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# A $\sigma$ -model for glassy dynamics

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Newton Institute, Cambridge, 30/05/2006

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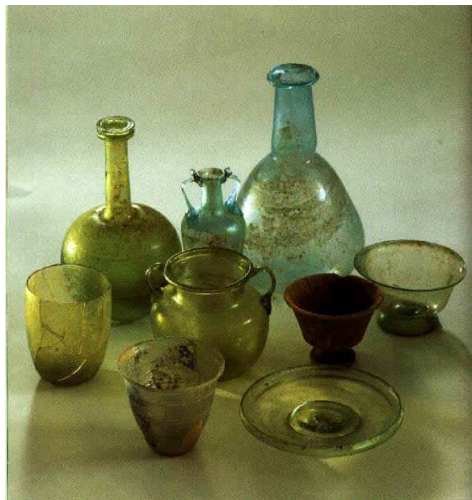
# Plan

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- Dynamics of mean-field models

Fully connected disordered spin systems &  
mode-coupling approximations

- **Describing fluctuations.**



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# Disordered spin models

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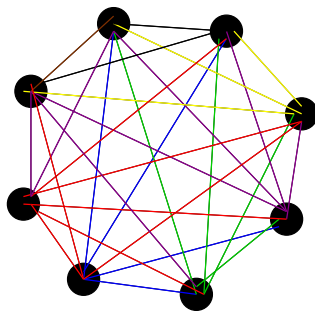
The fully-connected  $p$ -spin spherical model

$$E_J(\vec{s}) = \sum_{i_1 \neq \dots \neq i_p} J_{i_1 \dots i_p} s_{i_1} \dots s_{i_p} + z \left( \sum_i s_i^2 - N \right),$$

$z$  is a Lagrange multiplier,

$J_{i_1 \dots i_p}$  are quenched i.i.d. Gaussian random variables with  $[J_{i_1 \dots i_p}] = 0$

and  $[J_{i_1 \dots i_p}^2] \propto N^{1-p}$ .



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# Stochastic dynamics

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The system is coupled to its environment  $\Rightarrow$  stochastic dynamics for  $s_i$ ,  
e.g. Langevin dynamics in the overdamped limit

$$\gamma \dot{s}_i(t) = -\frac{\delta E_J(\vec{s})}{s_i(t)} + \xi_i(t) ,$$

with  $\xi$  a Gaussian white noise :

$$\begin{aligned} \langle \xi_i(t) \rangle &= 0 , \\ \langle \xi_i(t) \xi_j(t') \rangle &= 2\gamma k_B T \delta_{ij} \delta(t - t') . \end{aligned}$$

$\gamma$  is the friction coefficient,  $T$  is the temperature of the bath and  $k_B$  is the Boltzmann constant ( $k_B = 1$  henceforth).

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# Initial condition

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- Rapid **quench** from high temperature mimicked by random initial conditions,  $s_i(0)$ , taken, e.g. from a Gaussian pdf.
- Note : the initial condition is uncorrelated with the quenched randomness ( $J_{i_1 \dots i_p}$ ).

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# Key quantities

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In the limit  $N \rightarrow \infty$  the causal dynamics can be described with

- the global correlation function

$$C(t, t_w) = N^{-1} \sum_{i=1}^N [ \langle s_i(t) s_i(t_w) \rangle ]$$

- its associated linear response function

$$R(t, t_w) = N^{-1} \sum_{i=1}^N \left[ \left\langle \frac{\delta s_i(t)}{\delta h_i(t_w)} \right\rangle \right] \Big|_{h=0}$$

or its integral over time  $\chi(t, t_w) = \int_{t_w}^t dt' R(t, t')$ .

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# Dynamic equations

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In the  $N \rightarrow \infty$  limit exact causal Schwinger-Dyson equations

$$(\partial_t - z_t)C(t, t_w) = \int dt' [\Sigma(t, t')C(t', t_w) + D(t, t')R(t_w, t')] \\ + 2TR(t', t) ,$$

$$(\partial_t - z_t)R(t, t_w) = \int dt' \Sigma(t, t')R(t', t_w) ,$$

where the self-energy and vertex are functions of  $C$  and  $R$  :

$$D(t, t_w) = \frac{p}{2}C^{p-1}(t, t_w) , \quad \Sigma(t, t_w) = \frac{p(p-1)}{2}C^{p-2}(t, t_w) R(t, t_w) .$$

and the Lagrange multiplier  $z_t$  is fixed by  $C(t, t) = 1$ .

