Quantum disordered systems

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– Pasquale Calabrese (Università di Pisa, Italia)
– Laura Foini (Université de Genève, Suisse)
– Marco Schiró (IPhT, Saclay, France)
– Guilhem Semerjian (LPT-ENS, Paris, France)
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– Guilhem Semerjian (LPT-ENS, Paris, France) - quantum cavity
– Pasquale Calabrese (Università di Pisa, Italia) - quantum quenches
– Laura Foini (Université de Genève, Suisse) - quantum quenches
– Marco Schiró (IPhT, Saclay, France) - quantum quenches
Motivation

Why should one care about quantum fluctuations?

Physical

- High-energy physics
- Condensed matter clear
- Atomic physics

\[ \hbar \omega \gtrsim k_B T \]
Motivation

Why should one care about quantum fluctuations?

Physical

– High-energy physics
– Condensed matter
– Atomic physics & cold atom experiments

revived fundamental questions concerning equilibration in classical and quantum closed systems
Motivation

Why should one care about quantum fluctuations?

Physical

– High-energy physics
– Condensed matter clear
– Atomic physics
– Glassy oriented crowd?
Motivation: physics

Some putative quantum spin-glass phases

High Tc SCs: $\text{La}_{2-x}\text{Sr}_x\text{Cu}_2\text{O}_4$

Dipolar systems: $\text{LiHo}_x\text{Y}_{1-x}\text{F}_4$

M-H Julien et al. 03

G. Aeppli et al. 90s
Motivation: physics

Field-cooled vs zero field-cooled magnetisation

La$_{1.96}$Sr$_{0.04}$CuO$_4$

Chou et al. 95
Motivation: physics

Field-cooled vs. Zero field-cooled magnetisation

\[ \text{ZnCr}_{2(1-x)} \text{Ga}_{2x} \text{O}_4 \]

LaForge, Pulido, Cava, Chan, Ramírez 13

A geometrically frustrated magnet – no quenched disorder.

Proposals to realise quantum spin-glasses with atoms in optical cavities.
Motivation: physics

Methods from glassy physics

Statics

TAP Thouless-Anderson-Palmer \( \equiv \) fully-connected (complete graph)

Replica theory \( \equiv \) Gaussian approx. to field-theories

Cavity or Peierls approx. \( \equiv \) dilute (random graph)

Bubbles & droplet arguments \( \equiv \) finite dimensions

RG

Dynamics

Generating functional for classical field theories (MSRJD).

Perturbation theory, renormalization group techniques, self-consistent approximations, droplet methods.

Extensions?
Motivation: computer science

Quantum annealing

Goal: use quantum fluctuations to solve (hard) optimisation problems.

Idea: once mapped onto a classical physical Hamiltonian, find its ground state by following a well–chosen path in parameter space that takes the system into the quantum realm and then back to classical.

Quantum fluctuations are efficient to tunnel through tall (but not wide) barriers

while temperature fluctuations are efficient to jump over short (but possibly wide) barriers

Arrhenius

Tunneling
Motivation: computer science

Quantum annealing

Goal: use quantum fluctuations to solve (hard) optimisation problems.

Idea: similar to simulated annealing but in an ‘enlarged’ phase diagram.

Quantum tunneling & thermal activation
Motivation

Why should one care about quantum fluctuations?

Physical

– High-energy physics

– Condensed matter

– Atomic physics

– Glassy oriented crowd?

Understand these materials.

Use our toolbox to deal with these problems.

Develop formalism.
Plan

Quantum fluctuations

- Canonical equilibrium.
  - Classical disordered models & optimisation problems.
  - Quantum disordered models & optimisation problems.
  - The bath. Effects on equilibrium phase diagrams.

- Dynamics.
  - Closed systems and questions on equilibration.
  - Open systems, Markov vs. non-Markov dynamics.
  - A single dissipative quantum particle.
  - Quantum macroscopic dissipative systems.
Plan

Quantum fluctuations

- **Canonical equilibrium.** Preliminaries.
  - Classical disordered models & optimisation problems.
  - Quantum disordered models & optimisation problems.
  - The bath. Effects on equilibrium phase diagrams.

- **Dynamics.**
  - Closed systems and questions on equilibration.
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No need to solve the classical dynamic equations!

Under certain circumstances, *ergodic hypothesis*, after some equilibration time, \( t_{eq} \), the macroscopic observables can be, on average, obtained with a *static* calculation, as an average over all configurations in phase space weighted with a probability distribution function \( P(\{\vec{p}_i, \vec{x}_i\}) \):

\[
\langle A \rangle = \int \prod_i d\vec{p}_i d\vec{x}_i \ P(\{\vec{p}_i, \vec{x}_i\}) \ A(\{\vec{p}_i, \vec{x}_i\})
\]

Recipes for \( P(\{\vec{p}_i, \vec{x}_i\}) \) are given and depend upon the conditions under which the system evolves, whether it is isolated or in contact with an environment.

L. Boltzmann, late XIX
Ensembles

Isolated system $\Rightarrow$ total energy is conserved
\[ \mathcal{E} = H(\{C\}) = H(\{\vec{p}_i, \vec{x}_i\}) \]

Flat probability density
\[ P(\{C\}) \propto \delta(H(\{\vec{p}_i, \vec{x}_i\}) - \mathcal{E}) \]

Microcanonical distribution
\[ S_\mathcal{E} = k_B \ln V(\mathcal{E}) \quad \beta \equiv \frac{1}{k_B T} = \frac{\partial S_\mathcal{E}}{\partial \mathcal{E}} \bigg|_{\mathcal{E}} \]

Entropy \quad Temperature

\[ \mathcal{E} = \mathcal{E}_{\text{syst}} + \mathcal{E}_{\text{env}} + \mathcal{E}_{\text{int}} \]

Neglect $\mathcal{E}_{\text{int}}$ (short-range interact.)
\[ \mathcal{E}_{\text{syst}} \ll \mathcal{E}_{\text{env}} \]
\[ P(\{C\}) \propto e^{-\beta H(\{\vec{p}_i, \vec{x}_i\})} \]

Canonical ensemble
Quantum mechanics

Notation & reminder

Each dynamical variable or observable (e.g. position, translational momentum, etc.) is associated with a Hermitian operator, say $\hat{A}$.

The state a quantum system is represented by a vector in a Hilbert space, say $|a\rangle$.

The eigenvalues of the operator, $\hat{A}|a\rangle = a|a\rangle$, correspond to the possible values of the dynamical variable.

If the system is in a general state $|\psi\rangle$ the value $a$ is obtained with probability $p = |\langle a | \psi \rangle|^2$.

Observables associated to operators that do not commute are not simultaneously measurable, e.g. $\hat{p}$ and $\hat{x}$. 
Quantum mechanics

Notation & reminder

Take a quantum particle’s momentum, \( \hat{p} \), and position, \( \hat{x} \), operators satisfying the commutation relation \( [\hat{p}, \hat{x}] = -i\hbar \).

The system’s Hamiltonian is \( \hat{H}_{\text{syst}} = \frac{\hat{p}^2}{2m} + V(\hat{x}) \).

Take a quantum spin 1/2 such that \( \hat{S}_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle \).

The spin operator \( \hat{S}^a \) with \( a = x, y, z \) satisfies the commutation relations \( [\hat{S}^a, \hat{S}^b] = i\hbar \epsilon_{abc} \hat{S}^c \).

