Integrable systems' dynamics under noise & dissipation

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Ramin Golestanian & Etienne Fodor editors

e.g. HJ Kappen, *Path integrals and symmetry breaking for optimal control*

Plan

— Introduction : equilibrium, non-equilibrium & the goal

— Hamiltonian dynamics of a (simple) integrable model in the thermodynamic limit and the GGE measure

— *Control* stochastic evolution towards the GGE measure

— Summary

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Equilibrium Statistical Physics

Advantage

No need to solve the Newton dynamics!

Under the *ergodic hypothesis*, after some *equilibration time* $t_{\rm eq}$, a *macroscopic observable* A of a *macroscopic system* can be, on average, obtained with a *static calculation*, as an average over all configurations in phase space weighted with a probability distribution function $\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$

$$\langle A \rangle = \int \prod_i d\boldsymbol{p}_i d\boldsymbol{x}_i \; \rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \; A(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$$

$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\rm eq}}^{t_{\rm eq}+\tau} dt' \; A(\{\boldsymbol{p}_i(t'), \boldsymbol{x}_i(t')\})$

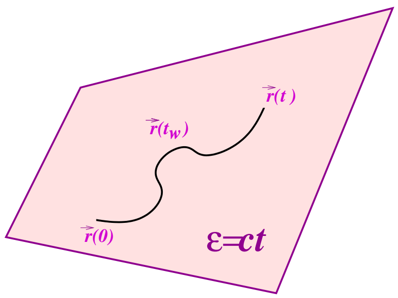
the *time average* typically measured experimentally

Ergodicity

Boltzmann, late XIX

Equilibrium Statistical Physics

Recipes for $\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$ according to circumstances



Microcanonical ensemble

$$\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto \delta(\mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) - \mathcal{E})$$

Flat probability density

Isolated system

$$\mathcal{E} = \mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct$$

$$S_{\mathcal{E}} = k_B \ln g(\mathcal{E})$$

Entropy

$$\beta \equiv \frac{1}{k_B T} = \left. \frac{\partial S_{\mathcal{E}}}{\partial \mathcal{E}} \right|_{\mathcal{E}}$$

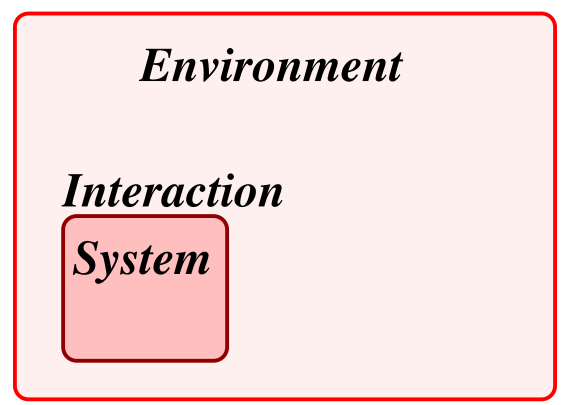
Temperature

$$\mathcal{E} = \mathcal{E}_{syst} + \mathcal{E}_{env} + \mathcal{E}_{int}$$

Neglect $\mathcal{E}_{int}$ (short-range interact.)

$$\mathcal{E}_{syst} \ll \mathcal{E}_{env} \qquad \beta = \frac{\partial S_{\mathcal{E}_{env}}}{\partial \mathcal{E}_{env}}$$

$$\rho(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto e^{-\beta \mathcal{H}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}$$



Canonical ensemble

Out of equilibrium

Three possible reasons

- The equilibration time goes beyond the experimentally accessible times in **macroscopic systems** in which $t_{\rm eq}$ grows with the system size,

$$\lim_{N\gg 1} t_{\rm eq}(N) \gg t$$

e.g., **diffusion, critical slowing down, coarsening, glassy physics**

- **Driven systems** Energy injection

$$\boldsymbol{F}_{\rm ext} \neq -\boldsymbol{\nabla} V(\boldsymbol{x}) \qquad \Gamma_1 \neq \Gamma_2$$

e.g., **active matter**

- **Integrability**

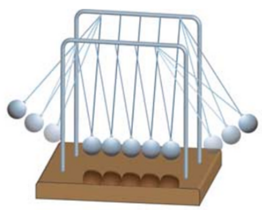
$$I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) = ct, \qquad \mu = 1, \dots, N$$

