Fluctuations in

non-equilibrium systems

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 We want to understand the out of equilibrium dynamics of macroscopic systems in interaction.

e.g. coarsening, critical relaxation, glassy dynamics.

- Relaxation of one-time observables, e.g. $\langle E(t) \rangle$, is insufficient.
- Averaged two-time correlation and linear-response

$$C(t,t_w) = \langle \phi(t)\phi(t_w) \rangle \qquad \chi(t,t_w) = \int_{t_w}^t dt' \left. \frac{\delta\langle \phi(t) \rangle}{\delta h(t')} \right|_{h=0}$$

have a much richer structure.

Separation of time scales : additive (non-vanishing order parameter), multiplicative (vanishing order parameter).

Relation between spontaneous and induced fluctuations via time-scale dependent fluctuation-dissipation relations, $\chi(C, t_w)$, with different limiting forms.

Langevin process

$$\dot{\phi} = -\frac{\delta F}{\delta \phi} + \xi \quad \Rightarrow \quad \phi = \mathcal{F}[\phi_0, \xi] \quad \text{implicit solution}$$

Fluctuating two-time composite fields

$$\hat{C}=\phi(t)\phi(t_w)$$
 "corr" $2T\hat{\chi}=\phi(t)\xi(t_w)$ "resp"*

*subtlety equal-times, see Corberi, Lippiello, Sarracino, Zannetti 11

Martin-Siggia-Rose generating functional

$$\mathcal{Z}_{dyn} = \int \mathcal{D}\phi \mathcal{D}i\hat{\phi} \; e^{S[\phi,\hat{\phi}]}$$

In this formalism $2T\hat{\chi} = \phi(t)i\hat{\phi}(t_w).$

Questions

How are these objects distributed?

 $P(\hat{C}, \hat{\chi}; t, t_w)$

- Does $P(\hat{C}, \hat{\chi}; t, t_w) = \tilde{P}(\hat{C}, \hat{\chi}; C, t_w)$ scale in the long t_w limit? ($C = \langle \phi \phi \rangle$ is here the averaged two time-correlation)
- Less ambitious : scaling of (some) moments.
 In particular, do averages involving factors of
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 \u03c6 in different combinations scale in the same way ? e.g.

$$V_{CC}(t,t_w) = \int d^d x \left\langle \hat{C}(\vec{x};t,t_w) \hat{C}(\vec{0};t,t_w) \right\rangle$$
$$V_{\chi\chi}(t,t_w) = \int d^d x \left\langle \hat{\chi}(\vec{x};t,t_w) \hat{\chi}(\vec{0};t,t_w) \right\rangle$$

Questions

• Generalized fluctuation-dissipation relations beyond the first moment?

- With the same effective temperature?
- Can one identify the **ruling mechanisms**?

Guiding symmetry? Castillo, Chamon, LFC & Kennett 02 Theoretic analysis numeric analysis

As usual, treat different dynamic classes in parallel :

Gaussian models – critical relaxation – coarsening – glasses

NB We focused on the aging part of the out of equilibrium relaxation while Franz, Parisi, Ricci-Tersenghi & Rizzo 11 are looking at the super-cooled equilibrium regime and fluctuations around the plateau.

Plan

Back to the analysis of the averaged correlation and linear response.

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glassy dynamics : the p-spin model.
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domain growth : the O(N) ferromagnet.
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Emerging symmetries in the asymptotic aging regime.

- Comments on the analysis of the **effective MSR actions**.
- Consequence on fluctuations.
- Massless scalar field and the critical phase of the 2d xy model. Work in progress, see Corberi's talk.

Global dynamic equations

Schwinger-Dyson equations

Quite generally, one can derive closed equations on the two-time global averaged correlation C and linear response R:

$$(\partial_t - \mathbf{z_t})C(t, t_w) = \int dt' \left[\Sigma(t, t')C(t', t_w) + D(t, t')R(t_w, t') \right] +2TR(t_w, t) ,$$
$$(\partial_t - \mathbf{z_t})R(t, t_w) = \delta(t - t_w) + \int dt' \Sigma(t, t')R(t', t_w) ,$$

where the self-energy $\Sigma(t, t')$ and vertex D(t, t') are model-dependent functionals of C and R.

Of course, it is difficulty is to compute them, but in some cases one can.

Global dynamic equations

p-spin models

The self-energy and vertex are

$$D(t,t') = \frac{p}{2}C^{p-1}(t,t') ,$$

$$\Sigma(t,t') = \frac{p(p-1)}{2}C^{p-2}(t,t') R(t,t') .$$

and the Lagrange multiplier $z_t \rightarrow z_\infty$ fixed by setting C(t,t) = 1.



