

NON EQ. MACROSCOPIC COMPLEX SYST

MANY ADJECTIVES

NON. EQ.: WE ALL TALK ABOUT OUT OF EQ
DYNAMICS

MICROSC DYN:

SOMETIMES CONSTRAINED BY PHYSICAL LAWS (GLASSES)
OR NOT (ACTIVE MATTER, SOCIOLOGY, ECONOMY, ...)

MACROSC: MANY-BODY

THERMODYN LIMIT IMPORTANT $\Rightarrow$

LAW OF LARGE NUMBERS $N \gg 1$

STAT PHYS. THERMODYN?

BUT WE START FROM SINGLE PART MICROSC. & WE
TRY TO DERIVE COLLECTIVE FEATURES.

COMPLEX: HARD, NOT EASY $\Rightarrow$ INTERESTING

WHAT DO WE WANT TO DESCRIBE ?

- PHYSICAL MANY-BODY SYSTEMS
eg. GLASSES
WITH WEIRD COLLECTIVE PHENOMENA
SPECIAL PROPERTIES
- BIOLOGICAL PROBLEMS
WITH EFFECTIVE MICROSC DYN WHICH
MAY VIOLATE PHYS LAWS LIKE
ACTION-REACTION
STILL COLLECTIVE PHENOMENA
BOTH
OPEN
- ALGORITHMIC DYN TO SOLVE HARD PROBLEMS
COMPUTER SCIENCE

- QUANTUM MATTER, LIKE ATOMIC GASES
WITH INTERESTING DYN EFFECTS

ALMOST PERFECTLY CLOSED

DRIVEN-DISSIPATIVE SYSTEMS

KIND OF LIGHT-MATTER INTER.

DRIVEN OUT OF EQ. OPEN

I'U USE CLASSICAL NOTATION

I MENTION QUANTUM

GENERICS

OPEN

CLOSED

DISSIPATIVE DYN.

CONSERV. DYN

NO CONST MOTION

A FEW CONST

N CONST

NON-INTES.

INTEG.

APPROACH TO EQUILIBRIUM : WHICH ONE ?

IS THERE SOMETHING LIKE
ERGODICITY

$$A(\{\vec{x}_i, \vec{p}_i\}(t)) \equiv$$

$$\lim_{\tau \rightarrow \infty} \lim_{t \gg t_{eq}} \frac{1}{\tau} \int_t^{t+\tau} dt A(\{\vec{x}_i, \vec{p}_i\}(t))$$

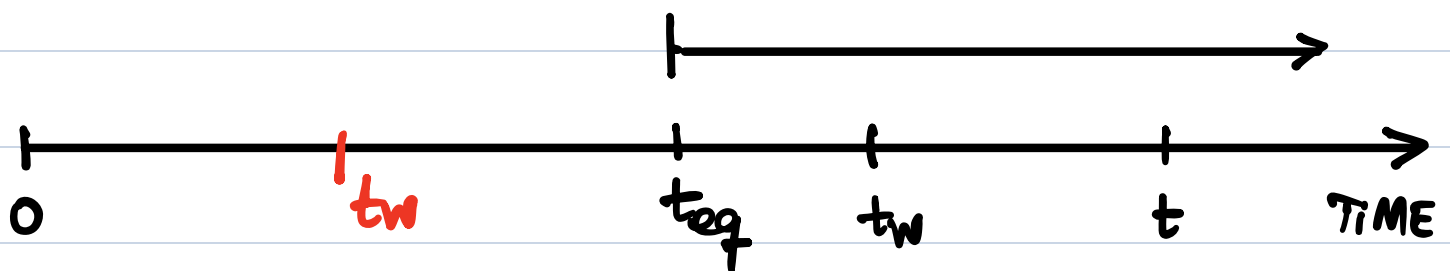
$$= \langle A(\{\vec{x}_i, \vec{p}_i\}) \rangle \equiv \int \mathcal{N} \times \mathcal{D} \rho \mathcal{P}(\{\vec{x}_i, \vec{p}_i\}) A(\{\vec{x}_i, \vec{p}_i\})$$

WITH ALL THE CONSEQUENCES THAT

$$P(\{\vec{x}_i, \vec{p}_i\}(t)) \longrightarrow P(\{\vec{x}_i, \vec{p}_i\})$$

$t \gg t_{eq}$

MAY HAVE ON THE TIME-DEP. PROP. ONE CAN
MEASURE - TO BE DERIVED - AFTER t



$$C(t_w, t) = \overline{A(\{\vec{x}_i, \vec{p}_i\}(t_w)) B(\{\vec{x}_i, \vec{p}_i\}(t))}$$

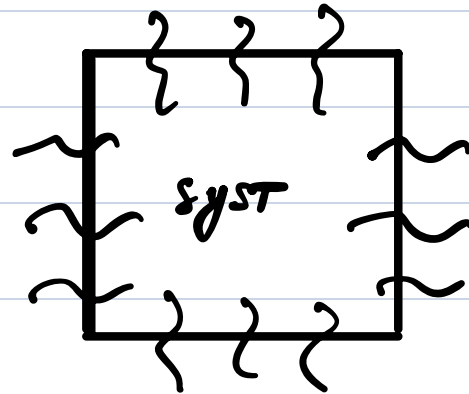
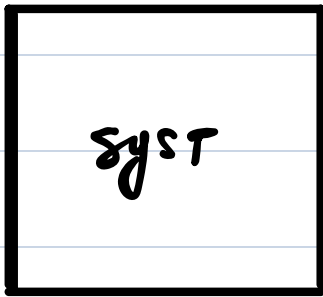
$$C(t_w, t) = \overline{A(\{\vec{x}_i, \vec{p}_i\}(t_w)) B(\{\vec{x}_i, \vec{p}_i\}(t))}$$

THEY MAY / WILL BEHAVE DIFFERENTLY

CLOSED

vs.

OPEN



CLASS. NOTATION

SURROUNDINGS

ENVIRONMENT - BATH

NOISE & DISSIPATION

$$E = H(\{\vec{x}_i, \vec{p}_i\}) = \text{CST.}$$

E FLUCTUATES

MAY $\exists$ OTHER CONSERVED
QUANTITIES, eg MOMENTUM,
ANG MOMENTUM, ETC.

EXCHANGES ∇
BATH ALSO N
OR SOMETHING ELSE

MICROCANONICAL

CANONICAL

GRAN. CANONICAL

PHYSICAL

EQUILIBRIUM

MEASURES

(NEWTON / HEISENBERG MICRASC DYN. ASSUMED)

MICROCANONICAL - CLOSED

$$P_{\text{micro}}(\{ \vec{x}_i, \vec{p}_i \}) = \frac{1}{\Omega} \delta(\mathcal{H}(\{ \vec{x}_i, \vec{p}_i \}) - E)$$

$N_c \neq \text{CT. MOTION}$ $\mathcal{H}_\mu(\{ \vec{x}_i, \vec{p}_i \})$ $\mu = 1, \dots, N_c$ $\frac{d\mathcal{H}_\mu}{dt} = 0$

$\mathcal{H}_\mu(0)$ THEIR INITIAL VALUES

CANONICAL - OPEN

$$P_{\text{can}}(\{ \vec{x}_i, \vec{p}_i \}) = \frac{1}{Z(\{ \beta_\mu \})} e^{- \sum_{\mu=1}^{N_c} \beta_\mu \mathcal{H}_\mu(\{ \vec{x}_i, \vec{p}_i \})}$$

eg. $\mathcal{H}_1 = H$, $\beta_1 = \beta$ INVERSE TEMP
 $\mathcal{H}_2 = N$, $\beta_2 = \mu\beta$ CHEMICAL POT.

DYNAMICS

CLOSED

NEWTON - HAMILTON

CLASSICAL MECHANICS

OPEN

LANGERIN

SINGLE NOISY TRAJECTORIES

FOKKER-PLANCK - MASTER EQ $P(\{\vec{x}_i, \vec{p}_i\}, t)$

MSR-JD GENERATING FUNCT TRAJ.

BIG QUESTION

DOES THE DYNAMICS OF AN $N \gg 1$ SYSTEM EVOLVE
TOWARDS A SITUATION WITH

$$P_{eq}(\{\vec{x}_i, \vec{p}_i\}) = \text{EQUIL. ONE}$$

AFTER SOME t_{eq} ?

EXTERNAL / INTERNAL
NON POTENTIAL FORCES

$$\begin{aligned} \vec{F} &\neq -\vec{\nabla} V && \text{ACTIVE MATTER} \\ \vec{f}_{ij} &\neq \vec{f}_{ji} && \text{NON-RECIPROCAL} \end{aligned}$$

EFFECTIVE INTERACTIONS
FOR PHYSICAL
OR "REAL" FOR OTHER
BRANCHES OF SCIENCE

NOT POSSIBLE $\mathcal{I}_{eq}(\{\vec{x}_i, \vec{p}_i\})$

BUT SOME $\mathcal{I}_{Ness}(\{\vec{x}_i, \vec{p}_i\})$?

PLAN OF THE TALKS

- DISCUSSION OF LANG EQS FROM A MICROSC. POINT OF VIEW
HOW IS THIS RELATED TO WHAT IS CALLED DMFT
 $C(t|tw)$ & $R(t|tw)$ A BIT OF GLASS PHENOMENOLOGY
- DOMAIN GROWTH FOR FM-PM TRANS. cfr GLASSES / ACTIVE MATTER
SIMPLEST POSSIBLE MACROSC OUT OF EQ SYST.
A SIMPLE MF MODEL FOR COARSENING
- CLASSICAL NEWTON DYNAMICS OF THE SAME MODEL AND GENERALIZED - GIBBS - ENS.

LANGENIN EQS

- M. WYART'S IN GENERATIVE MODELS
- ACTIVE BROWNIAN PARTICLES IN ACTIVE MATTER

BUT OF COURSE ALSO IN

TIME-DEPENDENT NOISY GINZBURG-LANDAU

AND MANY OTHER FIELDS

BASIS OF DMF OF GLASSY PHYSICS
DESCRIPTION AS WELL $\Rightarrow$

WHY?

LANGENIN FOR BROWNIAN MOTION ~ 1900
BEYOND?

JUSTIFICATION OF NOISE / DISSIP. IN LANGEVIN EQ.

BETTER & MORE GENERAL:

MODEL OF SYST H_{SYST} BUT ALSO
MODEL OF BATH & COUPLING

PHONONS, SIMPLEST IDEA

HARMONIC OSCILLATORS & LINEAR COUPLING

$$H_B = \sum_{a=1}^{N_0} \left(\frac{\pi_a^2}{2m_a} + \frac{m_a \omega_a^2}{2} q_a^2 \right)$$

$d=1$

TO SIMPLIFY

THE NOTATION

$$H_I = \sum_{a=1}^{N_0} c_a q_a \cdot f(x)$$

$$H_C = \frac{1}{2} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} (f(x))^2$$

(COUNTERTERM)

c_a COUPLING CONST. (WE'LL SCALE THEM WITH N_b)

$\{q_a, \pi_a\}$ OSCILLATOR'S PHASE SPACE VARIABLES

N_0 # OSCILLATORS (WILL GO TO ∞ TO MAKE A BATH)

SOLVABLE NEWTON/HAM. DYN FOR THE OSC.

LINEAR EDS. FORCED BY THE COUPLING TO x

VIA $f(x)$ USUALLY $f(x) = x$ SIMPLEST CASE
GEN. LATER

SOME VERY SIMPLE CALCULATIONS & USING
THE IMPORTANT ASSUMPTION

$P(\{q_a(0), \pi_a(0)\}; x(0))$

EQUILIBRIUM BATH
CANONICAL BOLZEMANN
(COUPLING TO $x(0)$)

TRICKY ISSUE ABOUT A COUNTERTERM TO BE DISCUSSED

TAKING $N_0 \rightarrow \infty$ A BATH: ENORMOUS

GAUSSIAN RANDOM INITIAL CONDITIONS $\Rightarrow$

INDUCE $\left\{ \begin{array}{l} \text{RANDOMNESS / STOCHASTICITY} \\ \text{FRICTION} \end{array} \right.$ $k_B T$

THE DYN NO LONGER CONSERVES ENERGY OR
OTHER CONST MOTION OF THE SYST

$\Rightarrow$

GENERALIZED LANGEVIN EQS

$$f(x) = x$$

$$m\ddot{x}(t) + \int_0^t dt' \Gamma(t-t') \dot{x}(t') = F[x(t)] + S(t)$$

NOISE

$$\xi(t) = - \sum_{a=1}^{N_0} c_a \left[\frac{\pi_a(0)}{m_a \omega_a} \sin \omega_a t + \left(q_a(0) + \frac{c_a x(0)}{m_a \omega_a^2} \right) \cos \omega_a t \right]$$

WITH

$$\Gamma(t-t') = \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} \cos[\omega_a(t-t')] \quad \text{DELAYED FRICTION}$$

SYMMETRIC FRICTION KERNEL $t \leftrightarrow t'$

FUNCTION FORM DEPENDS ON $\{c_a, m_a, \omega_a^2\}$

PROPS OF OSC. BATH

STATISTICS OF $\{q_a(0), \pi_a(0)\} \Rightarrow$ THE ONE OF ξ :

$$\langle \xi(t) \rangle = 0$$

$$\langle \xi(t) \xi(t') \rangle = k_B T \Gamma(t-t')$$

$\langle \dots \rangle$ MEANS AVERAGE WITH RESPECT TO
THE i.c. OF THE OSC.

KAWASAKI, ZWANZIG 60s-70s

usually
$$S(\omega) = \frac{1}{N_a} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a} \delta(\omega - \omega_a)$$

$$\Gamma(t-t') = \int_0^\infty d\omega \frac{S(\omega)}{\omega} \cos \omega(t-t')$$

THE CHARACTERISTICS OF THE OSCILLATORS
 $\Rightarrow$ NOISE, FRICTION PROPS.

FROM CLASSICAL MECHANICS TO STOCHASTIC PROCESSES

REDUCED SYSTEM
NON-MARKOVIAN-MEMORY

SPECTRAL FUNCTION LEADING TO POWER-LAWS

$$\frac{S(\omega)}{\omega} = \frac{2\gamma_0}{\pi} \left(\frac{|\omega|}{\tilde{\omega}} \right)^{\alpha-1} f_c \left(\frac{|\omega|}{\Lambda} \right)$$

USUAL CASES

$$\Gamma(t-t') = \begin{cases} 2\gamma_0 \delta(t-t') & \begin{array}{l} a=1 \\ \text{WHITE NOISE} \end{array} \\ \gamma_0 \exp\left(-\frac{|t-t'|}{\tau}\right) & \text{EXP. DECAY} \\ \gamma_0 \left|\frac{t-t'}{\tau}\right|^{-a} & \begin{array}{l} \text{POWER LAW} \\ \text{SUB/SUPER-OHMIC} \end{array} \end{cases}$$

ALL THE INFO ABOUT BATH IS IN $k_B T$, $\Gamma(t-t')$

MULTIPLICATIVE NOISE

$$f(x) \quad \text{GENERIC} \quad H_I = \sum_{a=1}^{N_0} c_a g_a f(x)$$

$$\begin{aligned} m\ddot{x}(t) + f'(x(t)) \int_0^t dt' \Gamma(t-t') \dot{x}(t') & \\ = F[x(t)] + S(t) f'(x(t)) & \end{aligned}$$

MENTIONED COUNTER TERM.

TO GET THE EQ. I WROTE, ADDED

$$H_C = \frac{1}{2} \sum_{a=1}^{N_0} \frac{C_a^2}{m_a \omega_a^2} \left(f(x) \right)^2$$

AND CHOOSE
INITIAL DIST OF
 $\{T_a(0), q_a(0)\}$
WITH NO WITHIN

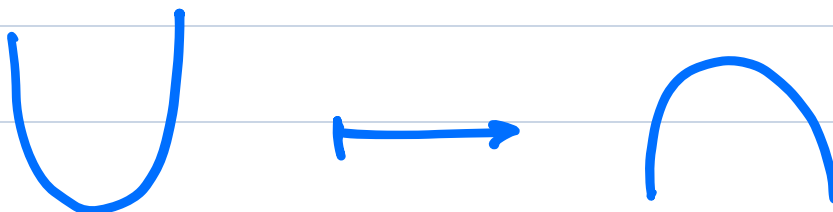
NEEDED TO AVOID A "RENORM." OF H_{Syst}

PHYSICAL CHOICE TO DO IT OR NOT

RENORMALIZATION OF "MASS" MEANS

THAT THE POTENTIAL OF THE SYSTEM IS
MODIFIED

$$V(x) \rightarrow V(x) - \sum_{a=1}^{N_0} \frac{C_a^2}{m_a \omega_a^2} x^2$$



COULD
HAPPEN

$\Rightarrow$ INSTABILITY

→ ADD IT BY ADDING THE COUNTERTERM.

OVERDAMPED LIMIT - USUAL IN STAT PHYS.