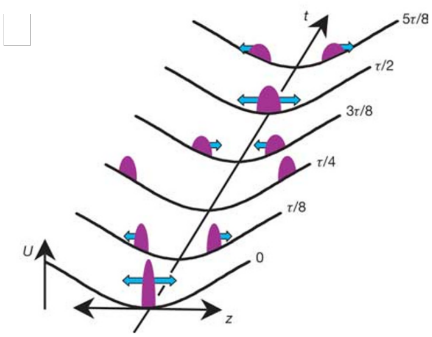
Too many constants of motion inhibit equilibration to the Gibbs ensembles

e.g., **$1d$ bosonic gases**

Motivation

Isolated quantum systems: experiments and theory $\sim$ 15y ago





A quantum Newton's cradle

experiment

cold atoms in isolation

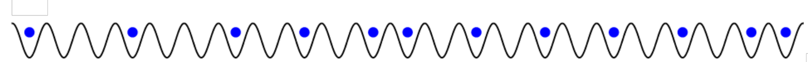
Kinoshita, Wenger & Weiss 06

(Conformal) field **theory** methods for quantum quenches

Calabrese & Cardy 06

Numerical study of

lattice hard core bosons



Rigol, Dunjko, Yurovsky & Olshanii 07

Mostly 1d systems

And many others

Steady state

of a macroscopic system in isolation

- In a **non-integrable** system there are a few constants of motion

$$I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\}) \qquad\qquad \mu = 1, \dots, n$$

and the microcanonical measure is

$$\boxed{\rho \propto \prod_{\mu=1}^{n} \delta(I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\}) - \mathcal{I}_\mu)}$$

with $I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\})$ fixed by the initial conditions to $\mathcal{I}_\mu$ for all $\mu = 1, \dots, n$

typically, just the **energy**, and the usual **microcanonical** measure

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- In an **integrable** system, there are $\mu = 1, \dots, N = \#$ d.o.f. constants $I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\})$

fixed by the initial conditions and

Yuzbashyan 18

$$\boxed{\rho \propto \prod_{\mu=1}^{N} \delta(I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\}) - \mathcal{I}_\mu)}$$

The canonical version

Open system

One hardly works with a microcanonical measure

Equivalence of ensembles $\Rightarrow$ canonical one,* but not obvious for integrable cases

Are local observables characterised by a canonical measure ?

Is it a Generalized Gibbs Ensemble :

$$\rho_{\rm GGE} \propto e^{-\beta \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})}$$

with $\beta\gamma_\mu$ **fixed requiring** $I_\mu(0^+) = I_\mu(t) = \langle I_\mu \rangle_{\rm GGE}$ **?**

* the proof of the equivalence cannot be simply applied in integrable cases

Statistical Physics

An ergodic hypothesis

No need to solve the Newton dynamics!

Under a *new ergodic hypothesis*, after some *characteristic time* $t_{\rm GGE}$, a *local observable* A of a *macroscopic system* can be, on average, obtained with a *static calculation*, as an average over all configurations in phase space weighted with $\rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$

$$\langle A \rangle = \int \prod_i d\boldsymbol{p}_i d\boldsymbol{x}_i \; \rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \; A(\{\boldsymbol{p}_i, \boldsymbol{x}_i\})$$

$\langle A \rangle$ should coincide with $\overline{A} \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_{t_{\rm GGE}}^{t_{\rm GGE}+\tau} dt' \; A(\{\boldsymbol{p}_i(t'), \boldsymbol{x}_i(t')\})$

the *time average* typically measured experimentally

Goal : sample $\rho_{\rm GGE}$

Stochastic dynamics : e.g., Langevin, Monte Carlo

The measure to sample is

$$\rho_{\rm GGE}(\{\boldsymbol{p}_i, \boldsymbol{x}_i\}) \propto e^{-\beta \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i,\, \boldsymbol{x}_i\})} = e^{-\beta \mathcal{H}_{\rm GGE}(\{\boldsymbol{p}_i,\, \boldsymbol{x}_i\})}$$