Separation of time-scales

The linear response in the long t_w limit



Fast

Eqs. for the slow relaxation $C_{ag} \equiv C < q$ and $\chi_{ag} \equiv \chi > (1-q)/T$

Approx. asymptotic time-reparametization invariance t
ightarrow h(t)

Separation of time-scales

Example : the eq. $(\partial_t - z_t)R = \delta + \Sigma R$ in the p-spin model

Approximations in the long t_w limit :

- Take $t t_w \gg t_w$.
- Assume $\partial_t R \ll$ terms in the right-hand-side.
- Assume $z_t \to z_\infty$.
- Separate the fast contributions to the integral $\int_{t_w}^t dt' \ \Sigma(t,t') R(t',t_w)$

and assume that the contributions from the fast relaxation are constants.

The aging equation becomes :

$$\tilde{z}_{\infty}R_{ag}(t,t_w) \sim \int_{t_w}^t dt' \ D'[C_{ag}(t,t')] \ R_{ag}(t,t') \ R_{ag}(t',t_w)$$
 (1)

Separation of time-scales

The p-spin model

A similar approximation is applied the equation "for" C.

The coupled remaining equations lead to

$$\begin{aligned} R_{st}(t,t_w) &\simeq (t-t_w)^{-a(T)-1} & \text{for} \quad t-t_w \to \infty \\ R_{ag}(t,t_w) &\simeq t^{-1} f_R\left(\frac{t}{t_w}\right) & \text{for} \quad t \propto t_w \end{aligned}$$

with a(T = 0) = 1/2 and $a(T_d) = 1$ as a (possible) solution to these equations.

Note that 1 + a(T) > 1

But this is not the only solution; there are infinitely many to the approximate equations :

Time-reparametrization

The transformation

$$t \to h_t \equiv h(t) \qquad \begin{cases} C_{ag}(t, t_w) \to C_{ag}(h_t, h_{t_w}) \\ R_{ag}(t, t_w) \to \frac{dh_{t_w}}{dt_w} R_{ag}(h_t, h_{t_w}) \end{cases}$$

with h_t positive and monotonic leaves eq. (1) **invariant** :

$$\tilde{z}_{\infty} R_{ag}(h_t, h_{t_w}) \sim \int_{h_w}^{h_t} dh_{t'} D'[C_{ag}(h_t, h_{t'})] R_{ag}(h_t, h_{t'}) R_{ag}(h_{t'}, h_{t_w})$$

One can compute analytically f_c and $\chi_{ag}(C_{ag})$ (consistent w/assumptions)

$$C_{ag}(t, t_w) \sim f_c \left(\frac{L(t)}{L(t_w)}\right) ,$$

$$\chi(t, t_w) \equiv \int_{t_w}^t dt' R(t, t') \sim \frac{1-q}{T} + \frac{1}{T_{\text{eff}}} \left[q - C_{ag}(t, t_w)\right]$$

but not the 'clock' L(t)

The $O(N \to \infty)$ model

Exact solution

$$\dot{\phi}_{\alpha}(\vec{x},t) = \nabla^2 \phi_{\alpha}(\vec{x},t) - \lambda |\phi^2/N - 1|\phi_{\alpha}(\vec{x},t) + \xi_{\alpha}(\vec{x},t)$$

Quadratic equation under the replacement $\phi^2(\vec{x},t) \rightarrow \langle \phi^2 \rangle \equiv z_t N$.

One finds

$$\phi_{\alpha}(\vec{k},t) = \mathcal{F}[\phi_{\alpha}(\vec{k},0),\xi_{\alpha}(\vec{k},t')]$$

and from here the two-time correlation and linear-response.

See, e.g., Corberi, Lippiello & Zannetti 02

A much more cumbersome route, closer to what has been done for the **p-spin model** is the following.

Chamon, LFC & Yoshino 06

The $O(N \to \infty)$ model

Invariance of the slow dynamic equations?

The Schwinger-Dyson equations act on $R(t,t') \equiv \int d^d r \ R(\vec{r},t,t')$ and $C(t,t') \equiv \int d^d r \ C(\vec{r},t,t')$.

The self-energy is

$$\Sigma(t,t') = \sum_{n=0}^{\infty} A_n \int dt_{n-1} \dots \int dt_1 \ R(t,t_1) R(t_1,t_2) \dots R(t_{n-1},t')$$

with the constants A_n fixed by the Fourier-mode density.

After a separation of time-scales and $t - t_w \gg t_w$ one has

$$\frac{\partial R_{ag}(t,t_w)}{\partial t} = -z_t R_{ag}(t,t_w) + \sum_{n=0}^{\infty} B_n(t-t_w)$$
$$\times \int dt_n \int dt_{n-1} \dots \int dt_1 R_{ag}(t,t_1) R_{ag}(t_1,t_2) \dots R_{ag}(t_n,t_w)$$

The $O(N \to \infty)$ model

Invariance of the slow dynamic equations?