The time-dependent state of the system is represented by a vector in a Hilbert space \( |\psi(t)\rangle \).

It evolves in time following Schrödinger’s equation
\[
i\hbar \frac{d}{dt} |\psi(t)\rangle = \hat{H}_{\text{syst}} |\psi(t)\rangle
\]
Quantum mechanics

Notation & reminder : density operator

Take a time-dependent state $|\psi(t)\rangle$ with expansion $|\psi\rangle = \sum_n a_n(t) |u_n\rangle$ in an orthonormal basis $|u_n\rangle$ and assume it is normalised.

The time-dependent density operator is defined as $\hat{\rho}(t) \equiv |\psi(t)\rangle \langle \psi(t)|$.

Since $|\psi(t)\rangle$ is normalised, $Tr \hat{\rho}(t)$ is normalised as well.

The quantum average of an operator $\hat{O}$ is given by $\langle \hat{O} \rangle = Tr \hat{\rho}(t) \hat{O}$.

The density operator evolves according to

$$i\hbar \frac{d}{dt} \hat{\rho} = [\hat{H}_{syst}, \hat{\rho}]$$

The density matrix elements are given by $\rho_{mn}(t) = \langle u_m | \hat{\rho}(t) | u_n \rangle = a_m(t) a_n^*(t)$. 
Quantum mechanics

Notation & reminder: statistical ensembles

The system may be in state $|\psi_n\rangle$ with probability $p_n$.

When if we prepare a system (an atom, say) many times.

The density operator is then $\hat{\rho} \equiv \sum_n p_n |\psi_n\rangle\langle \psi_n |$ with $|\psi_n\rangle$.

The density matrix $\langle \psi_n | \hat{\rho} | \psi_m \rangle$ is the quantum-mechanical analogue to a classical phase-space probability measure, $P(C)$ of statistical physics.

In canonical equilibrium the density operator is $\hat{\rho} \equiv \mathcal{Z}^{-1} e^{-\beta \hat{H}}$ with $\mathcal{Z} = \text{Tr} e^{-\beta \hat{H}}$.

One studies $\hat{\rho}$ to infer phase diagrams of open quantum systems.
Plan

Quantum fluctuations

- Canonical equilibrium.
  - Classical disordered models & optimisation problems.
  - Quantum disordered models & optimisation problems.
  - The bath. Effects on equilibrium phase diagrams.

- Dynamics.
  - Closed systems and questions on equilibration.
  - Open systems, Markov vs. non-Markov dynamics.
  - A single dissipative quantum particle.
  - Quantum macroscopic dissipative systems.
Disordered spin systems

Classical $p$-spin model

\[ H_{\text{syst}} = \sum_{i_1 < \cdots < i_p}^{N} J_{i_1 i_2 \cdots i_p} s_{i_1} s_{i_2} \cdots s_{i_p} \]

Ising, $s_i = \pm 1$, or spherical, $\sum_{i=1}^{N} s_i^2 = N$, spins. Drawing

Sum over all $p$-uplets on a complete graph: fully-connected model.

Random exchanges $P(J_{i_1 i_2 \cdots i_p}) \propto e^{-p! J_{i_1 i_2 \cdots i_p}^2 / (2N^{p-1}J^2)}$

Extensions to random graphs possible: dilute models.

$p = 2$ Ising: Sherrington-Kirkpatrick (SK) model for spin-glasses

$p = 2$ spherical $\approx$ mean-field ferromagnet.

$p \geq 3$ Ising or spherical: models for fragile glasses.
Disordered spin systems

Classical $p$-spin model & fragile glasses

$T_s$  $T_d$

T. Kirkpatrick, Thirumalai & Wolynes 80s
Disordered spin systems

Random $K$-sat problem

A clause is the ‘logical or’ between $K$ requirements imposed on Boolean variables $x_i$ chosen randomly from a pool of $N$ of them.

A formula is the ‘logical and’ between $M$ such clauses, $F = \bigwedge_{\ell=1}^{M} \bigvee_{i=1}^{K} x_i^{(\ell)}$. It is satisfied when all $M$ clauses are.

The search for a solution can be set as the search for the spin configuration(s) with vanishing energy

$$H_{syst} = \alpha 2^{-K} N + \sum_{R=1}^{K} (-1)^R \sum_{i_1 < \ldots < i_R} J_{i_1 i_2 \ldots i_R} s_{i_1} s_{i_2} \ldots s_{i_R}$$

with $\alpha = M/N$, Ising classical spins, $s_i = \pm 1$, and interactions $J_{i_1 \ldots i_R} = 2^{-K} \sum_{\ell=1}^{M} C_{\ell,i_1} \ldots C_{\ell,i_R}$

with $C_{\ell,i_k} = +,-$ for the condition $x_{i_k}^{(\ell)} = T,F$ and $C_{\ell,i_k} = 0$ otherwise.

Sum of classical dilute $p \leq K$-spin models
Optimisation problems

Status

Consensus: there exist families of cost functions of $N$ discrete variables such that no algorithm can find their global minimum by executing a number of operations smaller than some polynomial of $N$.

\[ \text{P} \neq \text{NP conjecture} \]

Consequence: classical algorithms need an exponentially large (in the system size) number of operations to solve hard instances in the NP class,

\[ t \approx e^{AN} \]

Such hard instances exist in Random $K \geq 3$-sat for special values of the parameter $\alpha$ (close to the threshold between satisfiable and unsatisfiable phases).
Challenges

Classical disordered systems & computer science

Glasses. Go beyond mean-field models (fully-connected and dilute) disordered spin systems and understand the behaviour of particle systems with short-range interactions.

Fully understand the glassy arrest.

Optimisation. Dilute spin models are the focus of study. Understand all their possible dynamics, physical and unphysical.

Find algorithms that solve hard instances in polynomial time, and disprove $P \neq NP$, or prove that this is not possible and then establish $P \neq NP$. 
Plan

Quantum fluctuations

- **Canonical equilibrium.**
  - Classical disordered models & optimisation problems.
  - **Quantum disordered models & optimisation problems.**
  - The bath. Effects on equilibrium phase diagrams.

- **Dynamics.**
  - Closed systems and questions on equilibration.
  - Open systems, Markov vs. non-Markov dynamics.
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  - Quantum macroscopic dissipative systems.
Disordered spin systems

Quantum $p$-spin model

$$\hat{H}_{\text{syst}} = \sum_{i_1 < \ldots < i_p}^N J_{i_1 \ldots i_p} \hat{\sigma}_{i_1}^z \ldots \hat{\sigma}_{i_p}^z + \Gamma \sum_{i=1}^N \hat{\sigma}_i^x$$

$\hat{\sigma}_i^a$ with $a = 1, 2, 3$ the Pauli matrices, $[\hat{\sigma}_i^a, \hat{\sigma}_i^b] = 2i\epsilon_{abc}\hat{\sigma}_i^c$.

$\Gamma$ transverse field. It induces quantum fluctuations.

In the limit $\Gamma \rightarrow 0$ the classical limit should be recovered.

Sum over all $p$-uplets on a complete graph (extensions to random graphs)

$$P(J_{i_1i_2\ldots i_p}) \propto e^{-p! J_{i_1i_2\ldots i_p}^2 / (2N^{p-1}J^2)}$$

$p \geq 2$ Ising: quantum Sherrington-Kirkpatrick and $p$-spin models.

