
From coupled logistic maps to the KPZ growth equation

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Plan

Deterministic chaos becoming stochastic noise ?

- Coupled logistic *deterministic* maps
- The Kardar-Parisi-Zhang (KPZ) *stochastic* growth equation

*Are they **connected** ?*

cfr. space and time dependent properties :

tests on the CML and its Lyapunov vector

Effective temperatures and fluctuation theorems

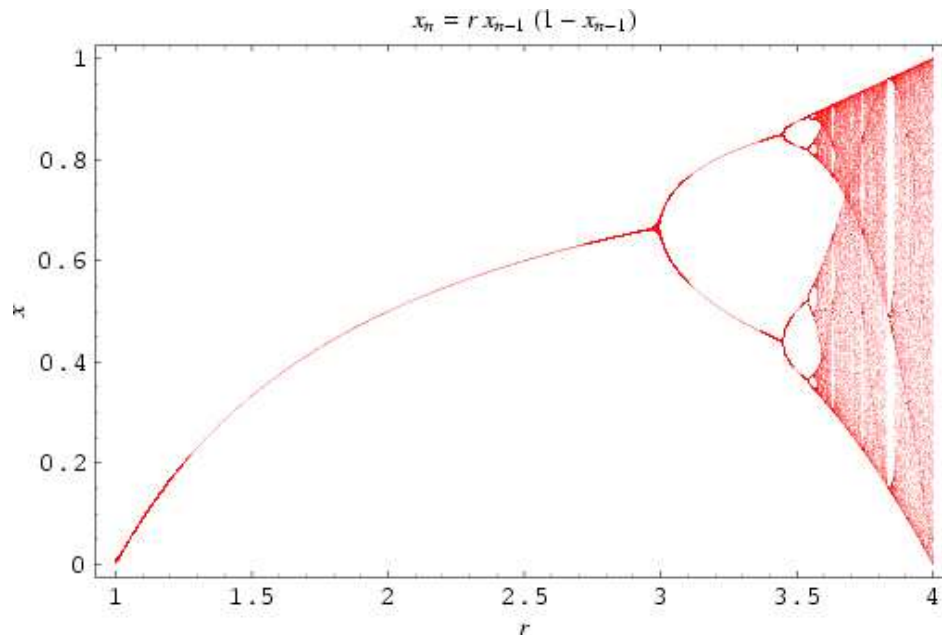
- In coupled map lattices

First part

CMLs & KPZ

The logistic map

$$x_n = g(x_{n-1}) \equiv r x_{n-1} (1 - x_{n-1}), \quad p(x_0) = \theta(x_0)\theta(1 - x_0)$$



$$0 < r \leq 4$$

S. Ulam and J. von Neumann, Bull. Am. Math. 53, 1120 (1947).

Proposal for a random number generator.

Coupled logistic maps (CLM)s

d -dimensional lattice of logistic maps with nearest-neighbour coupling

$$x_n^i = g(x_{n-1}^i) + \frac{\nu}{2} [g(x_{n-1}^{i-1}) - 2g(x_{n-1}^i) + g(x_{n-1}^{i+1})] ,$$

$$g(x_n^i) = r x_n^i (1 - x_n^i) \quad \text{in the chaotic regime}$$

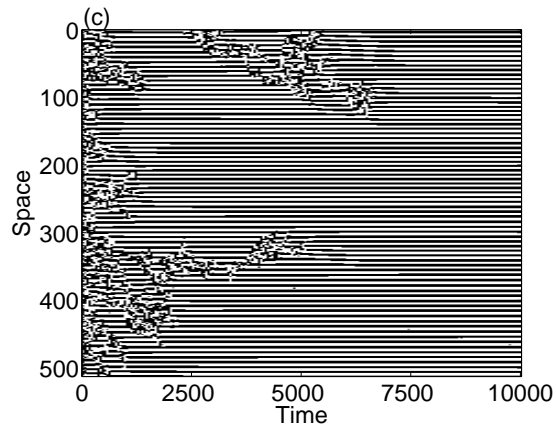
$$\left\{ \begin{array}{ll} \text{Small } \nu & \text{Individual chaotic trend} \quad \Rightarrow \quad \text{Inhomogeneity} , \\ \text{Large } \nu & \text{Locking of neighbouring maps} \quad \Rightarrow \quad \text{Homogeneity} . \end{array} \right.$$

Characterization of the phase diagram (r, ν)

see, e.g., **K. Kaneko, Physica D 34, 1 (1989).**

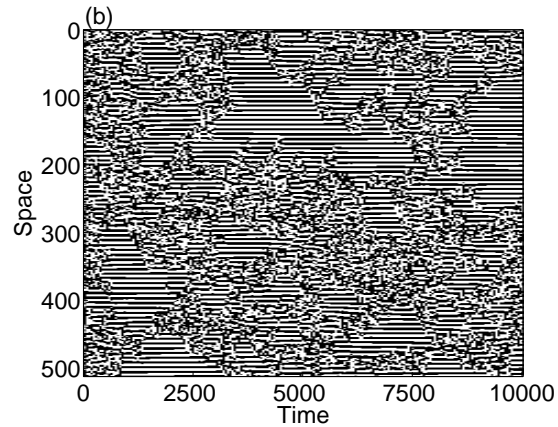
CLMs : space-time maps

Increasing r at fixed $\nu = 0.2$:



$$r = 3.8$$

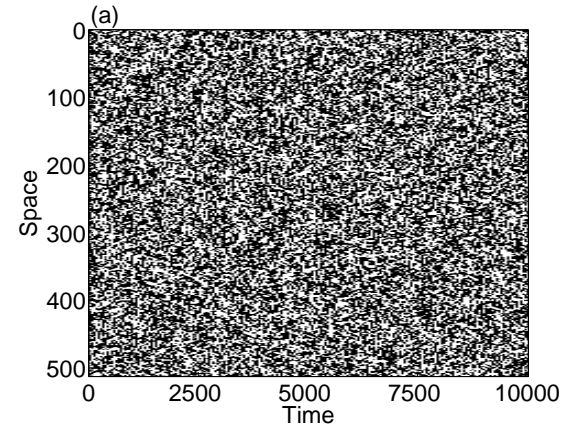
Frozen random pattern,



$$r = 3.82$$

After a transient :

