# Coarsening

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with

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# Plan

1. Review of Coarsening phenomena

   ex. ferromagnetic systems.

2. The scaling hypothesis.

   Analytical and numerical results.

3. New : details on the domain conformations.

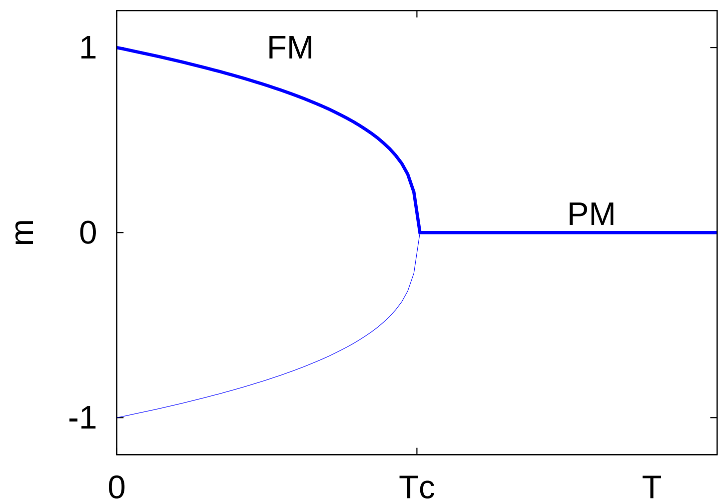
   Analytical and numerical results.

4. Work in progress.

5. Why should one look at this problem ?

# Ferromagnets in equilibrium

A ferromagnetic system in contact with a heat bath at temperature $T$ under no applied field ($h = 0$) acquires a magnetization density $m$ below a critical temperature $T_c$ :




$m$ is the order parameter.

**Curie-Weiss mean-field theory (1907), Ginzburg-Landau theory (1937, 1950), Wilson renormalization group (1971).**

# The standard Ising model

$$H = -J \sum_{\langle ij \rangle} s_i s_j \ ,$$

**Ising, 1925**

- The spins $s_i$ take bimodal values, $s_i = \pm 1$.
- The sum $\sum_{\langle ij \rangle}$ runs over nearest neighbours on a $d$ dimensional, typically hypercubic, lattice.
- $J > 0$ is the coupling strength.

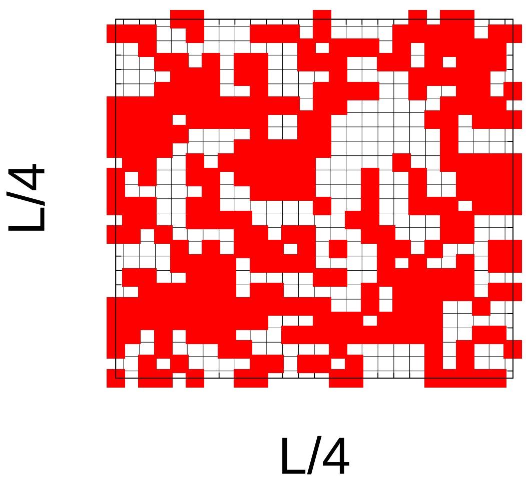
One finds

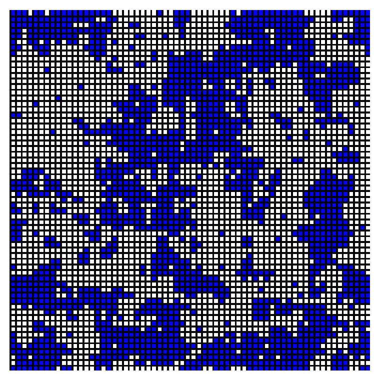
$$\frac{T_c}{J} \begin{cases} = 0 \ , & d = 1 \\ \sim 2.27 \ , & d = 2 \\ \sim 4.5 \ , & d = 3 \end{cases}$$

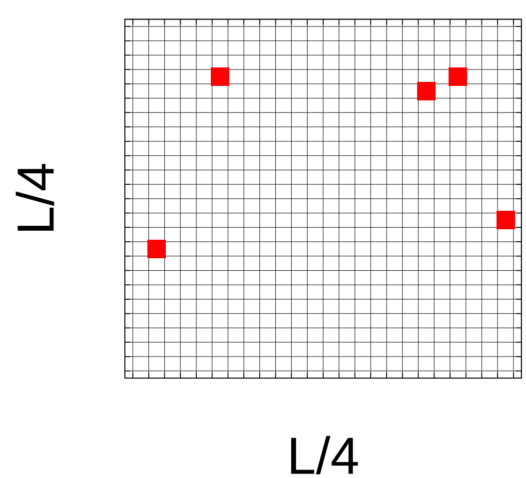
exact (**Ising, 1925**), $d=1$;
exact (**Onsager, 1944**), $d=2$;
num. (**D. P. Landau, 1976**), $d=3$.

# Equilibrium configurations

$2d$ slices of a $3d$ Ising model






| $T \to \infty$ | $T \sim T_c$ | $T < T_c$ |
|---|---|---|
| Random configuration | Structures of all sizes | Essentially ordered |
| $s_i = \pm 1$ | Self-similarity | Thermal fluct. ($m < 1$) |

# Ginzburg-Landau

Coarse-graining $\Rightarrow$ the local magnetization density

$$\phi(\vec{x}) = \frac{1}{\ell^d} \sum_{i \in V_{\vec{x},\ell}} s_i \; , \qquad Z = \sum_{\phi} e^{-\beta F(\phi)} \; .$$

Symmetry arguments ($\phi \rightarrow -\phi$) and $\langle \, \phi \, \rangle \sim 0$ at $T \sim T_c$ suggest

$$F(\phi) = \int d^d x \left[ \underbrace{\frac{c}{2} (\nabla \phi)^2}_{\text{Energy-cost domain-wall}} + \underbrace{\frac{T - T_c}{T_c} \phi^2 + \frac{\lambda}{4} \phi^4}_{\text{Symmetric double-well}} \right]$$

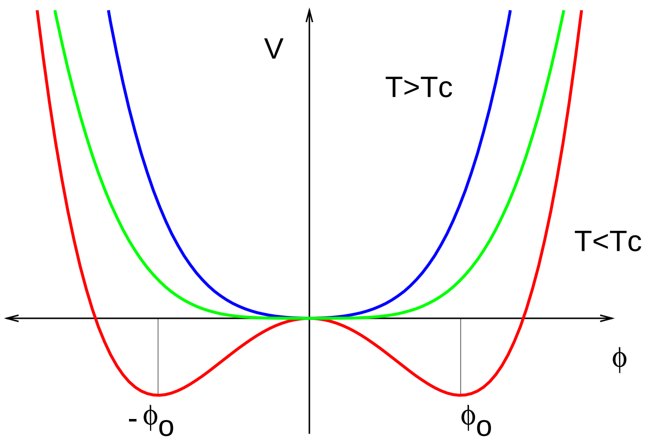
Energy-cost domain-wall

Symmetric double-well

# Ginzburg-Landau

Large volume limit

$F \approx L^d \Rightarrow$ saddle-point, mean-field or stationary phase approx.