VALID FOR LARGE FRICTION/MASS $\gamma_0/m \gg 1$

OFTEN TIME SCALE OF MOMENTUM
RELAX. ARE VERY SHORT, GAUSSIAN WITH

$$\langle v(t) \rangle \rightarrow v_0 \quad \langle v^2(t) \rangle \rightarrow \frac{k_B T}{m}$$

DID $m \ddot{x}$ FROM EQ. MOTION

OUT OF EQ. BATHS

eg. ACTIVE MATTER AOP, OTHERS

BREAK BALANCE BETWEEN FRICTION AND THERMAL
TERMS. eg. USE $\Gamma_f(t-t') \neq \Gamma_{th}(t-t')$

eg. USE COUPLING TO TWO BATHS, WITH DIFF Γ_S AND
DIFF TEMPS. WHITE FAST AT T_f COLOURED SLOW AT T_S



FRICTION $\Gamma_1(t-t') + \Gamma_2(t-t')$

NOISE - NOISE CORR $k_B T_1 \Gamma_1(t-t') +$
 $k_B T_2 \Gamma_2(t-t')$

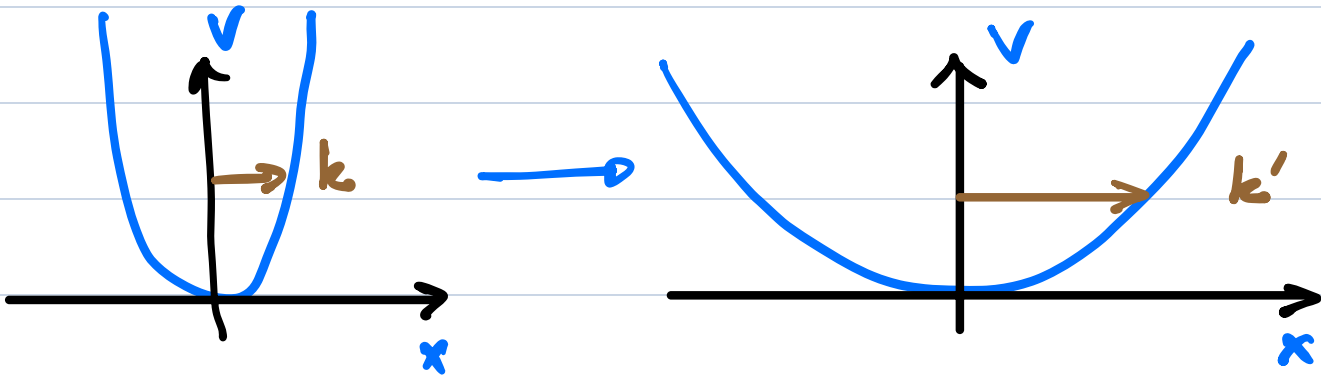
FOR THE SETTING IN WHICH EACH SYST
VARIABLE COUPLED TO BOTH BATHS.

TIME DEPENDENT FORCES VIA A CONTROL PARAM.

$$F[x(t), \lambda(t)]$$

← TIME DEP. PARAMETER

EX. QUENCHES AS CHANGES IN POTENTIAL,



NON-POTENTIAL FORCES

JUST ADD THEM IN $F[x(t)]$

BOTH WITH NON-EQ BATH & NON POT FORCES
ONE CAN INJECT ENERGY IN SYSTEM

ACTIVE BROWNIAN PARTICLES

$$m \ddot{\vec{x}}_i + \gamma \dot{\vec{x}}_i = \vec{f}_{\text{act},i}[\theta_i] + \vec{F}_i[\{\vec{x}_j\}] + \vec{\xi}_i$$

EQ. WHITE BATH AT TEMP. T

$$\vec{f}_{\text{act},i} = f_{\text{act}} (\cos \theta_i, \sin \theta_i)$$

FUNCTONATING
FORCE . DEP ON θ
NOT BALANCED W/
FRICTION

$$\dot{\theta}_i = \eta_i$$

OVERDAMPED BROWNIAN MOTION



$$\langle \hat{s}_i(t) s_j^b \rangle = 2\gamma k_B T \delta^{ab} \delta(t-t') s_j$$

$$\langle \eta_i(t) \eta_j(t') \rangle = \delta_{ij}$$

INVERSE PROBLEM

OBSERVE THE MOTION OF A TRACER

eg. BROWNIAN MOTION

CONCLUDE ABOUT THE PROPS OF THE BATH
THE FORCES IT INDUCES ON TRACER

LANGBEIN DID IT THIS WAY, EARLY 1900S

EXPERIMENTS, THEORY



TRACER IN A CELLULAR ENVIRONMENT



HOW TO GO FROM INFER FROM TRAJECTORIES

P. RONCEY (MANSIEU)

MIT GROUP EXPS.

MANY OTHERS

cfr MAURO'S INVERSE
ISING MODEL BASED ON
MAX ENT. IDEA

RECALL GENERALIZED LANGEVIN EQ.

$$m \ddot{x}(t) + f'(x(t)) \int_0^t dt' \Gamma(t-t') \dot{x}(t') = F[x(t), \lambda(t)] + S(t) f'(x(t))$$

causality

POSSIBLE TIME-DEP PARAM.

ADD INDICES TO MAKE IT VALID FOR

$$i = 1, \dots, N$$

$$a = 1, \dots, d$$

LOOK AT THE NOTES FOR FULL GENERALITY

INDEPENDENT

USUAL ADDITIVE NOISE CASE, ONE IDENTICAL

BATH PER

$$m \ddot{x}_i^a + \int_0^t dt' \Gamma(t-t') \dot{x}_i^a = \text{d.o.f.}$$

$$= F_i^a [\vec{x}_i(t), \vec{\lambda}_i(t)] + S_i^a(t)$$

$$\langle S_i^a(t) S_j^b(t') \rangle = k_B T \Gamma(t-t') \delta_{ij} \delta^{ab}$$

CONNECTION WITH USUAL

MULT. NOISE WHITE NOISE LANGEQ ($d=1$)

$$\dot{x} = \bar{f}(x) + g(x) \xi$$

TAKE $\Gamma(t-t') = 2\gamma_0 \delta(t-t')$

DROP $m \ddot{x}$ OVERDAMPED LIMIT

SET $\gamma_0 = 1$ BY A REDEF. OF TIME

$$\left(f'(x(t))\right)^2 \dot{x}(t)$$

$$= F[x(t), \lambda(t)] + S(t) f'(x(t))$$

DIVIDE BY $\left(f'(x(t))\right)^2$ AND REDEFINE THE
DETERM FORCE TERM $F / (f')^2 = \bar{f}$
AND

$$g = 1/f'$$

AND THAT'S IT.

REFS. DERIVATION & DISCUSSION OF LANG EQ

<http://www.lpthe.jussieu.fr/~leticia/SEMINARS/Langern.pdf>

on XiV

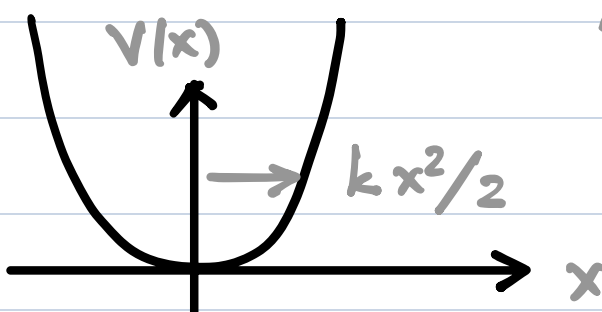
SKETCHY SLIDES

<http://www.lpthe.jussieu.fr/~leticia/SEMINARS/ictp-non-Markovian-bis.pdf>

YANG ET AL 03; MIN, LUO, CHERAYIL, KOU & XIE 05

$$C_x(t, t') = \langle x(t) x(t') \rangle$$

$$= \frac{1}{k} E_{\alpha, 1} \left(- \frac{k |t - t'|^\gamma}{g} \right)$$



MITTAG LEFFLER - FCT.

$$\gamma \simeq 0.5$$

SEPARATION OF TIME SCALES

- 1 ● $\neq$ VARIABLES EVOLVING IN $\neq$ TIME SCALES
- 2 ● SAME VARIABLE SHOWING $\neq$ TIME SCALES

1 ● JUSTIFIES USING OVER-DAMPED LIMIT
EASIER WAY TO SEE IT, FREE PARTICLE
WITH INERTIA.

SEE EX. IN NEXT PAGE

10

RANDOM WALK WITH INERTIA

$$\dot{x} = p/m$$

WHITE MULT. NOISE

BROWNIAN MOTION

$$\dot{p} = -\gamma_0/m p + \xi$$

$$p(t) = e^{-\gamma_0/m t} p(0) + \int_0^t dt' e^{-\gamma_0/m (t-t')} \xi(t')$$

$$\langle p(t) \rangle = e^{-\gamma_0/m t} p(0) \xrightarrow[t \gg m/\gamma_0]{} 0$$

$$\langle p^2(t) \rangle = \underbrace{e^{-2\gamma_0/m t} p^2(0)}_{\rightarrow 0}$$

$$+ 2\gamma_0 k_B T \int_0^t dt' e^{-2\gamma_0/m (t-t')}$$

$$= \cancel{2\gamma_0 k_B T} e^{-2\gamma_0/m t} \frac{1}{\cancel{2\gamma_0/m}} \left(e^{+2\frac{\gamma_0}{m} t} - 1 \right)$$

$$= m k_B T$$

MAXWELL ok

$$\gamma = m/\gamma_0 \ll 1$$

BUT, IF YOU LOOK AT THE POSITION MEAN-SQUARE DISPLACEMENT:

CALCULATE $\langle (x(t) - x(0))^2 \rangle$

AT VERY **SHORT** t NEGLECT ξ in $\dot{p}$ EQ
 $\Rightarrow p(t) \sim p(0) e^{-t/\tau}$

$$\sim p(0)$$

Now, in $\dot{x} = p/m$ EQ :

$$x(t) = x(0) + \frac{p(0)}{m} t$$

$$\langle x^2(t) \rangle \sim \frac{k_B T}{m} t^2$$

$$(x(t) - x(0))^2 \simeq \left(\frac{p(0)}{m} \right)^2 t^2$$

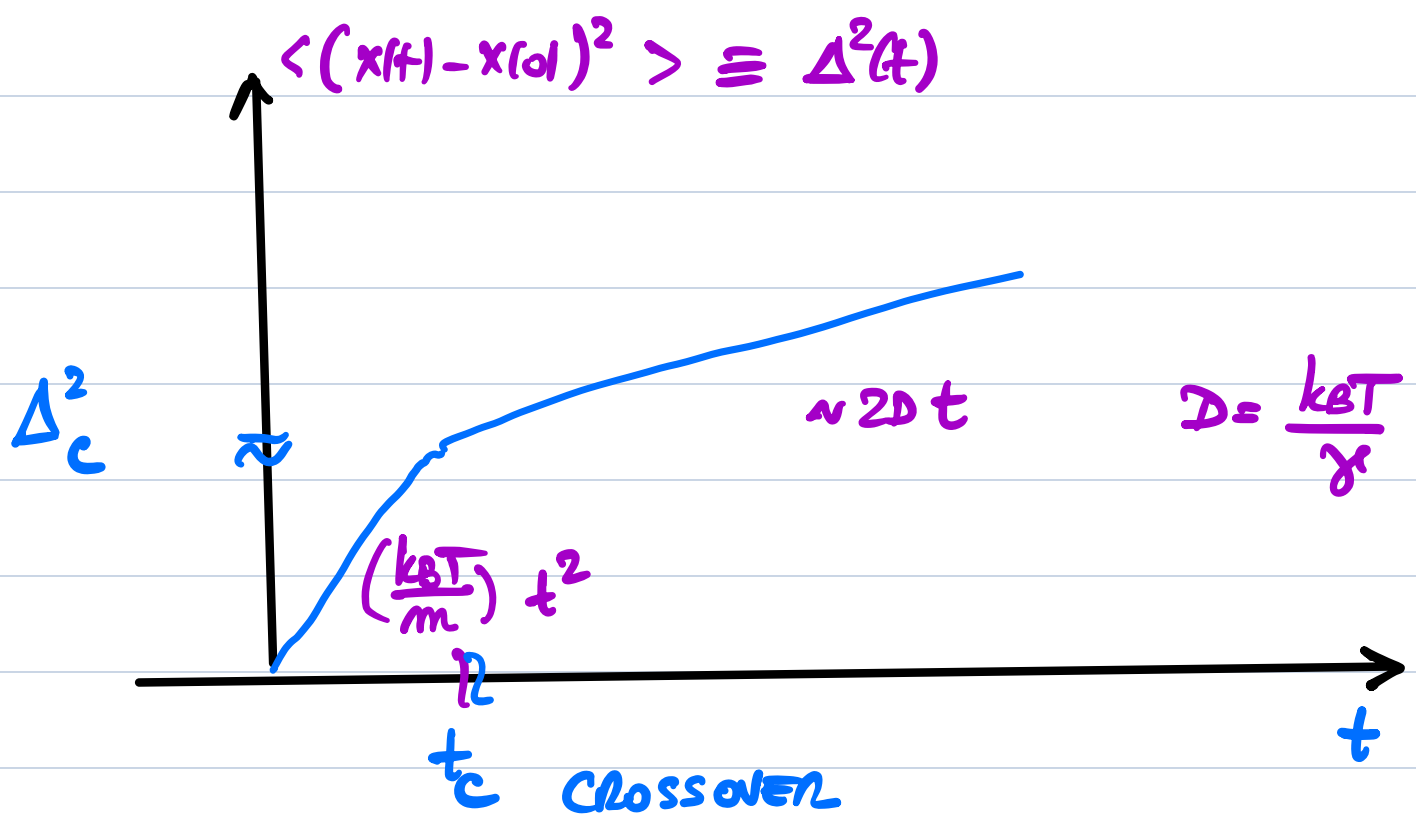
Ballistic

LONG t THE NOISE PART IS IMPORTANT!

$$\langle (x(t) - x(0))^2 \rangle \longrightarrow 2Dt$$

Diffusion

$$t \gg \tau$$



$$\frac{k_B T}{m} t_c^2 \approx \frac{k_B T}{\gamma} t_c \Rightarrow$$

$$t_c \sim \frac{m}{\gamma}$$

$$t_c \approx \frac{m}{\gamma}$$

VELOCITY
RELAX.
TIME

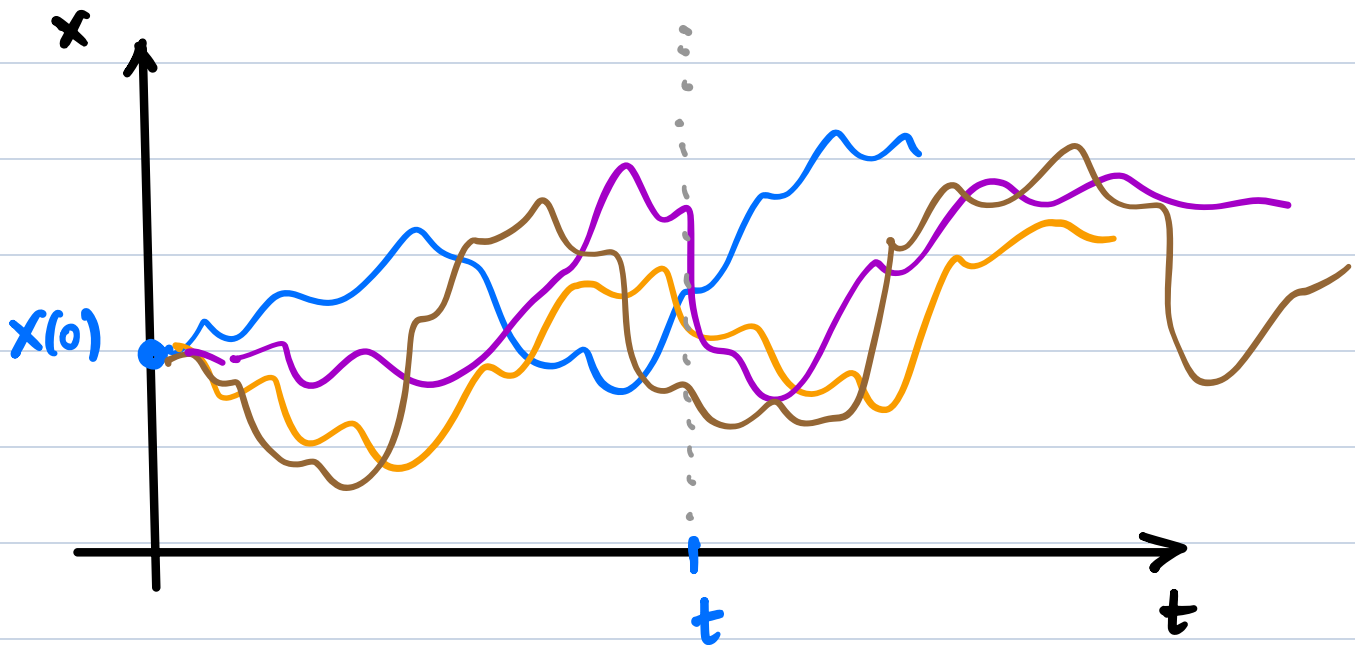
THERMAL
INITIAL KINETIC
ENERGY

KRAMERS / FOKKER-PLANCK EQUATION

$$P(\{\vec{x}, \vec{p}\}; t)$$

PROB. OF PARTICLES
HAVING $\{\vec{x}, \vec{p}\}$ AT
TIME t KNOWING
INITIAL PROB.

eg. JUST x (OVERDAMPED) $d=1$ $N=1$



READ $P(x, t)$ HERE

RESTRICTION

CLOSED DIFF EQ. ONLY FOR WHITE NOISE

KRAMERS EQ

BOTH MOMENTUM & POSITION

UNDERDAMPED

$$\frac{\partial P}{\partial t} + \frac{p}{m} \frac{\partial P}{\partial x} =$$

$$\gamma_0 \frac{\partial}{\partial p} (pP) + \frac{\partial}{\partial p} (-F P) + \gamma_0 k_B T \frac{\partial^2 P}{\partial p^2}$$

FOKKER-PLANCK

OVERDAMPED LIMIT $\Rightarrow$

JUST POSITIONS

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x} (F P) = \gamma_0 k_B T \frac{\partial^2 P}{\partial x^2}$$

THIS FORMALISM IS

USUALLY USED TO PROVE THAT IF

$$F = -V'(x) \Rightarrow$$

$$P(\{x, p\}) = \frac{e^{-\beta H_{\text{sys}}(\{x, p\})}}{\tilde{Z}(\beta)}$$

STAT SOLUTION

BUT IF WE DON'T HAVE KRAMERS / FP ?