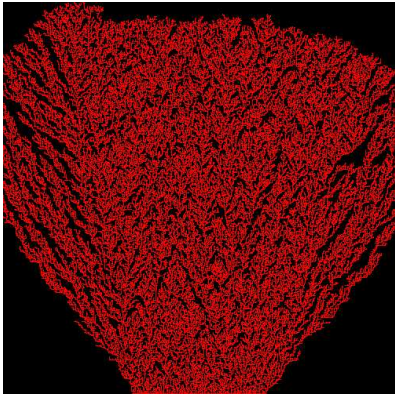
pattern formation,



$$r = 4.0$$

'Fully dev. turbulence'

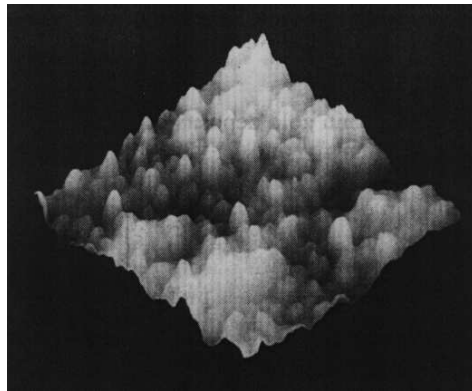
Surface growth



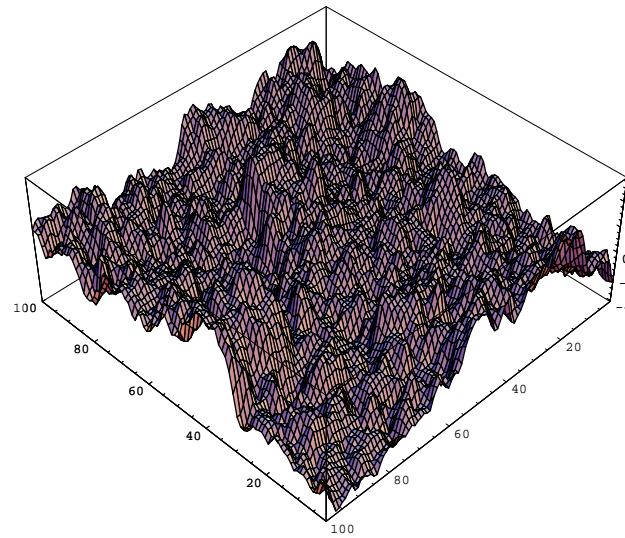
Ballistic deposition

simulation

$$d = 1$$



STM



KPZ equation

simulation

$$d = 2$$

The KPZ equation

$$\frac{\partial h(\vec{x}, t)}{\partial t} = \nu \nabla^2 h(\vec{x}, t) + \frac{\lambda}{2} [\vec{\nabla} h(\vec{x}, t)]^2 + \eta(\vec{x}, t)$$

surface tension

nonlinear term

noise

(growth along the local normal)

ν elastic coefficient

$\frac{\lambda}{2}$ non-linear parameter

Typically, Gaussian white noise $\eta \Rightarrow$ bath at temperature T

$$\langle \eta(\vec{x}, t) \rangle = 0 ,$$

$$\langle \eta(\vec{x}, t) \eta(\vec{x}', t') \rangle = 2k_B T \delta^d(\vec{x} - \vec{x}') \delta(t - t') .$$

M. Kardar, G. Parisi and Y-C Zhang, Phys. Rev. Lett. 56, 889 (1986).

KPZ : focus on $d = 1$

$$\bar{h}(t) \equiv \frac{1}{L} \int dx h(x, t) \quad \text{Mean height ,}$$

$$w^2(L, t) \equiv \frac{1}{L} \int dx [h(x, t) - \bar{h}(t)]^2 \quad \text{Mean-square fluctuations}$$

Family-Vicsek scaling

$$\sqrt{\langle w^2(L, t) \rangle} \sim L^\alpha f\left(\frac{t}{L^z}\right), \quad f(u) \sim \begin{cases} u^{\frac{\alpha}{z}} & u \ll 1 & \text{growth ,} \\ \text{ct} & u \gg 1 & \text{saturation .} \end{cases}$$

Exact in $d = 1$: roughness exponent $\alpha = \frac{1}{2}$; dynamic exponent $z = \frac{3}{2}$

Ballistic deposition : $\alpha = 0.47 \pm 0.02$; $2/z = 0.33 \pm 0.006$

Also interesting : $P(w^2) = \langle w^2 \rangle \Phi(w^2 / \langle w^2 \rangle)$.

Z. Rácz et al.

Continuum ('hydrodynamic') limit

e.g., stochastic crystal growth modelling

coarse-graining \Rightarrow continuum limit

discrete stochastic eq. \Rightarrow continuous stochastic eq.

usual trick to derive a **field theory** for, in particular, **non-equil. processes**

see, e.g. R. K. Zia *et al*

Observations at large distances ($l_{obs} \gg a$) and long times ($t_{obs} \gg t_0$)

We apply the same trick here

Deterministic chaotic evolution \Rightarrow **continuous stochastic eq. ?**

CLMs : continuum limit

Calling δt ($= 1$) the time-step and δx ($= 1$) the lattice spacing and using

$$\frac{\partial h}{\partial t} \leftrightarrow \frac{h_{n+1}^i - h_n^i}{\delta t}, \quad \frac{\partial h}{\partial x} \leftrightarrow \frac{h_n^{i+1} - h_n^i}{\delta x},$$
$$\frac{\partial^2 h}{\partial x^2} \leftrightarrow \frac{h_n^{i+1} - 2h_n^i + h_n^{i-1}}{(\delta x)^2},$$

the CML model becomes

$$\frac{\partial h}{\partial t} = \frac{\nu r}{2}(1 - 2h) \frac{\partial^2 h}{\partial x^2} - \nu r \left(\frac{\partial h}{\partial x} \right)^2 + (r - 1)h - rh^2$$
$$\frac{\partial h}{\partial t} \equiv \nu(h) \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{4} \left(\frac{\partial h}{\partial x} \right)^2 + \eta(h)$$

CML to KPZ but

- (i) field-dependent viscosity $\nu(h) \equiv \frac{\nu r}{2}(1 - 2h)$
- (ii) negative non-linear coupling $\lambda \equiv -\nu r < 0$.
- (iii) Does local chaos become a stochastic noise ?

Statistical properties of $\eta(h) \equiv (r - 1)h - rh^2$?