Assuming one knows the parameters $\{\beta\gamma_\mu\}$,

couple the system $\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu(\{\boldsymbol{p}_i,\ \boldsymbol{x}_i\})$ to an equilibrium bath at inverse temperature $\beta = {}^1\!/\!_{k_B T}$ and a Lagrange multiplier to stay on the sphere

Use, e.g., a white bath and Langevin stochastic dynamics

$$m\dot{x}_\mu = p_\mu \qquad \dot{p}_\mu(t) + \eta \dot{x}_\mu(t) = -\frac{\delta \mathcal{H}^z_{\rm GGE}}{\delta x_\mu(t)} + \xi_\mu(t)$$

with $\langle \xi_\mu(t) \rangle = 0$ and $\langle \xi_\mu(t) \xi_\nu(t') \rangle = 2\eta k_B T \delta_{\mu\nu} \delta(t-t')$

Newton dynamics after a quench of $\mathcal{H}$, the integrable model	"Ergodicity" & the **GGE ensemble** $P_{\rm GGE} \propto e^{-\beta \mathcal{H}_{\rm GGE}}$ also the GB measure of $\mathcal{H}_{\rm GGE}$	**Langevin relaxation** at β^{-1} of $\mathcal{H}_{\rm GGE}(\{\gamma_\mu\})$ $\leftarrow$ approaches the GB measure
$\overline{\langle I_\nu(t>0)\rangle_{i.c.}}$	$\langle I_\nu \rangle_{\rm GGE}$	$\lim_{t\to\infty} \langle I_\nu(t)\rangle$
$\mathcal{I}_\nu$ from initial conditions	$= I_\nu^{\rm GGE}(\{\gamma_\mu\})$	$= I_\nu^{\infty}(\{\beta\gamma_\mu\})$
	$\{\gamma_\mu\}$ fixed from $\mathcal{I}_\nu = I_\nu^{\rm GGE}$ to represent a quench of $\mathcal{H}$	
$\overline{\langle p_\mu^2(t>0)\rangle_{i.c.}}$	$\langle p_\mu^2 \rangle_{\rm GGE}$	$\lim_{t\to\infty} \langle p_\mu^2(t)\rangle_{\xi,i.c.}$
$\overline{\langle x_\mu^2(t>0)\rangle_{i.c.}}$	$\langle x_\mu^2 \rangle_{\rm GGE}$	$\lim_{t\to\infty} \langle x_\mu^2(t)\rangle_{\xi,i.c.}$

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Strategy

Choice of model

– Choose a simple classical *integrable interacting* model with

(not just harmonic oscillators)

