Fluctuations in
non-equilibrium systems

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Introduction

- We want to understand the out of equilibrium dynamics of macroscopic systems in interaction.

  e.g. coarsening, critical relaxation, glassy dynamics.

- Relaxation of one-time observables, e.g. $\langle E(t)\rangle$, is insufficient.

- Averaged two-time correlation and linear-response

$$C(t,t_w) = \langle \phi(t)\phi(t_w)\rangle \qquad \chi(t,t_w) = \int_{t_w}^{t} dt' \, \frac{\delta\langle\phi(t)\rangle}{\delta h(t')}\bigg|_{h=0}$$

  have a much richer structure.

  Separation of time scales : additive (non-vanishing order parameter), multiplicative (vanishing order parameter).

  Relation between spontaneous and induced fluctuations via time-scale dependent fluctuation-dissipation relations, $\chi(C,t_w)$, with different limiting forms.

Introduction

Langevin process

$$\dot{\phi} = -\frac{\delta F}{\delta \phi} + \xi \qquad \Rightarrow \qquad \phi = \mathcal{F}[\,\phi_0, \xi\,]$$ implicit solution

Fluctuating two-time composite fields

$\hat{C} = \phi(t)\phi(t_w)$ "corr" $\qquad 2T\hat{\chi} = \phi(t)\xi(t_w)$ "resp"*

*subtlety equal-times, see **Corberi, Lippiello, Sarracino, Zannetti 11**

Martin-Siggia-Rose generating functional

$$\mathcal{Z}_{dyn} = \int \mathcal{D}\phi \mathcal{D}i\hat{\phi}\; e^{S[\phi,\hat{\phi}]}$$

In this formalism $2T\hat{\chi} = \phi(t)i\hat{\phi}(t_w)$.

Introduction

Questions

How are these objects distributed ?

$$P(\hat{C}, \hat{\chi}; t, t_w)$$

- Does $P(\hat{C}, \hat{\chi}; t, t_w) = \tilde{P}(\hat{C}, \hat{\chi}; C, t_w)$ scale in the long t_w limit ?

  ($C = \langle \phi\phi \rangle$ is here the averaged two time-correlation)

- Less ambitious : scaling of (some) **moments**.

  In particular, do averages involving factors of $\hat{C}$ and $\hat{\chi}$ in different combinations scale in the same way ? e.g.

$$V_{CC}(t, t_w) = \int d^d x \, \langle \hat{C}(\vec{x}; t, t_w) \hat{C}(\vec{0}; t, t_w) \rangle$$

$$V_{\chi\chi}(t, t_w) = \int d^d x \, \langle \hat{\chi}(\vec{x}; t, t_w) \hat{\chi}(\vec{0}; t, t_w) \rangle$$

Introduction

Questions

- **Generalized fluctuation-dissipation relations** beyond the first moment ?
- With the same effective temperature ?
- Can one identify the **ruling mechanisms** ?

Guiding **symmetry** ? **Castillo, Chamon, LFC & Kennett 02**

Theoretic analysis **numeric analysis**

As usual, treat different **dynamic classes** in parallel :

Gaussian models – **critical relaxation** – **coarsening** – **glasses**

NB We focused on the aging part of the out of equilibrium relaxation while **Franz, Parisi, Ricci-Tersenghi & Rizzo 11** are looking at the super-cooled equilibrium regime and fluctuations around the plateau.

Plan

- Back to the analysis of the **averaged correlation and linear response**.

  glassy dynamics : the p-spin model.

  domain growth : the $O(N)$ ferromagnet.

  Emerging **symmetries** in the asymptotic aging regime.

- Comments on the analysis of the **effective MSR actions**.

- Consequence on fluctuations.

- **Massless scalar field and the critical phase of the $2d$ xy model**.

  Work in progress, see **Corberi's talk**.

Global dynamic equations

Schwinger-Dyson equations

Quite generally, one can derive closed equations on the two-time global averaged correlation C and linear response R :

$$(\partial_t - z_t)C(t,t_w) = \int dt' \; [\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')]$$
$$+2TR(t_w,t) \; ,$$
$$(\partial_t - z_t)R(t,t_w) = \delta(t-t_w) + \int dt' \; \Sigma(t,t')R(t',t_w) \; ,$$

where the **self-energy** $\Sigma(t,t')$ and **vertex** $D(t,t')$ are model-dependent functionals of C and R.

Of course, it is difficulty is to compute them, but in some cases one can.