- (iv) $h_n^i \in [0, 1]$ implies $h(x, t) \in [0, 1]$ too.

Thus, KPZ-like with a hard confining potential : $\eta(h) = -\frac{\partial V(h)}{\partial h}$?

Let's explore these points one by one.

The elastic coefficient

$$\nu(h) = -\frac{\nu r}{2}(1 - 2h)$$

- (i) The bare $\nu(h)$ has a weak dependence on the field h .
- (ii) The bare $\nu(h)$ is bounded (since h is bounded).
- (iii) $\nu(h) < 0$ when $h > 1/2$: **Instability**. In the Kuramoto-Sivashinsky equation

$$\frac{\partial h}{\partial t} = -\kappa h - \frac{\partial^2 h}{\partial x^2} - \frac{\partial^4 h}{\partial x^4} - h \frac{\partial h}{\partial x}$$

with h and κ real, a similar instability – plus **confinement** – generates ‘**noise**’ and allows for a link with the KPZ eq. in $d = 1$.

Y. Yakhot, Phys. Rev. A 24, 642 (1981).

V. L’vov and I. Procaccia, Phys. Rev. Lett. 69, 3543 (1992).

- (iv) $\nu(h)$ should be renormalized by the non-linearity λ .

The non-linearity

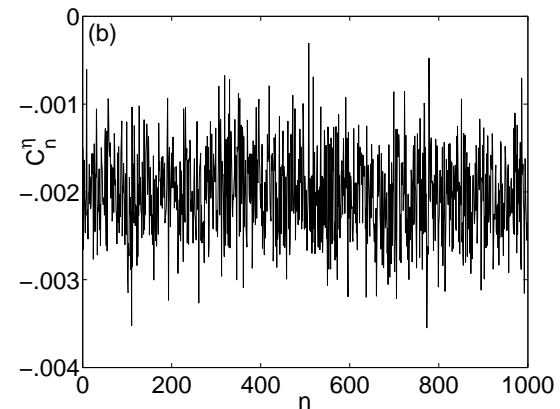
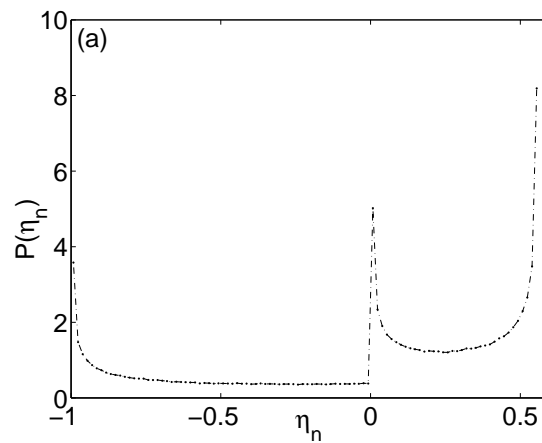
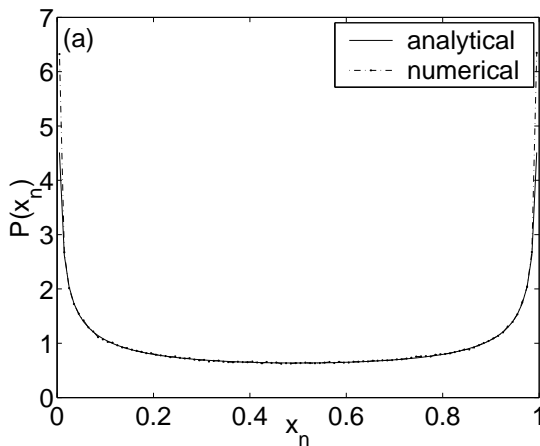
$$\frac{\lambda}{4} \left(\frac{\partial h}{\partial x} \right)^2 \text{ with } \lambda = -\nu r < 0$$

- (i) The non-linearity is identical to the one in the KPZ eq. :
it favors growth along the parts with high curvature.
- (ii) A negative λ in the KPZ eq. is not important.

$$h \rightarrow -h \quad \lambda \rightarrow -\lambda$$

η : a stochastic noise ?