an interesting *phase diagram* to atudy different *initial conditions*

and *quenches* across the *phase transition(s)*

– Solve the dynamics after the quenches

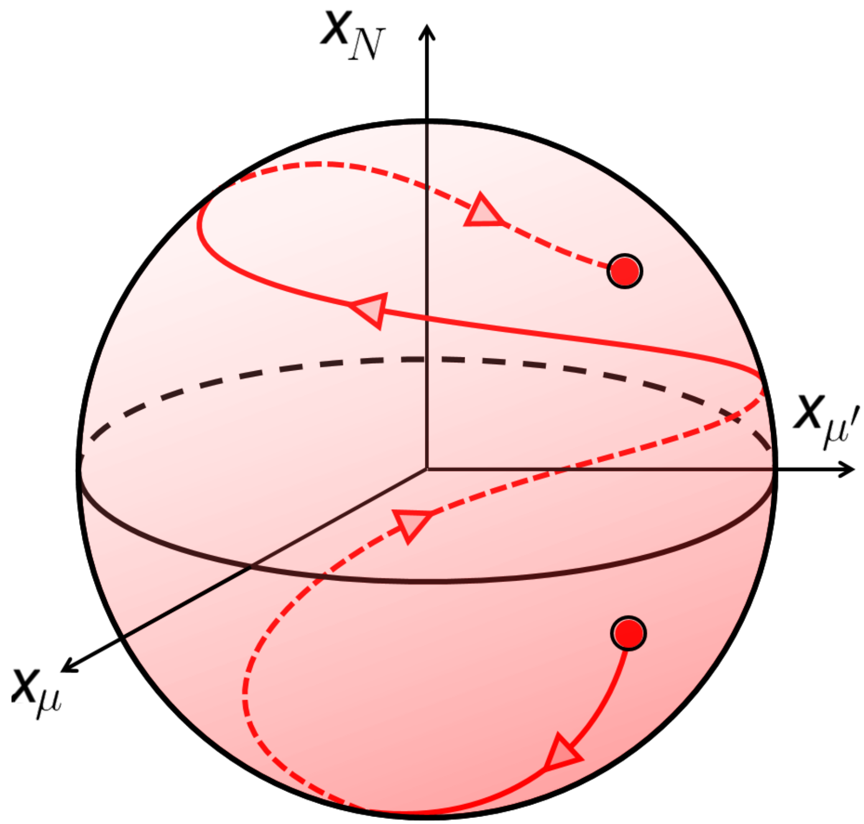
– Build a Generalised Gibbs Ensemble (GGE)

– Prove that the asymptotic limit of local observables is given by the GGE

Not trivial ! First interacting problem with phase transitions

The model

A particle moving on the sphere



Strict constraints

$$\phi : \sum_\mu x_\mu^2 - N = 0$$

$$\phi' : \sum_\mu x_\mu p_\mu = 0$$

$\mu = 1, \ldots, N$ label the coordinates

The model

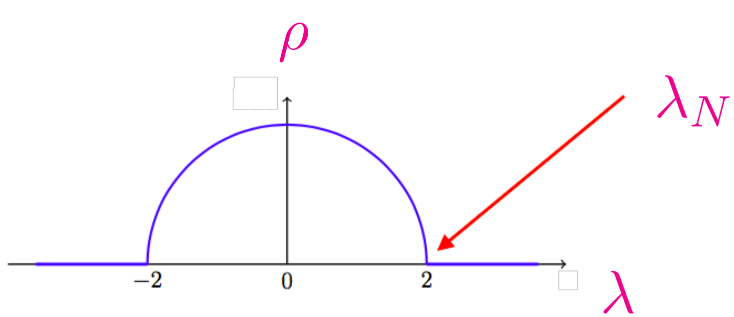
The potential and kinetic energies

Potential energy

$$V_J^{(z)}[\{x_\mu\}] = -\frac{1}{2}\sum_\mu \lambda_\mu \, x_\mu^2 + \frac{z(\boldsymbol{p},\ \boldsymbol{x})}{2}\left(\sum_\mu^N x_\mu^2 - N\right)$$

with $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ the eigenvalues of a matrix from the

GOE ensemble in the $N \to \infty$ limit : **Wigner's semi-circle** law*



and standard **kinetic energy** $K = \sum_\mu \frac{p_\mu^2}{2m}$ Newton dynamics

*handy choice, not essential. This model is also a **spin-glass** like problem $\Rightarrow$ Toolbox.

Neumann's model

1859

Journal

für die

reine und angewandte Mathematik.

In zwanglosen Heften.

Als Fortsetzung des von

A. L. Crelle

gegründeten Journals

herausgegeben

unter Mitwirkung der Herren

Steiner, Schellbach, Kummer, Kronecker, Weierstrass

von

C. W. Borchardt.

Mit thätiger Beförderung hoher Königlich-Preufsischer Behörde

Sechs und funfzigster Band.

In vier Heften.

Berlin, 1859.

Druck und Verlag von Georg Reimer.

Journal of Pure & Applied Math.