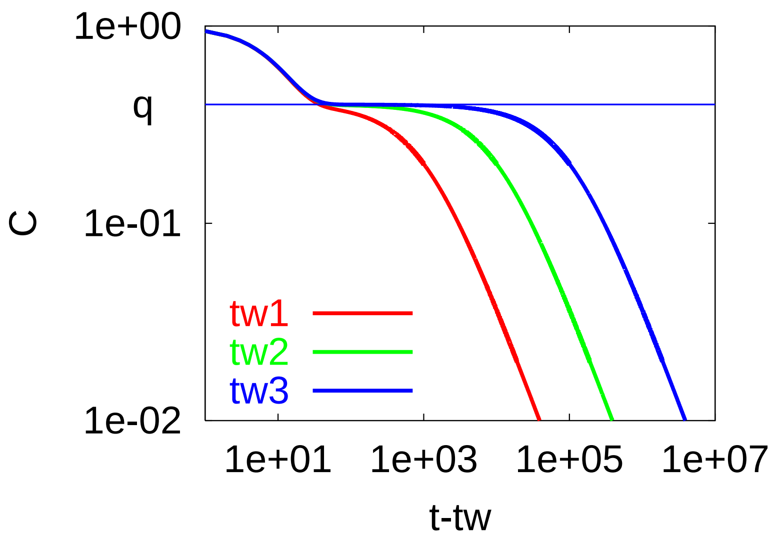
Global dynamic equations

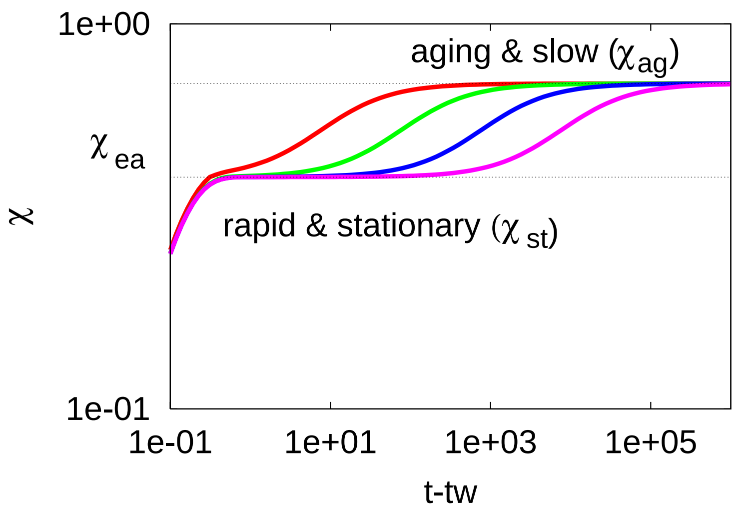
p-spin models

The self-energy and vertex are

$$D(t,t') = \frac{p}{2} C^{p-1}(t,t') \; ,$$
$$\Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t') \, R(t,t') \; .$$

and the Lagrange multiplier $z_t \to z_\infty$ fixed by setting $C(t,t) = 1$.

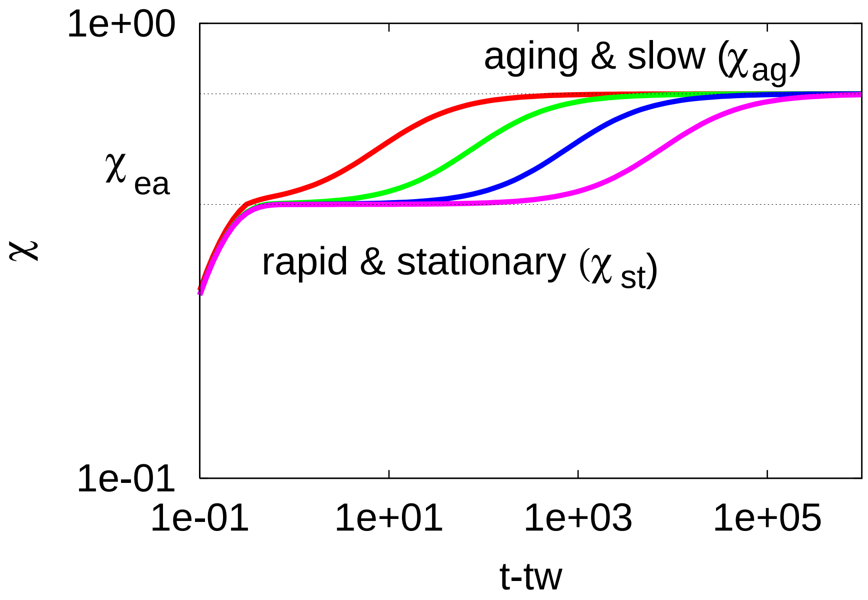




Separation of time-scales

The linear response in the long t_w limit

Fast



Slow

$$\chi_{ag}(t,t_w) \approx f_\chi\left(\frac{L(t)}{L(t_w)}\right)$$

$$\boxed{\chi_{ag}(t,t_w) > 0}$$

$$\partial_t \chi_{ag}(t,t_w) \ll \chi_{ag}(t,t_w)$$

log-log scale !