An isolated logistic map



$$p(x) = \frac{1}{\sqrt{\pi x(1-x)}}$$

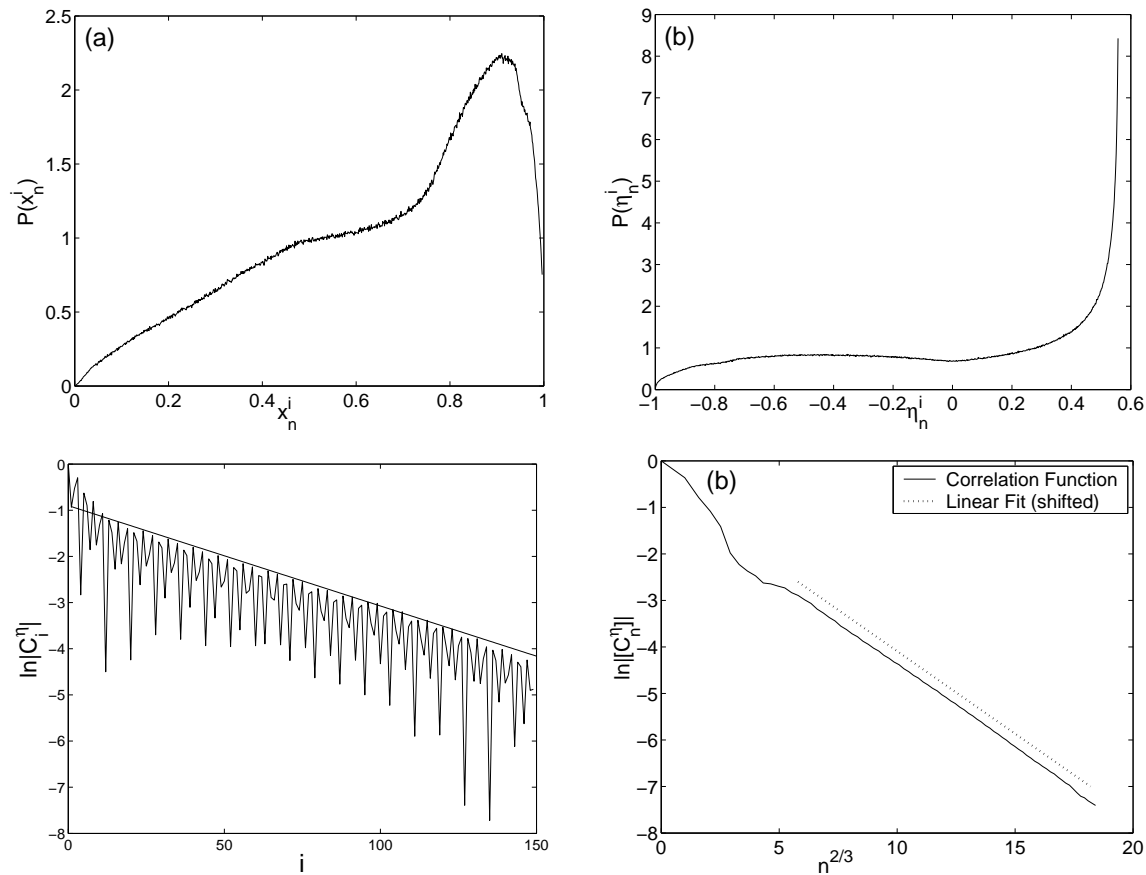
\Rightarrow

$$p(\eta)$$

No obvious time-correlations

η : a stochastic noise ?

Coupled logistic maps



$$\overline{C_i^\eta} \equiv \frac{1}{T} \sum_{k=1}^T \frac{1}{N-i} \sum_{j=1}^N \eta_k^i \eta_k^{i+j}$$

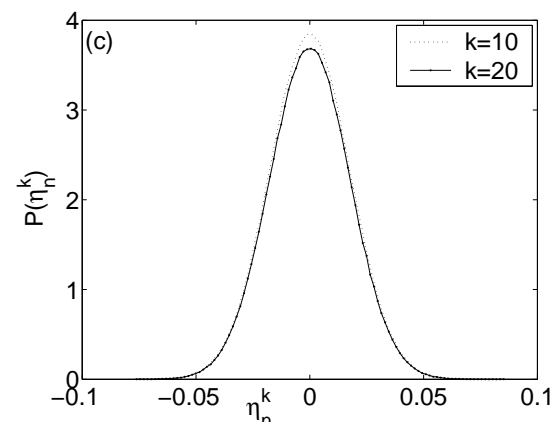
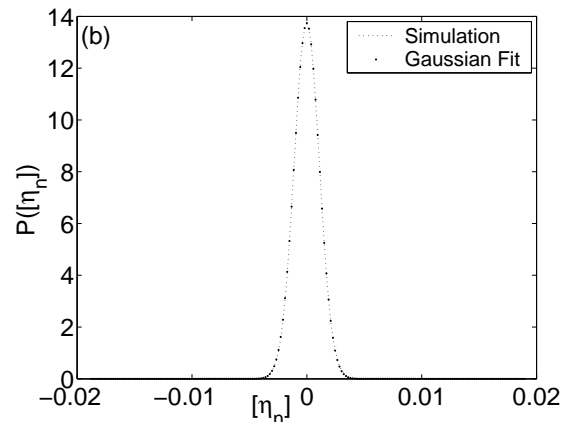
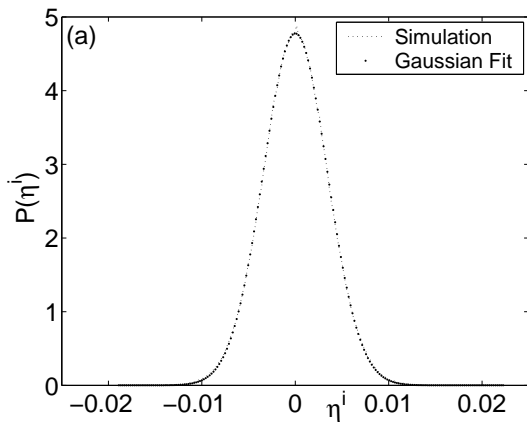
$$[C_n^\eta] \equiv \frac{1}{N} \sum_{i=1}^N \frac{1}{T-n} \sum_{k=1}^{T-n} \eta_n^i \eta_{n+k}^i$$

η : a stochastic noise ?

Short-range correlations in space and time ;

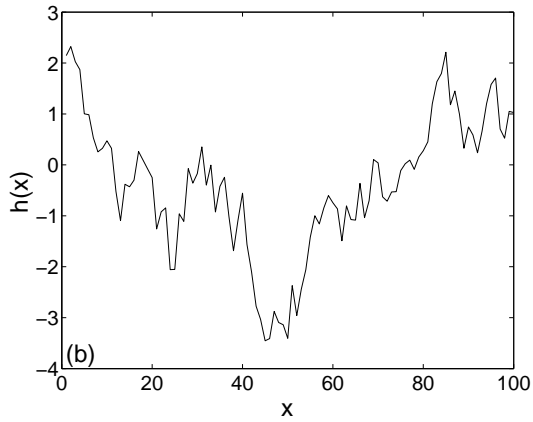
central limit theorem

Gaussian pdf's of $[\tilde{\eta}_n^k] \equiv \frac{1}{N} \sum_{i=1}^N \cos(2\pi ki) \eta_n^i$ and
 $\bar{\eta}^i \equiv \frac{1}{T} \sum_{n=1}^T \eta_n^i$

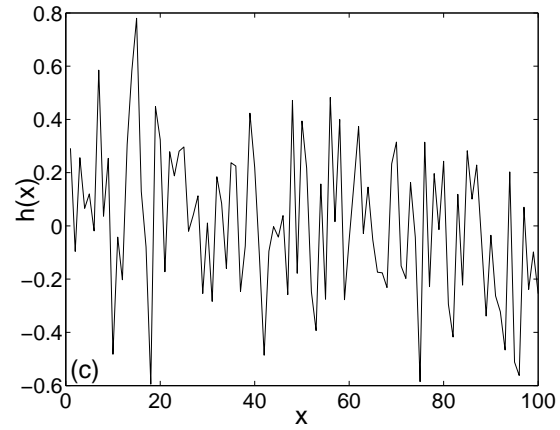


- Asymmetric non-Gaussian $p(\eta_n^i)$ with support in $[-1, 0.56]$ for $r = 4$.
- Short-range correlations in space and time.