$p \geq 2$ continuous variables: quantisation achieved by adding a kinetic energy.
Quantum systems

Quantum fluctuations

- Take an isolated quantum system with Hamiltonian $\hat{H}_i$
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $\hat{H}_i$.
- Evolve it with a different, possibly time-dependent, Hamiltonian $\hat{H}(t)$

$$i\hbar \frac{d}{dt}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle$$

Can these dynamics help reach the ground state of a cost function?

- For example, choose $\hat{H}(t)$ such that $\hat{H}(t_0) = \hat{H}_i$ and $\hat{H}(t_f) = H_f$

  the (classical) cost function in question, and try to find in this way its ground state.
Disordered spin systems

Quantum $p$-spin model and random $K$-sat problem

\[ \hat{H}_{\text{syst}}(t) = \frac{\alpha N}{2K} + \sum_{R=1}^{K} (-1)^R \sum_{i_1 < \ldots < i_R} J_{i_1 i_2 \ldots i_R} \hat{\sigma}_{i_1}^z \hat{\sigma}_{i_2}^z \ldots \hat{\sigma}_{i_R}^z + \Gamma(t) \sum_i \hat{\sigma}_i^x \]

with $\alpha = \frac{M}{N}$,

the Pauli matrices, $[\hat{\sigma}_i^a, \hat{\sigma}_j^b] = 2i \delta_{ij} \epsilon_{abc} \hat{\sigma}_i^c$,

and the interactions

$J_{i_1 \ldots i_R} = 2^{-K} \sum_{\ell=1}^{M} C_{\ell,i_1} \ldots C_{\ell,i_R}$

with $C_{\ell,i_k} = +, -$ for the condition $x_{i_k}^{(\ell)} = T,F$ and $C_{\ell,i_k} = 0$ otherwise.

**Sum of quantum dilute $p \leq K$-spin models**

Interpolate between $\Gamma(0) \gg 1$ and $\Gamma(t_f) = 0$ (easy to hard)
Optimisation problems
Adiabatic theorem and quantum annealing

If a quantum system is prepared in the ground state of a simple Hamiltonian, $\hat{H}_i$, and one gives a slow enough evolution to the Hamiltonian, $\hat{H}(t)$, the adiabatic theorem ensures that the system remains, with high probability, in the instantaneous ground state of $\hat{H}(t)$ at all subsequent times.

Purpose: use this property to take the system to the ground state of a desired (classical) Hamiltonian $H_f = \hat{H}(t_f)$.

Quantum annealing
Kadowaki & Nishimori 98
Dipolar spin-glass
G. Aeppli et al. 90s
Optimisation problems

Quantum annealing

Take, slowly, the system from the ground state of a simple Hamiltonian, $\hat{H}_i$, to the ground state of a desired (classical) Hamiltonian $H_f = \hat{H}(t_f)$.

But, how slow is slow?

The running time should be $t_f > \Delta_{\text{min}}^{-2}$

with $\Delta_{\text{min}} = E_1 - E_0$ the minimal gap between the energy of the first excited state, $E_1$, and the energy of the ground state, $E_0$, encountered along the evolution.

M. Born & V. Fock 31
Interesting optimisation problems have first order phase transitions when rendered quantum. Technical details below and in Semerjian’s talk.

At the first order phase transition the gap closes exponentially in the system size

$$\Delta_{\text{min}} \sim N e^{-aN}$$

Jörg, Krzakala, Kurchan & Maggs 08
Bapst, Foini, Krzakala, Semerjian & Zamponi 13

Therefore an exponentially long running time is also needed to follow the ground state.

$$t_f \sim \Delta_{\text{min}}^{-2} \sim e^{2aN}$$ No gain...
Motivation: physics

Methods from glassy physics

**Statics**

- TAP Thouless-Anderson-Palmer
- Replica theory
- Cavity or Peierls approx.
- Bubbles & droplet arguments
- RG

Fully-connected (complete graph)
Gaussian approx. to field-theories
dilute (random graph) [Semerjian]
finite dimensions [Friday]

**Dynamics**

Generating functional for classical field theories (MSRJD).
Perturbation theory, renormalization group techniques, self-consistent approximations, droplet methods.
Matsubara replica calculation

A sketch

\[ -\beta f = \lim_{N \to \infty} \frac{\ln Z}{N} = \lim_{N \to \infty} \lim_{n \to 0} \frac{[Z^n] - 1}{Nn} \]

\( Z^n \) partition function of \( n \) independent copies of the system: replicas.

Quantum mechanically, \( Z = \text{Tr} \ e^{-\beta \hat{H}} \)

and \( Z = \int \{s_i(\beta \hbar)\} \mathcal{D}s_i(\tau) \ e^{-\frac{1}{\hbar} S^e_{\text{syst}}[\{s_i(\tau)\}]} \)

or \( Z = \sum_{s_i(\tau_k) = \pm 1} \ e^{-\frac{1}{\hbar} S^e_{\text{syst}}[\{s_i(\tau_k)\}]} \)

the form of the Euclidean action, \( S^e_{\text{syst}} \), depends on whether we use

truly SU(2) quantum spins or the ‘spherical’ version of the model.

Feynman-Matsubara construction of functional integral over imaginary time.
Matsubara replica calculation

A sketch

\[-\beta f = \lim_{N \to \infty} \frac{\ln Z}{N} = \lim_{N \to \infty} \lim_{n \to 0} \frac{[Z^n] - 1}{Nn}\]

Self-averageness average over disorder

Quantum mechanically, \( Z = \text{Tr} \ e^{-\beta \hat{H}} \Rightarrow \{ \int, \sum \} e^{-\frac{1}{\hbar} S_{\text{syst}}^e[\{s_i\}]} \)

No i contrary to the dynamic path-integral (that will appear later).

Mapping to \( d + 1 \) classical statistical physics problem with anisotropic (imaginary-time \( \neq \) spatial) interactions.

Feynman-Matsubara construction of functional integral over imaginary time.
Matsubara replica calculation

A sketch

Average over disorder \Rightarrow coupling between replicas

\[
\sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} \int d\tau \sum_a s^a_{i_1}(\tau) \ldots s^a_{i_p}(\tau) \Rightarrow \int d\tau \int d\tau' \sum_{a,b} \left( \sum_i s^a_i(\tau) s^b_i(\tau') \right)^p
\]

One introduces the auxiliary two-time dependent replica matrix

\[
\delta \left( Q_{ab}(\tau, \tau') - N^{-1} \sum_i s^a_i(\tau) s^b_i(\tau') \right)
\]

In terms of the replica indices \( Q_{ab}(\tau, \tau') \) is still a \( 0 \times 0 \) matrix.

Slightly intricate imaginary-time & replica index structure. Recipes to deal with them: Bray & Moore 80 and the Parisi Ansatz.
Matsubara replica calculation

Spherical case

$Q_{ab}(\tau, \tau')$ can be evaluated by saddle-point if one exchanges the limits $N \to \infty$ $n \to 0$ with $n \to 0$ $N \to \infty$.

Stationary behaviour expected. The equation to solve is

$$\left( -\frac{1}{\Gamma} \frac{\partial^2}{\partial \tau^2} + z \right) Q_{ab}(\tau)$$

$$= \delta_{ab} \delta(\tau) + \frac{p}{2} \int_{0}^{\beta \hbar} d\tau' \sum_{c} Q_{ac}^{(p-1)}(\tau - \tau') Q_{cb}(\tau')$$

with periodic boundary conditions, $Q_{ab}(\beta \hbar) = Q_{ab}(0)$.