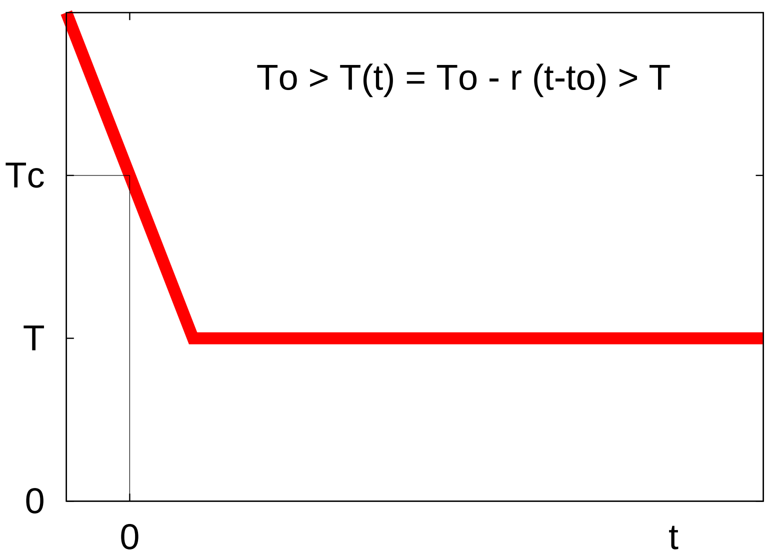


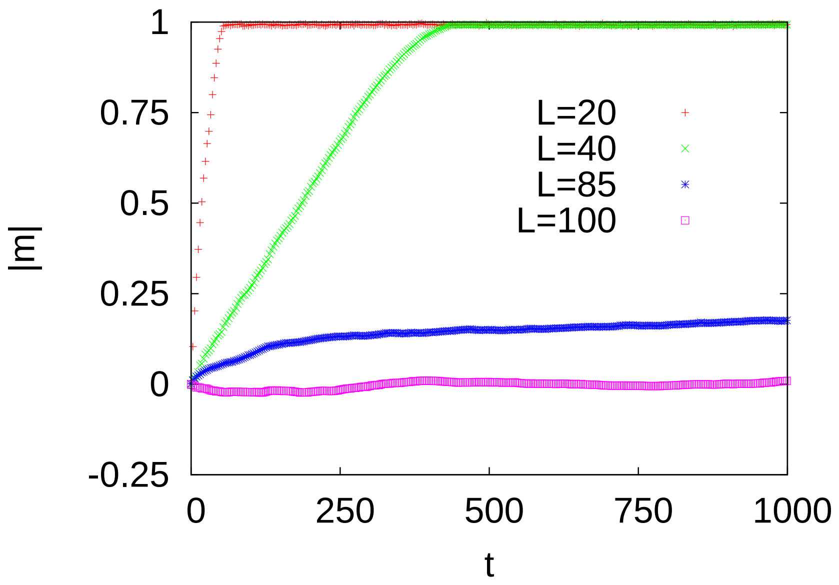

$$\langle \phi(\vec{x}) \rangle = \phi_o \propto (T_c - T)^{\frac{1}{2}} \, , \qquad \beta = \frac{1}{2} \, .$$

Essentially correct but for the critical region (*e.g.* $\beta \sim 1/3$).

# Evolution

A rapid quench



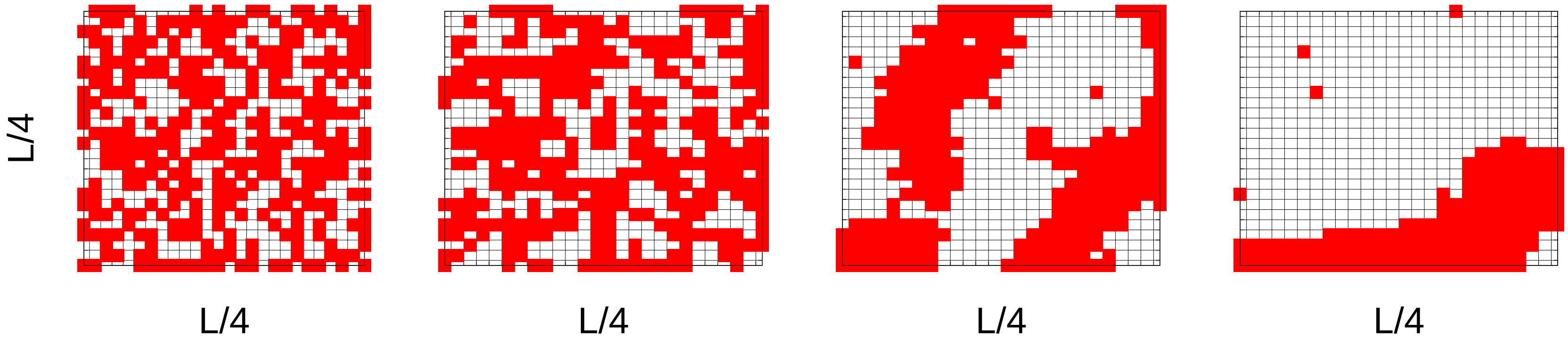

Stochastic dynamics ; Monte Carlo updates

Note : the order parameter ($m$) is not conserved.

Slow dynamics

# Domain growth

After a rapid quench




$t = 0$ MCs $t = 1$ MCs $t = 10$ MCs $t = 10^2$ MCs

# Time-dep. Ginzburg-Landau

$$\phi(\vec{x}) \to \phi(\vec{x},t) = \frac{1}{\ell^d} \sum_{i \in V_{\vec{x},\ell}} s_i(t)$$

Langevin dynamics in $F(\phi)$, model A **(Hohenberg & Halperin 1977).**

$$\begin{aligned} \gamma \frac{\partial \phi(\vec{x},t)}{\partial t} &= -\frac{\delta F(\phi)}{\delta \phi(\vec{x},t)} + \eta(\vec{x},t) \\ &= \nabla^2 \phi(\vec{x},t) + a\phi(\vec{x},t) - \lambda \phi^3(\vec{x},t) + \eta(\vec{x},t) \ , \end{aligned}$$

with $\gamma = t_0^{-1}$ and $\eta$ a Gaussian white noise,
$\langle \, \eta \, \rangle = 0$ and $\langle \, \eta(\vec{x},t)\eta(\vec{x}',t') \, \rangle = 2k_B T \gamma \, \delta(\vec{x}-\vec{x}')\delta(t-t')$.

# Scaling theory

*At late times there is a single length-scale, the typical radius of the domains* $R(T,t)$*, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by* $R(T,t)$*, e.g.*

$$C(r,t) \equiv \langle\, s_i(t) s_j(t)\,\rangle\big|_{|\vec{x}_i-\vec{x}_j|=r} \sim m_{eq}^2(T)\; f\left(\frac{r}{R(T,t)}\right),$$

$$C(t,t_w) \equiv \langle\, s_i(t) s_i(t_w)\,\rangle \sim m_{eq}^2(T)\; g\left(\frac{R(T,t)}{R(T,t_w)}\right),$$

*etc.* when $r \gg \xi(T)$, $t, t_w \gg t_0$ and $C < m_{eq}^2(T)$.