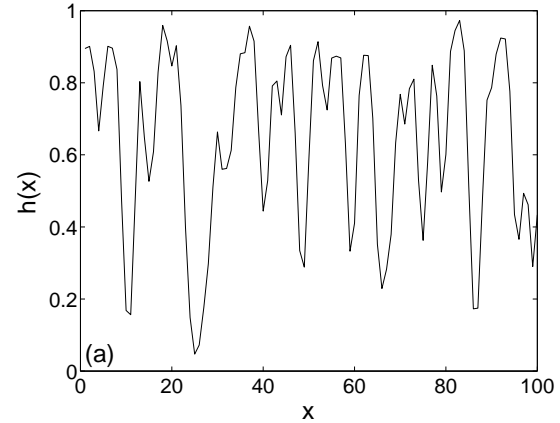
Surfaces



Usual KPZ



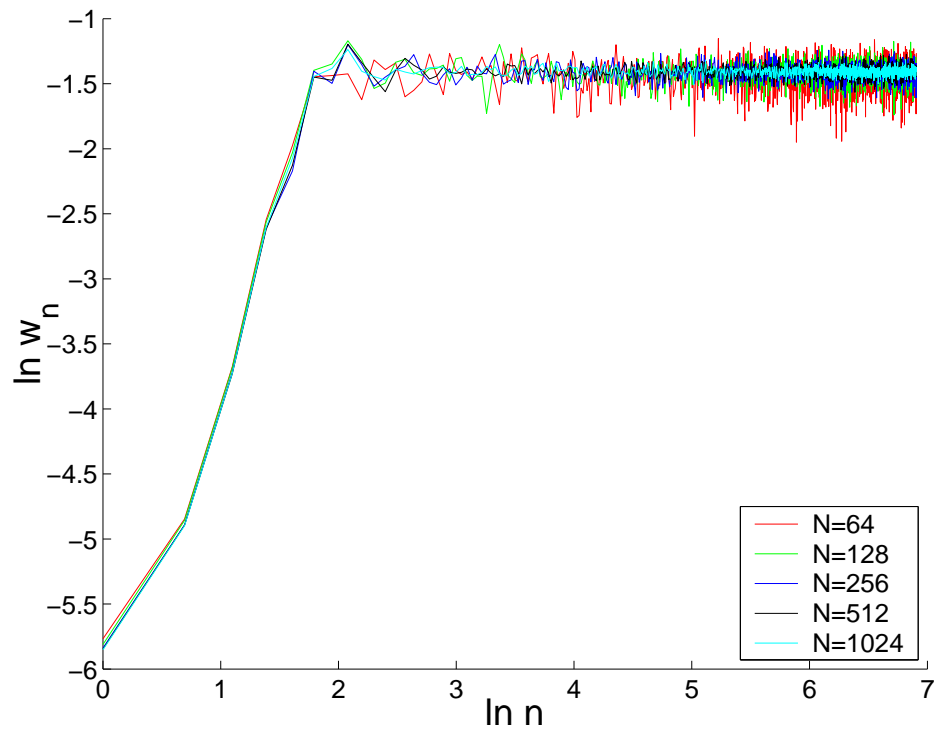
Confined KPZ



CML

Very different roughness

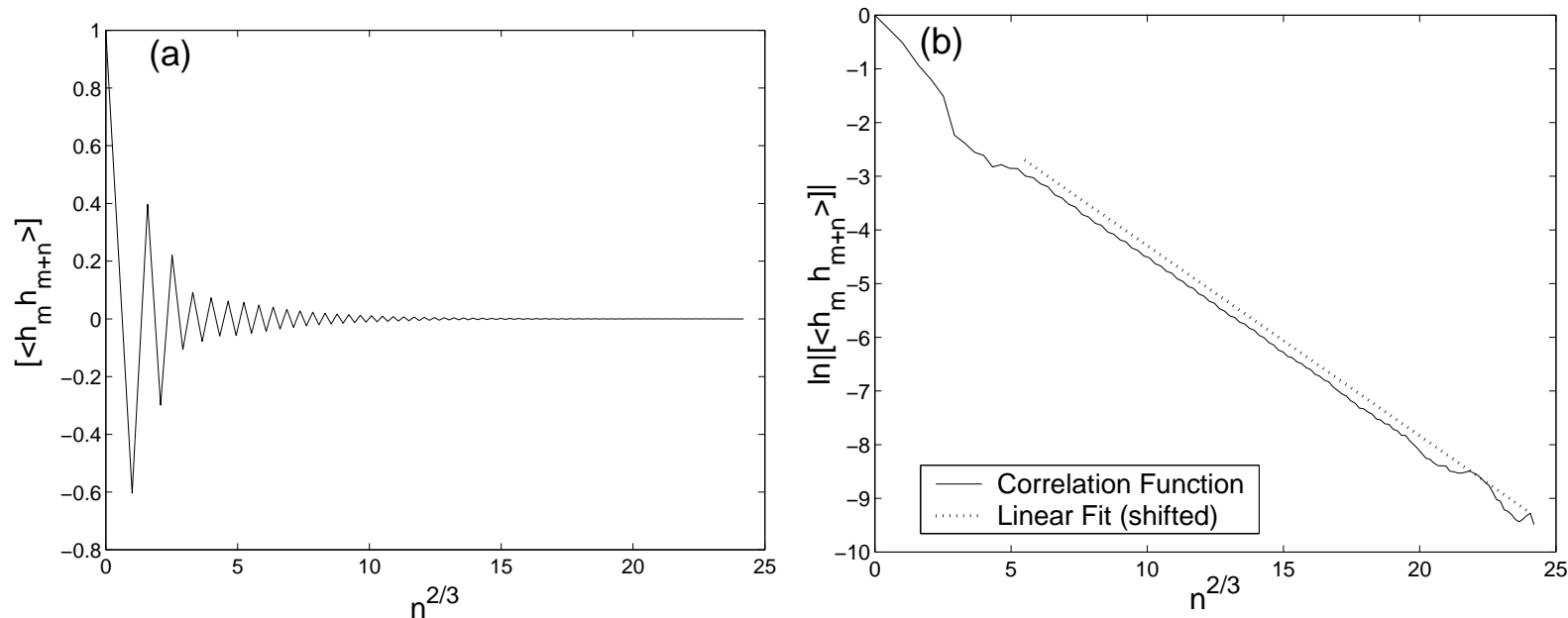
Scaling – exponents



Trivial due to confinement :

$$\alpha = 0, \quad z = 0$$

Time correlations



Stretched exponential relaxation even in the most chaotic regime !