FUNCTIONAL FORMALISM

ANALYSIS CAN BE DONE AT THE LEVEL OF GEN FUNCT

$$\mathcal{Z}_{\text{dyn}}[\lambda] = \int \prod_a \mathcal{D}q_a \prod_a \mathcal{D}\pi_a \int \mathcal{D}x \int \mathcal{D}p \quad e^{\int_0^t dt' \lambda(t') x(t')} \\ \exp \left\{ -S[\{x, p; q_a, \pi_a\}] \right\} \quad \lambda \text{ SOURCE}$$

INTEGRATE AWAY THE $\{ \pi_a, q_a \}$

$$\mathcal{Z}_{\text{dyn}} = \int \mathcal{D}x \int \mathcal{D}p \quad \exp \left\{ -S[\{x, p\}; \hbar\beta T, \Gamma] \right\}$$

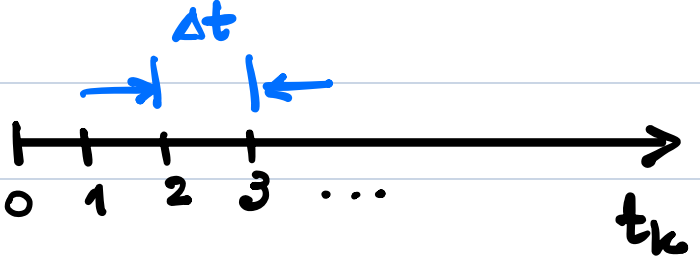
$$S = S_{\text{sys}}[\{x, p\}] + \underbrace{S_{\text{INF}}[\{x\}; \hbar\beta T, \Gamma]}$$

BATH EFFECT

THIS IS THE FORMALISM WHICH CAN BE GENERALIZED
QUANTUM MECHANICALLY.

FEYNMAN - VERNOU 60s

HINTS ON HOW TO OBTAIN THE ACTION S FOR LANGEVIN PROC. AND SOME TRICKY ISSUES.

- DISCRETIZE TIME
- 

- DISCRETIZE LANGEVIN EQ.

RECALL PRESCRIPTION HAS TO BE CHOSEN

- IF INERTIA PRESENT OR COLOURED NOISE
≠ PRESCRIPTIONS ARE EQUIVALENT FOR
 $\Delta t \rightarrow 0$

- IF OVERDAMPED & WHITE NOISE :
CAREFUL !

SPECIALLY WHEN CONSTRUCTING THE GEN.

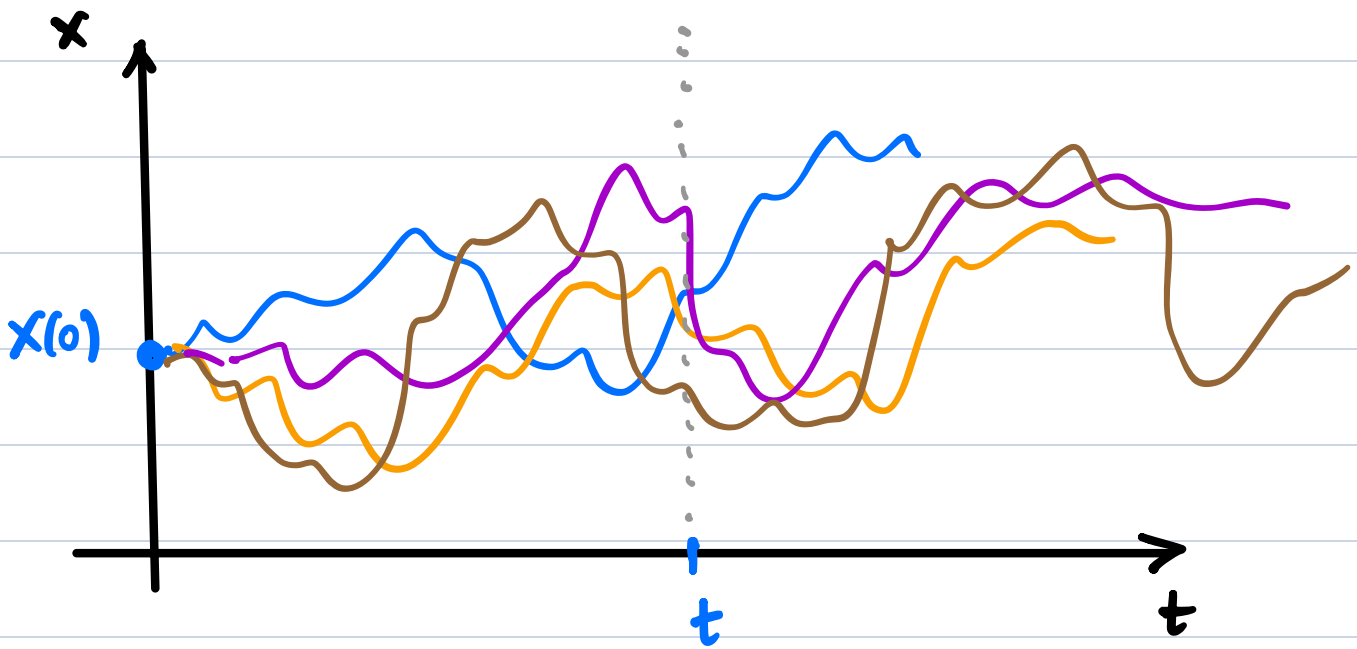
FUNCT. ALMOULX DE PIREY, LFC,

LECOMTE, VAN WIJLAND 22

SKETCH : SUM OVER ALL POSSIBLE TRAJ.

GENERATED BY THE LANGEVIN EQ.

UNDER ITS NOISE



$$Z_{\text{dyn}}(\lambda) = \left\langle e^{\int_0^t dt' \lambda(t') x_{\xi}(t')} \right\rangle_{\xi}$$

MEANS A SUM OVER ALL REALIZ. OF ξ
WITH THEIR

$$P[\xi(t)]$$

AND $x_{\xi}(t)$ SOL. OF LANGE EQ.

- IMPOSE THIS COND WITH A FUNCT δ -DIRAC
- EXPONENTIATE THE δ WITH AUX $\{i\tilde{x}(t)\}$
- AVERAGE OVER $\{\xi(t)\}$

FOR EXAMPLE, FOR A SINGLE VARIABLE BILINEARLY
COUPLED TO AN OSCILLATOR BATH

CLASSICALLY

$$S[x, i\hat{x}] = \underbrace{\ln P_0(x(0), \dot{x}(0))}_{\text{INITIAL COND.}} - \underbrace{\int_0^t dt' i\hat{x}(t') \left\{ m\ddot{x}(t') + \int_0^{t'} dt'' \Gamma(t'-t'') \dot{x}(t'') - F[x(t')] \right\}}_{\text{LANGE EQ. (APART FROM NOISE)}} + \underbrace{\frac{k_B T}{2} \int_0^t dt' \int_0^t dt'' i\hat{x}(t') \Gamma(t'-t'') i\hat{x}(t'')}_{\text{INTEGRATION OVER NOISE}}$$

$$+ \underbrace{\ln J[x]}$$

JACOBIAN - HAS TO DO WITH DISCRETIZATION
USED TO DEFINE THE STOCH PROCESS AND
BUILD THE PATH INTEGRAL

WHY IS THIS FORMALISM USEFUL?

- PERTURB. TH.

- SADDLE-POINT / MEAN-FIELD

- SYMMETRIES $\rightarrow$ WARD IDENTITIES, FUNCT THS. eg.

CORRELATIONS

$$\langle x(t) x(t') \rangle = \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S[\]} x(t) x(t')$$

LINEAR RESPONSE

$$\left. \frac{\delta \langle x(t) \rangle_h}{\delta h(t')} \right|_{h=0} = \frac{\delta}{\delta h(t')} \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S_h[\]} x(t) \Big|_{h=0}$$

↑ in:

$$F_h[x(t)] = F[x(t)] + h(t) x(t)$$

RECALL $e^{\int_0^t dt'' i \hat{x}(t'') F_h[x(t'')]} \quad \text{IN PATH INTEG.}$

$$\Rightarrow \left. \frac{\delta \langle x(t) \rangle_h}{\delta h(t')} \right|_{h=0} = \int \mathcal{D}x \mathcal{D}\hat{x} e^{-S[\]} x(t) i \hat{x}(t')$$

$$= \langle x(t) i \hat{x}(t') \rangle$$

CAUSALITY
 $\neq 0 \quad t \geq t'$

RESPONSE FIELD 36

SYMMETRIES

PROVE FDT IF

$P_0(x_{i0}, i\dot{x}_{i0})$ MAXWELL-BOLTZM.

$F[x] = -V'[x]$ POTENTIAL FORCE
WITH THE SAME
 $V[x]$ USED TO DRAW i.c.

$\Gamma(t-t')$ THE SAME IN FRICTION AND NOISE CORR.

PROOF. IT'S BASED ON THE INVARI. OF ACTION
UNDER SIMULT TRANSF OF $\{x(t), i\hat{x}(t)\}$

$$S[x(t), i\hat{x}(t)] = S[x(-t), i\hat{x}(-t)] + \underbrace{\beta \frac{d}{dt} x(-t)}_{\text{TRANSL.}}$$

↑ ↑
TIME REVERSAL TRANSL.

ALSO THE MEASURE $\mathcal{D}x \mathcal{D}i\hat{x}$ IS INVARIANT
Σ INTEGRATION INTERVAL TOO

CONSEQUENCE ON
CORRELATIONS Σ LINEAR RESP.²

NEWTON

INFLUENCE

NOTE THAT S_{sys} & S_{diss} ARE SEPARATELY
+ JAC IN VARIANT

$$\langle x(-t) | \hat{x}(-t') \rangle =$$



ALL THE $\langle \dots \rangle$ OPERATION IS INV. SINCE
S & MEASURE ARE

$$\langle x(t) \left[i \hat{x}(t') - \beta \frac{d}{dt'} x(t') \right] \rangle \Rightarrow$$

$$\langle x(t) | \hat{x}(t') \rangle = \langle x(-t) | \hat{x}(-t') \rangle + \beta \langle x(t) | \dot{x}(t') \rangle$$

FDT

$$R(t, t') = R(-t, -t') + \beta \frac{\partial}{\partial t'} C(t, t')$$

THEREFORE, IF FDT $\Rightarrow$ EQUIL. A LA
MAXWELL-BOLTZMANN cfr J. KUNCHAN'S
THEN THESE PROCESSES DO TAKE SYST
TO EQUIL. UNDER ABOVE CONDITIONS

QUANTUM

VERY SIMILAR !



COMPLEX PLANE

FEYNMAN'S

PATH INT $\rightarrow$
STAT MECH.

TIME

$$e^{-\beta \hat{H}} / \mathcal{Z} = \langle \hat{S} \rangle$$

$$-i\hbar \rightarrow -\beta$$

$$t = -i\tau$$

$$\tau = \beta$$

IN ORDER TO CALCULATE CORREL. LIKE

$$\langle \hat{A}(t) \hat{B}(t') + \hat{B}(t') \hat{A}(t) \rangle$$

NO TIME ORDERED, TRICK GO FORTH & BACK.

$$\frac{x^+(t) + x^-(t)}{2} \xrightarrow{\hbar \rightarrow 0} x(t)$$

$$\frac{x^+(t) - x^-(t)}{\hbar} \xrightarrow{\hbar \rightarrow 0} i\dot{x}(t)$$

DYNAMIC MEAN-FIELD THEORY

FOCUS ON "ONE VARIABLE"

ONE PART., ONE SPIN, WHATEVER

ESTIMATE THE EFFECT OF THE REST,
COUPLED TO IT,
IN SOME APPROX.

SIMILAR TO INTEGRATING AWAY THE OSC.
ONLY THAT, OF COURSE, THEY ARE NOT QUANT.
VARIABLES

DERIVE A GENERALIZED LANG EQ

$$m\ddot{x}(t) + \gamma_0 \dot{x}(t) + \int_0^t dt' \sum (C(t,t'), R(t,t')) x(t') \\ = F[x(t)] + \xi(t) + \eta(t)$$

$$\langle \eta(t) \eta(t') \rangle = D(C(t,t'), R(t,t'))$$

$$C(t, t') = \frac{1}{N} \sum_i \langle x_i(t) x_i(t') \rangle$$

$$R(t, t') = \frac{1}{N} \sum_i \frac{\delta \langle x_i(t) \rangle}{\delta h_i(t')}$$

$$\sum (C, R) = D'(C) R$$

FOR USUAL POTENTIAL FORCES

BUT COULD BE "INDEP" FOR OTHER CASES

$D(C)$ SOME FUNCTION OF C

FOR EX. A POWER LAW $\gamma_p^2 \frac{C^p}{2}$

↑
POLLMEIER

IN EQUILIBRIUM

$$\left\{ \begin{array}{l} Z = \frac{\partial D}{\partial t'} \\ R = \frac{1}{T} \frac{\partial C}{\partial t'} \end{array} \right. \quad \begin{array}{l} \text{INTEGRATE BY PARTS} \Rightarrow \\ \text{USUAL FRICTION EXPRESSION} \end{array}$$

EXACT? TOTAL "MF MODEL" eq.

$$H = \sum_{i_1, \dots, i_p} J_{i_1, \dots, i_p} x_{i_1} \dots x_{i_p} \quad p \text{ PARAM.}$$

RANDOM

HOW TO DERIVE THIS EQ?

- MANIPULATE Zdyn LES HOUCHES 02
- CAVITY-LIKE METHOD
ALTIERI, CAMALOTTA, BIROLI

SUMMARY 1ST LECTURE

- I JUSTIFIED THE GENERALIZED LANGEVIN EQ.
BEYOND BROWNIAN MOTION
GENERIC MEMORY KERNEL & NOISE-CORR
RELATED BY FDT IF EQ BATH
- SKETCHED (I DID NOT DO IT IN CLASS)
THE CLASSICAL GEN. FUNCT. CONSTRUCTION
- DISCUSSED ITS TIME-REV. SYMM. $\Rightarrow$ FDT
& THEN EQUIL. (NOT DONE IN CLASS)

$$G_d(\omega) = \frac{1}{\omega - \epsilon_d - Z(\omega)}$$

cfr EDUARDO'S
EQ. QUANTUM
NO EXT BATH
SIMILAR TO BM FT

$$\hat{x}(t) = e^{i\hat{H}t/\hbar} \hat{x} e^{-i\hat{H}t/\hbar}$$

HEISENBERG

IMPORTANT

$$F = -V'$$
 AND

ALL THESE FORMALISMS SHOULD ADMIT (IF BATH IN EQ.)

TO TAKE SYST FOR $t \gg t_{eq}$ (PARAM.) TO CANON. EQUIL

$$\frac{e^{-\beta H_{\text{SYST}}}}{Z(\beta)}$$

NOT EASY IN GENERAL

→ OK FOR FOKKER PLANCK / KRAMERS

→ SYMM. IN Z_{dyn} FOR COLOURED NOISE

$t_{eq} ??$

COULD BE VERY LONG AND SYST SIZE-DEPENDENT

$\Rightarrow$ LONG OUT OF EQ. RELAXATION

STILL FOR EQ. BATH $\&$ $F = -V'$

LET'S DISCUSS SOME PHYSICAL PROBLEMS NOW.

1- SEP. of TIME SCALES p vs. $x \Rightarrow$ ex.

I DISCUSSED IT IN 1ST LECTURE

2- DROP INERTIA, JUST x . SEPARATION OF TIME SCALES IN ITS OWN BEHAVIOUR.