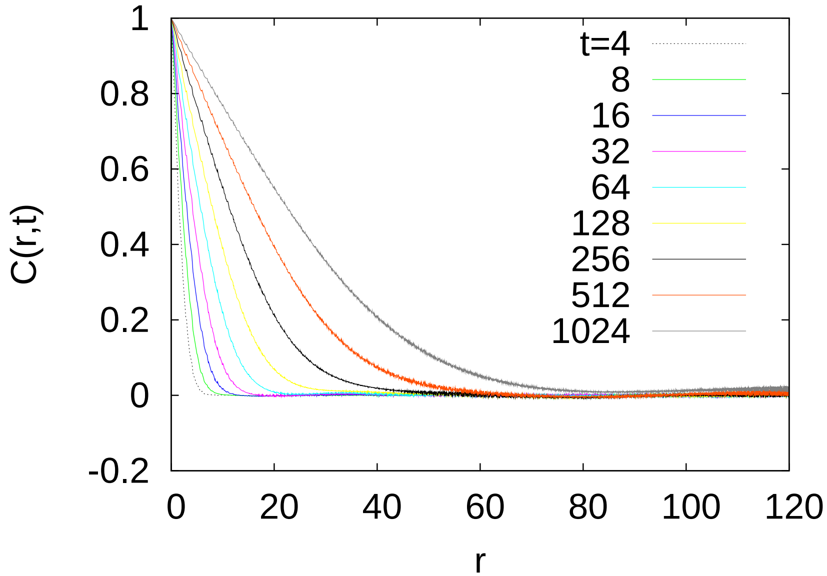
Suggested by experiments and numerical simulations. Proven in

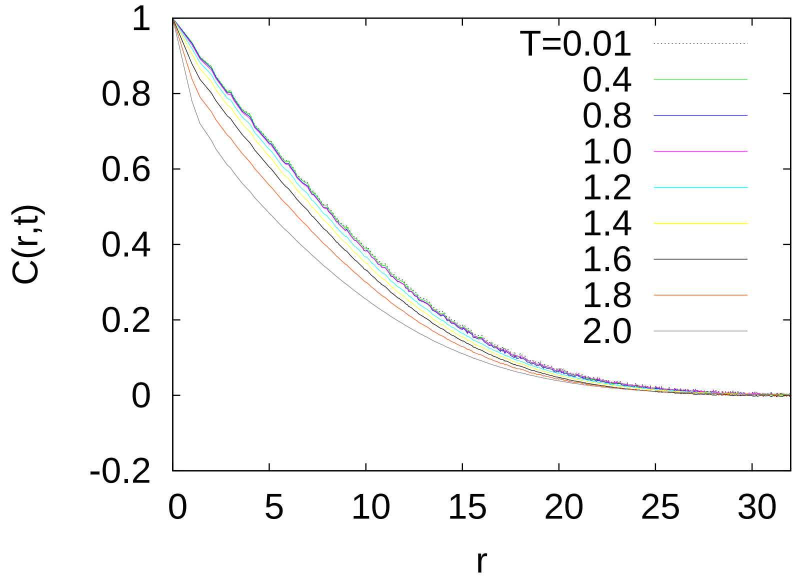
- Ising chain with Glauber dynamics.
- Langevin dynamics of the O($N$) model with $N \to \infty$, and the spherical ferromagnet.

**Review A. J. Bray, 1994.**

# MC dynamics 2dIM

Equal-times spatial correlation



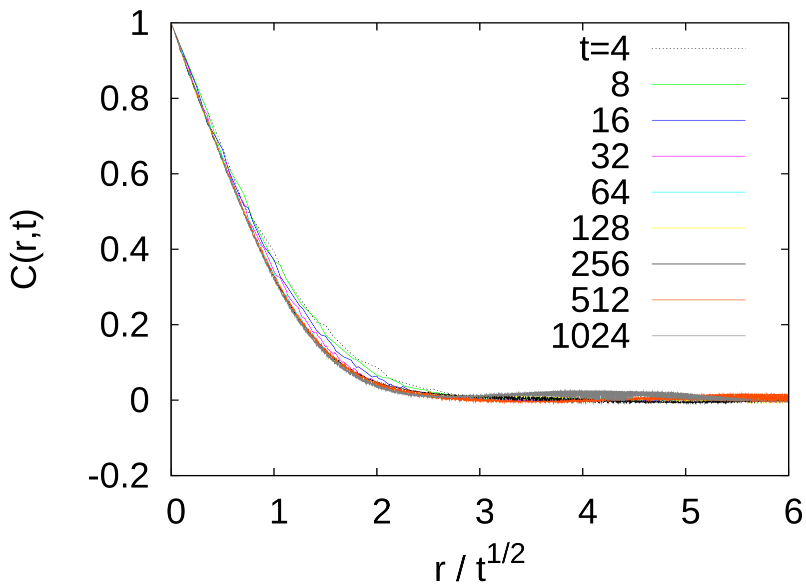




$T = 0.5$ $\qquad$ $t = 128$ MCs

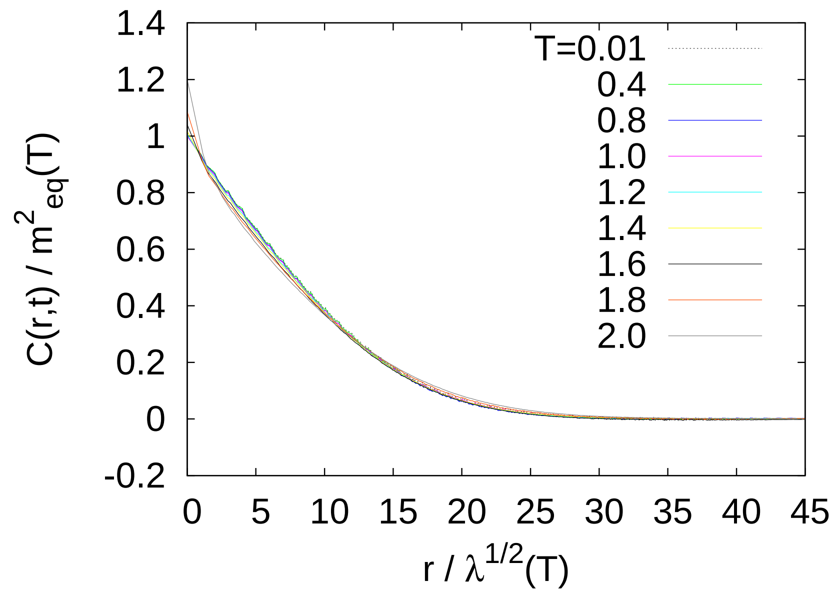
$$C(r,t) \equiv \langle\, s_i(t) s_j(t)\,\rangle|_{|\vec{x}_i - \vec{x}_j| = r} \sim m_{eq}^2(T)\; f\left(\frac{r}{R(T,t)}\right)$$