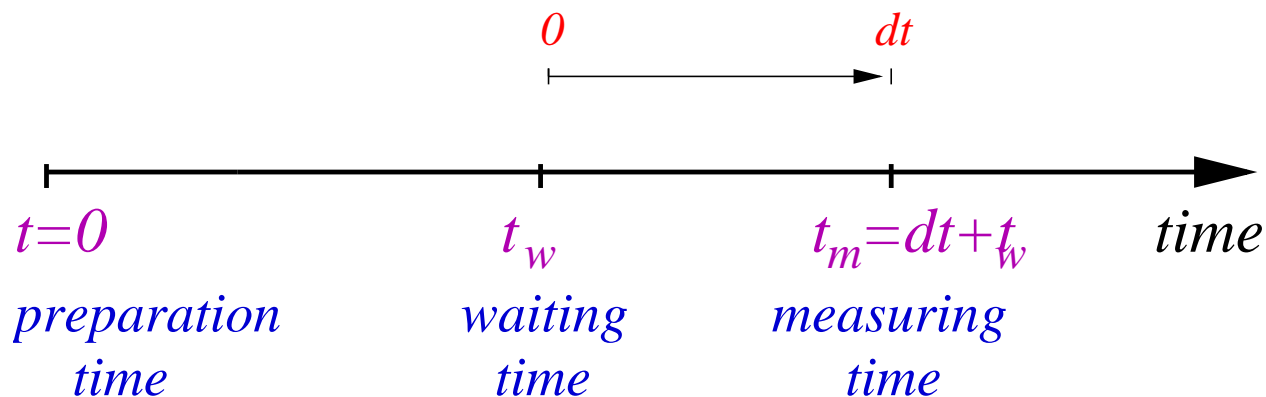
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# Glassy dynamics below $T_d$

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Two-time quantities age, *i.e.* the stationary relaxation is lost

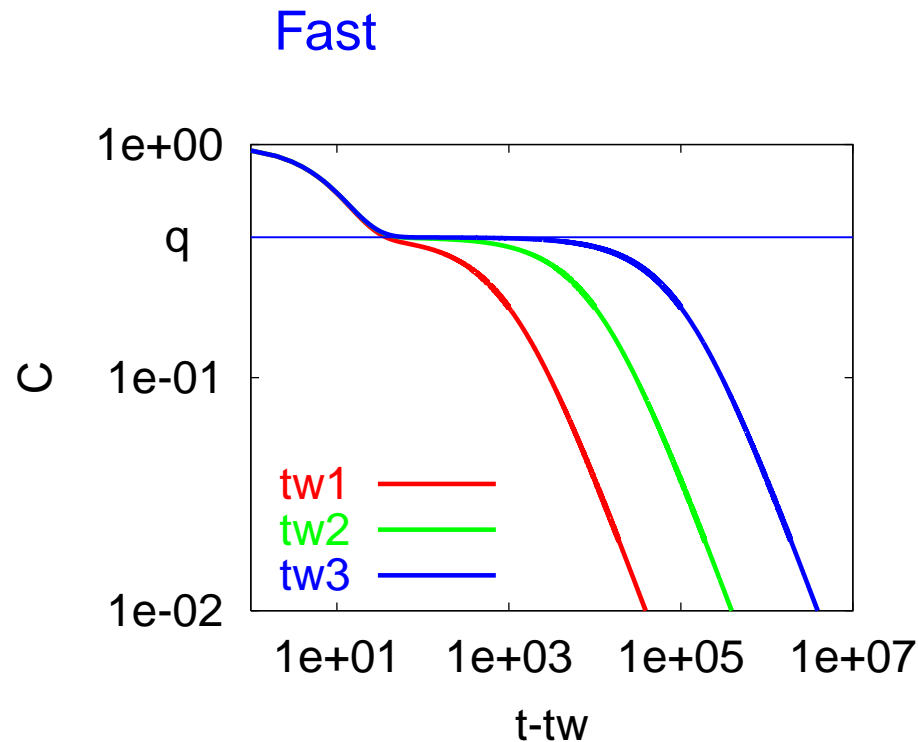
separation of time-scales, rapid-slow, controlled by  $t_w$ .