Knowing the exact R one can plug in R_{ag} to find that, apart from a function $g(t/t_w)$,

• the time-derivative behaves as $\left| \partial_t R_{ag} \simeq t^{-1-d/2}
ight|;$

- the Lagrange multiplier z_t decays as $t^{-1}\,;$ then $\left| \, z_t \, R_{ag} \sim t^{-1-d/2} \, \right|$ too ;

- the coefficients B_n (stationary contributions) do not approach constants !

INSTEAD $B_n(t - t_w) \sim (t - t_w)^{-1 + n(1 - d/2)}$.

- The integral factors go as $I_n \sim t^{-d/2 - n(1 - d/2)}$ in such a way that

 $B_n I_n \sim t^{-1-d/2}$ as well.

No time-reparametrization invariance, just scale invariance $t \to \zeta t$

Classification

Invariance of the slow dynamic equations?

 The key to the difference seems to be in the bad separation of timescales in the linear response :

 $R_{st}(t-t_w) \simeq (t-t_w)^{-d/2} \& R_{ag}(t,t_w) \simeq t^{-d/2} f_R(t_w/t)$

in the O(N) model while

 $R_{st}(t - t_w) \simeq (t - t_w)^{-a(T)-1}$ & $R_{ag}(t, t_w) \simeq t^{-1} f_R(t_w/t)$

with $a(T) \in [1/2, 1]$ in the **p-spin model**.

 The same analysis can be performed at the level of the MSR generating functional; separate the field into fast and slow components as done by Corberi, Lippiello & Zannetti 02

Chamon, LFC & Yoshino 06

Massless fluctuations

Scaling of the slow part of the global correlation

$$C^{s}(t, t_{w}) \approx \mathbf{f_{c}} \left(\frac{L(t)}{L(t_{w})}\right)$$

Time-reparametrization invariance $\Rightarrow C_r^s(t, t_w) \approx \mathbf{f_c}\left(\frac{h_r(t)}{h_r(t_w)}\right).$

Example :

$$h_{r_1} = e^{\ln^a \left(\frac{t}{t_0}\right)}$$
 ('fast') $h_{r_3} = \frac{t}{t_0}$ ('normal'), $h_{r_2} = \ln\left(\frac{t}{t_0}\right)$ ('slow').



Same t_w , slower and faster decays on different regions labeled by r_1 , r_3 , r_2 ,

Castillo, Chamon, LFC, Iguain, Kennett 02, 03

Consequences

Easier to measure consequences

Time-reparametrization invariance implies that the moment of \hat{C} and $\hat{\chi}$ should scale in the same way, e.g.

$$V_{CC}(t,t_w) = \int d^d x \left\langle \hat{C}(\vec{x};t,t_w) \hat{C}(\vec{0};t,t_w) \right\rangle$$
$$V_{\chi\chi}(t,t_w) = \int d^d x \left\langle \hat{\chi}(\vec{x};t,t_w) \hat{\chi}(\vec{0};t,t_w) \right\rangle$$

Chamon, Corberi & LFC 11

This should not be the case for system breaking this symmetry asymptotically, such as **coarsening systems**, if we believe that the O(N) should be extended to non-mean-field cases.

Variances

3dEA



All "variances" scale in the same way.

See Corberi's talk for more.

Consequences

- We argued in favor of time-reparametrization invariance as the guiding symmetry that controls fluctuations in glassy samples.
 We checked the consequences with various numeric simulations, mostly on the 3d Edwards-Anderson model.
- We analyzed the non-equilibrium dynamics of the ${\cal O}(N)$ model with the same ideas.

We found that the symmetry is reduced to rescaling of time.

The moments of the distribution do not scale in the same way.

- We are currently working on Gaussian models as the massless scalar field and the 2d xy model (critical relaxation).
- This framework we can get a full understanding of fluctuations in the aging regime of non-equilibrium macroscopic systems.

Gaussian Langevin process & critical KT phase

$$\dot{\phi} = -\frac{\delta F}{\delta \phi} + \xi \quad \Rightarrow \quad \phi = \mathcal{F}[\phi_0, \xi] \quad \text{linear functional}$$

(e.g., massless scalar field, angle in spin-wave approximation to 2d xy model, height in Edwards-Wilkinson interface)

 $P(\hat{C},\hat{\chi};t,t_w)=$ known analytically

Generalized fluctuation-dissipation relations beyond the first moment.

Corberi & LFC, in preparation