# MC dynamics 2dIM

Equal-times spatial correlation : scaling




$T = 0.5$

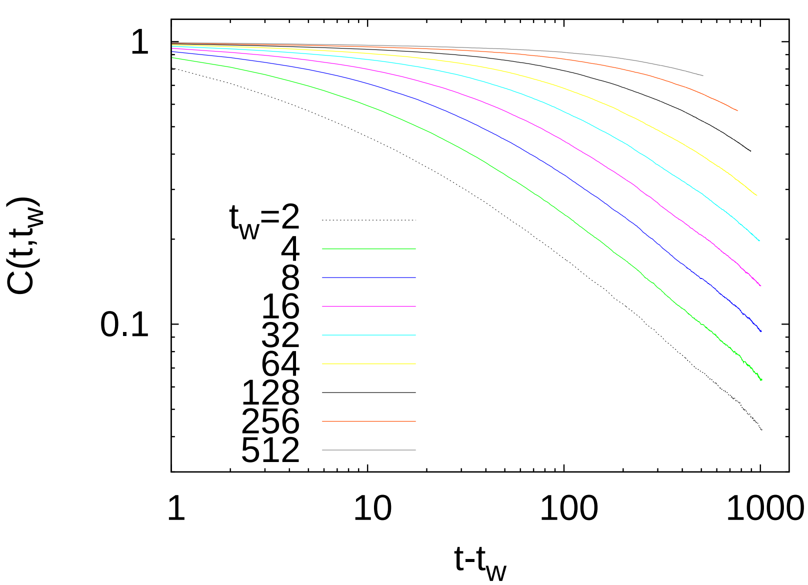



$t = 128$ MCs

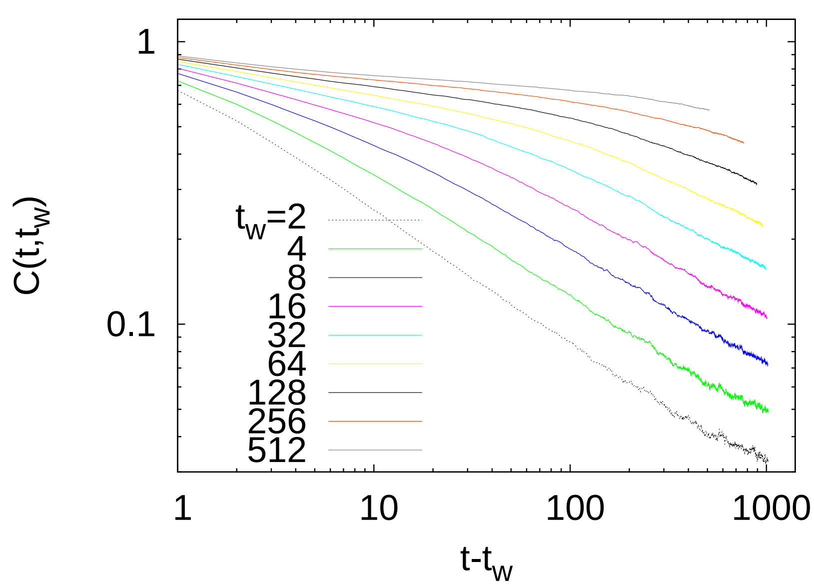
$$C(r,t) \sim m_{eq}^2(T)\, f\left(\frac{r}{R(T,t)}\right) \qquad \text{with} \qquad R(T,t) \sim [\,\lambda(T) t\,]^{1/2}$$

# MC dynamics 2dIM

Two-times local correlation




$T = 0.5$

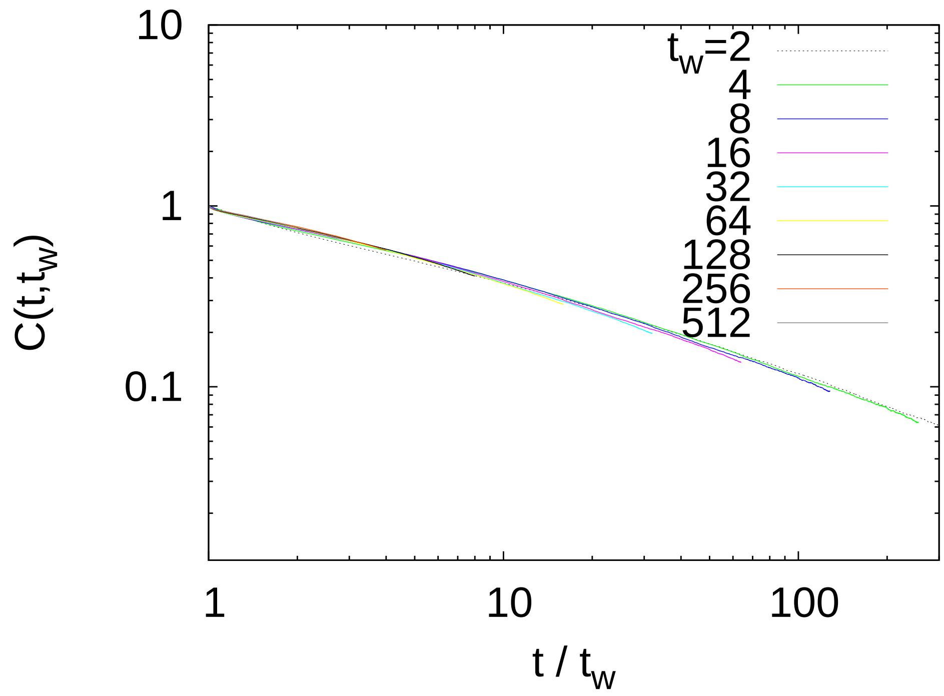



$T = 2$

$$C(t,t_w) = N^{-1}\sum_{i=1}^{N}\langle\, s_i(t)s_i(t_w)\,\rangle \sim m_{eq}^2(T)\; g\left(\frac{R(T,t)}{R(T,t_w)}\right)$$

# MC dynamics 2dIM

Two-times local correlation



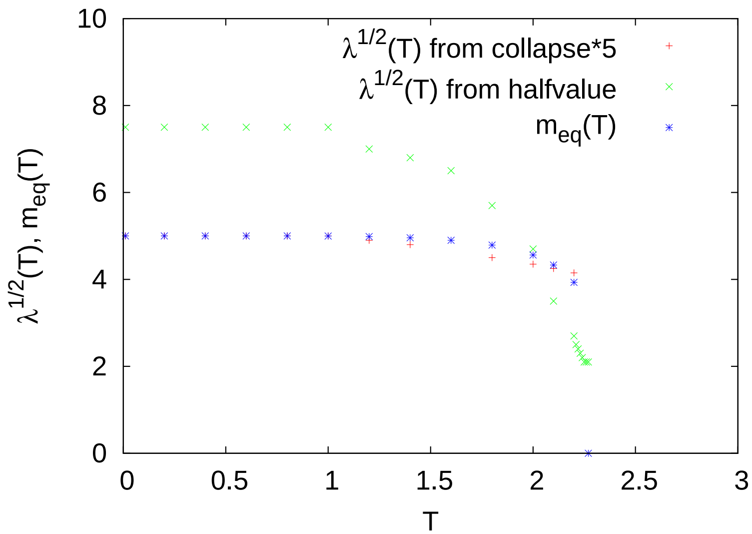

$$T = 0.5$$

$$C(t, t_w) \sim m_{eq}^2(T)\; g\left(\frac{R(T,t)}{R(T,t_w)}\right)$$ for $$C < m_{eq}^2(T) \sim 1$$

# MC dynamics 2dIM

The typical length-scale ⇔ a typical area

$$R(T,t) \sim \sqrt{\lambda(T)\, t} \qquad \Leftrightarrow \qquad A(T,t) \sim \lambda(T)\, t$$




NB the exponent $\frac{1}{2}$ is independent of $T$ and the details of the dynamics, lattice, *etc.* as long as the order parameter is non-conserved.

The $T$-dependence in $\lambda(T)$ is due to the roughening of the domain walls.

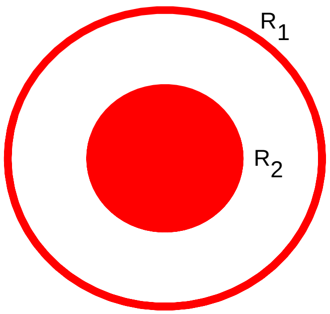
# Fluctuations

What happens locally ?

Basic question : what does $R$ really mean ?

- How many domains ?
- Which sizes ?