Numeric and analytic solution

# Separation of time-scales

In the long  $t_w$  limit



Eqs. for the slow relaxation  $C^s \equiv C < q$  :

Approx. asymptotic time-reparametization invariance

$$t \rightarrow h(t)$$

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# Time-reparametrization

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Example : the equation  $(\partial_t - z_t)R(t, t_w) = \int dt' \Sigma(t, t')R(t', t_w)$

- Take  $t - t_w \gg t_w$  use  $z_t \rightarrow z_\infty$ , drop  $\partial_t R$  separate the fast contributions to the integral :

$$\tilde{z}_\infty R^s(t, t_w) \sim \int_{t_w}^t dt' D'[C^s(t, t')] R^s(t, t') R^s(t', t_w) . \quad (1)$$

- The transformation

$$t \rightarrow h_t \equiv h(t) , \quad \begin{cases} C^s(t, t_w) \rightarrow C^s(h_t, h_{t_w}) , \\ R^s(t, t_w) \rightarrow \frac{dh_{t_w}}{dt_w} R^s(h_t, h_{t_w}) . \end{cases}$$

with  $h_t$  positive and monotonic leaves eq. (1) **invariant** :

$$\tilde{z}_\infty R^s(h_t, h_{t_w}) \sim \int_{h_w}^{h_t} dh_{t'} D'[C^s(h_t, h_{t'})] R^s(h_t, h_{t'}) R^s(h_{t'}, h_{t_w}) .$$

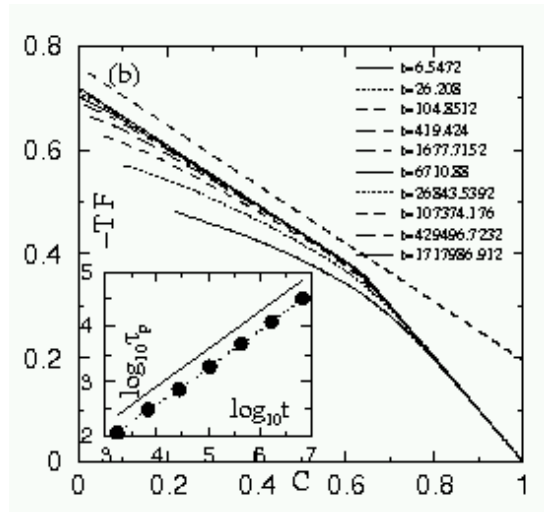
# Time reparametrization invariance

One can compute analytically  $f_c$  and  $\chi^s(C^s)$

$$C^s(t, t_w) \sim f_c \left( \frac{L(t)}{L(t_w)} \right),$$

$$\chi(t, t_w) \equiv \int_{t_w}^t dt' R(t, t') \sim \frac{1-q}{T} + \frac{1}{T_{eff}} C^s(t, t_w)$$

but not the 'clock'  $L(t)$



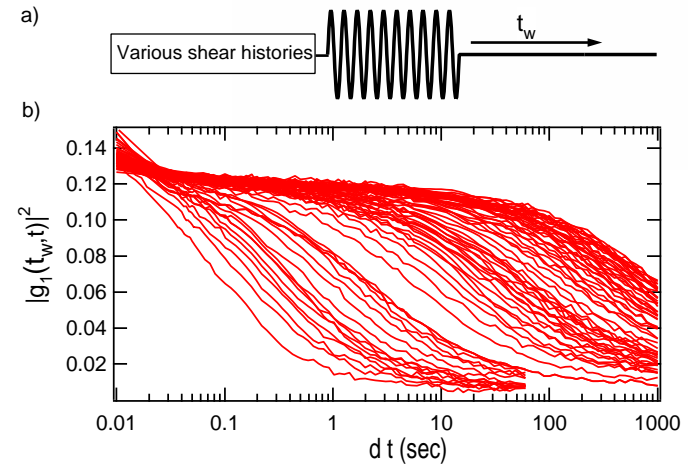
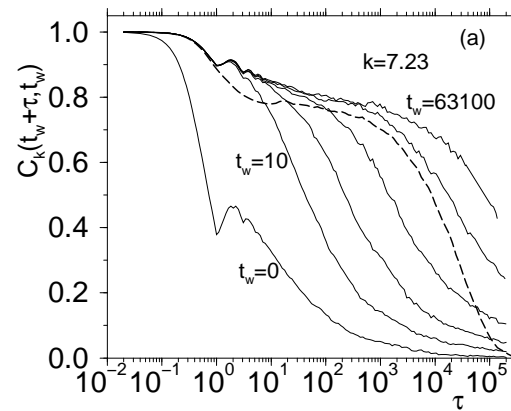
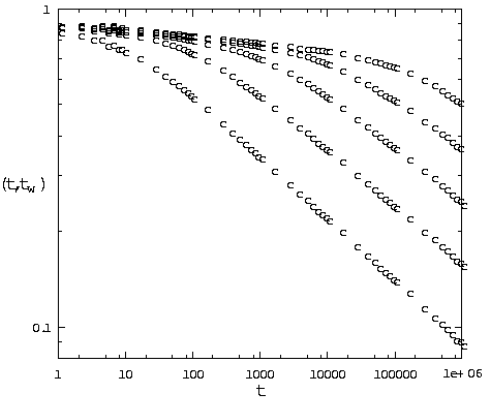
NOTE : the slow part of  $\chi$  is finite ( $T_{eff} < +\infty$ ).