NOT SO SURPRISING. THINK ABOUT RW AGAIN, FOR

$t < t_{eq}^{(p)} \Rightarrow$ BALLISTIC $x(t) \approx \frac{p}{m} t$

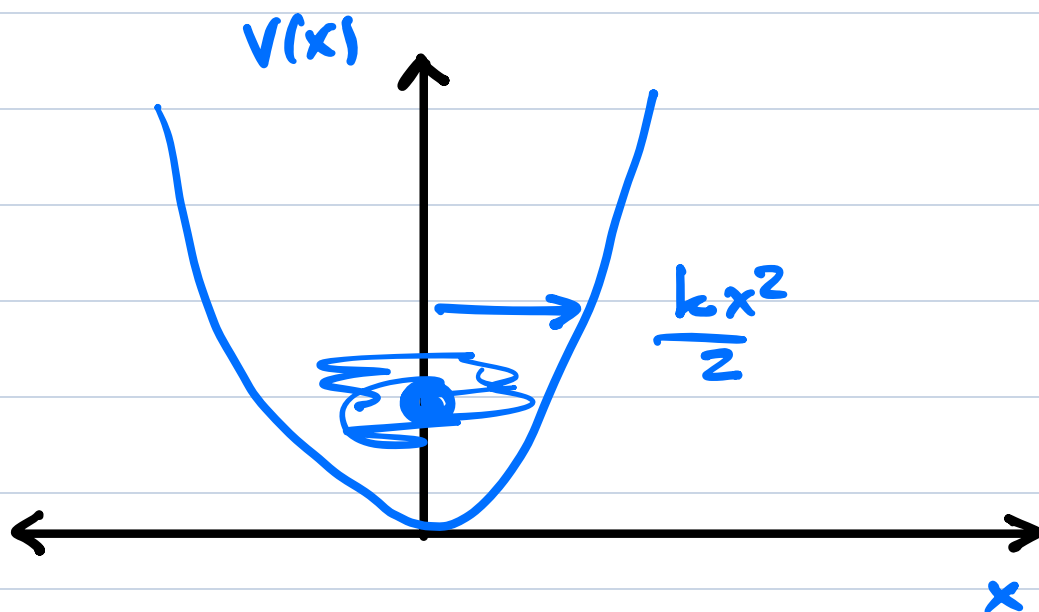
$t > t_{eq}^{(p)} \Rightarrow$ DIFFUSIVE WHEN NOISE CONTROLS

FOCUS ON $t > t_{eq}^{(p)}$

LET'S LOOK AT ANOTHER SIMPLE EX IN THE OVERDAMPED LIMIT

$$\gamma_0 \dot{x} = -kx + S + h \rightarrow \text{TO MEASURE LINEAR RESP.}$$

A PARTICLE IN A HARM WEL UNDER ADD. WHITE NOISE



$$x(t) = x_0 e^{-\frac{k}{\gamma_0} t} + \frac{1}{\gamma_0} \int_0^t dt'' e^{-\frac{k}{\gamma_0} (t-t'')} [S(t'') + R(t'')]$$

$x_0 = 0$

AVERAGE POSITION

$$\langle x(t) \rangle_R = \frac{1}{\gamma_0} \int_0^t dt'' e^{-\frac{k}{\gamma_0} (t-t'')} R(t'')$$

NON-ZERO DUE TO R

SELF-CORRELATION

$$x(0) = x_0 = 0$$

SIMPLER

$$\langle x(t) x(t') \rangle_{h=0} = \frac{k_B T}{k} \left[e^{-\frac{k}{\gamma_0} |t-t'|} - \underbrace{e^{-\frac{k}{\gamma_0} (t+t')}}_{\substack{\rightarrow 0 \\ t+t' \rightarrow 0 \\ \text{iff } k \neq 0}} \right]$$

STATIONARY $\uparrow$

SAME $\tau = \frac{\gamma_0}{k}$

TIME DERIVATIVE

$$\frac{\partial \langle x(t) x(t') \rangle}{\partial t'} \xrightarrow{t > t'} \frac{k_B T}{k} \cancel{\frac{k}{\gamma_0}} e^{-\frac{k}{\gamma_0} (t-t')}$$

$$\partial_{t'} C(t, t') = \frac{k_B T}{\gamma_0} e^{-\frac{k}{\gamma_0} (t-t')}$$

STATIONARITY

$$\hat{x}(t) = e^{i\hat{H}t/\hbar} \hat{x} e^{-i\hat{H}t/\hbar} \quad \text{HEISENBERG}$$

USE IT TO ARGUE STAT-TTI IN EQ.

$$\langle \hat{A}(t) \hat{B}(t') \rangle = \text{Tr} \left(\hat{A}(t) \hat{B}(t') e^{-\beta \hat{H}} \right)$$

↓ $\hat{\rho}_{\text{eq}}$

$$= \text{Tr} e^{i\hat{H}t/\hbar} \hat{A} e^{-i\hat{H}t/\hbar} e^{i\hat{H}t'/\hbar} \hat{B} e^{-i\hat{H}t'/\hbar} e^{-\beta \hat{H}}$$

$e^{-i\hat{H}(t-t')/\hbar}$

USE PERMUTABILITY
OF TRACE

$$e^{i\hat{H}(t-t')/\hbar}$$

$$= C(t-t')$$

IMPORTANT: SAME $\hat{H}$ IN $\hat{\rho}_{\text{eq}}$
AND IN EVOLUTION

LINEAR RESPONSE

$$R(t, t') = \frac{\delta \langle x(t) \rangle_R}{\delta h(t')} = \frac{1}{\gamma_0} e^{-\frac{k}{\gamma_0} (t-t')} \theta(t-t')$$

$$\tau \equiv \frac{\gamma_0}{k}$$

STATIONARY
CAUSAL

TIME SCALE FOR RELAX.

NB $\tau \xrightarrow{k \rightarrow 0} \infty$ $\tau \xrightarrow{\gamma_0 \rightarrow 0} 0$

cfr

$$R(t, t') = \frac{1}{k_B T} \partial_{t'} C(t, t')$$

FDT

ex. CHECK WHERE THIS FAILS IF $k=0$ BEFORE LONG TIMES

$$\chi(t, t') = \int_{t'}^t dt'' R(t, t'') = \frac{1}{k_B T} [C(t, t) - C(t, t')]$$

WE SHOWED THIS IDENTITY BETWEEN
 $R \perp \partial_t C$

FOR A SIMPLE PROBLEM, THE HARM FOT
w/ OVER-DAMPED DYN

IT HOLDS $\forall$ SYST EVOLVING IN EQUIL.

FOT

HOW TO PROVE IT ?

SEVERAL WAYS. THE MOST POWERFUL
ONE USING GEN FUNCT, PATH INT,
AND USING TIME REVERSAL.

LIKE THAT HIGHER ORDER ONES

AS WELL

IMPORTANT : STRONG TEST OF EQUIL !

WHAT HAPPENS IF YOU COUPLE THE SAME PART TO TWO BATHS ?

$$\Gamma_f(t-t') = 2\gamma_f \delta(t-t') \quad k_B T_f$$

$$\Gamma_s(t-t') = \frac{\gamma_s}{\tau_s} e^{-|t-t'|/\tau_s} \quad k_B T_s$$

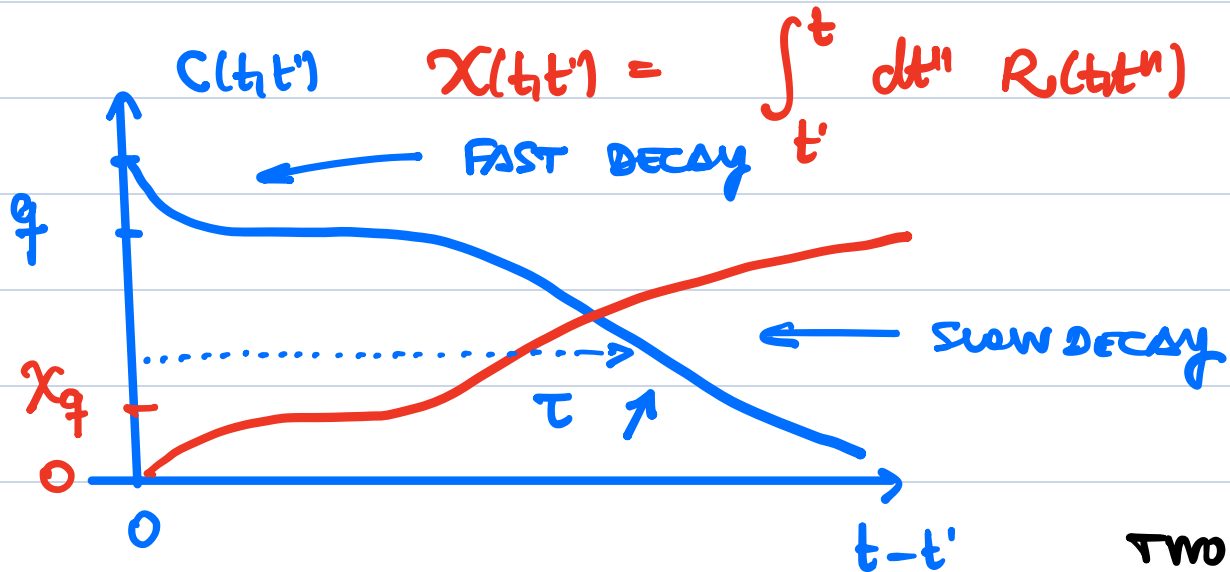
WE HAD $\tau = \frac{\gamma_0}{k}$ BUT NOW IT'S CLEAR THAT IT'LL BE MORE COMPLEX

$$\int_0^t dt' \left[\Gamma_s(t-t') + 2\gamma_0 \delta(t-t') \right] \ddot{x}(t') =$$
$$= -kx(t) + S(t) + h(t)$$

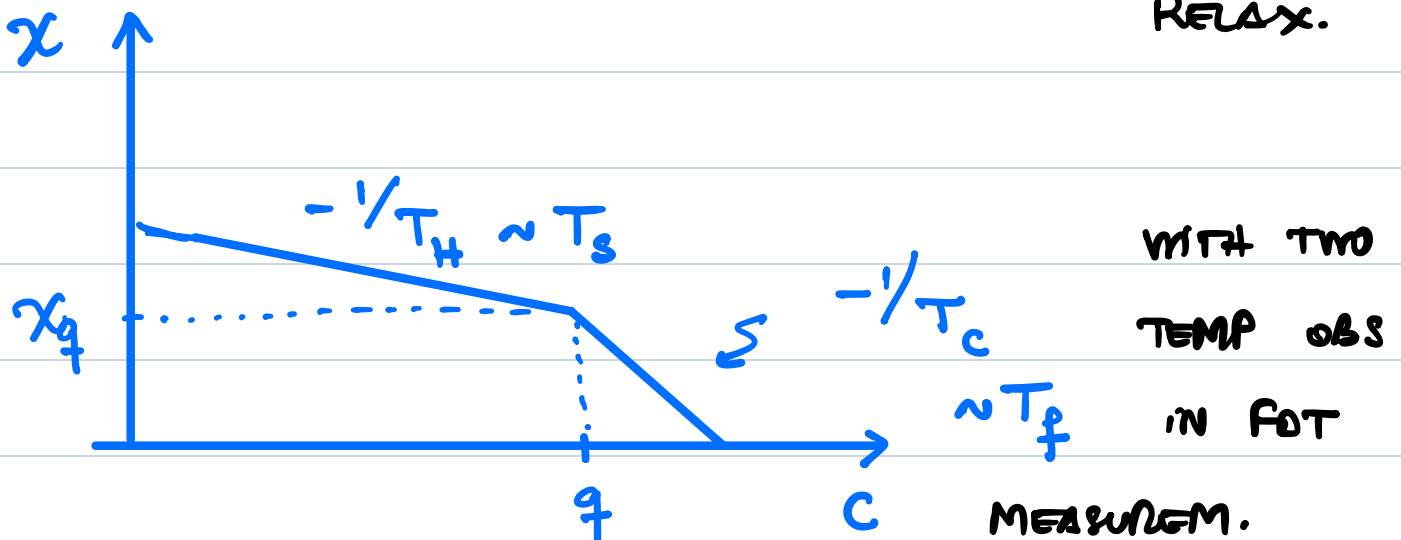
STILL A LINEAR 1st ORDER DIFF EQ.
SOLVABLE ANALYTICALLY.

TRICK: SET $t_0 = 0$ TO $t_0 = -\infty$, $\gamma = \theta(t-t') \Gamma(t-t')$
IN FRICTION AND USE FOURIER TRANSF.

DEPENDING ON PARAMETERS



TWO STEP
RELAX.



32

$$\frac{1}{T_{eff}(\omega)} = \frac{\gamma_f (1 + \omega^2 \tau_s^2) + \gamma_s}{T_f \gamma_f (1 + \omega^2 \tau_s^2) + T_s \gamma_s}$$

A SKETCH OF WHAT HAPPENS IN GLASSY (MF)
MODELS

RECALL LANG EQ WITH

- EXT BATH AT TEMP. T AND WHITE
- INT BATH SELF-GENERATED AT ITS OWN T_{eff} .

T_{eff} KNOWS ABOUT THE INITIAL CONDITION
IN THE SENSE THAT

$$T_{\text{eff}} > T \quad \text{if } T_0 > T$$

$$T_{\text{eff}} < T \quad \text{if } T_0 < T$$

FINALLY COMPLEX SYSTEMS, BEYOND SINGLE VARIABLE.

DMFM WHAT IS IT ?

METHODS TO DEAL WITH DYNAMICS IN MEAN-FIELD.

GROSSO MODO. SINGLE OUT ONE VARIABLE OF YOUR WHOLE SYSTEM AND IDENTIFY SELF-CONSISTENTLY WHICH IS THE FRICTION & NOISE-NOISE KERNEL THAT SHOULD GO IN THE GEN. LANG. EQ.

$$m \ddot{x} + \gamma_0 \dot{x} = \int_0^t dt' \mathcal{I}(t, t') x(t') + \xi + \eta$$

$$\langle \eta(t) \eta(t') \rangle = \mathcal{D}(t, t')$$

NB $\mathcal{I}$ is like $\frac{d\Gamma(t-t')}{dt'}$ A RESPONSE

$\mathcal{D}$ is like $\Gamma(t-t')$ A CORR

WHO ARE I AND D ?

IN "MEAN-FIELD" MODELS THEY ARE FCTS OF
 $\{C, R\}$

$$C = \langle x(H) x(H') \rangle$$

$$R = \frac{\delta \langle x(H) \rangle}{\delta h(H')}$$

OF COURSE, THIS IS A MUCH HARDER
PROBLEM TO SOLVE!

- SIMILAR STRUCT. CLASSICALLY & QUANTUM

- CAN BE PUT IN A COMPUTER

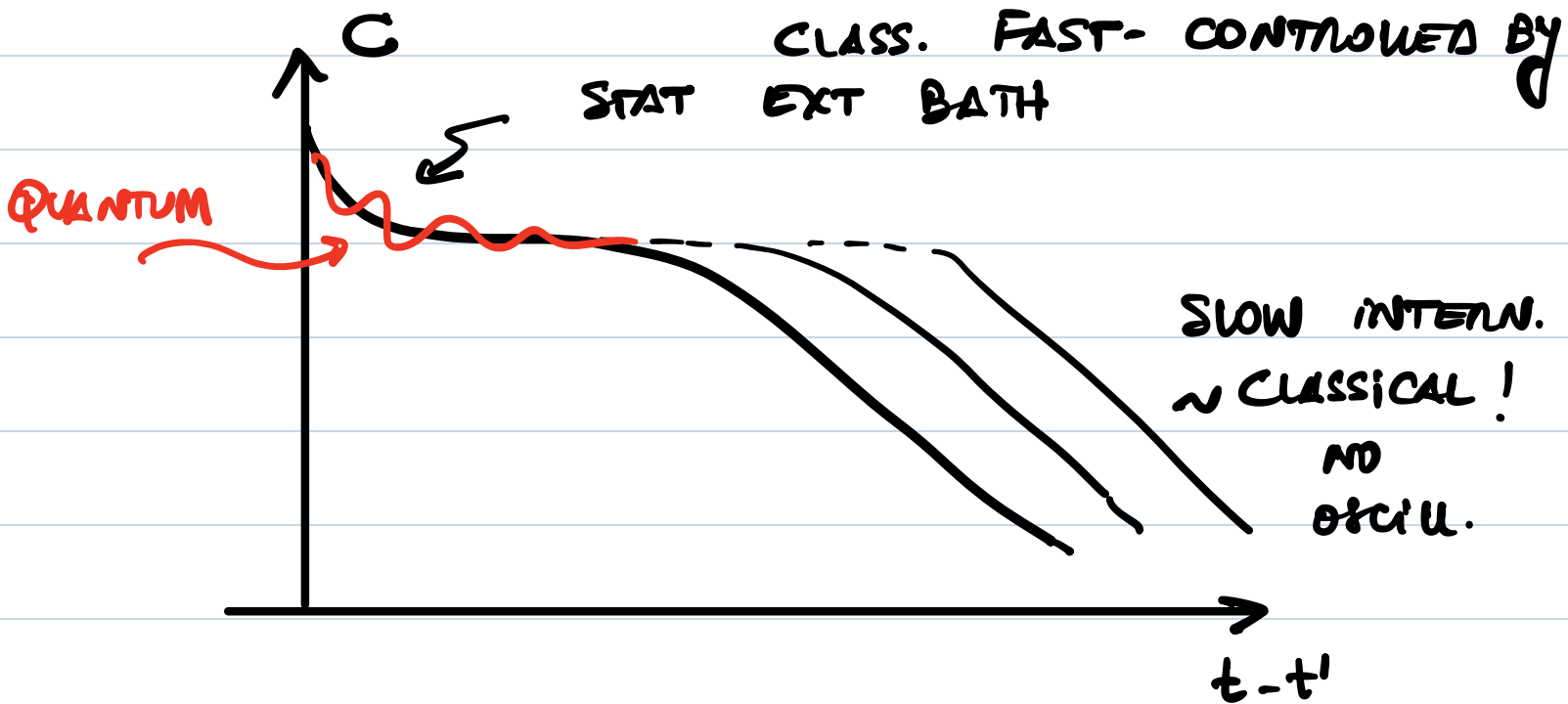
- CAN BE USED TO WRITE CLOSED EQS.

FOR $\{C, R\} \rightarrow$ SCHWINGER-DYSON EQS.

BBGKY HIERARCHY CLOSED FOR MF MODELS

PHYSICS ? GLASSY MODELS.

TWO (OR MORE) SCALE RELAX. $\Rightarrow$
SELF INDUCED.



$$\chi(c) = \lim_{t' \rightarrow \infty} \chi(t, t') \\ C(t, t') \text{ FIXED}$$

CLASSICAL

$$-\frac{1}{T_{eff}(c)} = \frac{d\chi(c)}{dc}$$

$c > q \quad \frac{1}{T}$

$c < q \quad \frac{1}{T^*}$

QUANTUM

$C > g \Rightarrow$ QUANTUM FDT

$C < g \Rightarrow$ CLASSICAL! $-\frac{1}{T^*}$

CAN I SHOW YOU A SOLVABLE NON-TRIVIAL
MODEL WHERE THIS IS REALIZED?