In terms of the replica indices $Q_{ab}(\tau)$ is still a $0 \times 0$ matrix.

Bray & Moore 80, just $q_{d}(\tau)$, and the Parisi Ansatz for $a \neq b$

Note the similarity with the MCT equations.
Quantum TAP & cavity method

Quantum TAP

Legendre transform of $f$ with respect to $\{m_i(\tau)\}$ and $C'(\tau - \tau')$ with $m_i(\tau) = \langle s_i(\tau) \rangle$ and $C'(\tau - \tau') = N^{-1} \sum_i \langle s_i(\tau) s_i(\tau') \rangle$.

In fully-connected models one finds the exact free-energy functional $f(m_i(\tau), C'(\tau - \tau'))$ and the saddle-point equations.

Derivation & analysis of this functional for quantum $p$-spin disordered models Biroli & LFC 01; Andreanov & Müller 12 (SK)

Quantum cavity methods allow one to deal with dilute quantum spin models Krzakala, Rosso, Semerjian & Zamponi 08, Laumann, Moessner, Scardicchio & Sondhi 09
Quantum TAP

Legendre transform of $f$ with respect to $\{m_i(\tau)\}$ and $C(\tau - \tau')$ with $m_i(\tau) = \langle s_i(\tau) \rangle$ and $C'(\tau - \tau') = N^{-1} \sum_i \langle s_i(\tau)s_i(\tau') \rangle$. $f(m_i(\tau), C'(\tau - \tau'))$. 

The TAP equations for the quantum $p$-spin disordered (spherical) models

$$ \Gamma^{-1} \partial^2_{\tau} C(\tau) = -\frac{p}{2} \int_0^{\beta h} d\tau' [C_{p-1}(\tau - \tau') - q^{p-1}][C(\tau') - q] + z[C(\tau) - q] - \delta(\tau) $$

$$ zm_i = \sum_{i_2 < \ldots < i_p} J_{i_1 \ldots i_p} m_{i_2} \ldots m_{i_p} + \frac{p}{2} m_i \int_0^{\beta h} d\tau' [C_{p-1}(\tau') + (p-2)q^{p-1} - (p-1)C(\tau')q^{p-2}] $$

$$ q = N^{-1} \sum_i m_i^2. $$

Biroli & LFC 01; Andreanov & Müller 12 (SK)
Quantum $p$-spin models

Some results

$$
\hat{H}_{\text{syst}} = \sum_{i_1 < \ldots < i_p}^{N} J_{i_1 \ldots i_p} \hat{\sigma}^z_{i_1} \ldots \hat{\sigma}^z_{i_p} + \Gamma \sum_{i=1}^{N} \hat{\sigma}^x_i
$$

$\hat{\sigma}^a_i$ with $a = 1, 2, 3$ the Pauli matrices, $[\hat{\sigma}^a_i, \hat{\sigma}^b_i] = 2i\epsilon_{abc}\hat{\sigma}^c_i$.

$\Gamma$ transverse field. It induces quantum fluctuations.

In the limit $\Gamma \rightarrow 0$ the classical limit should be recovered.

Sum over all $p$-uplets on a complete graph (extensions to random graphs)

$$
P(J_{i_1 i_2 \ldots i_p}) \propto e^{-p! J_{i_1 i_2 \ldots i_p}^2/(2N^{p-1}J^2)}
$$

$p \geq 2$ Ising: quantum Sherrington-Kirkpatrick and $p$-spin models.

$p \geq 2$ continuous variables: quantisation achieved by adding a kinetic energy.
1st order phase transition

Quantum fully-connected $p \geq 3$ spin model

Focus on the thin dashed and solid inner lines: static phase transition. Jump in the susceptibility across the dashed part of the critical line.

LFC, Grempel & da Silva Santos 00

In dilute disordered $p \geq 3$ models, review:

Bapst, Foini, Krzakala, Semerjian & Zamponi 13
Combinatorial optimisation

\( K \)-satisfiability is written in terms of \( p(\leq K) \)- spin models on a random (hyper-)graph.

**Quantum annealing**

Kadowaki & Nishimori 98

Dipolar spin-glass

G. Aeppli et al. 90s

1st order transitions: trouble for quantum annealing techniques as

\[
t_f \sim e^{aN}
\]

Jörg, Krzakala, Kurchan, Maggs, Pujos 08-09
1st order phase transition?

Dipolar glasses

Non-linear susceptibility \( \chi_3 \)

The divergence disappears at low \( T \)

Out of phase linear susceptibility \( \chi_1'' \)

\[ \Gamma \]

Wu et al 93
Focus on the thick dashed and solid inner lines: dynamic phase transition.

Found with *marginality condition* (replicon vanishing)

LFC, Grempel & da Silva Santos 00

In dilute disordered $p \geq 3$ models, review:

Bapst, Foini, Krzakala, Semerjian & Zamponi 13
Plan

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  - Classical disordered models & optimisation problems.
  - Quantum disordered models & optimisation problems.
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Dissipative systems

Aim

Interest in describing the **statics** and **dynamics** of a classical or quantum physical system coupled to a classical or quantum environment.

The Hamiltonian of the ensemble is

\[ H = H_{syst} + H_{env} + H_{int} \]

The dynamics of all variables are given by Newton or Heisenberg rules, depending on the variables being classical or quantum.

The total energy is conserved, \( E = ct \), but each contribution is not, in particular, \( E_{syst} \neq ct \), and we’ll take \( E_{syst} \ll E_{env} \).
Reduced system

Model the environment and the interaction

*E.g.*, an ensemble of harmonic oscillators and a linear in $q_\alpha$ and non-linear in $x$, via the function $\mathcal{V}(x)$, coupling:

\[
H_{env} + H_{int} = \sum_{\alpha=1}^{N} \left[ \frac{\pi^2}{2m_\alpha} + \frac{m_\alpha \omega_\alpha^2}{2} q_\alpha^2 \right] + \sum_{\alpha=1}^{N} c_\alpha q_\alpha \mathcal{V}(x)
\]

**Equilibrium.** Imagine the whole system in contact with a megabath at inverse temperature $\beta$. Compute the reduced classical partition function or quantum density matrix by tracing away the bath degrees of freedom.

**Dynamics.** Classically (coupled Newton equations) and quantum (easier in a path-integral formalism) elimination of the bath variables.

In all cases one can integrate out the oscillator variables as they appear only quadratically, for this choice of $H_{env} + H_{int}$
Reduced system

Statistics of a classical system

Imagine the coupled system in canonical equilibrium with a megabath

\[ Z_{syst} + env = \sum_{env, syst} e^{-\beta H} \]

Integrating out the environmental (oscillator) variables

\[ Z_{red syst} = \sum_{syst} e^{-\beta \left( H_{syst} - \frac{1}{2} \sum_a \frac{c_a^2}{m_a \omega_a^2} [\mathcal{V}(x)]^2 \right)} \neq Z_{syst} = \sum_{syst} e^{-\beta H_{syst}} \]

One possibility: assume weak interactions and drop the new term.