# Finite dimensions

- **Slow dynamics** : observed
- **Separation of time-scales** : observed, though less clear-cut.

mixture

Exp. colloidal susp.



H. Rieger (92)

J-L Barrat & W. Kob (99)

B. Viasnoff & F. Lequeux (03)

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# Finite dimensions

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Analytically

$$S = S_{slow} + S_{fast} + S_{int}$$

- RG argument based on separation of time-scales allows one to show the approximate asymptotic invariance of  $S_{slow}$  under *global* time-reparametrizations

$$t \rightarrow h_t \equiv h(t), \quad \begin{cases} C_r^s(t, t_w) \rightarrow C_r^s(h_t, h_{t_w}), \\ R_r^s(t, t_w) \rightarrow \frac{dh_{t_w}}{dt_w} R_r^s(h_t, h_{t_w}). \end{cases}$$

Symmetry breaking terms become less important as  $t_w, t - t_w \rightarrow \infty$ .

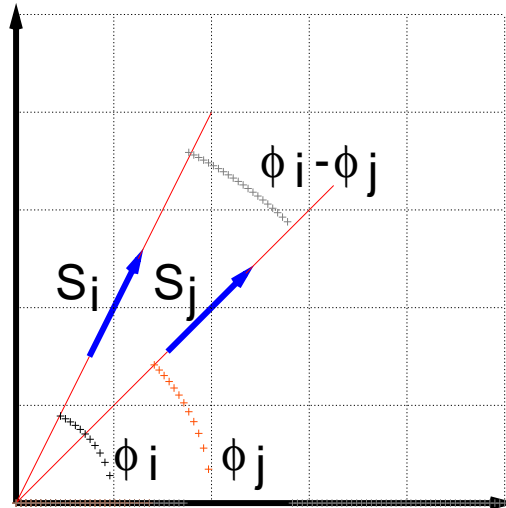
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# Parenthesis : an analogy

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# The Heisenberg ferromagnet

An analogy : a nuisance turned into a model



Landau free-energy

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda [m^2(\vec{r}) - m_0^2]^2 \right\} .$$

Invariant under the global rotation  $m^a(\vec{r}) = R^{ab} m^b(\vec{r})$ .

(Global time-reparametrization invariance)

# Statics of the Heisenberg ferro

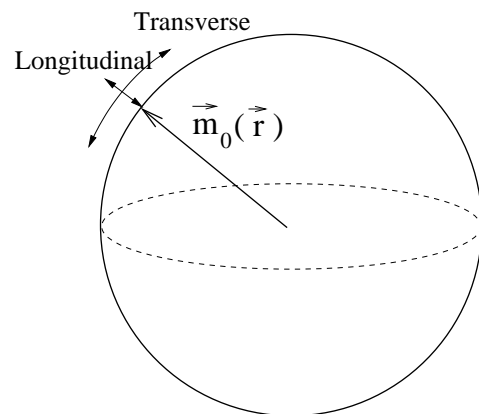
Ground state :  $\vec{m}(\vec{r}) = \vec{m}_0$  for all  $\vec{r}$ .

Fluctuations :  $\vec{m}(\vec{r}) = \vec{m}_0 + \delta\vec{m}(\vec{r})$ .

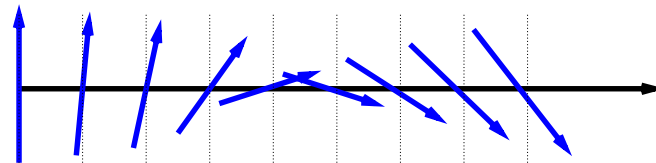
Longitudinal (easy) &

transverse (hard) fluctuations.

Spin-waves



Low energy excitation



(Time-reparametrization waves)

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# Explicit symmetry breaking

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A pinning field

Landau free-energy in a field

$$F = \int d^d r \left\{ [\nabla \vec{m}(\vec{r})]^2 + \lambda [m^2(\vec{r}) - m_0^2]^2 - \vec{h} \vec{m}(\vec{r}) \right\} .$$

No longer invariant under the global rotation

A particular direction is selected by the field :  $\vec{m}(\vec{r}) = m_0 \hat{h}$ .