PHASE ORDERING KINETICS

DYN ACROSS 2nd ORDER PHASE TRANS.

REFS BRAY, PURI, KRAPINSKY - REDNER, ONUKI, LFC

MODELS (OPEN, COUPLED TO A BATH)

TAKE A CLASSICAL ISING MODEL w/ STOCH MC DYN

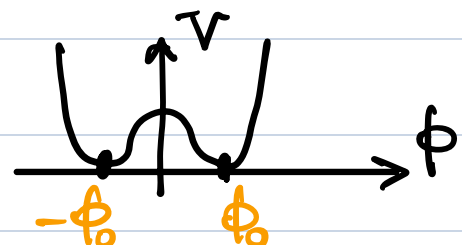
$$H_J[\{s_i\}] = -J \sum_{\langle i,j \rangle} s_i s_j$$

OR IT'S t-DEP. GINZBURG LANDAU VERSION

$$\frac{\partial \phi(\vec{x}, t)}{\partial t} = \nabla^2 \phi(\vec{x}, t) - \frac{\delta V}{\delta \phi(\vec{x}, t)} + \xi(\vec{x}, t)$$

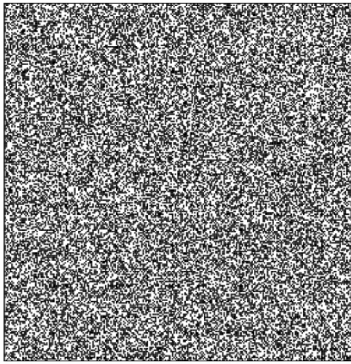
$$\langle \xi(\vec{x}, t) \rangle = 0 \quad \langle \xi(\vec{x}, t) \xi(\vec{y}, t') \rangle = 2\gamma k_B T \delta(\vec{x} - \vec{y}) \delta(t - t')$$

$$V(\phi) = \frac{\tau}{2} \phi^2 + \frac{\lambda}{4} \phi^4$$

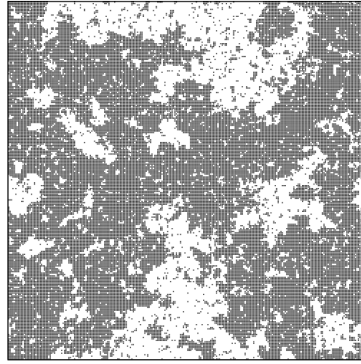


ϕ_0 MINIMA OF DOUBLE WELL ($\tau < 0$)

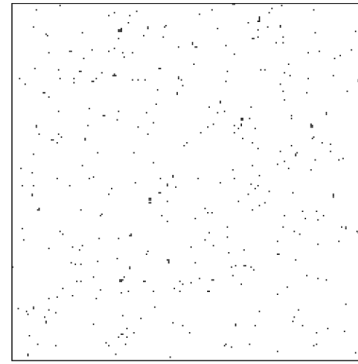
TYPICAL EQUILIBRIUM CONF. AT $\neq$ TEMP.



$$T_0 \rightarrow \infty$$



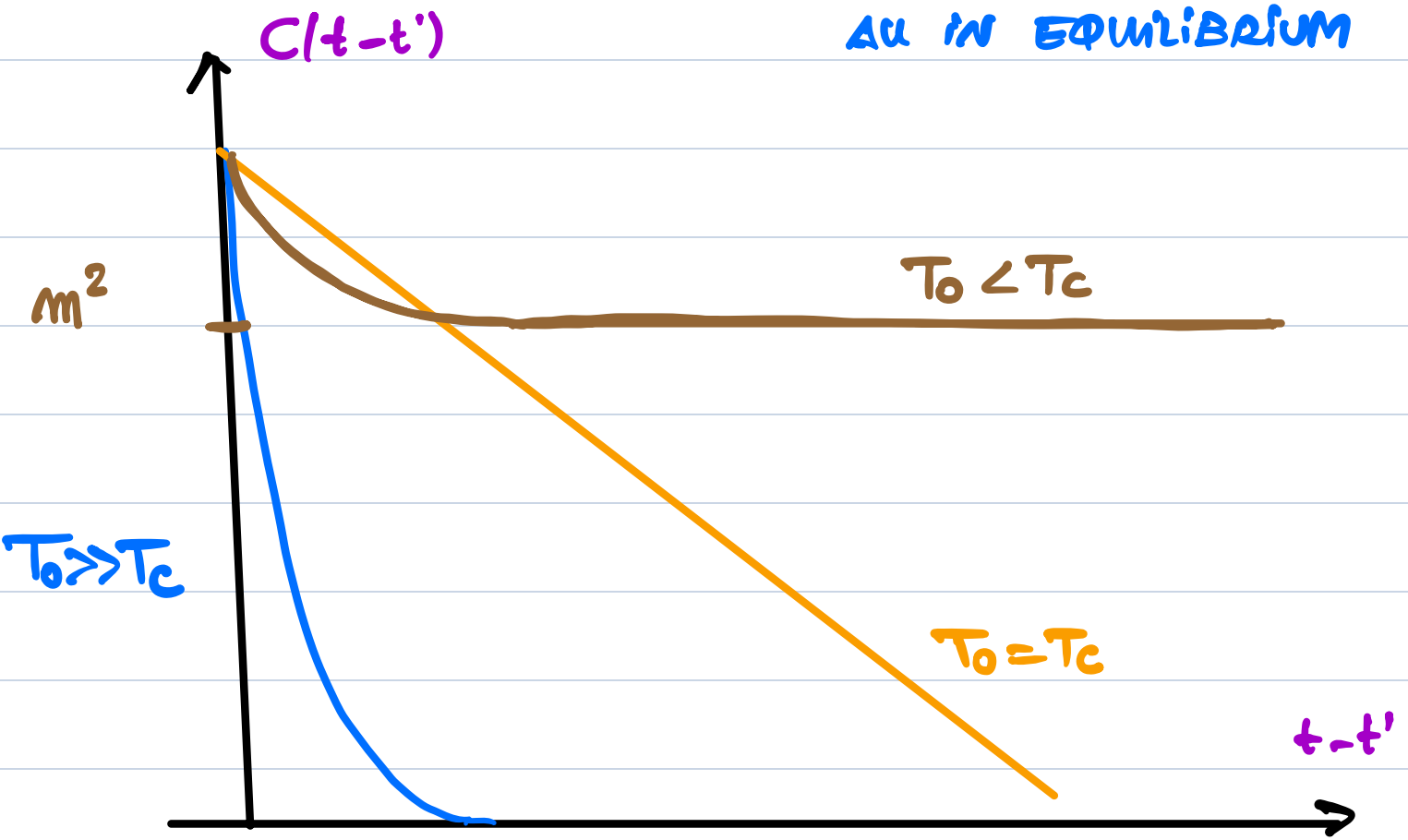
$$T_0 = T_c$$



$$T_0 < T_c$$

DECORRELATION AT THE SAME TEMP. T_0

ALL IN EQUILIBRIUM



INSTANTANEOUS QUENCH

RANDOM ic. $s_i(0) = \pm 1$ prob = $1/2$ or

$\phi(\vec{x}; t)$ GAUSSIAN pdf ZERO MEAN $T_0 \rightarrow \infty$

AT TIME $t=0$ A PARAMETER IS SUDDENLY CHANGED

$T_0 \rightarrow \infty \longrightarrow T$ BATH

$J_0 = 0 \longrightarrow J$ IN POTENTIAL
 $\Gamma_0 > 0 \longrightarrow \Gamma < 0$ ENERGY

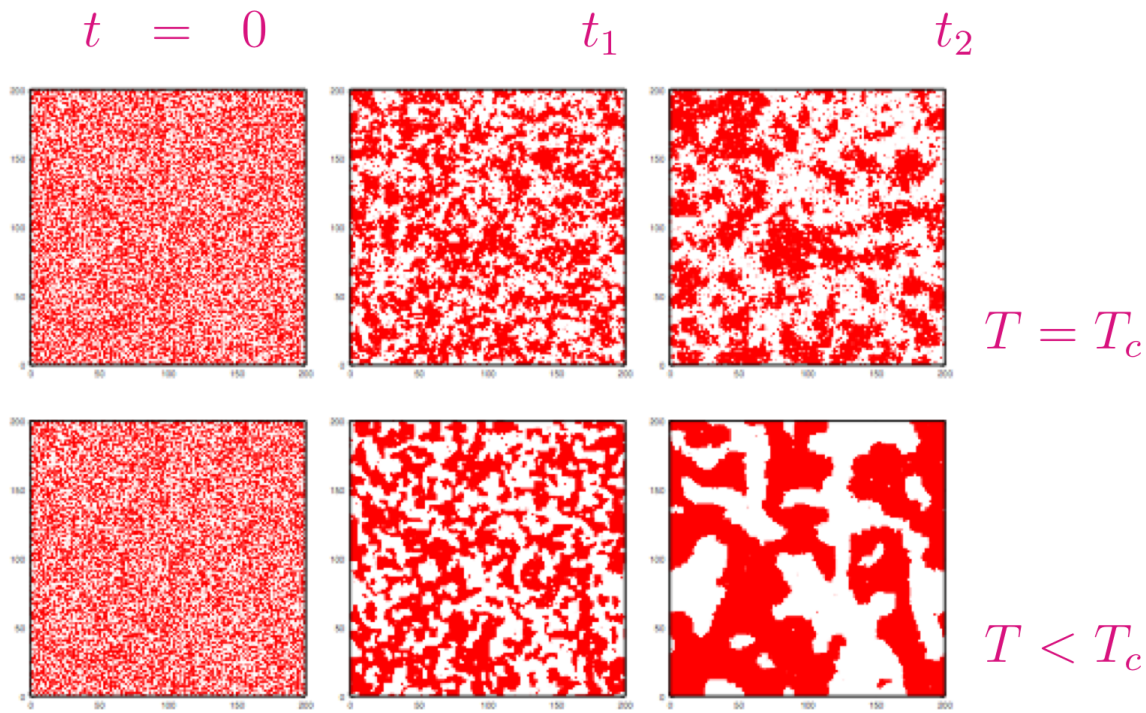
COULD SET THE EVOLUTION EXACTLY AT CRITICALITY
OR IN THE SYMMETRY BROKEN PHASE

BOTH J AND $\nabla^2 \phi$

PENALIZE CHANGES IN $\{s_i\}$ OR ϕ NEARBY

MECHANISM $\longrightarrow$ ERASE WALLS $\Rightarrow$ DECREASE ENERGY

CRITICAL DYNAMICS & SUB-CRIT COARSENING



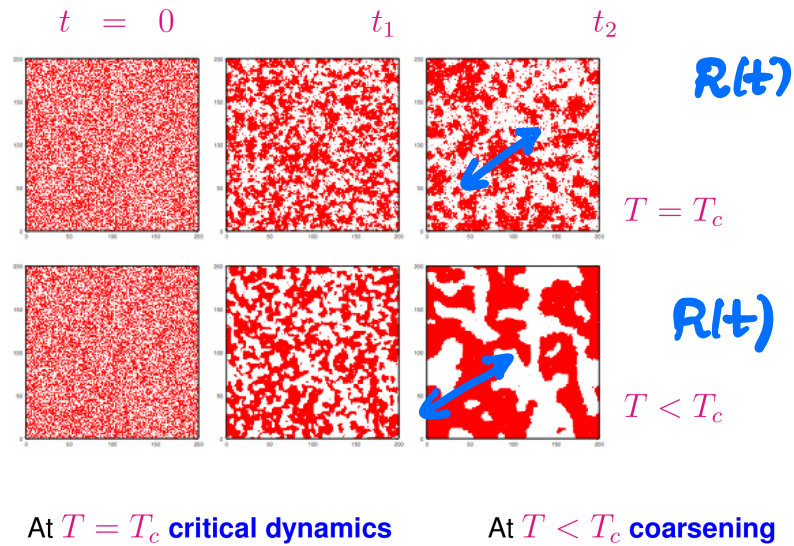
At $T = T_c$ critical dynamics

At $T < T_c$ coarsening

GROWTH of CRIT STRUCT AT T_c

GROWTH OF DOMAINS BELOW T_c

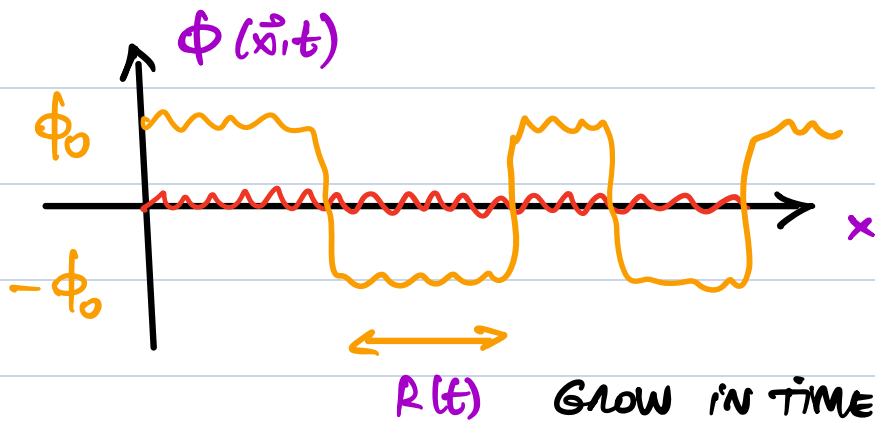
GROWING LENGTH



JUST VISUALLY ONE SEES THE STRUCT.
GROWING IN TIME.

ONE CAN MAKE THE ANALYSIS QUANTITAT.

→ FM ORDERING BELOW A CRIT. POINT, PHASE WITH
SPONT SYMM BREAK.



AND CRIT. FLUCT AT CRIT. POINT ⇒

- GROWTH OF CRIT EQUIL REGIONS AT CRIT POINT
- " " ORDERED DOMAINS IN ORD. PHASE

SIZE OF FLUCT DEPEND ON TEMP. T

THE EXCESS ENERGY WRT EQUIL ONE IS
CONCENTRATED ON THE INTERFACES

FROM t -DEP RG, SIMU :

$$R_c(t) \sim t^{1/z_c} \quad \text{HOFFENBERG - HALPERIN} \quad z_c \gtrsim 2 \quad \text{CRITICALITY}$$

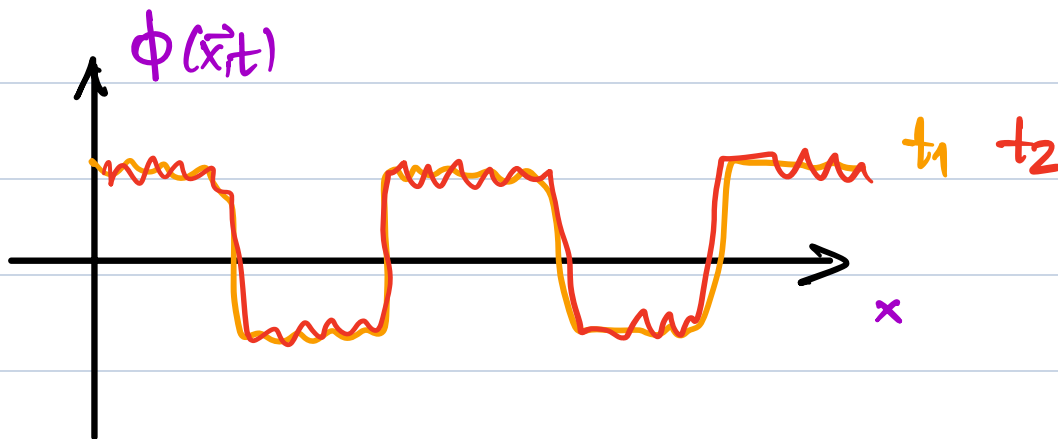
FROM SCALING ARGUMENTS, SIMU, APPROX :

$R(t) \sim t^{1/2}$ $z=2$ BELOW CRIT, SYMM BROKEN

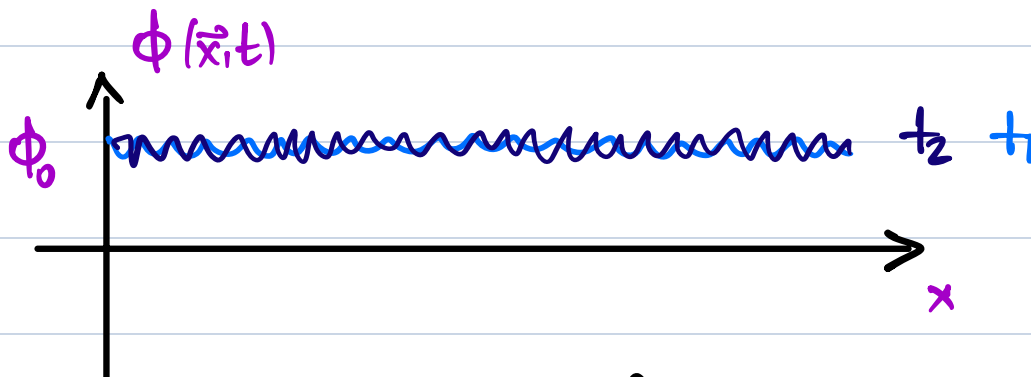
ALLEN CAHN (70's) BRAY (90's)

PHENOMENON

IF ONE COMPARES THE SYST CONFS AT TWO TIMES
SUCH THAT THE WALLS HAVE NOT MOVED $\Rightarrow$
ONLY THERMAL FLUCT WITHIN DOMAINS



AS IN EQUIL DYN THE WALLS ARE "UNSEEN"



THERMAL FLUCT. IN AN ORDERED SYST

QUANTITATIVE MEASUREMENTS

ORDER PARAMETER

$$m(t) = \frac{1}{N} \sum_{i=1}^N \langle s_i(t) \rangle = 0$$

TWO REASONS:

THERMAL Σ

SPATIAL AVER.

$$\tilde{\phi}(t) = \frac{1}{V} \int d^d r \langle \phi(\vec{r}, t) \rangle = 0$$

USELESS!