Eqs. for the slow relaxation $C_{ag} \equiv C < q$ and $\chi_{ag} \equiv \chi > (1-q)/T$

Approx. asymptotic time-reparametization invariance $\quad t \to h(t)$

Separation of time-scales

Example : the eq. $(\partial_t - z_t)R = \delta + \Sigma R$ in the p-spin model

Approximations in the long t_w limit :

- Take $t - t_w \gg t_w$.
- Assume $\partial_t R \ll$ terms in the right-hand-side.
- Assume $z_t \to z_\infty$.
- Separate the fast contributions to the integral $\int_{t_w}^{t} dt' \, \Sigma(t,t')R(t',t_w)$

  and assume that the contributions from the fast relaxation are **constants**.

  The aging equation becomes :

$$\tilde{z}_\infty R_{ag}(t,t_w) \sim \int_{t_w}^{t} dt' \; D'[C_{ag}(t,t')] \, R_{ag}(t,t') \, R_{ag}(t',t_w) \tag{1}$$

Separation of time-scales

The p-spin model

A similar approximation is applied the equation "for" C.

The coupled remaining equations lead to

$$\begin{aligned} R_{st}(t,t_w) &\simeq (t-t_w)^{-a(T)-1} & \text{for} \quad & t-t_w \to \infty \\ R_{ag}(t,t_w) &\simeq t^{-1} f_R\left(\frac{t}{t_w}\right) & \text{for} \quad & t \propto t_w \end{aligned}$$

with $a(T=0)=1/2$ and $a(T_d)=1$ as a (possible) solution to these equations.

Note that $\boxed{1+a(T)>1}$

But this is not the only solution ; there are infinitely many to the approximate equations :

Time-reparametrization

The transformation

$$t \to h_t \equiv h(t) \qquad \left\{ \begin{array}{l} C_{ag}(t,t_w) \to C_{ag}(h_t,h_{t_w}) \\ R_{ag}(t,t_w) \to \frac{dh_{t_w}}{dt_w} R_{ag}(h_t,h_{t_w}) \end{array} \right.$$

with h_t positive and monotonic leaves eq. (1) **invariant** :

$$\tilde{z}_\infty R_{ag}(h_t,h_{t_w}) \sim \int_{h_w}^{h_t} dh_{t'}\, D'[C_{ag}(h_t,h_{t'})] R_{ag}(h_t,h_{t'})\, R_{ag}(h_{t'},h_{t_w})$$

One can compute analytically f_c and $\chi_{ag}(C_{ag})$ (consistent w/assumptions)

$$C_{ag}(t,t_w) \sim f_c\left(\frac{L(t)}{L(t_w)}\right) ,$$

$$\chi(t,t_w) \equiv \int_{t_w}^{t} dt'\, R(t,t') \sim \frac{1-q}{T} + \frac{1}{T_{\text{eff}}}\,[q - C_{ag}(t,t_w)]$$

but not the 'clock' $L(t)$.

The $O(N \to \infty)$ model

Exact solution

$$\dot{\phi}_\alpha(\vec{x},t) = \nabla^2 \phi_\alpha(\vec{x},t) - \lambda|\phi^2/N - 1|\phi_\alpha(\vec{x},t) + \xi_\alpha(\vec{x},t)$$

Quadratic equation under the replacement $\phi^2(\vec{x},t) \to \langle \phi^2 \rangle \equiv z_t N$.

One finds

$$\phi_\alpha(\vec{k},t) = \mathcal{F}[\, \phi_\alpha(\vec{k},0), \xi_\alpha(\vec{k},t') \,]$$

and from here the two-time correlation and linear-response.

See, e.g., **Corberi, Lippiello & Zannetti 02**

A much more cumbersome route, closer to what has been done for the **p-spin model** is the following.

Chamon, LFC & Yoshino 06

The $O(N \to \infty)$ model

Invariance of the slow dynamic equations ?

The Schwinger-Dyson equations act on $R(t,t') \equiv \int d^dr \ R(\vec{r},t,t')$ and $C(t,t') \equiv \int d^dr \ C(\vec{r},t,t')$.