[ $\partial_t C$  and  $\partial_t R$  select the 'clock'  $L(t)$   
but they become less and less important as  
the system ages &  $t - t_w$  increases!]

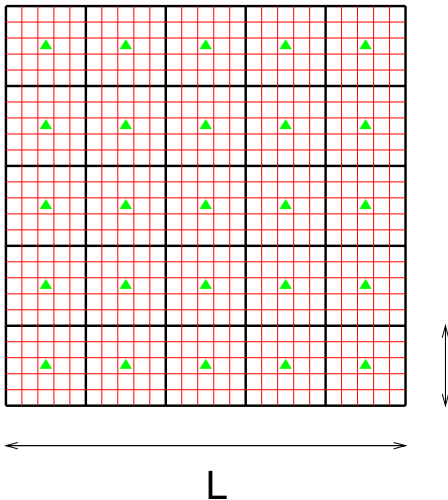
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# End of the parenthesis

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## Next

Use the time-reparametrization symmetry  
to characterize the fluctuations



$$C_r(t, t_w) \equiv \frac{1}{V_r} \sum_{i \in V_r} s_i(t) s_i(t_w) ,$$

$$\chi_r(t, t_w) \equiv \frac{1}{V_r} \sum_{i \in V_r} \int_{t_w}^t dt' \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0} .$$

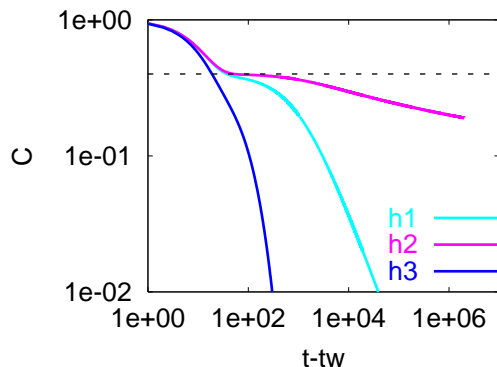
# Leading fluctuations

Scaling of the **slow** part of the **global** correlation

$$C^s(t, t_w) \approx f_c \left( \frac{L(t)}{L(t_w)} \right) .$$

The **global time-reparametrization invariance**  $\Rightarrow C_r^s(t, t_w) \approx f_c \left( \frac{h_r(t)}{h_r(t_w)} \right) .$

Ex.  $h_{r1} = \frac{t}{t_0}$ ,  $h_{r2} = \ln \left( \frac{t}{t_0} \right)$ ,  $h_{r3} = e^{\ln^a \left( \frac{t}{t_0} \right)}$  on different regions



Same  $t_w$ , slower and faster decays

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# Turn it useful : $\sigma$ model

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Easy fluctuations  $t \rightarrow h_r(t)$ .

- Ideally : **derive** the action  $S[h_r(t)]$ .

Doable in quasi-mean-field models, C. Chamon, LFC, S. Franz, in progress.

- In practice : **propose** the action  $S[h_r(t)]$  ;

derive predictions from  $S[h_r(t)]$ ,

e.g.  $\rho[C_r^s; t, t_w]$  ;  $\rho[R_r^s; t, t_w]$  ;  $\rho[C_r^s, R_r^s; t, t_w]$

that can be checked **numerically & experimentally**.

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# $\sigma$ -model

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Slow decay in terms of  $h_r(t) \equiv e^{-\varphi_r(t)}$

$$C_r^s(t, t_w) \approx f_c \left( \frac{h_r(t)}{h_r(t_w)} \right) = f_c \left( e^{-\int_{t_w}^t dt' \partial_{t'} \varphi_r(t')} \right)$$

The simplest

- (i) global time-reparametrization invariant ;
- (ii) local in space ;
- (iii) positive definite ( $\partial_t h_r(t) > 0 \Rightarrow \partial_t \varphi_r(t) > 0$ ) ;
- (iv) invariant under  $\varphi_r(t) \rightarrow \varphi_r(t) + \Phi(r)$  as  $C_r^s$  effective action is

$$\mathcal{A} = K \int d^d r \int dt \frac{[\nabla \partial_t \varphi_r(t)]^2}{\partial_t \varphi_r(t)}$$

Similar to a Gaussian surface after a redefinition of time and field.

