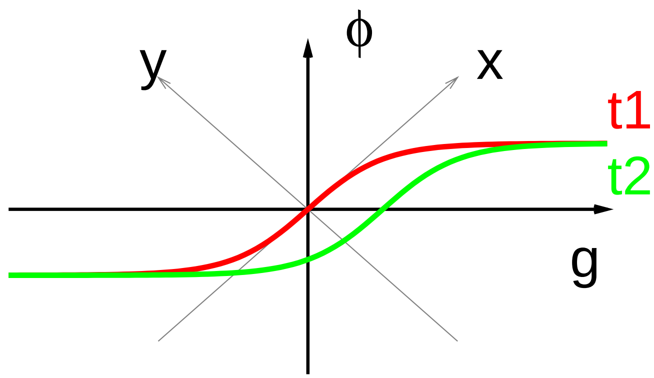
The time-variation of the *hull area*, $\frac{dA}{dt} = \oint \vec{v} \wedge d\vec{\ell} = \oint v d\ell$ , in the case $v = -\frac{\lambda}{2\pi}\kappa$, with $\kappa$ the *geodesic curvature*, is *also* constant

$$\frac{dA}{dt} = -\lambda$$

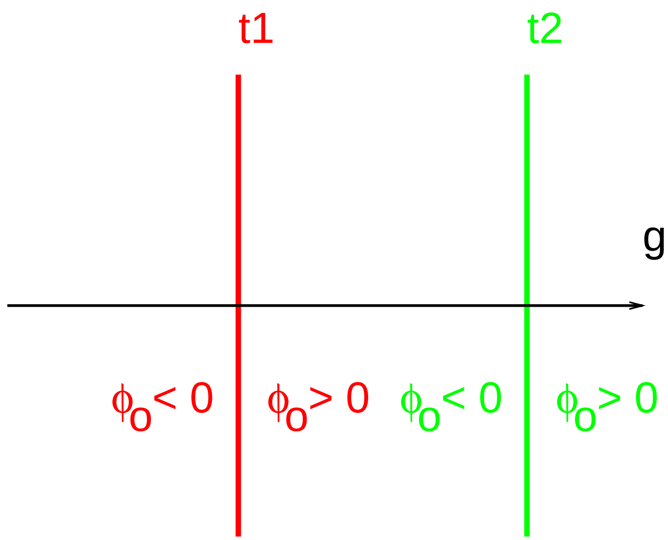
due to the Gauss-Bonnet theorem $\int_A K dA + \int_{\partial A} \kappa d\ell = 2\pi\chi(A)$ that simply becomes $\oint \kappa d\ell = 2\pi$ for a planar $2d$ manifold with no holes.

# T=0 argument

Domain wall profile




View from the top




$$\frac{\partial \phi(\vec{x},t)}{\partial t} = -\left.\frac{\partial \phi(\vec{x},t)}{\partial g}\right|_t \left.\frac{\partial g}{\partial t}\right|_\phi , \qquad \vec{\nabla}\phi(\vec{x},t) = \left.\frac{\partial \phi(\vec{x},t)}{\partial g}\right|_t \hat{g} ,$$

$$\nabla^2 \phi(\vec{x},t) = \left.\frac{\partial^2 \phi(\vec{x},t)}{\partial g^2}\right|_t + \left.\frac{\partial \phi(\vec{x},t)}{\partial g}\right|_t \vec{\nabla}\cdot\hat{g}.$$

Using $\frac{\partial^2 \phi(\vec{x},t)}{\partial g^2}|_t = V'(\phi)$ in the GL equation : $v \equiv \partial_t g|_\phi = -\vec{\nabla}\cdot\hat{g}$.

# Some consequences

- Temporal scaling of the pdf of local correlations dictated by the global correlation $\rho(C_r; t, t_w, \ell) = \rho[C_r; C^{ag}(t, t_w)]$ in the long-times limit.

- Negatively-skewed, non-Gaussian $\rho(C_r; C^{ag})$ for $0 < C^{ag} < q_{ea}$.

- The two-time dependent correlation length $\xi(t, t_w)$,

$$\left[\sum_i C_i^{ag}(t, t_w) C_j^{ag}(t, t_w)\right]_c \approx e^{-|\vec{r}_i - \vec{r}_j|/\xi(t,t_w)} ,$$

should diverge with $t$ and $t_w$.

- Constant of motion. $\rho[C_r, \chi_r; t, t_w]$ should follow the global FDT rel. :

$$\lim_{t_w \to \infty; C(t,t_w)=C} \chi(t, t_w) = \chi(C) .$$

**All can be tested with simulations & experiments.**