As in **KPZ** : Colaiori & M. A. Moore (01); M. Schwartz & S. F. Edwards (02); E. Katzav & M. Schwartz (04)

Not necessarily related to **trapping times** : N. Mousseau *et al* (02)

Second part

Effective temperatures and fluctuation relations

What is the FDT ?

In **equilibrium**

the dynamics is **stationary**, $\Delta_{AB}(t, t_w) \equiv \langle (A(t) - B(t_w))^2 \rangle = f(t - t_w)$,

and the fluctuation (F) dissipation (D) theorem (T) relates

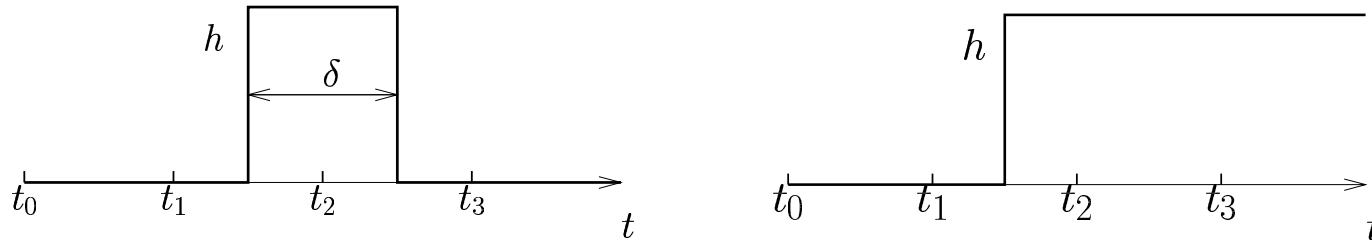
spontaneous Δ_{AB} & induced $R_{AB}(t, t_w) \equiv \left\langle \frac{\delta A(t)}{\delta h(t_w)} \right\rangle \Big|_{h=0}$ fluctuations :

$$R_{AB}(t, t_w) = -\frac{1}{2k_B T} \frac{\partial \Delta_{AB}(t, t_w)}{\partial t_w} \theta(t - t_w) ,$$

$$\chi_{AB}(t, t_w) = -\frac{1}{2k_B T} [\Delta_{AB}(t, t) - \Delta(t, t_w)] \theta(t - t_w) .$$

Model independent relations, Einstein, Nyquist, *etc.*

Response to perturbations



The **perturbation** couples **linearly** to the observable B

$$E \rightarrow E - hB(\{\vec{r}_i\})$$

The **linear instantaneous response** of another observable A is

$$R_{AB}(t, t_w) \equiv \left\langle \frac{\delta A(t)}{\delta h(t_w)} \Big|_{h=0} \right\rangle$$

The **linear integrated response** or **dc susceptibility** is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^t dt' R_{AB}(t, t')$$

and out of equilibrium ?

Can one identify a “family” of non-equilibrium systems for which the modification of

$$R_{AB}(\tau) = \frac{1}{2k_B T} \frac{d\Delta_{AB}(\tau)}{d\tau} \theta(\tau), \quad \tau \equiv t - t_w$$

is

- Easy;
- of the same form within the family.
- Can we understand what it might mean ?

Where to look for ?

There is a wide variety of non-equilibrium systems.

Focus on those with

- Slow dynamics

say, slower than exponential decay of correlation functions

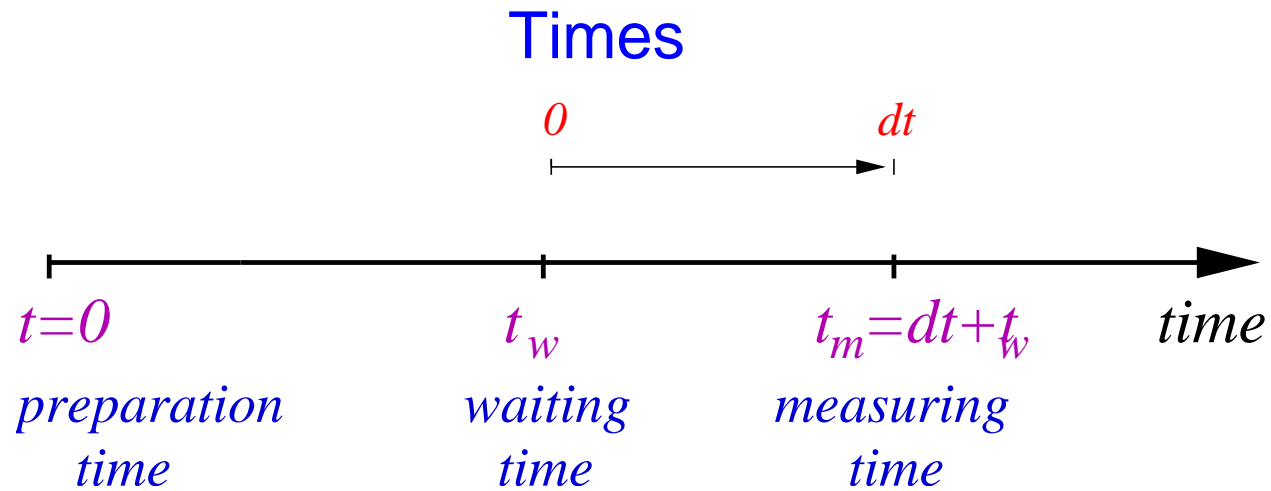
e.g. glasses,

weakly driven complex liquids, weakly tapped granular matter,

driven vortices, *etc.*

- Coupled maps ?