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# Some consequences

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- Temporal scaling of the pdf of local correlations dictated by the global correlation  $\rho(C_r; t, t_w) = \rho[C_r; C^s(t, t_w)]$ .
- Negatively-skewed, non-Gaussian  $\rho(C_r; C^s)$  for  $0 < C^s < q$ .
- The two-time dependent correlation length  $\xi(t, t_w)$ ,

$$\left[ \sum_i C_i^s(t, t_w) C_j^s(t, t_w) \right]_c \approx e^{-|\vec{r}_i - \vec{r}_j| / \xi(t, t_w)},$$

should diverge with  $t$  and  $t_w$ .

- Constant of motion.  $\rho[C_r, \chi_r; t, t_w]$  should follow the global FDT rel. :

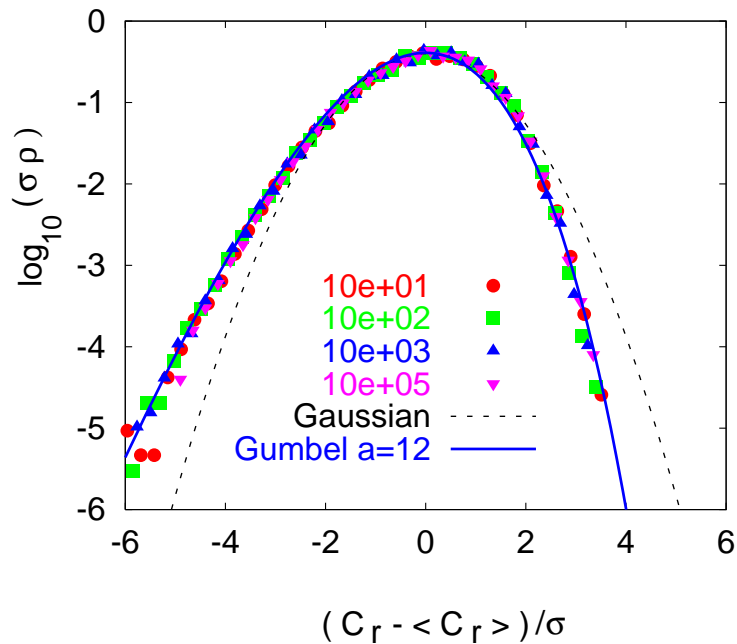
$$\lim_{t_w \rightarrow \infty; C(t, t_w) = C} \chi(t, t_w) = \chi(C).$$

All can be tested with simulations & experiments.

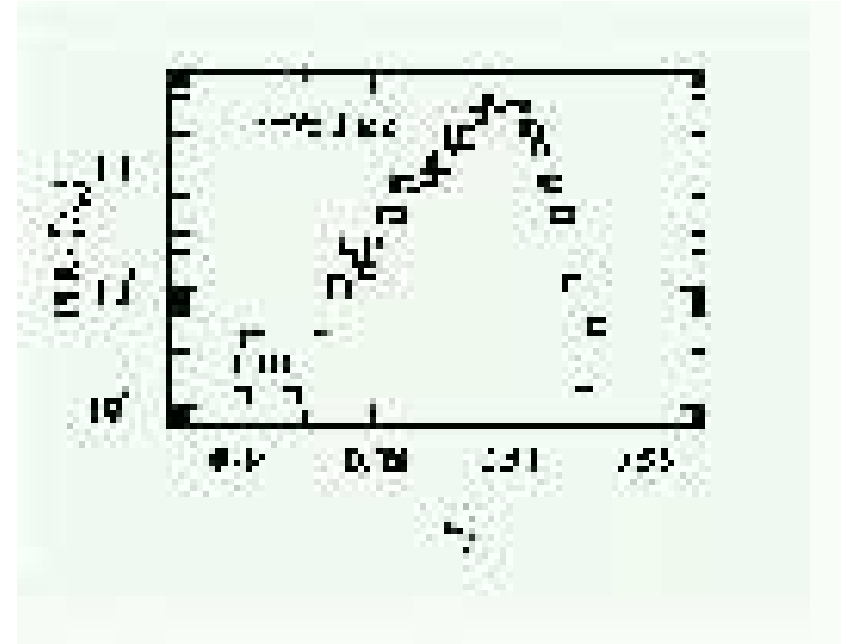
# pdf of local correlations

Kinetically constrained model ; four  $(t, t_w) / C(t, t_w) = 0.8$ .

Similar results for the  $3d$  spin-glass.



Chamon, Charbonneau, LFC, Reichman & Sellitto (04)



a micellar polycrystal

A. Duri, P. Ballesta, L. Cipelletti, H. Bissig, & V. Trappe (04)

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# How general is this ?