Trick: add \( H_{counter} \) to the initial coupled Hamiltonian, and choose it in such a way to cancel the quadratic term in \( \mathcal{V}(x) \) to recover

\[ Z_{red syst} = Z_{syst} \]

i.e., the partition function of the system of interest.
Reduced system

Model the quantum environment and the interaction

An ensemble of quantum harmonic oscillators and a bi-linear coupling, again using the single particle notation

\[ \hat{H}_{env} + \hat{H}_{int} = \sum_{\alpha=1}^{N} \left[ \frac{\hat{\pi}_{\alpha}^2}{2m_{\alpha}} + \frac{m_{\alpha}\omega_{\alpha}^2}{2} \hat{q}_{\alpha}^2 \right] + \sum_{\alpha=1}^{N} c_{\alpha} \hat{q}_{\alpha} \hat{x} \]

Quantum mechanically (easier in a Matsubara path-integral formalism) one can also integrate out the oscillator variables.

One obtains a reduced density operator, \( \hat{\rho}_{syst}^{red} \).
One integrates the oscillator’s degrees of freedom to get the reduced density matrix

$$\rho_{syst}^{red}(x'', x') = Z_{red}^{-1} \int_{x'}^{x''} Dx(\tau) e^{-\frac{1}{\hbar} \left[S_{syst}^{e} - \int_{0}^{\beta\hbar} d\tau \int_{0}^{\tau} d\tau' x(\tau)K(\tau-\tau')x(\tau')\right]}$$

Even choosing the counter-term to cancel a quadratic term in $x^2(\tau)$ a non-local (possibly long-range) interaction with kernel

$$K(\tau) = \frac{2}{\pi\hbar\beta} \sum_{n=-\infty}^{\infty} \int_{0}^{\infty} d\omega \frac{I(\omega)}{\omega} \frac{\nu_n^2}{\nu_n^2 + \omega^2} e^{i\nu_n\tau}$$

remains.

No obvious ‘weak-coupling’ argument can be used to drop it.

What are the effects of this term?
Noise-dependent transitions

Quantum $p = 3$-spin model with $I(\omega) = \eta \omega$

Magnetic susceptibility

Averaged entropy density

$\eta = 0, 0.5, 1$

$\eta$ is the parameter measuring the strength of the coupling to the bath

LFC, Grempel, Lozano, Lozza & da Silva Santos 02

Same kind of phenomena for $p = 2$, SU(2) spins, rotors, fermion bath, etc.
Static & dynamic phase diagram

Quantum $p = 3$-spin model with $I(\omega) = \eta\omega$

dashed = 1st order, solid = 2nd order  thin = static, bold = dynamic

$\eta = 0, 0.5$  LFC, Grempel, Lozano, Lozza & da Silva Santos 02

The ordered phase is stabilized by the environment
Static & dynamic phase diagram

Quantum \( p = 3 \)-spin model with \( I(\omega) = \eta \omega \)

dashed = 1st order, solid = 2nd order   thin = static, bold = dynamic

Recall RFOT
for fragile glasses

\( T_s \neq T_d \)
No \( \eta \)-dependence at \( \Gamma \rightarrow 0 \)

\( \eta = 0, 0.5 \)  LFC, Grempel, Lozano, Lozza & da Silva Santos 02

The ordered phase is stabilized by the environment
Goal: use the coupling to an engineered bath to take the system to a desired, glassy or ordered, phase and then switch-off the bath.
Summary

Statics of quantum disordered systems

- We introduced quantum $p$-spin disordered models.
- We very briefly mentioned that the TAP and replica approaches as well as the cavity method can be applied to them.
- We showed that these models have first order phase transitions in the low temperature limit.
- Problems for quantum annealing methods.
- A quantum environment induces long-range interactions in the imaginary-time direction and can have a highly non-trivial effect quantum mechanically.

Similar results for quantum Ising chains. For dilute models?

SK model & connection to electron glasses: talk to Müller
Plan

Quantum fluctuations

- Canonical equilibrium.
  - Classical disordered models & optimisation problems.
  - Quantum disordered models & optimisation problems.
  - The bath. Effects on equilibrium phase diagrams.

- Dynamics.
  - Closed systems and questions on equilibration.
  - Open systems, Markov vs. non-Markov dynamics.
  - A single dissipative quantum particle.
  - Quantum macroscopic dissipative systems.
Isolated systems

Dynamics of classical systems

A few particles: dynamical systems
Many-body: foundations of statistical physics

Questions:

Does the dynamics of a particular system reach a flat distribution over the constant energy surface in phase space?

Ergodic theory (∈ mathematical physics at present).

Can some part of the system, say modes, be taken as a bath with respect to others?

Etc.
Isolated quantum systems

Quantum quenches

- Take an isolated quantum system with Hamiltonian $\hat{H}_i$.
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $\hat{H}_i$.
- Unitary time-evolution with $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ with a Hamiltonian $\hat{H}$.

Does the system reach some steady state?

Note that it is the ergodic theory question posed in the quantum context.

Motivated by cold-atom experiments & exact solutions of $1d$ quantum models.

After a quantum quench, i.e. a rapid variation of a parameter in the system, are at least some observables described by thermal ones? When, how, which?

Calabrese, Foini & Schiró
Plan

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Reduced system

Model the classical environment and the interaction

E.g., an ensemble of harmonic oscillators and a bi-linear coupling:

\[
H_{env} + H_{int} = \sum_{\alpha=1}^{N} \left[ \frac{\pi_{\alpha}^{2}}{2m_{\alpha}} + \frac{m_{\alpha}\omega_{\alpha}^{2}}{2} q_{\alpha}^{2} \right] + \sum_{\alpha=1}^{N} c_{\alpha} q_{\alpha} V(x)
\]

Classical dynamics (coupled Newton equations)

Assuming the environment is coupled to the sample at the initial time, \( t_0 \), and that its variables are characterized by a Gibbs-Boltzmann distribution or density function at inverse temperature \( \beta \)

One finds a colored Langevin equation with multiplicative noise
The system, \( p, x \), coupled to an equilibrium environment evolves according to the multiplicative noise non-Markov Langevin equation

\[
\dot{m} \ddot{x}(t) + V'(x(t)) \int_{t_0}^{\infty} dt' \gamma(t - t') \dot{x}(t') V'(x(t')) = -\frac{\delta V(x)}{\delta x(t)} + V'(x(t)) \xi(t)
\]

The friction kernel is \( \gamma(t - t') = \Gamma(t - t') \theta(t - t') \)

The noise has zero mean and correlation \( \langle \xi(t) \xi(t') \rangle = k_B T \Gamma(t - t') \) with \( T \) the temperature of the bath and \( k_B \) the Boltzmann constant.
Reduced system

Dynamics of a classical system: general Langevin equations

The system, $p, x$, coupled to an equilibrium environment evolves according to the multiplicative noise non-Markov Langevin equation

$$\begin{align*}
m\ddot{x}(t) + V'(x(t)) \int_{t_0}^{\infty} dt' \gamma(t-t') \dot{x}(t') V'(x(t')) &= \delta V(x) \delta x(t) + V'(x(t)) \xi(t) \\
&\quad + V'(x(t)) \xi(t) \\
\mathcal{E}_{syst}(t) &\neq ct
\end{align*}$$
Separation of time-scales

Additive classical white noise

In classical systems one usually takes a bath kernel with the smallest relaxation time, \( t_{env} \ll \) all other time scales.

The bath is approximated by the white form \( \Gamma(t - t') = 2\gamma\delta(t - t') \).