SELF-CORRELATION FUNCTION

$$C(t_1, t_2) \equiv \frac{1}{N} \sum_{i=1}^N \langle s_i(t_1) s_i(t_2) \rangle$$

AVERAGE OVER THERMAL NOISE AND/OR i.c.

$$C(t_1, t_2) = \frac{1}{V} \int d^d r \langle \phi(\vec{r}, t_1) \phi(\vec{r}, t_2) \rangle$$

LOCAL IN SPACE, NON-LOCAL IN TIME INTERESTING

DISTINGUISH TIMES t_1, t_2

t_2

SUFFICIENTLY LARGE

TO DEVELOP DOMAINS

$$\lim_{\underbrace{t_2 \rightarrow \infty}} C(t_1, t_2)$$

IN PRACTICE MEANS "LARGE"

$t_1 - t_2$ "FINITE" WRT t_2

SO THAT

DOMAINS DON'T MOVE

THEN,

$$\lim_{t_2 \rightarrow \infty} C(t_1, t_2) = C_{eq}(t_1 - t_2)$$

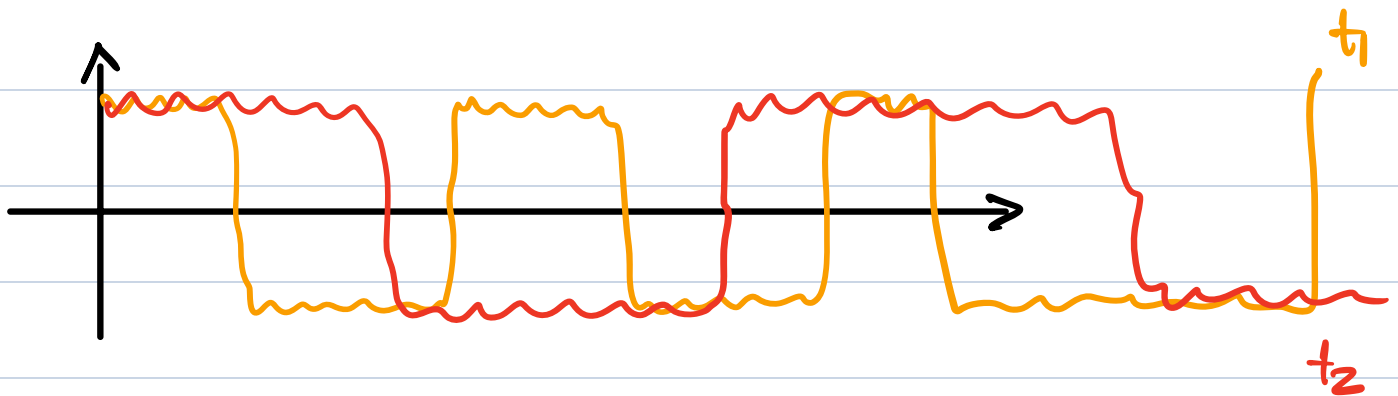
NB. IT'S STATIONARY

IF, ONE FURTHER TAKES $t_1 - t_2 \rightarrow \infty$

$$\lim_{t_1 - t_2 \rightarrow \infty} \lim_{t_2 \rightarrow \infty} C(t_1, t_2) = \phi_0^2$$

$$\lim_{t_1 - t_2 \rightarrow \infty} C_{eq}(t_1 - t_2) = \phi_0^2$$

HOWEVER, IF ONE LET'S THE WALLS MOVE
THERE IS A FURTHER DECORRELATION



DYNAMIC SCALING IN THIS $t_1 \approx t_2$ LIMIT

SIMILAR TO EQUI SCALING

$\exists$ A SINGLE GROWING LENGTH $R(t)$ WHICH
FOR $a \ll R(t) \ll L$ SCALERS ALL CORREL. FCT

$$\lim_{\substack{t_1, t_2 \rightarrow \infty \\ R(t_1)/R(t_2) \text{ FINITE}}} C(t_1, t_2) = C_{eq}(t_1, t_2) = f_c \left(\frac{R(t_1)}{R(t_2)} \right)$$

it's NOT STATIONARY

AGING

$$\partial_{t_1} C_{ag}(t_1, t_2) = f'_c \left(\frac{R(t_1)}{R(t_2)} \right) \frac{\dot{R}(t_1)}{R(t_2)}$$

$$= f'_c \left(\frac{R(t_1)}{R(t_2)} \right) \frac{R(t_1)}{R(t_2)} \frac{\dot{R}(t_1)}{R(t_1)}$$

$$g \left(\frac{R(t_1)}{R(t_2)} \right) \underbrace{\frac{d \ln R(t_1)}{dt_1}}_{\downarrow}$$

$$\text{if } R(t_1) \sim t_1^{1/2} \Rightarrow \frac{1}{2} \frac{1}{t_1} \xrightarrow{t_1 \rightarrow \infty} 0$$

- GIVEN t_2 FIXED, THE DERIVATIVE OF $C(t_1, t_2)$ WRT t_1 GETS SMALLER AND SMALLER

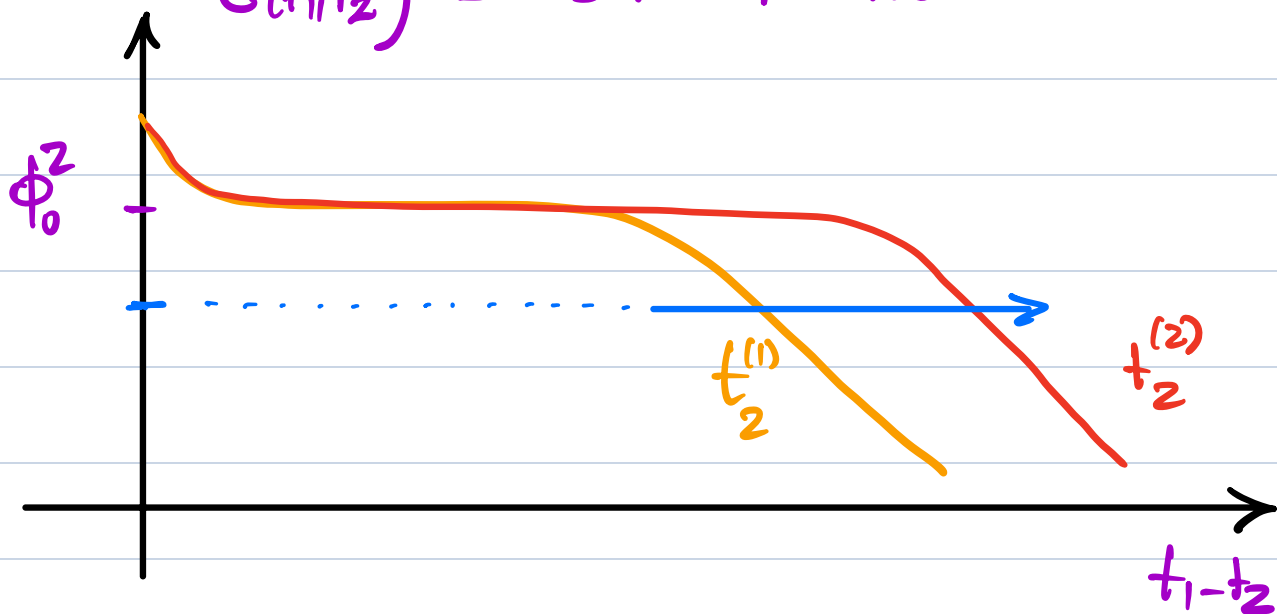
$$\partial_{t_2} C_{ag}(t_1, t_2) = f'_c \left(\frac{R(t_1)}{R(t_2)} \right) \frac{R(t_1)}{R^2(t_2)} \dot{R}(t_2)$$

$$= g \left(\frac{R(t_1)}{R(t_2)} \right) \frac{d \ln R(t_2)}{dt_2}$$

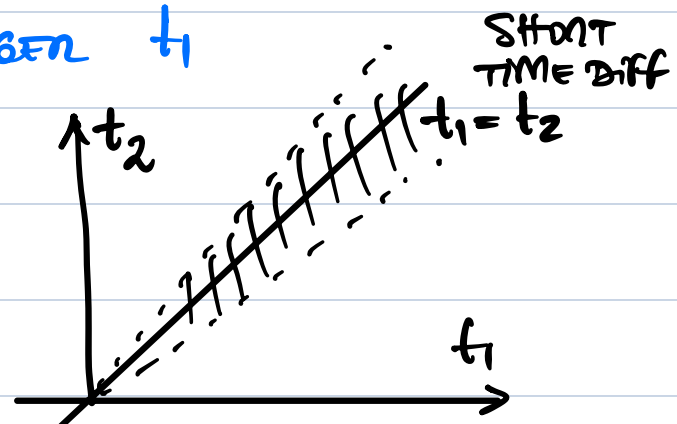
if $R|t_2| \sim t_2^{1/2} \Rightarrow \frac{1}{2} \frac{1}{t_2} \xrightarrow{t_2 \rightarrow \infty} 0$

- GIVEN t_1 FIXED, THE DERIVATIVE OF $C(t_1, t_2)$ WRT t_2 GETS SMALLER AND SMALLER

$C(t_1, t_2) = \text{SUM OF TWO CONTRIB.}$



TO REACH THE SAME C VALUE BELOW ϕ_0^2 ONE NEEDS TO GO TO A LONGER t_1 FOR t_2 LONGER.



A WAY TO WRITE IT

$$C(t_1, t_2) = C_{\text{STAT}}(t_1 - t_2) + C_{\text{eq}}(t_1, t_2)$$

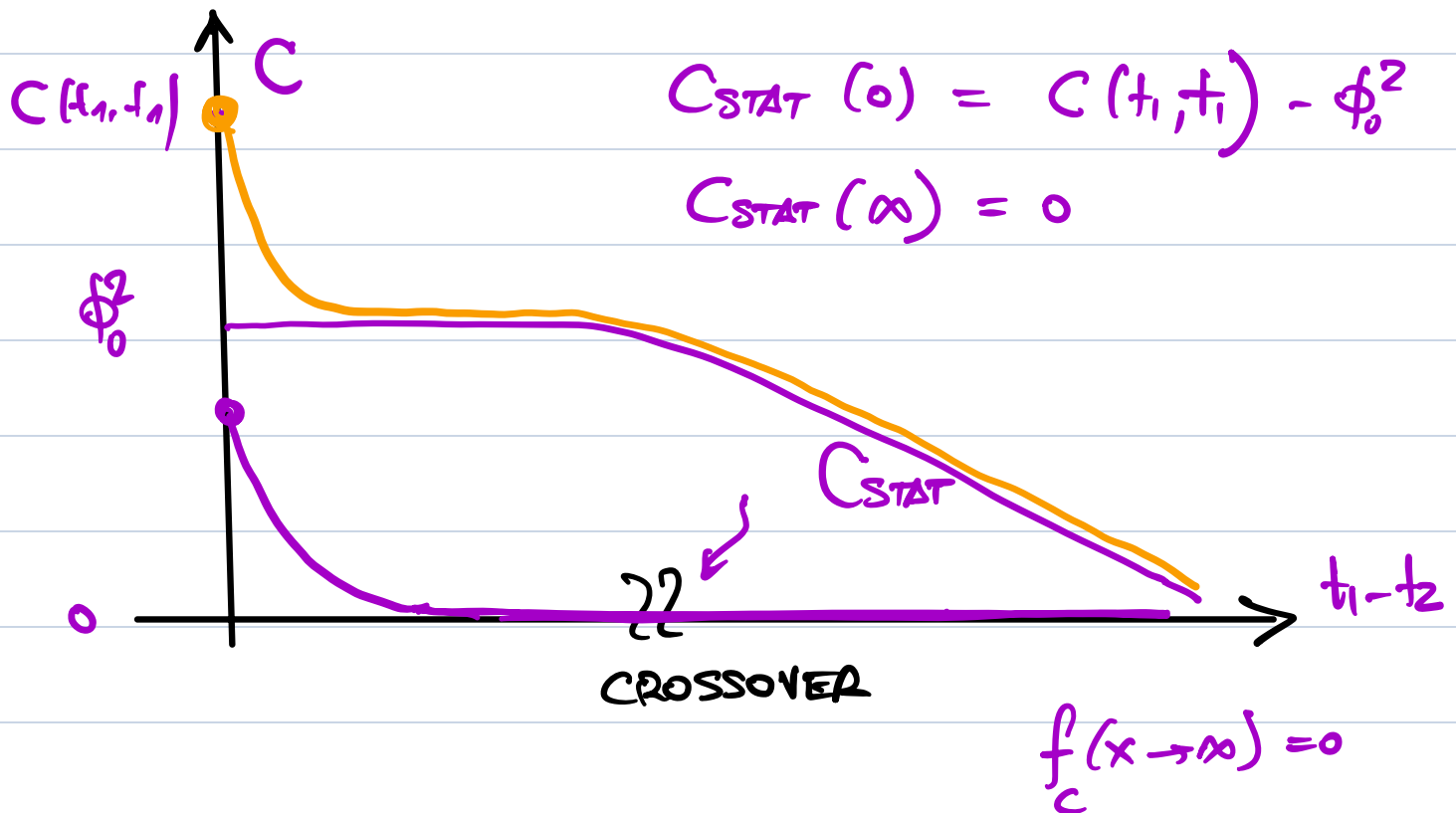
$$\lim_{t_1 - t_2 \rightarrow \infty} C_{\text{STAT}}(t_1 - t_2) = 0$$

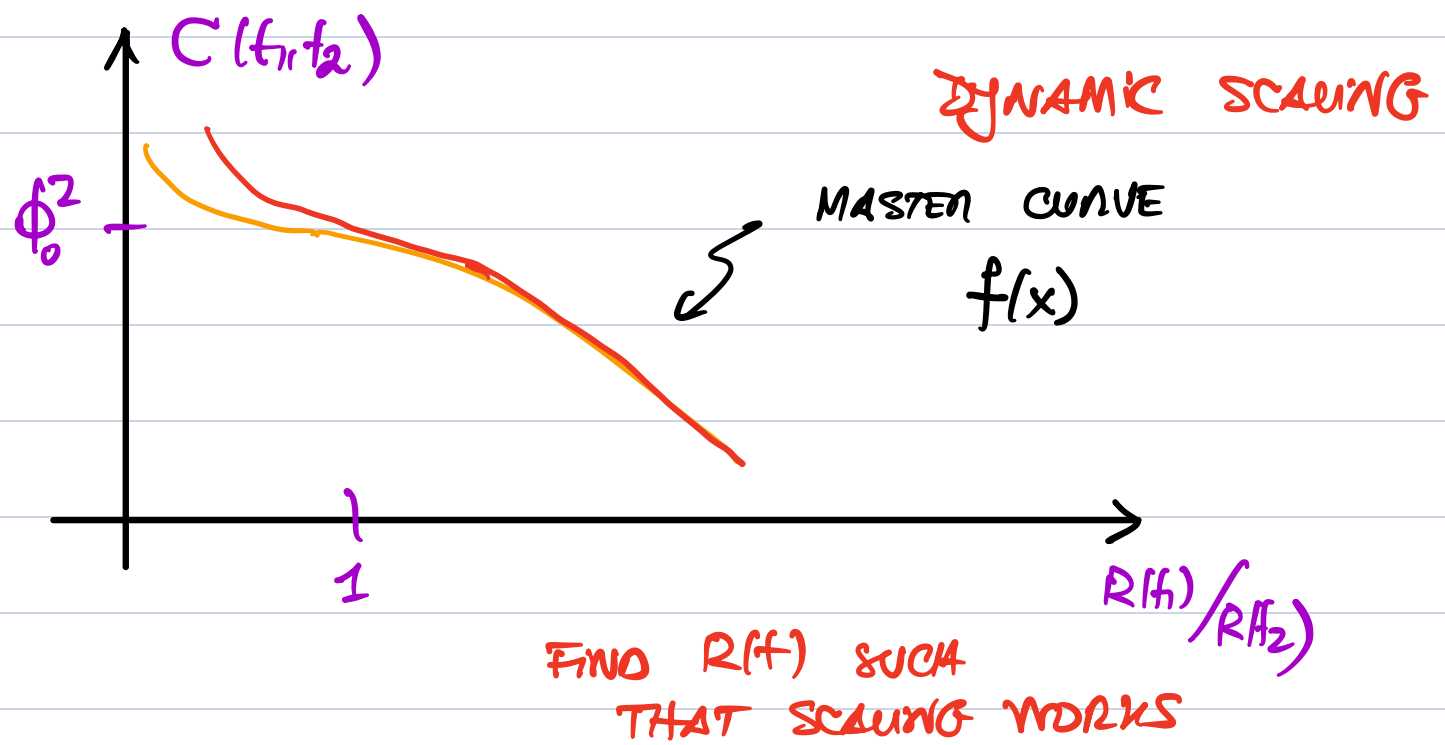
TWO
POSITIVE

$$\lim_{t_2 \rightarrow t_1^-} C_{\text{eq}}(t_1, t_2) = \phi_0^2$$

CONTRIB
IN FULLY
RELAX.
SYST.