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- Coarsening – domain growth

e.g. the  $d$ -dimensional  $O(N)$  model in the large  $N$  limit (continuous space limit of the Heisenberg ferro with  $N \rightarrow \infty$ )

$$\dot{\phi}_\alpha(\vec{r}, t) = \nabla^2 \phi_\alpha(\vec{r}, t) + \lambda |N^{-1} \phi^2(\vec{r}, t) - 1| \phi_\alpha(\vec{r}, t) + \xi_\alpha(\vec{r}, t)$$

with  $\phi(\vec{k}, 0)$  Gaussian distributed with variance  $\Delta^2$ . The solution is

$$\begin{aligned} \phi(\vec{k}, t) = & e^{-k^2 t - \int_0^t dt' z_{t'}} \phi(\vec{k}, 0) \\ & + \int_0^t dt' e^{-k^2(t-t') - \int_{t'}^t dt'' z_{t''}} [\xi(\vec{k}, t') + h(\vec{k}, t')], \end{aligned}$$

for any component  $\alpha$ , with  $z_{t'} = -\lambda |N^{-1} \phi^2(\vec{r}, t) - 1|$  for all  $r$ .

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# The large $N$ $O(N)$ model

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The global correlation and linear response are

$$C(t, t_w) \sim \frac{(tt_w)^{d/4}}{(t + t_w)^{d/2}} \left[ \Delta^2 + 2T \int_0^{t_w} dt'' \frac{e^{2 \int_0^{t''} dt' z_{t'}}}{\left[ 1 - \frac{2t''}{(t+t_w)} \right]^{d/2}} \right],$$

$$R(t, t_w) \sim \left( \frac{t}{t_w} \right)^{d/4} (t - t_w + t_0)^{-d/2},$$

with  $t_0 \equiv \Lambda^{-2}$  and  $t + t_w \gg t_0$ .

NOTE : the global correlation behaves as for the  $p$ -spin model but the slow integrated response vanishes as  $\chi^s(t, t_w) \sim t^{1-d/2} f_\chi(t/t_w)$  for all  $d > 2$ .

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# The large $N$ $O(N)$ model

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## Invariance of the slow dynamic eqs. ?

The full dynamic equation for  $R(t, t') \equiv \int d^d r R(\vec{r}, t, t')$  is

$$\frac{\partial R(t, t')}{\partial t} = -z_t R(t, t') + \sum_{n=0}^{\infty} A_n \int dt_n \int dt_{n-1} \dots \int dt_1 R(t, t_1) R(t_1, t_2) \dots R(t_n, t')$$

with  $A_n$  fixed by the Fourier-mode density. After a separation of time-scales one has

$$\frac{\partial R_s(t, t')}{\partial t} = -z_t R_s(t, t') + \sum_{n=0}^{\infty} B_n(t - t_w) \times \int dt_n \int dt_{n-1} \dots \int dt_1 R_s(t, t_1) R_s(t_1, t_2) \dots R_s(t_n, t')$$

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# The large $N$ $O(N)$ model

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## Invariance of the slow dynamic eqs. ?

Knowing the exact  $R$  one can plug in  $R_s$  to find that :

- The time-derivative  $\partial_t R_s$  behaves as  $t^{1-d/2}$ .
- The Lagrange multiplier  $z_t$  decays as  $t^{-1}$  ; then  $z_t R_s \sim t^{1-d/2}$  too.
- The coefficients  $B_n$  (stationary contributions) do not approach constants !

$$\text{INSTEAD } B_n(t - t_w) \sim (t - t_w)^{-[1+n(1-d/2)]}.$$

- The integrals factors go as  $I_n \sim t^{d/2-n(1-d/2)}$  in such a way that

$$B_n I_n \sim t^{1-d/2} \text{ as well.}$$

There is no time-reparametrization invariance, just scale invariance

$$t \rightarrow \zeta t.$$

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# The large $N$ $O(N)$ model

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$d > 2$  : a different mechanism

Due to the asymptotic absence of a **slow response** ( $\chi_s \rightarrow 0$ )

or the extreme violation of the **fluctuation-dissipation** equilibrium relation between correlations and responses ( $T_{eff} \rightarrow \infty$ ),

and the ‘bad’ separation of time-scales.

$d = d_l = 2$  : more like critical dynamics.

Chamon, LFC, Yoshino (06)

Is it this way for **all coarsening systems** ?

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# A doable problem ?