The case of glasses



- $t_w > t_{eq}$: equilibrium relaxation
- $t_w < t_{eq}$: out of equilibrium dynamics

Stationarity is lost : **Aging**

Fluctuation-dissipation

Dynamics of high dimensional disordered models : $\lim_{t, t_w \rightarrow \infty} \lim_{N \rightarrow \infty}$

- separation of time-scales $\Delta(t, t_w) = \Delta_{st}(t - t_w) + \Delta_{ag}(t, t_w)$ and

$$\chi(t, t_w) = \chi_{st}(t - t_w) + \chi_{ag}(t, t_w)$$

- For non-equilibrium systems, relaxing slowly towards an “asymptote” (cfr. the **threshold** in p spin models) such that one-time quantities approach a finite value [in particular, $\lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} \mathcal{E}(t) = \mathcal{E}_\infty$]

$$\lim_{t_w \rightarrow \infty, \Delta(t, t_w) = D} \chi(t, t_w) = f_\chi(D)$$

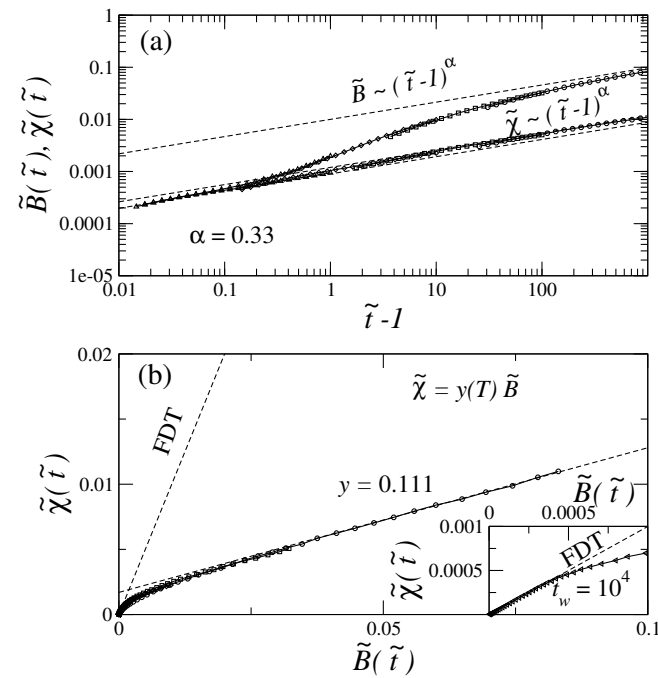
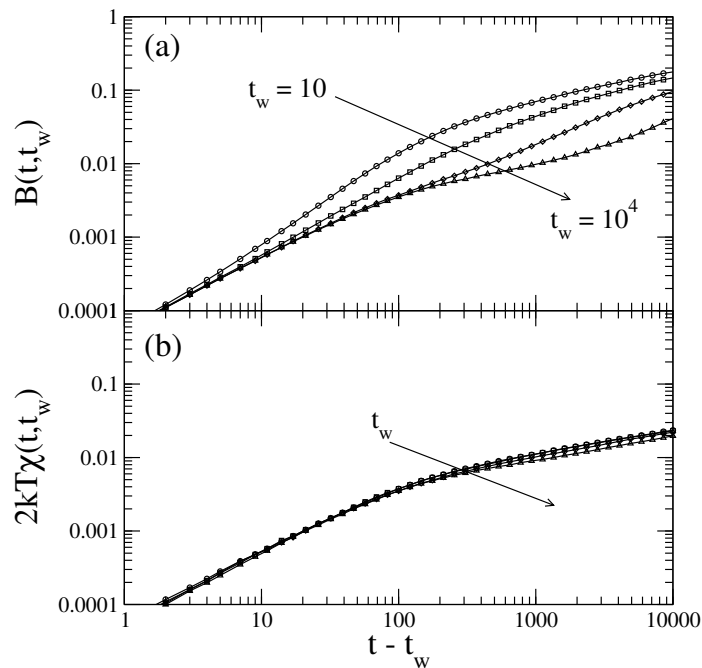
For weakly forced (ϵ) non-equilibrium systems in the limit of small work

$$\lim_{\epsilon \rightarrow 0, \Delta(t, t_w) = D} \chi(t, t_w) = f_\chi(D)$$

LFC & J. Kurchan, PRL 71, 173 (93) ; J. Phys. A (94).

FDT in relaxing glasses

Simulations : Vortex glass



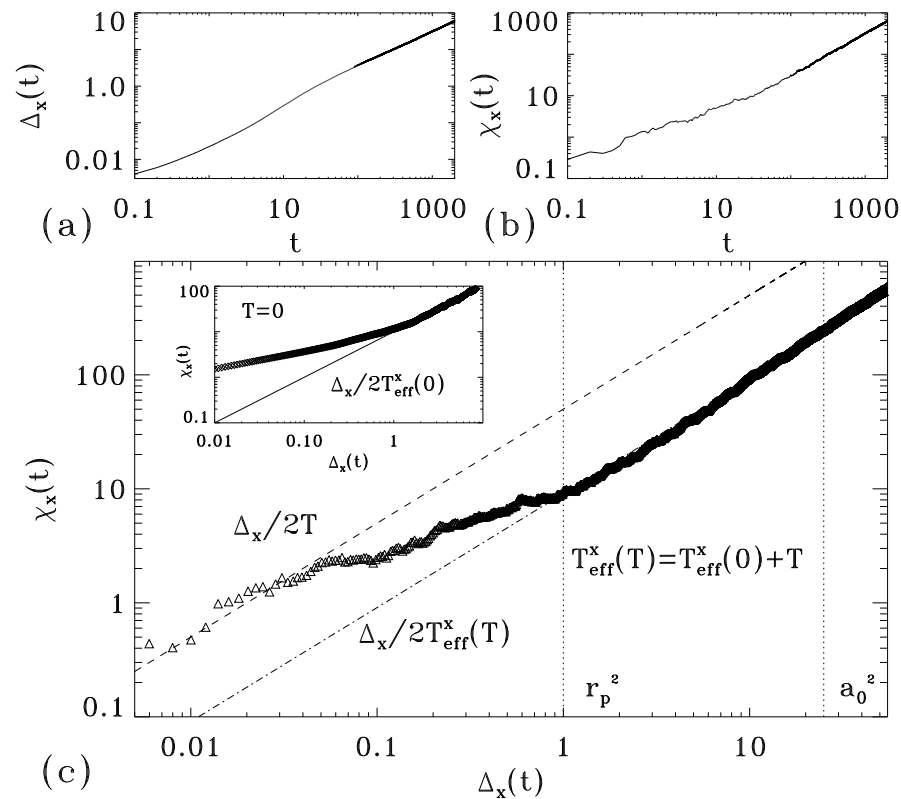
Disp & int. response

Reduced time & parametric plot

S. Bustingorry, LFC & D. Domínguez, Phys. Rev. Lett. (2005)

FDT in driven systems

Simulations : driven Bragg glass

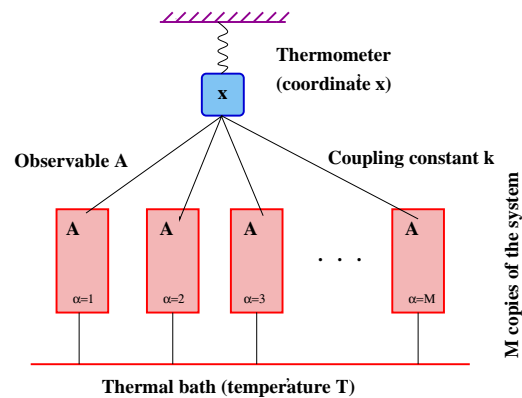


A. Kolton, R. Exartier, LFC, D. Domínguez, Grönbech-Jensen, PRL (2003)

FDT & effective temperatures

Can one interpret the slope as a temperature ?