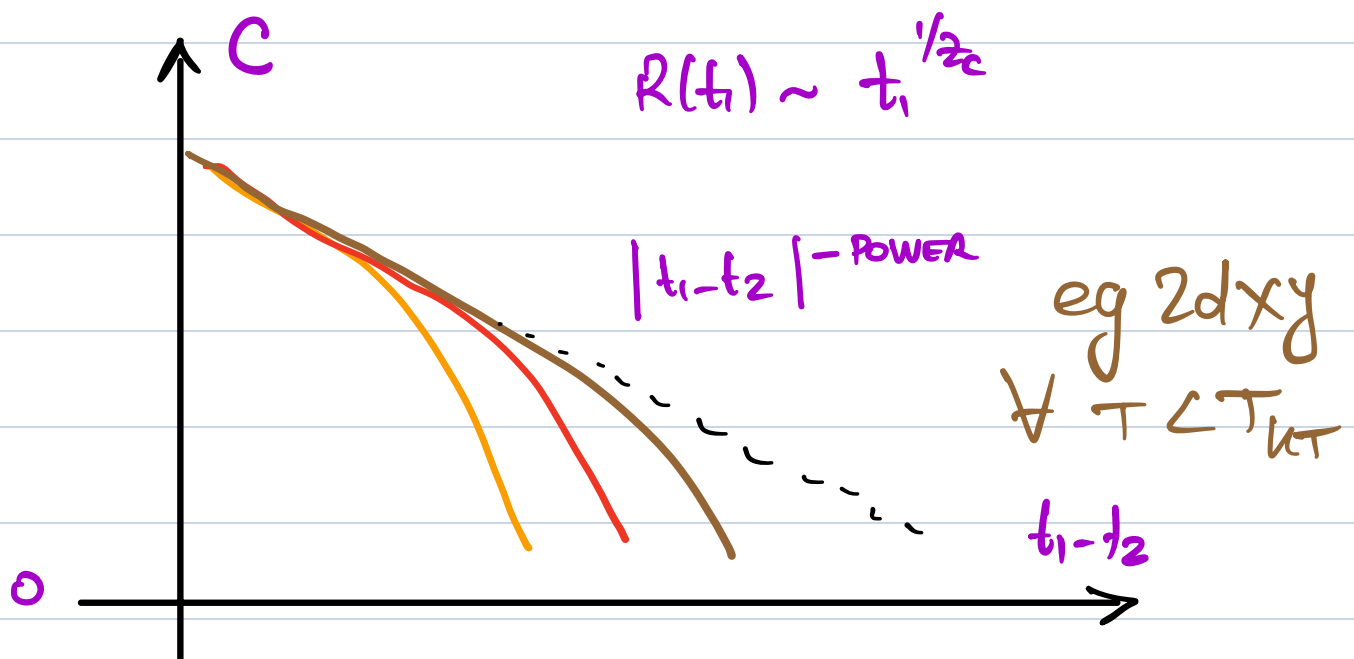
$$f_c(1) = \phi_0^2$$





QUENCHES AT CRITICAL POINT. WHAT IS DIFF.?

SINCE $\phi_0 = 0$ THE APPROACH TO THE PLATEAU IS REPLACED BY A POWER LAW DECAY TOWARDS ZERO.



CONCEPTS

- CO-EXISTENCE OF FAST (VELOCITIES) AND SLOW (POSITIONS) VARIABLES
- SEPARATION OF TIME SCALES FOR SLOW VARIABLES IN MACROSC. SYST.
- AGING OF SLOW VARIABLES
- DYN. SCALING: (BEYOND EQUIL. FLUCT.)
A SINGLE GROWING LENGTH $R(t)$
SCALING ALL CORREL. FUNCT.
ALSO OBTAINED FROM $C(r, t)$

VERY NICE TESTS OF SCALING IN
RECENT EXPS

DE LAMERDA & TAKEUCHI

ALL THESE QUALITATIVE FEATURES ARE
SHARED BY GLASSY SYST.

APART FROM

- $R(t)$ MAYBE NOT UNIQUE $\Rightarrow$
- MANY REGIMES $\&$
NOT CLEAR TO "SEE" $R(t)$

AND, OF COURSE, THE QUANTITATIVE
ONES

eg. $R(t)$ DOESN'T NEED TO BE $t^{1/2}$

BACK TO QUENCHES BELOW CRIT POINT.

THINK IN TERMS OF **WAVE-VECTORS**

UNIFORM ORDERING $\Rightarrow \phi(\vec{r}, t) = \phi_0$ (APART FLUCT.)

IN FOURIER MEANS $\tilde{\phi}(\vec{k}, t) = \phi_0 \delta_{\vec{k}, 0}$

(ALGEBRAIC) GROWTH OF ORDER MEANS

(ALGEBRAIC) GROWTH OF $\vec{k}=0$ MODE $\langle \tilde{\phi}(\vec{k}=0, t) \rangle$

MEAN-FIELD MODEL

SPHERICAL $p=2$

$$H = -\frac{1}{2} \sum_{i \neq j} J_{ij} s_i s_j = -\frac{1}{2} \sum_{\mu} \lambda_{\mu} x_{\mu}^2$$

+ CONSTRAINT

MEAN-FIELD IN THE SENSE THAT FULLY CONNECTED

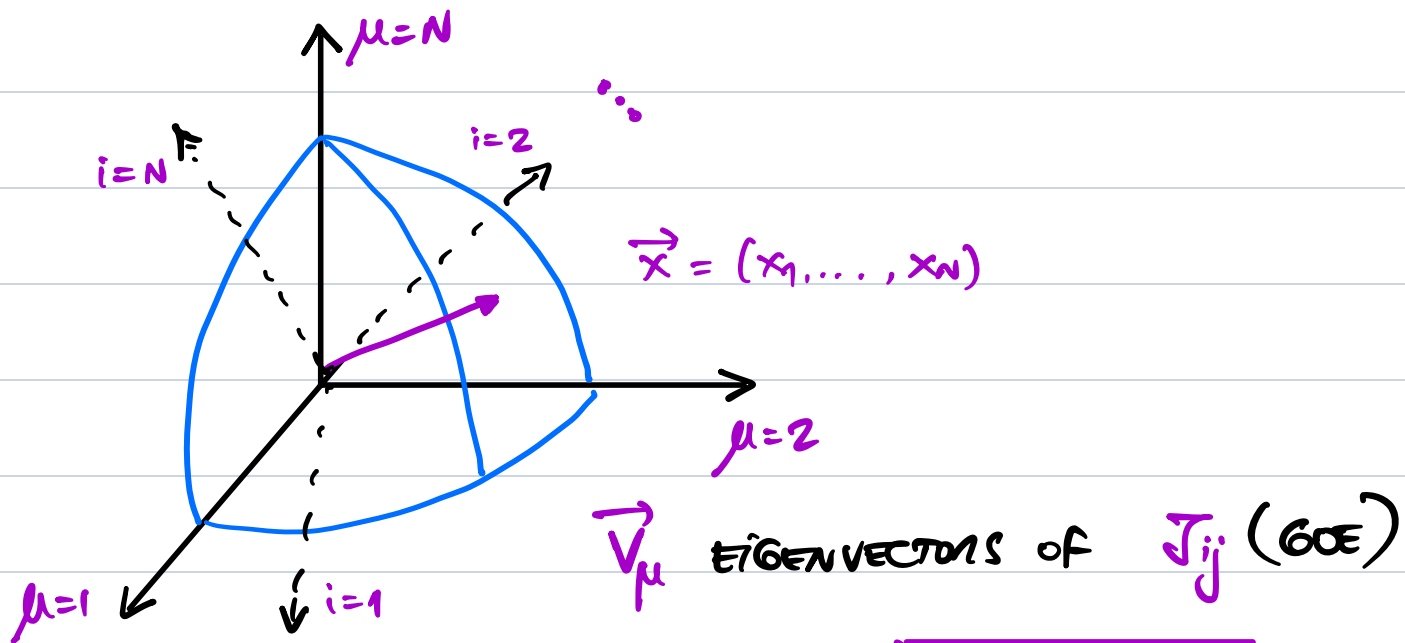
(ALL i INTERACT WITH ALL OTHER j , ETC.)

COMING FROM SPIN-GLASS THEORY

$s_i \in \mathbb{R}$

NEXT GLASS RPT

NOW MUCH MORE



$$\rho(\lambda_\mu) = \frac{1}{\sqrt{2\pi J^2}} \sqrt{4J^2 - \lambda_\mu^2}$$

EIGENVALUES

$$x_\mu = \vec{S} \cdot \vec{V}_\mu$$

$$\langle x_\mu \rangle_{eq} = 0 \quad \forall \mu \quad T \geq T_c$$

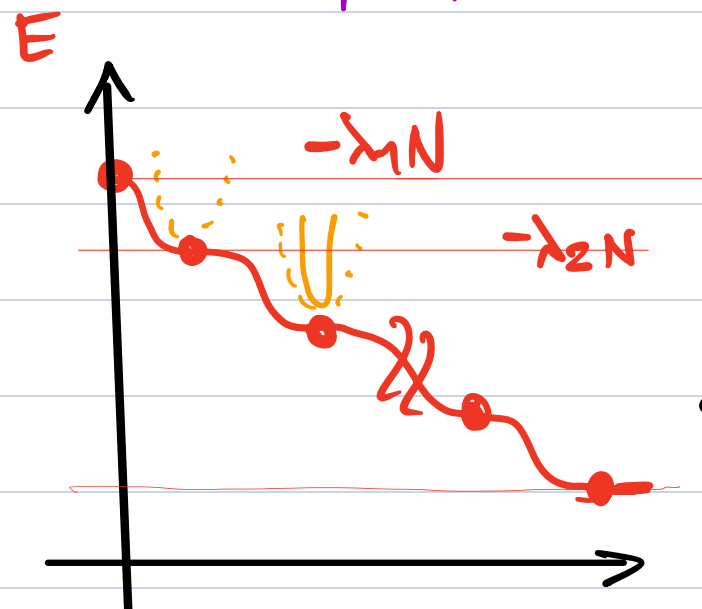
ENERGY LANDSCAPE

$$\langle x_N \rangle_{eq} = \sqrt{(1 - T/J)N} \quad T < T_c$$

$$\vec{x} = \vec{V}_\mu \quad \mu = 1 \dots N$$

STAT. STATES

(BEC)



$$\vec{V}_{\mu=N} = \text{MAX}$$

ABS. MAX

$$\vec{V}_{\mu=N-1}$$

...

← 1st EXC. W/ ONE UNST DIR

$$\vec{V}_1 - \lambda_N N$$

ABS. MINIMUM

LANGERIN SYN

ALMOST INDEP HARMONIC
OSCILLATIONS

$$\ddot{x}_\mu = -\lambda_\mu x_\mu + z_t x_\mu + s_\mu$$

z_t FIXED FROM $\frac{1}{N} \sum_\mu \langle x_\mu^2(t) \rangle = 1$

SOLVABLE ANALYTICALLY $x_\mu(t; z_t)$

$\langle x_{\mu_{\max}} \rangle$ GROWS AS A POWER LAW t^{POWER}

WHILE ALL OTHER MODES TEND TO VANISH EXP.

$$\langle x_\mu \rangle \sim t^{\text{POWER}} e^{-(\lambda_n - \lambda_\mu)t}$$

$$\lambda_n \geq \lambda_\mu$$

HOMOGENEOUS

$k=0$ MODE

GROWS IN

TIME $\equiv$

$\langle x_n \rangle$ GROWS
IN TIME

PROJECTION ON N-th DIRECTION

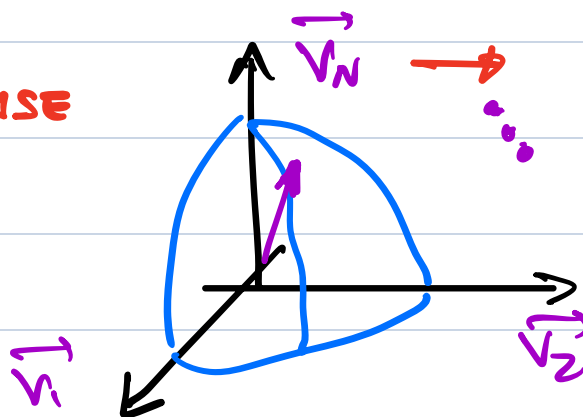
ALL OTHER MODES DECREASE
IN TIME EXP.

HOWEVER

$$\langle x_n \rangle(t) \sim t^{\text{POWER}} \sim N^{1/2} \Rightarrow$$

FOR EQ

$$t_{eq} = N^{1/\text{POWER}}$$



$C(t_1, t_2) \longrightarrow$ SUM OF STAT / EQUIV CONT +
AGING TERM

$$C_{\text{STAT}}(t_1 - t_2) + \underbrace{C_{\text{AG}}(t_1, t_2)}$$

$$f_c(t_1/t_2)$$

$R(t) \sim t^{1/2}$ ALL THIS FOUND ANALYTICALLY
EVEN SCALING

NOTE THERE IS NO SPACE IN THIS PROBLEM

CAN'T REALLY FIX THE POWER LOOKING AT $C(t, t')$

BUT IT DOESN'T MATTER.

IT'S A POWER.

LFC & DEAN '95

A BETTER DEF $N\phi^4$ IN LARGE N LIMIT
BRAY '94 SOME SPATIAL INFO

$$\phi(\vec{r}, t)$$

WHY IS THIS MODEL OUT OF EQUILIBRIUM ?

$$\gamma_0 \dot{x}_\mu = + \lambda_\mu x_\mu - z(t) x_\mu + \xi_\mu$$

AT $T < T_c$ $z(t) \rightarrow \lambda_N$

SO ALL MODES WITH $\lambda_\mu \neq \lambda_N$
HAVE A HARMONIC POT

$$(\lambda_N - \lambda_\mu) \frac{x_\mu^2}{2} = V_\mu$$

BUT THE N-th MODE DOESN'T !

$$\lambda_N - \lambda_N = 0$$

FLAT DIRECTION. THIS ONE CANNOT EQUIL.
RECALL RANDOM WALK

$$V(x) = \frac{kx^2}{2} \quad \text{FOR } k \rightarrow 0; \quad t^{(x)} \rightarrow \infty$$

MECHANISM FOR NON-EO DYN $\Rightarrow$ ZERO MODE.

$T^* \rightarrow \infty$ CAN BE CALCULATED.

HOW CAN THE SYST EQUIL ?

$t(N)$ SCALE TIME WITH N
AND THE TAKE $N \rightarrow \infty$

OR IN PRACTICE $N \gg 1$

D. BARBIER, P. FREITAS, LFC &
D. STAZIOLO

WHAT ABOUT THE NEWTONIAN DYN OF
THIS SAME MODEL?

RECALL QUANTUM QUENCHES

$|\psi_0\rangle$ EVOLVE WITH $\hat{A}$ ($\neq \hat{H}_0$)

$|\psi(t)\rangle$ WHAT HAPPENS?



DOES THE REST ACT AS A BATH
ON S?

IN WHICH CASES?

NON INTEGRABLE

VS.

INTEGRABLE



SHOULD



?

RESTRICTED BOLZMANN

GBE

$$\frac{e^{-\beta \hat{H}_{\text{sys}}}}{\mathcal{Z}}$$

$$\frac{e^{-\sum_{\mu} \gamma_{\mu} \hat{I}_{\mu}}}{\mathcal{Z}(\{\gamma_{\mu}\})}$$

WHY NOT ASK THE SAME QUESTION CLASSICALLY?

WE CAN CONSIDER THE SAME KIND OF MODELS
THAT YIELD GLASSY DYN WHEN OPEN,
NOW CLOSED.

$$H_J[\{x_\mu\}] = - \sum_{\mu \neq \nu \neq \rho} J_{\mu\nu\rho} x_\mu x_\nu x_\rho \\ + \frac{1}{2} \left(\sum_{\mu} x_\mu^2 - N \right)$$

$$[J_{\mu\nu\rho}] = 0$$

$$[J_{\mu\nu\rho}^2] \propto \frac{J^2}{N^2}$$

WE KNOW THAT
WHEN COUPLED TO
WHITE NOISE BATH



GLASSY BELOW T_d

GIVE THEM NEWTON DYN EXPECTED

NON INTEGRABLE

$\mathcal{P}_0(\{x_\mu(0), p_\mu(0)\})$?