Moreover, one assumes the coupling is bi-linear, \( H_{int} = \sum_a c_a q_a x \).

The Langevin equation becomes

\[
m\ddot{x}(t) + \gamma \dot{x}(t) = -\frac{\delta V(x)}{\delta x(t)} + \xi(t)
\]

with \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t)\xi(t') \rangle = 2k_B T \gamma \delta(t - t') \).
Brownian motion

First example of dynamics of an *open system*

The system: the Brownian particle

The bath: the liquid

Interaction: collisional or potential

‘*Canonical setting*’

A few Brownian particles or tracers imbedded in, say, a molecular liquid.

Late XIX, early XX (Brown, Einstein, Langevin)
Interesting effects

**Multiplicative noise**

$D_{\parallel}(h)$

$D_{\perp}(h)$

**Colored noise**

Varying diffusion constant

Carbajal-Tinoco et al. 07

Non-exponential relaxation

Yang et al. 03
Formulation

Dissipative quantum dynamics

- **Path-integral Schwinger-Keldysh** formalism.

  $\int T \ldots$

- Choose the system+reservoir initial density matrix at $t = 0$.
  Could be a factorized density operator

  $\hat{\rho}(0) = \hat{\rho}_{syst}(0) \otimes \hat{\rho}_{env}(0)$

  or not.

- Integrate out the bath degrees of freedom

- Obtain an effective action

  $S = S_{syst} + S_{influence}$

  $S_{influence}$ is non-local in time.
Markov limit

in dissipative quantum physics?

A very delicate question of time-scales and coupling constants. \( t_{syst}, t_{env} \) and \( \eta \).

Spohn 80, Gardiner 90s, Girvin - Les Houches 11

Search for a local differential equation, a master equation, for the reduced density operator

\[
\frac{i\hbar}{\Delta t} \rho_{\text{red}} = \left[ \hat{H}_{\text{syst}}, \rho_{\text{red}} \right] + \hat{L}(\rho_{\text{red}})
\]

Unitary  Non-unitary evolution

Lindblatt operators

OK in quantum optics, quantum machines not in atomic physics, cond-mat

NB no closed Fokker-Planck eq. for a Langevin process with coloured noise.
Plan

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Quantum dynamics

Non-trivial effects under Ohmic dissipation $I(\omega) = \eta \omega$

$P_{\text{tunn}} \rightarrow 0$

Suppression of tunnelling or

Localisation in a double well potential

at $k_B T = 0$ for $\eta > 1$

Bray & Moore 82, Leggett et al 87

Slowed-down diffusion

$\langle \hat{x}^2(t) \rangle \rightarrow \begin{cases} \frac{2k_B T}{\eta} t & \text{Classical } k_B T \neq 0 \\ \frac{\hbar}{\pi \eta} \ln t & \text{Quantum } k_B T = 0 \end{cases}$

Schramm-Grabert 87

Other non-trivial effects at $T \sim 0$ or non-Ohmic, $I(\omega) \sim \omega^\alpha$ baths.
A quantum impurity in a one dimensional harmonic trap

K atom : the impurity (1.4 on average per tube)

Rb atoms : the bath (180 on average per tube)

tall confined in one dimensional tubes

\[ T \approx 350 \text{ nK} \]

\[ \hbar \beta \sqrt{\frac{\kappa_0}{m}} \approx 0.1 \]

Catani et al. 12
Initially, the impurity is localized at the centre of the harmonic potential.

At $t = 0$, the impurity is released.

It subsequently undergoes quantum Brownian motion in the quasi $1d$ harmonic potential.
Protocol

A quench of the system

Initial equilibrium of the coupled system:

\[ \hat{\rho}(t_0) \propto e^{-\beta \hat{H}_i} \]

with

\[ \hat{H}_i = \hat{H}^i_{\text{syst}} + \hat{H}_{\text{env}} + \hat{H}_{\text{int}} \]

and

\[ \hat{H}^i_{\text{syst}} = \frac{1}{2m} \hat{p}^2 + \frac{1}{2} \kappa_0 \hat{x}^2 \]

At time \( t_0 = 0 \) the impurity is released, the laser blade is switched-off and the atom only feels the wide confining harmonic potential \( \kappa_0 \rightarrow \kappa \) as well as the bath made by the other species.

What are the subsequent dynamics of the particle? Use it to characterise the environment.
Obtain the generating functional

\[ Z_{\text{red}}[\zeta] = \int \mathcal{D}\text{variables} \ e^{\frac{i}{\hbar} S[\zeta]} \]

with the action given by

\[ S = S_{\text{det}} + S_{\text{init}} + S_{\text{diss}} + S_{\text{sour}}[\zeta] \]

where \( S_{\text{det}} \) characterises the deterministic evolution, \( S_{\text{init}} \) the initial density matrix, \( S_{\text{diss}} \) the dissipative and fluctuating effects due to the bath, and \( S_{\text{sour}} \) the terms containing the sources \( \zeta \).

Correlations between the particle and the bath at the initial time \( t_0 = 0 \) are taken into account via \( \hat{\rho}(t_0) \) and then \( S_{\text{init}} \).

Once written in this way, the usual field-theoretical tools can be used. In particular, the minimal action path contains all information on the dynamics of quadratic theories.
The model

The bath in the experiment

The environment is made of interacting bosons in one dimension that we model as a Luttinger liquid.

The local density operator is \( \hat{n}(x) = \rho_0 - \frac{1}{\pi} \frac{d}{dx} \hat{\phi}(x) \).

A canonical conjugate momentum-like operator \( \hat{\Pi}(x) \) is identified.

One argues

\[
\hat{H}_{\text{env}} = \frac{\hbar}{2\pi} \int dx \left[ \frac{u}{K} \left( \frac{d\hat{\phi}(x)}{dx} \right)^2 + \frac{uK\pi^2}{\hbar^2} \hat{\Pi}^2(x) \right]
\]

The sound velocity \( u \) and LL parameter \( K \) are determined by the microscopic parameters in the theory. For, e.g., the Lieb-Liniger model of bosons with contact potential \( \hbar \omega_L \sum_{i<j} \delta(\hat{x}_i - \hat{x}_j) \), one finds \( u(\gamma)K(\gamma) = \hbar\pi\rho_0/m_b \) and an expression for \( K(\gamma) \) with \( \gamma = m_b\omega_L/\hbar\rho_0 \).

Catani et al. 12

\( \gamma_{\text{exp}} \simeq 1 \)

Peotta et al. 13

\textit{t-DMRG of Bose-Hubbard model confirmation for } \hbar w \text{ small and } \hbar \omega_L \text{ large}
The model

The interaction in the experiment

- The interaction is \( \hat{H}_{\text{int}} = \int dr dr' \ U(|r - r'|) \ \delta(\hat{x} - r') \ \rho(r) \) with \( \tilde{U}(p) = \hbar \omega e^{-p/p_c} \), quantized wave-vectors \( p \to p_n = \pi \hbar n / L \), and \( L \) the ‘length’ of the tube.

- After a transformation to ladder operators \( \hat{b}_n^\dagger, \hat{b}_n \) for the bath, the coupling \( \hat{H}_{\text{int}} \) becomes \( \hat{H}_{\text{int}} \propto \sum p_n i p_n \tilde{U}(p_n) e^{-ip_n \hat{x} / \hbar} \hat{b}_{p_n} + \text{h.c.} \)

- One constructs the Schwinger-Keldysh path-integral for this problem.