# Domains and hulls




Two hulls

$$A_1 = \pi R_1^2$$
$$A_2 = \pi R_2^2$$

Two domains

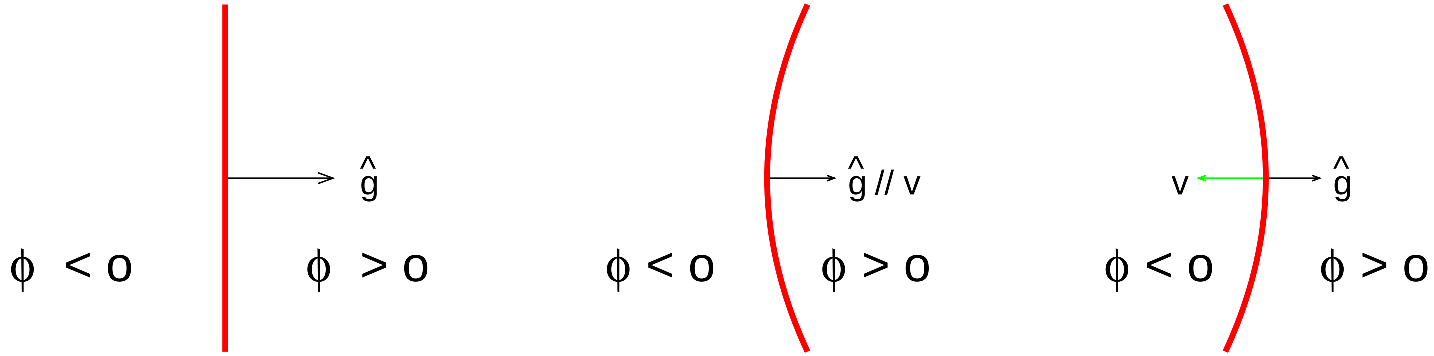
$$A_1 = \pi(R_1^2 - R_2^2)$$
$$A_2 = \pi R_2^2$$

*Hull : the interior of a domain boundary.*

- Typically hulls tend to be larger than domains ($A_1^h > A_2^h$).
- There are as many hulls as domains (two).
- Each spin belongs to one and only one domain (*e.g.* spin at the center).
- A spin can belong to more than one hull (*e.g.* spin at the center).

# Velocity of a quasi-planar wall

Time-dependent Ginzburg-Landau




$$v = -\vec{\nabla} \cdot \hat{g} = -K$$

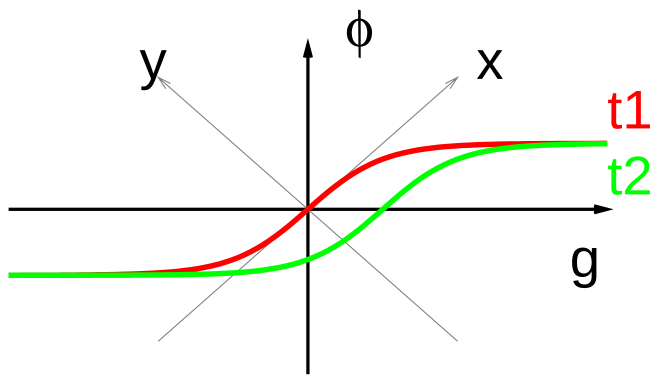
where $\hat{g}$ points in the direction $\phi > 0$ and

$K$ is the mean curvature measured from the phase $\phi < 0$.

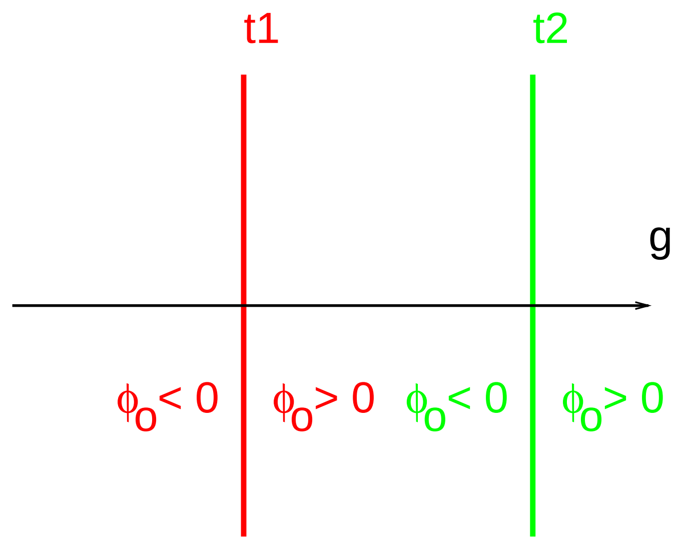
**S. M. Allen & J. W. Cahn, Acta Metall. 27, 1085 (1979).**

# T=0 argument

Domain wall profile




View from the top




$$\frac{\partial \phi(\vec{x},t)}{\partial t} = - \left.\frac{\partial \phi(\vec{x},t)}{\partial g}\right|_t \left.\frac{\partial g}{\partial t}\right|_\phi , \qquad \vec{\nabla}\phi(\vec{x},t) = \left.\frac{\partial \phi(\vec{x},t)}{\partial g}\right|_t \hat{g} ,$$

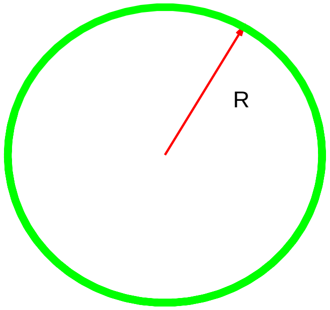
$$\nabla^2 \phi(\vec{x},t) = \left.\frac{\partial^2 \phi(\vec{x},t)}{\partial g^2}\right|_t + \left.\frac{\partial \phi(\vec{x},t)}{\partial g}\right|_t \vec{\nabla}\cdot\hat{g}.$$

Using $\frac{\partial^2 \phi(\vec{x},t)}{\partial g^2}|_t = V'(\phi)$ in the GL equation : $v \equiv \partial_t g|_\phi = -\vec{\nabla}\cdot\hat{g}$.

# Proof II

A spherical hull in $d = 2$

Take a sphere with radius $R$,
area $A = \pi R^2$ and perimeter
$L = 2\pi R$.

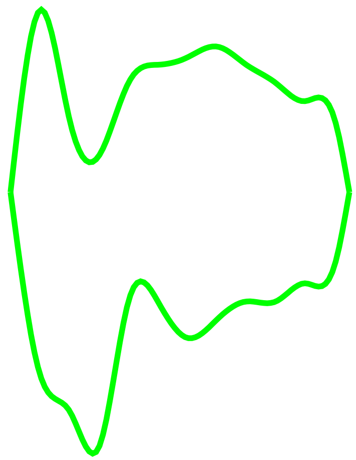



The time-variation of the *hull area*, $\frac{dA}{dt} = 2\pi R \,\frac{dR}{dt} = L\,v$ , in the case $v = -\frac{\lambda}{2\pi}\kappa$, with the curvature $\kappa = \frac{1}{R}$, is just constant

$$\frac{dA}{dt} = -\lambda$$

# Proof III

A generic hull in $d = 2$



with radius $R$, area $A$ and perimeter $L$.

The time-variation of the *hull area*, $\frac{dA}{dt} = \oint \vec{v} \wedge d\vec{\ell} = \oint v d\ell$ , in the case $v = -\frac{\lambda}{2\pi}\kappa$, with $\kappa$ the *geodesic curvature*, is *also* constant

$$\frac{dA}{dt} = -\lambda$$

due to the Gauss-Bonnet theorem $\int_A K dA + \int_{\partial A} \kappa d\ell = 2\pi\chi(A)$ that simply becomes $\oint \kappa d\ell = 2\pi$ for a planar $2d$ manifold with no holes.

# Proof IV

A spherical hull in $d = 3$

Take a sphere with radius $R$, volume $V = \frac{4}{3}\pi R^3$ and surface $A = 4\pi R^2$.