The self-energy is

$$\Sigma(t,t') = \sum_{n=0}^{\infty} A_n \int dt_{n-1} \dots \int dt_1 \ R(t,t_1)R(t_1,t_2)\dots R(t_{n-1},t')$$

with the constants A_n fixed by the Fourier-mode density.

After a separation of time-scales and $t - t_w \gg t_w$ one has

$$\frac{\partial R_{ag}(t,t_w)}{\partial t} = -z_t R_{ag}(t,t_w) + \sum_{n=0}^{\infty} B_n(t-t_w)$$
$$\times \int dt_n \int dt_{n-1} \dots \int dt_1 \ R_{ag}(t,t_1)R_{ag}(t_1,t_2)\dots R_{ag}(t_n,t_w)$$

The $O(N \to \infty)$ model

Invariance of the slow dynamic equations ?

Knowing the exact R one can plug in R_{ag} to find that, apart from a function $g(t/t_w)$,

- the time-derivative behaves as $\boxed{\partial_t R_{ag} \simeq t^{-1-d/2}}$;
- the Lagrange multiplier z_t decays as t^{-1}; then $\boxed{z_t\, R_{ag} \sim t^{-1-d/2}}$ too ;
- the coefficients B_n (stationary contributions) do not approach constants !

  INSTEAD $B_n(t-t_w) \sim (t-t_w)^{-1+n(1-d/2)}$.
- The integral factors go as $I_n \sim t^{-d/2-n(1-d/2)}$ in such a way that

  $\boxed{B_n I_n \sim t^{-1-d/2}}$ as well.

No time-reparametrization invariance, just scale invariance $\boxed{t \to \zeta t}$

Classification

Invariance of the slow dynamic equations ?

- The key to the difference seems to be in the bad separation of time-scales in the linear response :

$$R_{st}(t-t_w) \simeq (t-t_w)^{-d/2} \quad \& \quad R_{ag}(t,t_w) \simeq t^{-d/2} f_R(t_w/t)$$

in the $O(N)$ **model** while

$$R_{st}(t-t_w) \simeq (t-t_w)^{-a(T)-1} \quad \& \quad R_{ag}(t,t_w) \simeq t^{-1} f_R(t_w/t)$$

with $a(T) \in [1/2, 1]$ in the **p-spin model**.

- The same analysis can be performed at the level of the MSR generating functional ; separate the field into fast and slow components as done by **Corberi, Lippiello & Zannetti 02**

Chamon, LFC & Yoshino 06

Massless fluctuations

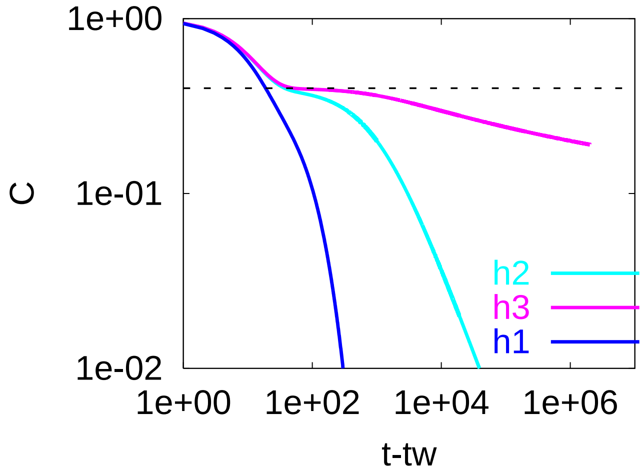
Scaling of the slow part of the global correlation

$$C^s(t,t_w) \approx \mathbf{f_c}\left(\frac{L(t)}{L(t_w)}\right) .$$

Time-reparametrization invariance $\Rightarrow$ $C^s_r(t,t_w) \approx \mathbf{f_c}\left(\frac{h_r(t)}{h_r(t_w)}\right)$.

Example :

$h_{r_1} = e^{\ln^a\left(\frac{t}{t_0}\right)}$ ('fast') $h_{r_3} = \frac{t}{t_0}$ ('normal'), $h_{r_2} = \ln\left(\frac{t}{t_0}\right)$ ('slow') .