- Low-energy expansion: \( e^{i p_n x / \hbar} \) to quadratic order, the action becomes the one of a particle coupled to a bath of harmonic oscillators with coupling constants determined by \( p_n \). The spectral density \( S(\nu)/\nu \) is fixed. A further approximation, \( L \to \infty \), is to be lifted later.
Impurity motion

Schwinger-Keldysh generating functional

The effective action has delayed quadratic interactions (both dissipative and noise effects) mediated by

$$\Sigma^K_B(t - t') = 2 \int_0^\infty d\nu \frac{S(\nu)}{\nu} \cos[\nu(t - t')]$$

with the (Abraham-Lorentz) spectral density ($\hbar = 1$)

$$S(\nu) = \frac{\pi}{2L} \sum_{p_n} \frac{K}{2\pi} |p_n|^3 |\tilde{U}(p_n)|^2 \delta(\nu - u|p_n|)$$

$$\rightarrow \eta \left(\frac{\nu}{\omega_c}\right)^3 e^{-\nu/\omega_c}$$

continuum limit for $L \rightarrow \infty$

$$\eta = K w^2 \omega_c^3 / u^4$$ with $\omega_c = up_c$  \textbf{Super-Ohmic diss.} $\alpha = 3$

$K$ LL parameter, $u$ LL sound velocity, $\hbar w$ strength of coupling to bath, $\omega_c$ high-freq. cut-off
The model

**Schwinger-Keldysh generating functional**

The action is *quadratic* in all the impurity variables.

The *generating functional* of all expectation values and correlation functions can be computed by the stationary phase method (exact in this case) as explained in, e.g., Grabert & Ingold’s review with some extra features: rôle of initial condition, quench in harmonic trap, non-Ohmic spectral density, possible interest in many-time correlation functions.

A *polaron effect* (mass renormalisation) and the potential renormalisation due to the fact that the bath itself is confined are also taken into account.

The equal-times correlation $C_x(t, t) = \langle \hat{x}^2(t) \rangle$ is thus calculated.
Breathing mode

Theory vs. experiment

\[ \eta = w/\omega_L = 1 \]

\[ \eta = w/\omega_L = 4 \]

Dynamics with \( m^* \) and \( \kappa^* \), interpolation to \( \lim_{t \to \infty} \langle \hat{x}^2(t) \rangle \to k_B T/\kappa^* \):

\[
\langle \hat{x}^2(t) \rangle = \frac{\hbar^2 \kappa_0}{4k_B T} \mathcal{R}(t) - \frac{\kappa^*}{k_B T} C_{eq}^2(t) + \frac{k_B T}{\kappa^*} + (1 - e^{-\Gamma t}) \left( \frac{k_B T}{\kappa} - \frac{k_B T}{\kappa^*} \right)
\]

Bonart & LFC EPL 13
Plan

Quantum fluctuations

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Classical dynamics

Two-time correlation

\[ r(t) \]

\[ \tilde{r}(t) \]

\[ \tilde{r}(t_w) \]

\[ \tilde{r}(0) \]

\[ 0 \]

\[ dt \]

\[ t=0 \]

\[ t_w \]

\[ t=dt+t_w \]

\[ \text{preparation time} \]

\[ \text{waiting time} \]

\[ \text{measuring time} \]

\[ t_w \] not necessarily longer than \( t_{eq} \).

Correlations

The two-time correlation between \( A(\vec{r}(t)) \) and \( B(\vec{r}(t_w)) \) is

\[ C_{AB}(t, t_w) \equiv \langle A(\vec{r}(t)) B(\vec{r}(t_w)) \rangle \]

the average is over realizations of the stochastic dynamics (random numbers in a MC simulation, thermal noise in Langevin dynamics, etc.)
Classical dynamics

The perturbation couples linearly to the observable $E \rightarrow E - hB(\{\vec{r}_i\})$

The linear instantaneous response of another observable $A(\{\vec{r}_i\})$ is

$$R_{AB}(t, t_w) \equiv \left\langle \frac{\delta A(\{\vec{r}_i\})(t)}{\delta h(t_w)} \right|_{h=0} \right\rangle$$

The linear integrated response or dc susceptibility is

$$\chi_{AB}(t, t_w) \equiv \int_{t_w}^{t} dt' R_{AB}(t, t')$$
Real-time quantum dynamics

Two-time observables

\[ C(t + t_w, t_w) \equiv \langle [\hat{O}(t + t_w), \hat{O}(t_w)]_+ \rangle \]

Correlation

\[ R(t + t_w, t_w) \equiv \frac{\delta \langle \hat{O}(t + t_w) \rangle}{\delta h(t_w)} \bigg|_{h=0} = \langle [\hat{O}(t + t_w), \hat{O}(t_w)]_- \rangle \]

Linear response
Real-time dynamics in equilibrium

If after $\tau_{eq}$ the system is in equilibrium with its environment:

- **One-time quantities** reach their equilibrium values,
  \[ \langle \hat{A}(t) \rangle \rightarrow \langle \hat{A} \rangle \]

- **All time-dependent correlations** are stationary,
  \[ \langle \hat{A}(t_1) \hat{A}(t_2) \cdots \hat{A}(t_n) \rangle = \langle \hat{A}(t_1 + \Delta) \hat{A}(t_2 + \Delta) \cdots \hat{A}(t_n + \Delta) \rangle \]
  for any number of observables, $n$, and time-delay, $\Delta$.
  In particular, $C(t + t_w, t_w) = C(t)$.

Classical glassy systems do not satisfy the second property and are out of equilibrium.
Real-time dynamics out of equilibrium

In classical glassy systems $\tau_{eq} \gg \tau_{exp}$ and the system does not equilibrate with its environment; it ages.

Quantum glassy systems?
Spherical model

A particle in a random potential

\[ \hat{H}_{\text{syst}} = \hat{H}_J(\{\hat{S}\}) + \sum_i \frac{\hat{\Pi}_i^2}{2M} \]

- Potential energy
- Kinetic energy

\[ [\hat{\Pi}_i, \hat{S}_j] = -i\hbar \delta_{ij} \]

- Canonical commutation rules

\[ \sum_i \langle \hat{S}_i^2 \rangle = N \]

- Spherical constraint

\[ \Gamma \equiv \hbar^2/(JM) \]

- Strength of quantum fluctuations

Coupled to a bath of quantum harmonic oscillators.

Results for the Ohmic case.
Real-time dynamics

Paramagnetic phase

Symmetric correlation

Linear response

Dependence on the quantum parameter $\Gamma$

LFC & Lozano 98-99
Real-time dynamics

Glassy or coarsening phases

Symmetric correlation

LFC & Lozano 98-99

Aron, Biroli & LFC 09
Real-time dynamics

Dependence on the coupling to the bath

Symmetric correlation  Linear response

Comparison between $\eta = 0.2$ (PM) and $\eta = 1$ (SG)

LFC, Grempel, Lozano, Lozza & da Silva Santos 02
Localization

the Caldeira-Leggett problem

A quantum particle in a double-well potential coupled to a bath of quantum harmonic oscillators in equilibrium at $T = 0$.