The time variation of the *hull* volume, $\frac{dV}{dt} = 4\pi R^2 \ \frac{dR}{dt}$, in the case $v = -\frac{\lambda}{2\pi}\kappa$, with $\kappa$ the *mean* curvature, *is not* constant :

$$\frac{dV}{dt} = -2R \propto -V^{1/3} \ .$$

Guess : $\frac{dV}{dt} \sim -V^{1/3}$ for generic geometries.

# The hull area distribution

$$d = 2$$

$\frac{dA}{dt} = -\lambda \Rightarrow$ all hulls tend to disappear at the same speed $-\lambda$.

- hulls with initial area smaller than $\lambda t$ will have disappeared at $t$.
- hulls with initial area larger than $\lambda t$ will have decreased by $\lambda t$.

The full hull area distribution is advected uniformly to the left at rate $\lambda$.

The number of hulls, per unit area of the system, with area greater than $A$ satisfies

$$N_h(A, t) = N_h(A + \lambda t, 0) \ .$$

# The hull area distribution II

The initial condition

- Quench from an infinite temperature $\Leftrightarrow$ random initial condition,

  $s_i = \pm 1$ with $p = \frac{1}{2}$ : critical point of percolation in $d = 2$.

$$N(A, 0) \approx \frac{2c}{A} \quad \text{with} \quad c = \frac{1}{8\pi\sqrt{3}} \qquad (a^2 \ll A \ll L) \; .$$

- Quench from equilibrium at $T_c$ : Ising cluster hulls at criticality.

$$N(A, 0) \approx \frac{c}{A} \quad \text{with} \quad c = \frac{1}{8\pi\sqrt{3}} \qquad (a^2 \ll A \ll L) \; .$$

Conformal field theory, scaling & numerical checks

**J. Cardy and R. M. Ziff, J. Stat. Phys. 110, 1 (2003).**

# The prediction

$$N_h(A,t) = \frac{2c}{A+\lambda t}\ , \qquad n_h(A,t) \equiv -\frac{\partial N_h(A,t)}{\partial A} = \frac{2c}{(A+\lambda t)^2}\ ,$$

with the expected scaling forms

$$N_h(A,t) = (\lambda t)^{-1} f\left(A/\lambda t\right)\ , \qquad n_h(A,t) = (\lambda t)^{-2} f'\left(A/\lambda t\right)\ .$$

# Numerical simulations

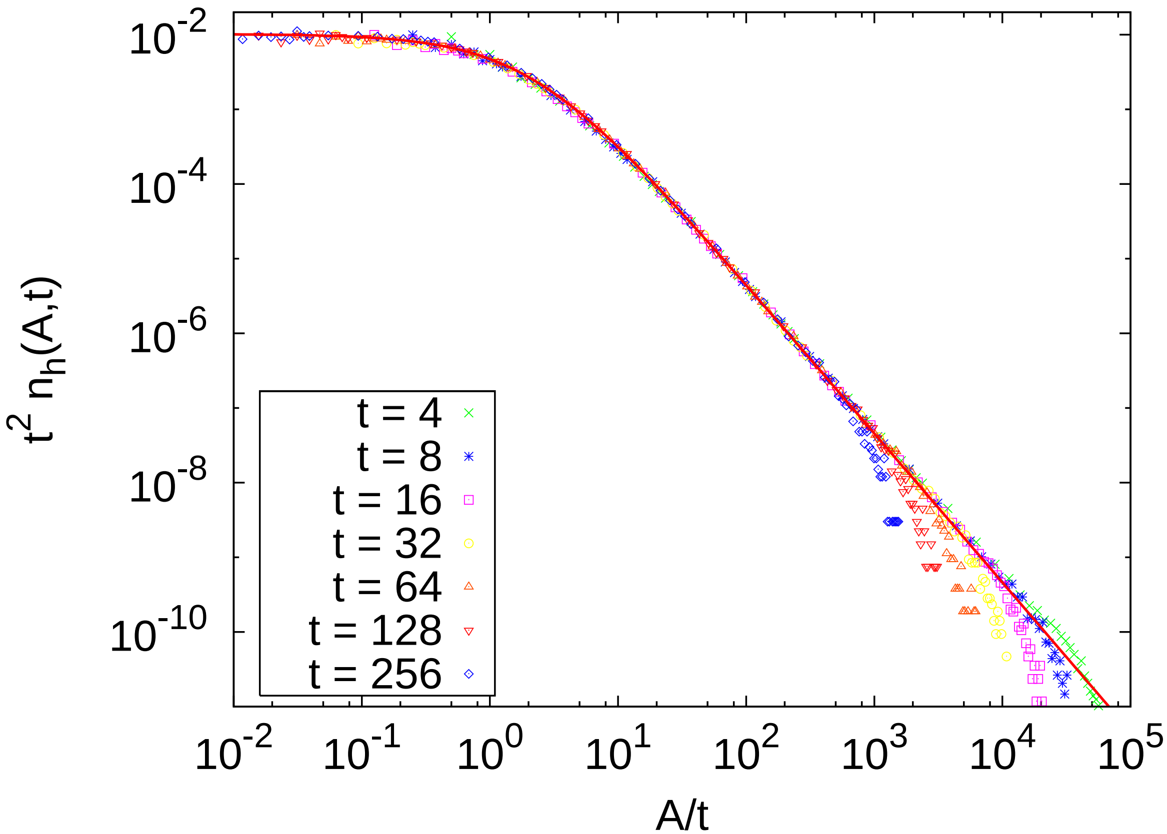
- $2d$ Ising model on a square lattice with periodic boundary conditions.
- Monte Carlo (MC) dynamics with heat-bath updates.
- $L = 10^3$, $2 \times 10^3$ samples, one time step corresponds to a MC sweep.
- Critical initial conditions generated with the Swendsen-Wang cluster algorithm to avoid critical slowing down.
- Hoshen-Kopelman algorithm to identify the domains.
- Our algorithm to identify the hulls inspired by the one used in