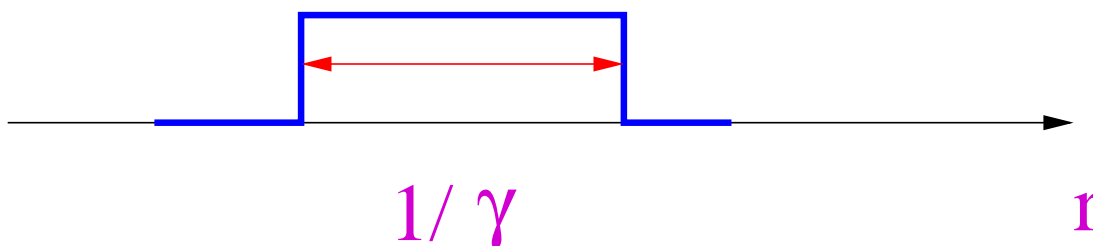
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## The Kac $p$ -spin spherical model

$$E_J(\vec{s}) = \sum_{i_1 \neq \dots \neq i_p} J_{i_1 \dots i_p} s_{i_1} \dots s_{i_p} + z_x \left( \sum_{i \in V_x} s_i^2 - N_x \right),$$

$z_x$  is a semi-local Lagrange multiplier,

$J_{i_1 \dots i_p}$  are quenched i.i.d. Gaussian random variables such that interactions exist only when the sites are at distance  $r \leq \gamma^{-1}$ .



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# A doable problem ?

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## The Kac $p$ -spin spherical model

Program : in order to compute  $P(C_x, R_x)$  we have to solve :

$$(\partial_t + \mu_x(t))C = DR^\dagger + \Sigma C + 2TR^\dagger + 2T\zeta^{R^\dagger}C$$

$$(\partial_t + z_x(t))R = \delta + DQ + \Sigma R + 2TQ + 2T\zeta^{R^\dagger}R$$

$$(-\partial_t + z_x(t))R^\dagger = \delta + \Sigma^\dagger R^\dagger + \Delta C + (\zeta^C + 2T\zeta^R\zeta^{R^\dagger})C + 2T\zeta^{R^\dagger}R^\dagger$$

$$(-\partial_t + z_x(t))Q = \Sigma^\dagger Q + \Delta R + (\zeta^C + 2T\zeta^R\zeta^{R^\dagger})R + 2T\zeta^{R^\dagger}Q$$

$$R = R^* - C\zeta^R \quad D = \hat{\psi}f'(\hat{\psi}C) \quad \Sigma = \hat{\psi}[f''(\hat{\psi}C) \bullet \hat{\psi}R]$$

$$\Delta = \hat{\psi}[f''(\hat{\psi}C) \bullet \hat{\psi}Q + f'''(\hat{\psi}C) \bullet \hat{\psi}R \bullet \hat{\psi}R^\dagger]$$

$$2f(x) = x^p$$

The normal product involve  $\int_0^\infty dt'$  ;  $\hat{\psi}$  has the spatial information.

**Causality is lost !**

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# Summary

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Theory for the nonequilibrium dynamics in the glassy phase.

dictated by (the assumption of)

Global time reparametrization invariance

*cfr.* Spin-waves in Heisenberg ferromagnets.

Predictions for the behaviour of local correlations and responses,

in rather good agreement with simulations in **disordered spin models**

and **kinetically constrained models** ;

experiments on **colloidal systems** on their way

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# Summary

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## Classification of non-equilibrium systems ?

The theory suggests a strong link between  $T_{eff}$  and the fluctuations.

- Structural and spin glasses – aging,  $T_{eff} < +\infty$ .
- Domain growth – aging in the correlations but ‘no memory’,  $T_{eff} \rightarrow \infty$
- Critical dynamics – interrupted aging, no asymptotic  $T_{eff}$ .

with different properties of the fluctuations.

to be confirmed !

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# $\sigma$ -model

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Using the ‘proper’ time  $\tau(t) \equiv \ln L(t)$

with  $L(t)$  the “growth” law in the global corr.

& the change of variables  $\psi_r^2(\tau) \equiv \partial_\tau \varphi_r(\tau)$

$$C_r^s(t, t_w) \approx f_c \left( e^{-\int_{\ln L(t_w)}^{\ln L(t)} d\tau' \psi_r^2(\tau')} \right)$$

$$\mathcal{A} = K \int d^d r \int d\tau [\nabla \psi_r(\tau)]^2$$

Chamon, Charbonneau, LFC, Reichman & Sellitto (04)

cfr. Bramwell, Holdsworth & Pinton (98) [xy-model – spin waves](#) ;

Antal & Rácz (94) [Edwards-Wilkinson manifold](#).