Quantum tunneling for $0 < \eta < 1/2$

‘Classical tunneling’ for $1/2 < \eta < 1$

Localization in initial well for $1 < \eta$

Bray & Moore 82

The same behaviour for a dissipative SU(2) spin in a transverse field

Leggett et al. 87
Real-time dynamics

Interactions against real-space localization

LFC, Grempel, Lozano, Lozza & da Silva Santos 02

Notation: $\alpha$ is the coupling to the bath here, that we called $\eta$ in the rest of the talk
Real-time dynamics

Fluctuation-dissipation theorem in classical glassy systems

Focus on the time-integrated linear response

\[ \chi(t + t_w, t_w) \equiv \int_{t_w}^{t+t_w} dt' \, R(t + t_w, t') \]

In equilibrium:

\[ \chi(t + t_w, t_w) = \frac{1}{T}[C(t_w, t_w) - C(t + t_w, t_w)] \]

In glasses: breakdown of the above FDT.

\[ \chi(t + t_w, t_w) = \text{cst} - \frac{1}{T_{\text{eff}}} C(t + t_w, t_w) \]

in the long \( t_w \) and \( t \gg t_w \) limit.

LFC & Kurchan 93
Real-time dynamics

Fluctuation-dissipation theorem in quantum glassy systems

The equilibrium FDT

\[ R(t + t_w, t_w) = \frac{i}{\hbar} \int_{-\infty}^{\infty} \frac{d\omega}{\pi} \, e^{-i\omega t} \tanh \left( \frac{\beta \hbar \omega}{2} \right) C(\omega, t_w) \]

becomes

\[ \chi(t + t_w, t_w) \approx \text{cst} - \frac{1}{T_{\text{eff}}} C(t + t_w, t_w) \quad t \gg t_w \]

if the integral is dominated by \( \omega t \ll 1 \) and \( T \to T_{\text{eff}} \) such that \( \beta_{\text{eff}} \hbar \omega \to 0 \).

LFC & G. Lozano 98-99
Real-time dynamics
Fluctuation-dissipation theorem in quantum glassy & coarsening systems

Parametric plot $\chi(C')$.

LFC & G. Lozano 98-99
Aron, Biroli & LFC 09
FDT & effective temperature

Can one interpret the slope as a temperature?

Yes, in classical glassy mean-field models

LFC, Kurchan, Peliti 97

(1) Measurement with a thermometer with

- Short internal time scale $\tau_0$, fast dynamics is tested and $T$ is recorded.
- Long internal time scale $\tau_0$, slow dynamics is tested and $T^*$ is recorded.

(2) Partial equilibration

(3) Direction of heat-flow

Quantum mechanically?
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Quantum quench

Setting

- Take a quantum closed system and suddenly change a parameter.

- *E.g.*, the quantum Ising chain

\[
H_{\Gamma_0} = - \sum \sigma_i^x \sigma_{i+1}^x + \Gamma_0 \sum \sigma_i^z
\]

Rieger & Iglói 90s

- Questions:
  
  *Does the system reach a thermal equilibrium measure?*
  
  *Under which conditions?*
  
  *(e.g., integrable vs. non-integrable systems; sub vs. critical quenches)*

  Calabrese & Cardy; Rossini et al., etc.

- Is there some kind of emerging effective bath?
Quantum quench

Previous studies

- Definition of $T_e$ from time-independent observables:

  \[ \langle H_\Gamma \rangle_{\Gamma_0} = \langle H_\Gamma \rangle_{T_e} \]
  \[ \langle M_{\Gamma}^x \rangle_{\Gamma_0} = \langle M_{\Gamma}^x \rangle_{T_e}, \text{ etc.} \]

  (We know these can be very misleading in glassy systems.)

- Definition of $T_e$ from the functional form of correlation functions:

  \[ C(r) \equiv \langle \sigma_i^x(t)\sigma_j^x(t) \rangle_{\Gamma_0} \text{ vs. } C_{eq}(r) \equiv \langle \sigma_i^x(t)\sigma_j^x(t) \rangle_{T_e}, \text{ etc.} \]

  (Again, they can be misleading.)

- Proposal: put qFDTs to the test to check whether $T_{\text{eff}}$ exists.
**Fluctuation-dissipation theorem**

### Classical dynamics in equilibrium

The classical FDT for a stationary system with $\tau \equiv t - t_w$ reads

\[
\chi(\tau) = \int_{0}^{\tau} dt' \ R(t') = -\beta [C(\tau) - C(0)] = \beta [1 - C(\tau)]
\]

choosing $C(0) = 1$.

**Linear relation** between $\chi$ and $C$

### Quantum dynamics in equilibrium

The quantum FDT reads

\[
\chi(\tau) = \int_{0}^{\tau} d\tau' \ R(\tau') = \int_{0}^{\tau} d\tau' \int_{-\infty}^{\infty} \frac{id\omega}{\pi \hbar} \ e^{-i\omega\tau'} \ \tanh \left( \frac{\beta \hbar \omega}{2} \right) C(\omega)
\]

**Complicated relation** between $\chi$ and $C$
Quantum quench

$T_{\text{eff}}$ from transverse spin $\hat{\sigma}_i^z$ and $\hat{M} = N^{-1} \sum_i \hat{\sigma}_i^z$ qFDTs?

$\text{Im} R^z(\omega) = \tanh \left( \frac{\beta^z_{\text{eff}}(\omega) \omega \hbar}{2} \right) C^z_+ (\omega)$

But $\beta^z_{\text{eff}}(\omega) \neq \beta^M_{\text{eff}}(\omega) \neq ct$
Quantum quench

$T_{\text{eff}}$ from longitudinal spin $\sigma_i^x$ qFDT?

$C^{x}(\tau) \simeq A_C e^{-\tau/\tau_C} [1 - a_C \tau^{-2} \sin(4\tau + \phi_C)]$

$R^{x}(\tau) \simeq A_R e^{-\tau/\tau_R} [1 - a_R \tau^{-2} \sin(4\tau + \phi_R)]$
Quantum quench

\[ T_{\text{eff}} \text{ from longitudinal spin } \sigma^x_i \text{ qFDT?} \]

For sufficiently long-times such that one drops the power-law correction

\[
- \beta^x_{\text{eff}} \sim \frac{R^x(\tau)}{d\tau C^x_+(\tau)} \sim - \frac{\tau C A_R}{A_C}
\]

A constant consistent with a classical limit but

\[ T^x_{\text{eff}}(\Gamma_0) \neq T_e(\Gamma_0) \]

A complete study in the full time and frequency domains confirms that

\[ T^x_{\text{eff}}(\Gamma_0) \neq T^z_{\text{eff}}(\Gamma_0) \neq T_e(\Gamma_0) \]

(though the values are close).

Fluctuation-dissipation relations as a probe to test thermal equilibration

No equilibration for generic \( \Gamma_0 \)
Quantum quench

No $T_{\text{eff}}$ from FDT

A quantum quench $\Gamma_0 \rightarrow \Gamma_c = 1$ of the isolated Ising chain

Foini, LFC & Gambassi 11, 13

More in the talk by Foini
Summary

Dynamics of quantum disordered systems

- We very briefly mentioned the Schwinger-Keldysh functional formalism & the delayed interactions induced by the coupling to a bath.
- The Markov limit & Lindblatt equation.
- An experimental realisation of quantum Brownian motion & its modelling.
- The real-time dynamics of dissipative quantum $p$-spin models.
- Quantum ageing and FDTs
- We used FDT ideas to check for equilibration in closed quantum systems.