**R. M. Ziff, cond-mat/0510633, StatPhys22.**.

# Numerical tests

Number density of (finite) *hulls* per unit area

$T = 0$ dynamics after a quench from $T \to \infty$

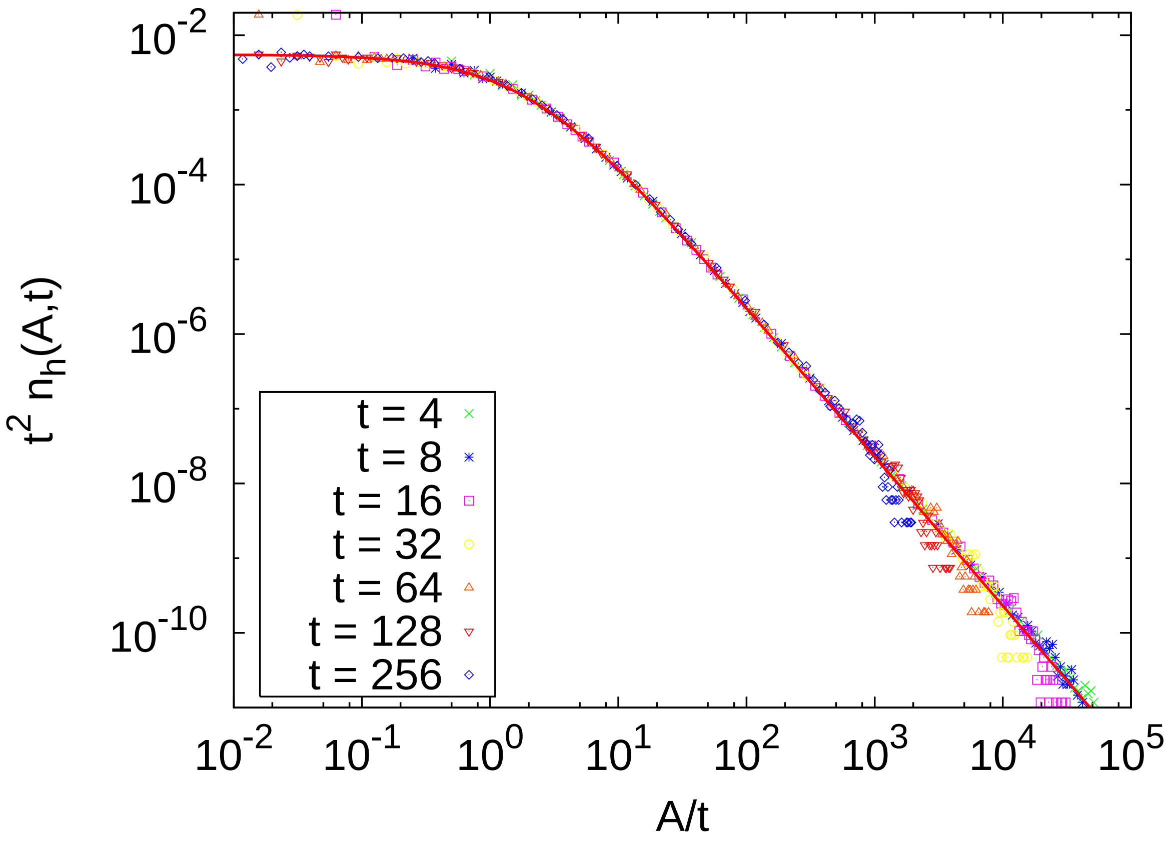



The bending is a finite size effect due to the percolating hulls.

# Numerical results

Number density of (finite) *hulls* per unit area

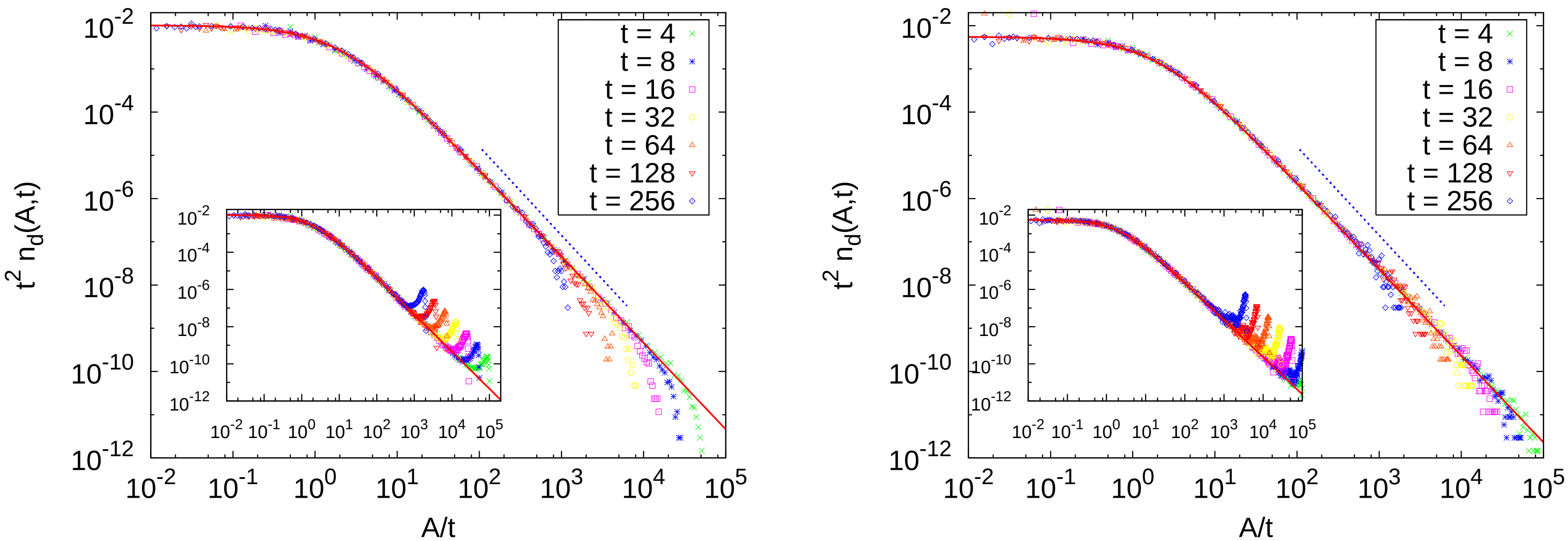
$T = 0$ dynamics after a quench from equilibrium at $T_c$



# Numerical results

Number density of *domains* per unit area

$T = 0$ dynamics after a quench from $T \to \infty$ (left) and $T_c$ (right)

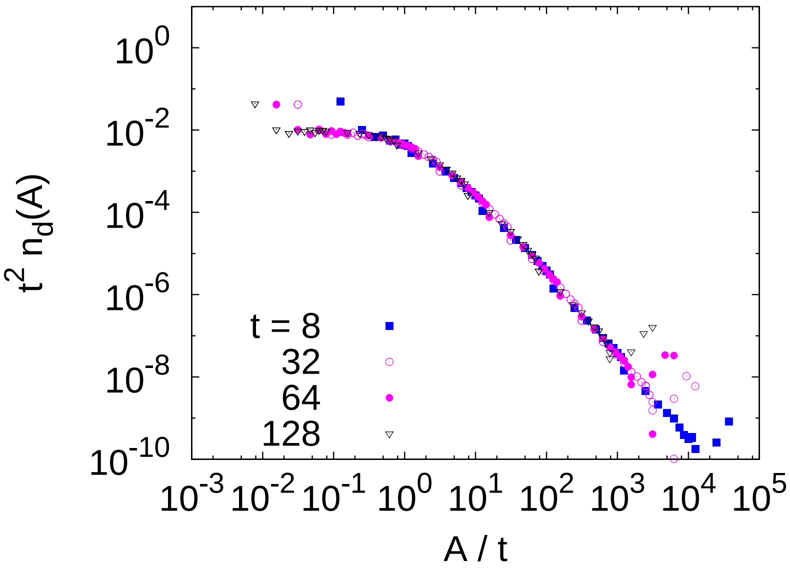



The insets include the contribution from the percolating cluster (hump).