DRAW i.c. FROM $\mathcal{P}_0$

eg. MAXWELL-BOLTZMANN AT T_0

EVOLVE w/ H ($\neq H_0$) : A QUENCH

eg

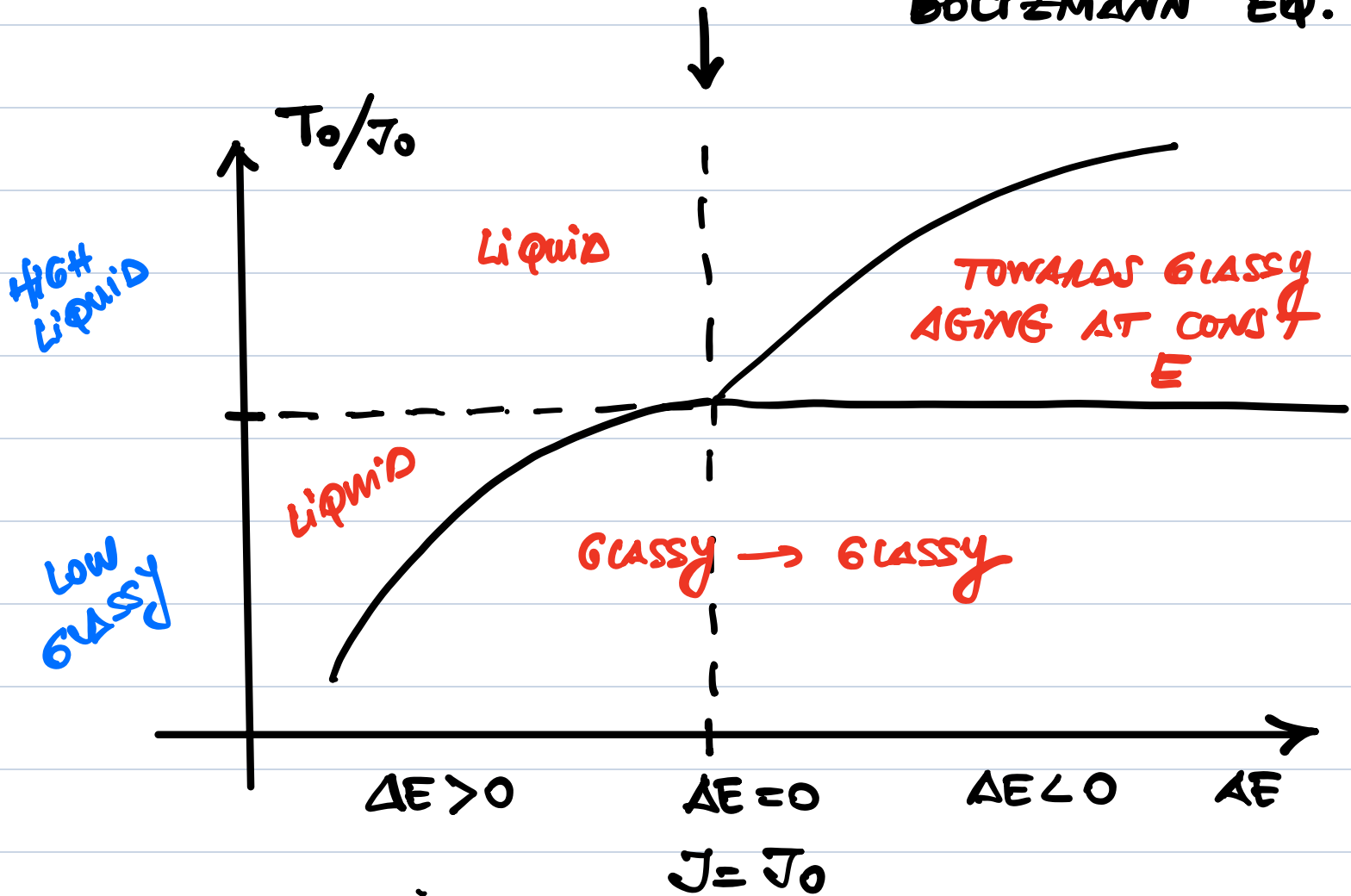
$J_0 \rightarrow J$

IN INTERNAL PARAM.

IT'S WHAT QUANTUM PEOPLE DO.

- WHAT HAPPENS ?

NORMAL EQUIL. MAXWELL
BOLTZMANN EQ.



THE NEXT CALC. IS ABOUT FDT IN
CLOSED INT SYST (NOT DONE)

$$H = \sum_a^{N_0} \frac{m\omega_a^2}{2} x_a^2 + \sum_a^{N_0} \frac{p_a^2}{2m} \quad \left(-\hbar \sum_a x_a\right)$$

UNCOUPLED HARM OSC. $\Rightarrow$ INTEGRABLE!

N_0 CONST OF MOTION, THE N_0 ENERGIES OF EACH OSCILLATOR

$$E_a(0) = \frac{m\omega_a^2}{2} x_a^2(0) + \frac{p_a^2(0)}{2m} = \text{CONST.}$$

SOLUTION TO NEWTON EQS.

$$m \ddot{x}_a = -m\omega_a^2 x_a + f_a \quad \leftarrow \begin{array}{l} \text{FORCING} \\ \text{TO CALCULATE} \\ \text{LINEAR RESPONSE} \end{array}$$

$$x_a(t) = \text{i.c. DEP}_a + \int_0^t dt' \underbrace{G_a(t-t')}_{\frac{\sin \omega_a(t-t')}{m\omega_a}} f_a(t')$$

$$R_a(t, t') = G_a(t-t') \quad \leftarrow$$

$$\overline{C_a(t, t')} = \frac{1}{2m\omega_0^2} \underbrace{\left(m\omega_0^2 x_0 + \frac{p_0^2}{m\omega_0^2}\right)}_{2E_a(0)} \cos \omega_a(t-t')$$

AFTER AVERAGING OVER A TIME WINDOW
AROUND t' TO GET RID OF NON-STAT TERMS

CLASSICAL FDT IN FOURIER

$$\underbrace{\text{Im } \tilde{R}(\omega)} = \frac{1}{2k_B T} \underbrace{\omega \tilde{C}(\omega)}$$

WHAT DO WE GET HERE IF WE COMPARE ?

$$\text{Im } \tilde{R}(\omega) = \frac{\pi}{2m\omega_a} \delta(\omega - \omega_a)$$

$$\omega \tilde{C}(\omega) = \frac{\pi\omega}{2m\omega_a^2} z e_a(0) \delta(\omega - \omega_a)$$

$$\frac{2\text{Im } \tilde{R}(\omega)}{\omega \tilde{C}(\omega)} = \frac{1}{e_a(0)} = \frac{1}{k_B T_{\text{eff}}}$$

 AT $\omega = \omega_a$

AND IF WE CALCULATE β_{GBE} ?

$$P_{GGE}(\{x, p\}) = \frac{e^{-\beta_{GGE} H(\{x, p\})}}{\mathcal{Z}}$$

CONDITION

$$e(o) = \int dx \int dp P_{GGE}(\{x, p\}) H(\{x, p\})$$

$$\Rightarrow e(o) = \beta_{GGE}^{-1} = k_B T_{GGE}$$

SO WE GET

$$T_{eff} = T_{GGE}$$

IF WE REPEAT FOR AN ENSEMBLE OF OHC
w/ $\neq \omega$ WE'LL GET

$$T_{eff}(\omega) = T_{GGE}(\omega)$$

AND ENSEMBLE OF OHC IS INTEGRABLE

NB ω "LABELS" THE CONS QUANT $\Sigma_o(\omega)$

OBSERVED QUANTUM FOR TRANSV ISING
CHAIN $\Rightarrow$ MAPPABLE TO FREE FERMIONS

A NON TRIVIAL CLASSICAL INTEGRABLE MODEL
WITH A GGE DESCRIPTION

$$\dot{x}_\mu = p_\mu / m$$

NEWTON/
HAMILTON

$$\dot{p}_\mu = \lambda_\mu x_\mu - \mathbb{Z}(\vec{x}, \vec{p}) x_\mu$$

WITH $\sum_\mu x_\mu^2 = N$

SAY $\lambda_1, \lambda_2 \dots \lambda_N$

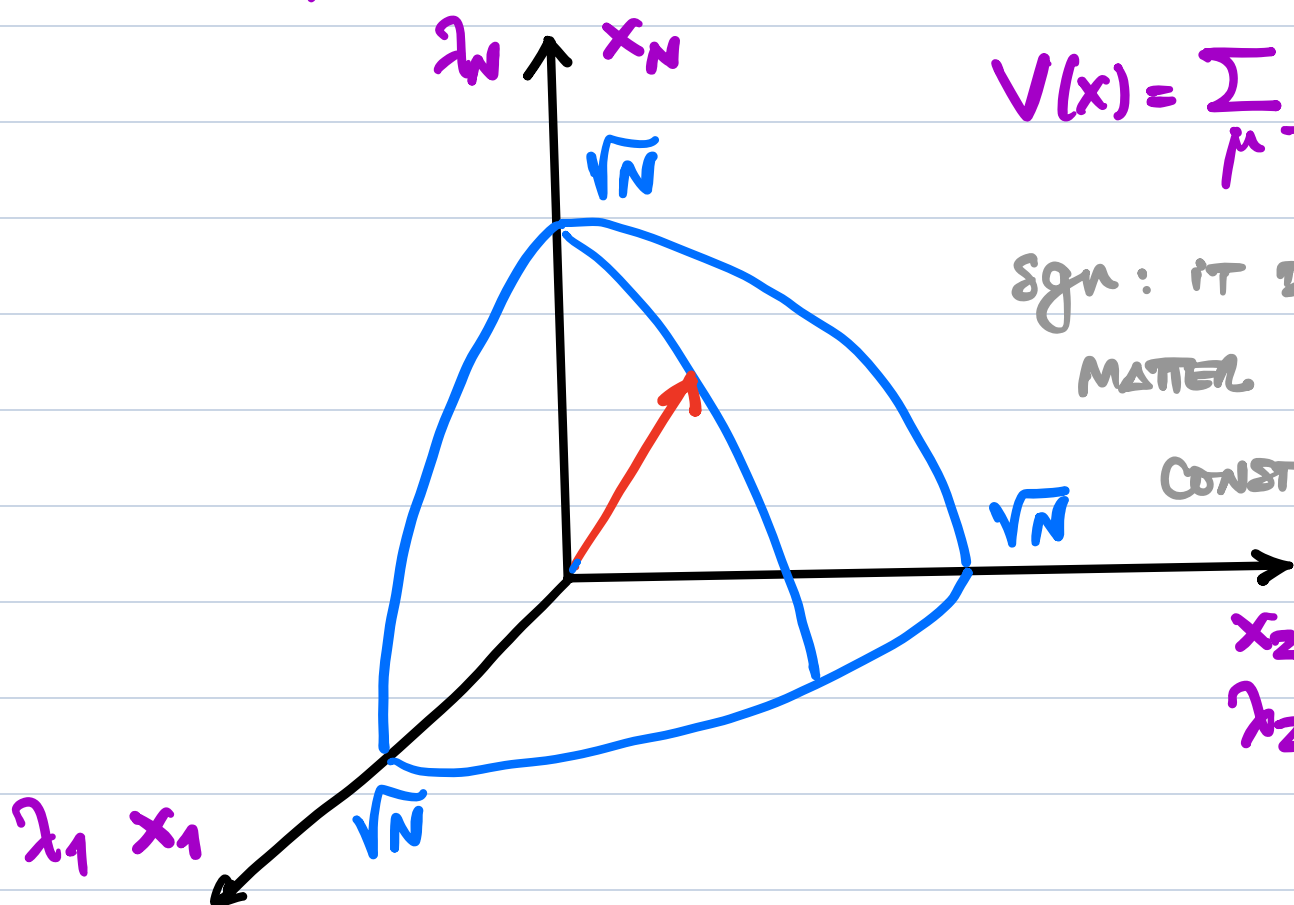
$$V(x) = \sum_\mu \frac{\lambda_\mu}{2} x_\mu^2$$

Sgn: IT DOESN'T
MATTER $\lambda_\mu \geq 0$

CONSTRAINT

BOUND

MOTION



$$\mathcal{Z}(\vec{x}, \vec{p}) = \frac{p^2}{2m} - V(x) = \frac{p^2}{2m} + \sum_{\mu} \frac{\lambda_{\mu}}{2} x_{\mu}^2$$

THIS IS NOT A LINEAR SET OF EQ
IT IS NOT A SET OF INDEP. HARM. OSC.

INTEGRABLE MODEL

$$I_{\mu} = x_{\mu}^2 + \frac{1}{mN} \sum_{\nu \neq \mu} \frac{(x_{\mu} p_{\nu} - x_{\nu} p_{\mu})^2}{\lambda_{\nu} - \lambda_{\mu}}$$

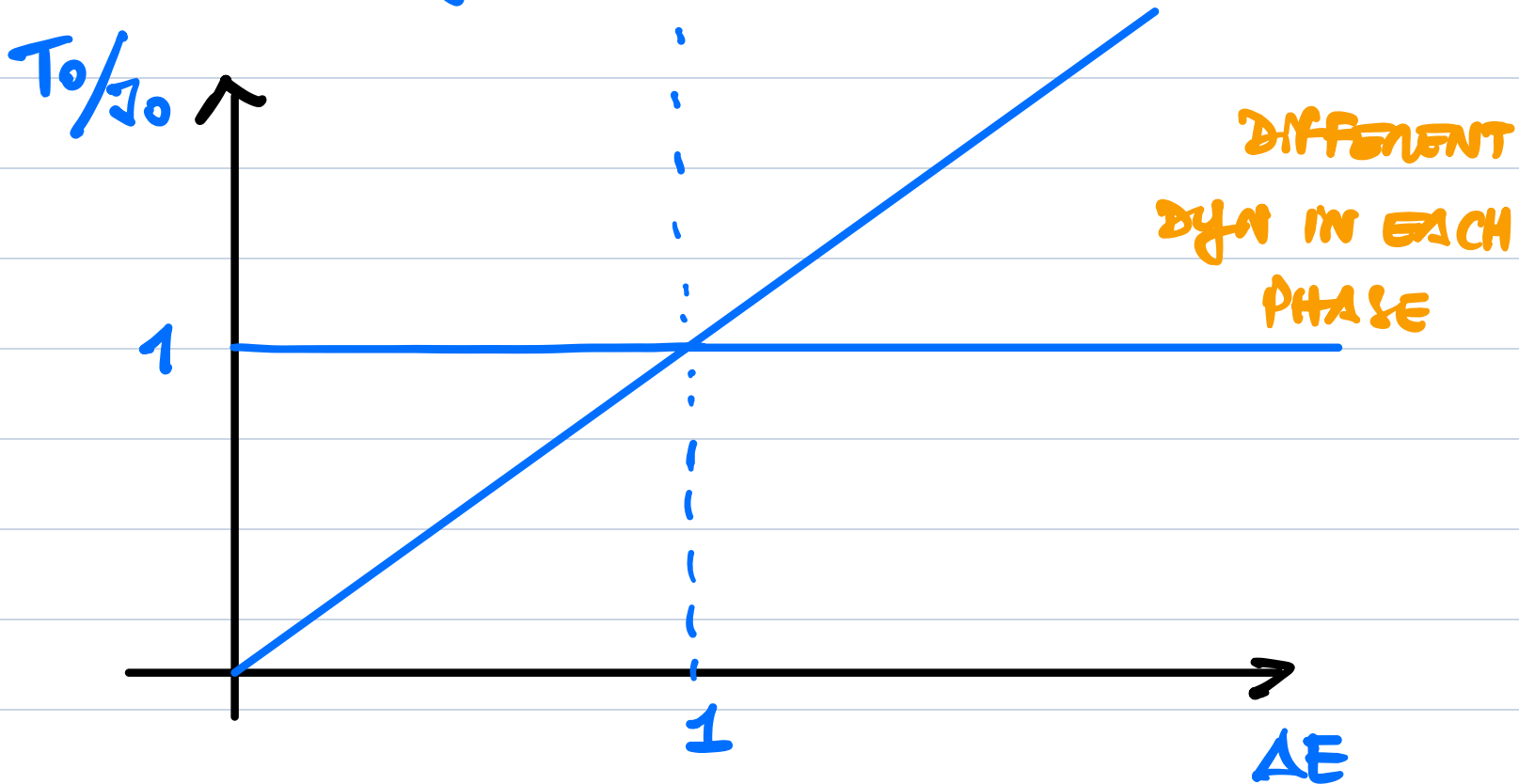
$$\frac{dI_{\mu}}{dt} = 0$$

WHAT HAPPENS UNDER QUENCHES?

$$P_0(\vec{x}, \vec{p}) = \frac{\exp \{-\beta_0 H_0(\vec{x}, \vec{p})\}}{\mathcal{Z}}$$

EVOLVE WITH H $\{ \lambda_{\mu}^{(0)} \rightarrow \lambda_{\mu} \}$

SOLVE DYN



J_0 SCALE OF $\{x_{\mu}^{(0)}\}$

$\Delta E = J/J_0$ J SCALE OF $\{x_{\mu}\}$

$$\lim_{t \rightarrow \infty} \overline{x_{\mu}^2(t)} \xleftarrow{\text{TIME AVERAGE}} = \langle x_{\mu}^2 \rangle_{\text{GBE AVERAGE}}$$

$\forall \mu$ ALSO PA!

$$P_{66E}(\vec{x}, \vec{p}) = \frac{e^{-\sum_{\mu} \gamma_{\mu} I_{\mu}}}{Z}$$

γ_{μ} FIXED FROM

$$\langle I_{\mu} \rangle_{66E} = I_{\mu}(0) \quad \forall \mu$$

SO, THE LONG TIMES LIMIT APPROACHES
THE 66E

WHAT ABOUT GEN. FDT ?

$$\frac{2 \operatorname{Im} \tilde{A}(\omega)}{\omega \tilde{C}(\omega)} = \beta_{\text{eff}}(\omega) = \frac{1}{k_B T_{\text{eff}}(\omega)}$$

CALCULATE IT DURING
DYNAMICS

AND EVALUATE IT AT

$$m \omega_{\mu}^2 = \left[\lim_{t \rightarrow \infty} Z(t) - \lambda_{\mu} \right] \geq 0 \quad \checkmark$$

EITHER FOR $\mu=1 \dots, N$ FINITE
OR FOR $N \rightarrow \infty$

COMPARE TO

$$m \langle p_\mu^2 \rangle_{GGE} = k_B T_\mu^{GGE}$$

IT WORKS ! IN ALL PHASES.

IN THE END THE MODES GET UNCOUPLED.

EACH W/ ITS OWN T_μ BUT FIXED BY
ALL THE OTHERS.

SOLVABLE - NON TRIVIAL EX FOR WHICH

GGE DESCRIPTION WORKS

PHASE TRANSITIONS & NON-TRIVIAL DYN

$T_{eff}(\omega) \rightarrow$ ALSO HAS A MEANING