Same t_w, slower and faster decays on

different regions labeled by r_1, r_3, r_2,

Castillo, Chamon, LFC, Iguain, Kennett 02, 03

Consequences

Easier to measure consequences

Time-reparametrization invariance implies that the moment of $\hat{C}$ and $\hat{\chi}$ should scale in the same way, e.g.

$$V_{CC}(t, t_w) = \int d^d x \, \langle \hat{C}(\vec{x}; t, t_w) \hat{C}(\vec{0}; t, t_w) \rangle$$

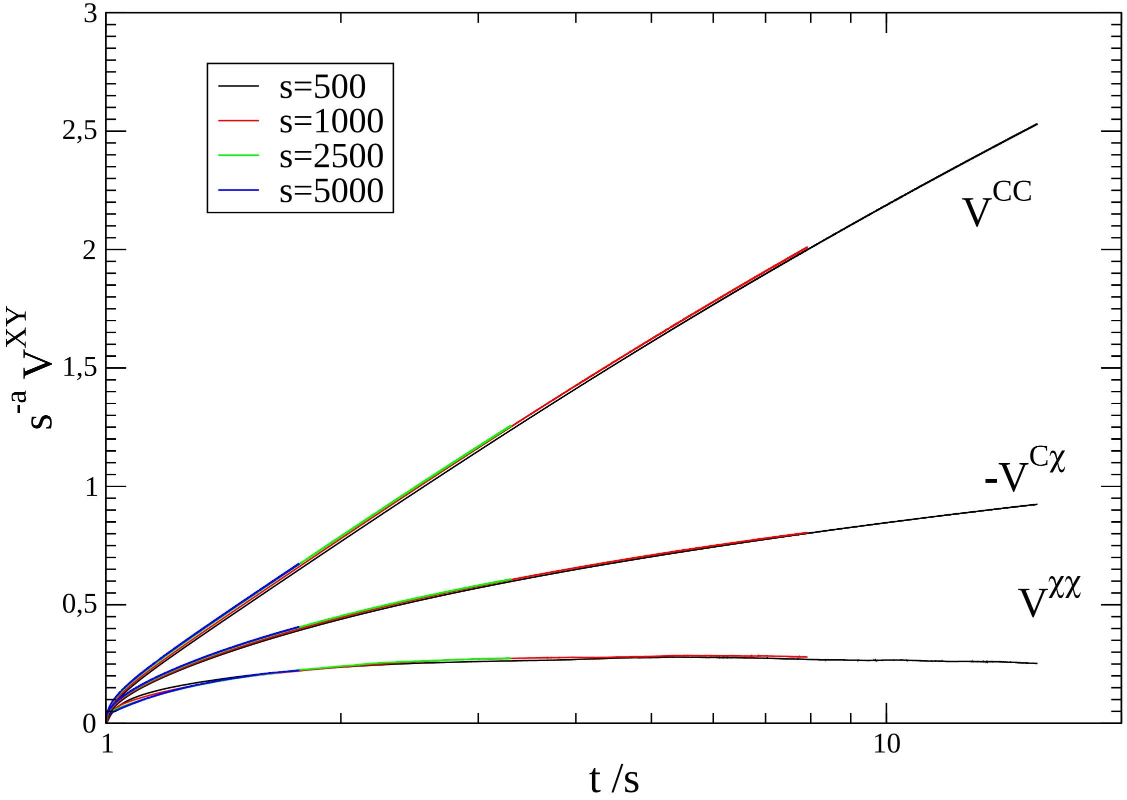
$$V_{\chi\chi}(t, t_w) = \int d^d x \, \langle \hat{\chi}(\vec{x}; t, t_w) \hat{\chi}(\vec{0}; t, t_w) \rangle$$

Chamon, Corberi & LFC 11

This should not be the case for system breaking this symmetry asymptotically, such as **coarsening systems**, if we believe that the O(N) should be extended to non-mean-field cases.

Variances

3dEA



All "variances" scale in the same way.

See **Corberi's talk for more.**

Consequences

- We argued in favor of time-reparametrization invariance as the guiding symmetry that controls fluctuations in glassy samples.

  We checked the consequences with various numeric simulations, mostly on the $3d$ Edwards-Anderson model.

- We analyzed the non-equilibrium dynamics of the $O(N)$ model with the same ideas.

  We found that the symmetry is reduced to rescaling of time.

  The moments of the distribution do not scale in the same way.

- We are currently working on Gaussian models as the massless scalar field and the $2d$ xy model (critical relaxation).

- This framework we can get a full understanding of fluctuations in the aging regime of non-equilibrium macroscopic systems.

Introduction

Gaussian Langevin process & critical KT phase

$$\dot{\phi} = -\frac{\delta F}{\delta \phi} + \xi \qquad \Rightarrow \qquad \phi = \mathcal{F}[\,\phi_0, \xi\,]$$ linear functional

(e.g., massless scalar field, angle in spin-wave approximation to $2d$ xy model, height in Edwards-Wilkinson interface)

$$P(\hat{C}, \hat{\chi}; t, t_w) =$$ known analytically

- Generalized fluctuation-dissipation relations beyond the first moment.
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Corberi & LFC, in preparation