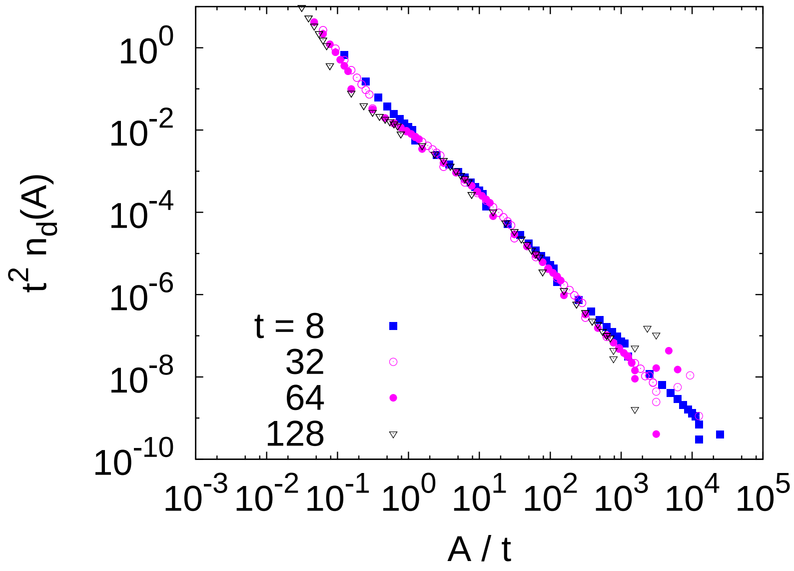
# Numerical results

Number density of domains per unit area

Finite $T$ dynamics after a quench from $T \to \infty$




$T = 0.5$



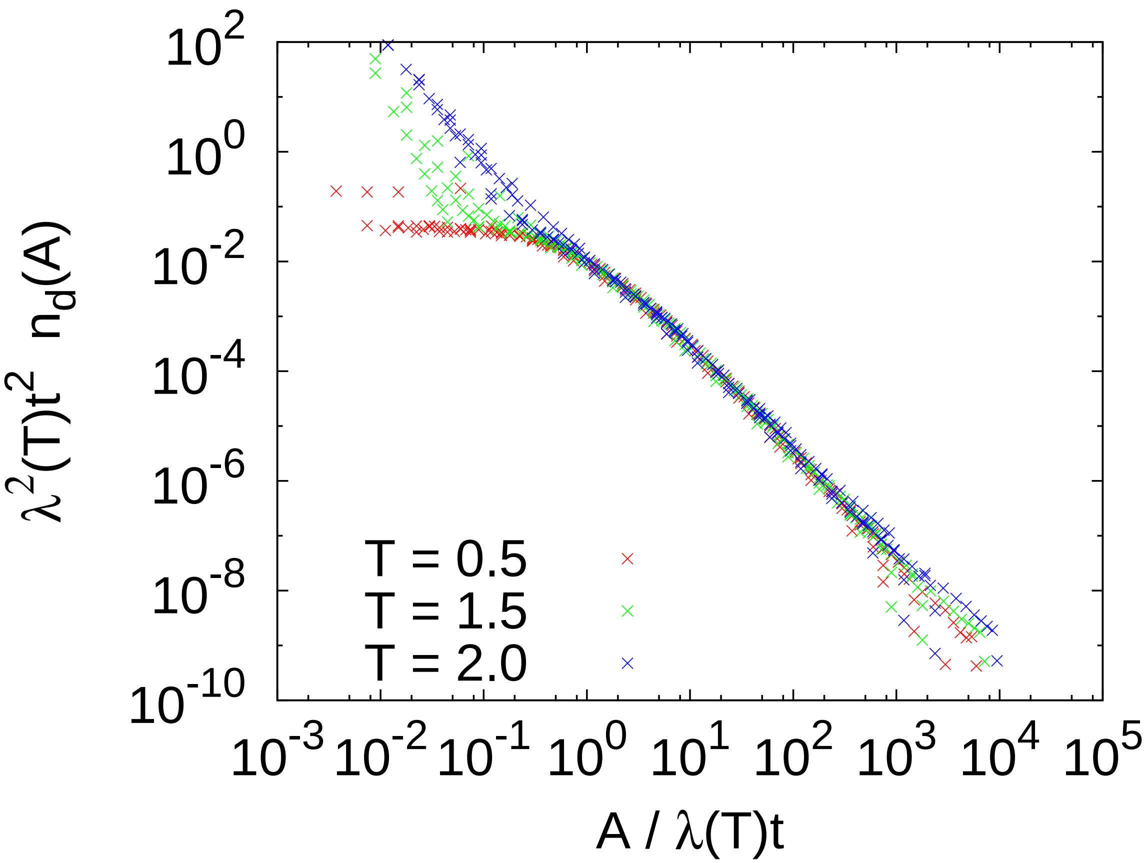

$T = 2$

# Numerical results

Number density of domains per unit area

Finite $T$ dynamics after a quench from $T \to \infty$

Scaling

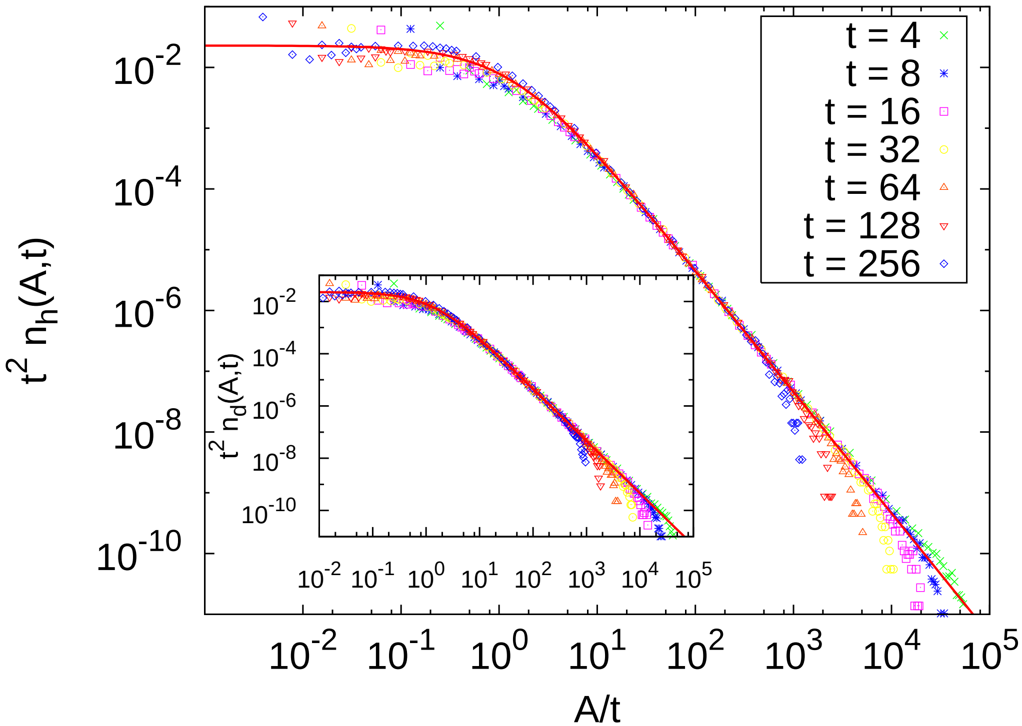


# Random ferromagnet

$$H = \sum_{\langle ij \rangle} J_{ij} s_i s_j \ , \qquad J_{ij} \text{ uniform distributed in } [0.75, 1.25]$$

Number density of *hulls* and *domains* (inset) per unit area

$T = 0.5$ dynamics ; random initial conditions



# Summary of results

- Exact results for hull pdfs.
- We proved scaling !
- The typical length-scale is not so typical after all :

  power-law tails in $N_h$ and $n_h$ (as well as $N_d$ and $n_d$).

# Future work

- Finite $T$ dynamics, numerical checks.
- $3d$ Ising model.
- Conserved order parameter (model B, Kawasaki dynamics) ; applications to phase separation.
- Potts model ; application to soap films and adsorbed atoms.
- Quenched randomness, *e.g. random ferromagnets, random field Ising model* ; application to hysteresis and the Barkhausen noise.
- Effect of annealing or finite cooling rates. Applications in cosmology : study of density of defects after a second-order phase transition.
- Understanding fluctuations in glassy systems.