LFC, J. Kurchan & L. Peliti, PRE (97)



- The “thermodynamics” of a glass can be described by the temperature of the bath and the effective temperature $T_{eff}(C)$ (98).
- $T_{eff} \approx T_g$: the glass remembers the structure it had when $T \approx T_g$.
- Similar interpretation as T_{fict} (40s) but T_{eff} has concrete definition.
- Link with microcanonical definition from configurational entropy (95).

Fluctuation theorems

Non-equilibrium steady states

The fluctuations of entropy production

$$p \equiv (\tau \sigma_+)^{-1} \int_{-\tau/2}^{\tau/2} dt \sigma(S_t)$$

where S_t is trajectory of the system starting from a given initial condition,

$\sigma(S_t) \equiv T^{-1}W(S_t)$ is the entropy production, with σ_+ its average, T the temperature of the environment and $W(S_t)$ the work done by the external forces, satisfy :

$$\xi(p) - \xi(-p) = p\sigma_+ \quad \text{with} \quad \xi(p) \equiv \lim_{\tau \rightarrow \infty} \tau^{-1} \ln \pi_\tau(p)$$

and π_τ the probability density of p .

Cohen, Morriss & Evans (90) ; Gallavoti & Cohen (93) ; Kurchan (98)

Fluctuation theorems

In driven glassy systems ?

- Remember that there is no meaning to T but there is to T_{eff} .
- What is $\sigma(S_t)$? Proposal : $T^{-1} \int_{-\tau/2}^{\tau/2} dt W(S_t)$ replaced by

$$\int_{-\tau/2}^{\tau/2} dt \frac{W(S_t)}{T_{eff}(S_t)}$$

with $T_{eff}(S_t)$ the **effective temperature** of the **unperturbed relaxing system**.

- With this definition the fluctuation theorem should hold.

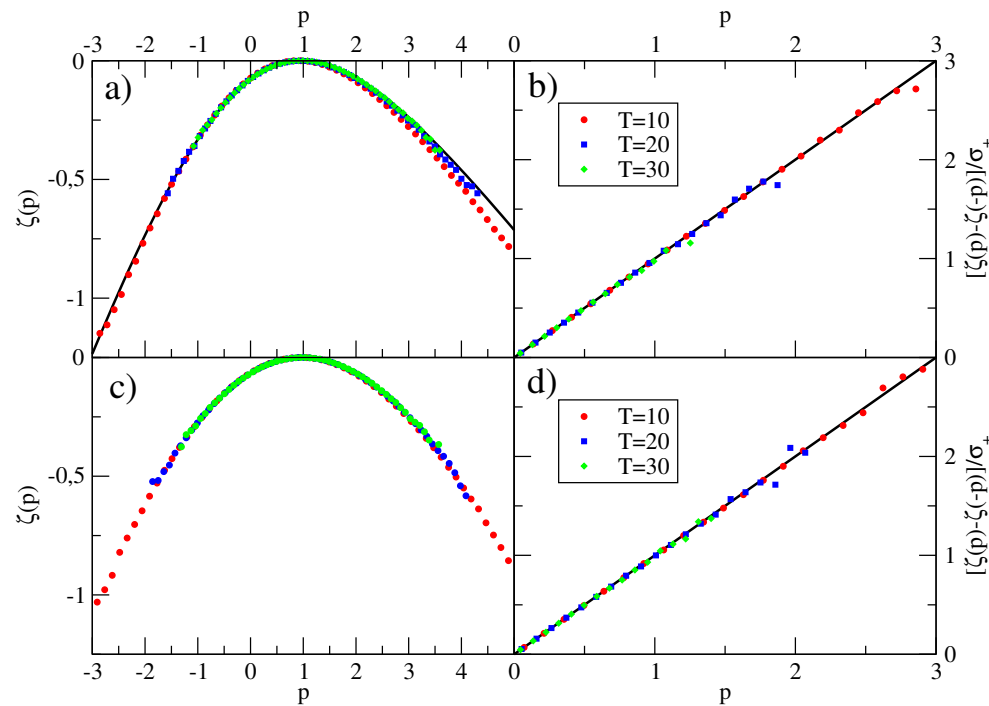
NB : when there is separation of time-scales (T_{eff} takes two values T and T^*) this reduces to the proposal by **Sellitto (99)**

Fluctuation theorems

Driving glassy systems

Check

A (harmonic & anharmonic) oscillator coupled to two baths with different time-scales and temperatures.



Fluctuation theorems

Question :

Is T_{eff} appearing in the fluctuation relation as applied to other non-equilibrium systems ?

Conjecture :

it should, in those in which the unperturbed relaxing system is far from equilibrium, in a 'state' with a non-trivial T_{eff} .

- **F. Zamponi**, revisiting simulations of Lennard-Jones mixtures.
- Is T_{eff} playing a role in CMLs ? **Hohenberg & Shraiman (89)**
- Does T_{eff} appear in a fluctuation relation applied to CMLs ?

LFC, E. Katzav & F. Zamponi, work in progress.

Conclusions

- **CMLs and glassy systems :**

Kaneko, Shraiman, Mousseau, Manrubia, Vulpiani, Robledo,...

We plan to revisit the connection,

focusing on dynamical non-equilibrium aspects of glasses, e.g. T_{eff} .

- **CMLs and KPZ :**

For CML itself : confined KPZ growth with slow stretched exponential relaxation in time but trivial spatial dependence.

For the Lyapunov vector : KPZ with a noise with a large distribution,

change in exponents !

Prikovsky, Politi,...

Conclusions – suite

- **Fluctuation relation :**

a way to test metastable non-equilibrium states ?

a way to measure T_{eff}

Sellitto ; Crisanti & Ritort ; Semerjian, LFC & Montanari