Crelle Journal

Newton dynamics on a sphere

under an anisotropic

harmonic potential

$$-\frac{1}{2}\sum_\mu \lambda_\mu x_\mu^2$$

with spring constants

$$\lambda_\mu \neq \lambda_\nu$$

De problemate quodam mechanico, quod ad primam integralium ultraellipticorum classem revocatur.

(Auctore *C. Neumann*, Hallae.)

§. 1.

Problema proponitur.

Sint puncti mobilis Coordinatae orthogonales x, y, z; sit

$$x^2+y^2+z^2 = 1$$

Neumann's model

Integrable

N constants of motion in involution $\{I_\mu, I_\nu\} = 0$ fixed by the initial conditions

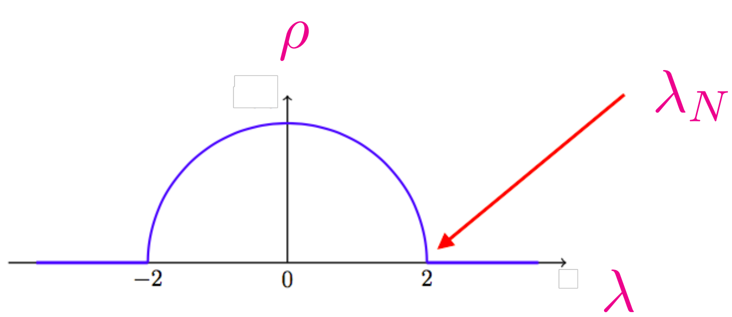
$$I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

K. Uhlenbeck 80s

Studies by **Avan, Babelon and Talon 90s** and many others for finite N

Thermodynamic $N \to \infty$ limit ?

Draw the $\lambda_1 < \lambda_2 < \cdots < \lambda_N$ from **Wigner's semi-circle** law



Kosterlitz, Thouless & Jones 76

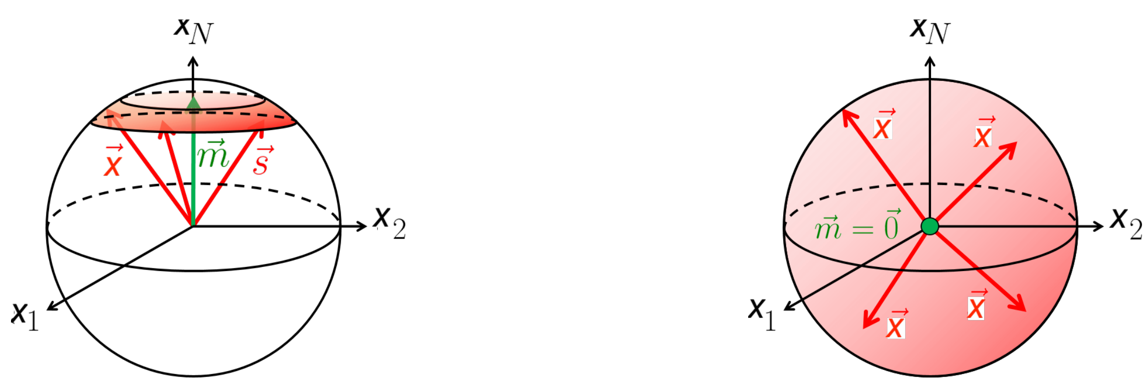
Initial conditions

Drawn from canonical equilibrium with $\lambda_\mu^{(0)}$ at T_0

$\langle x_N^2 \rangle_{\text{eq}} = qN$ | $\langle x_N^2 \rangle_{\text{eq}} = \mathcal{O}(1)$

condensed | **extended** | T_0/J_0

$T_0/J_0 < 1$ | $T_0/J_0 > 1$



Condensed on Nth direction, with largest $\lambda_\mu^{(0)}$ **Extended**

Relation to BEC

Kosterlitz, Thouless & Jones 76, Zannetti 15, Crisanti, Sarracino & Zannetti 19

Instantaneous quench

Global rescaling of all spring constants

At time $t = 0$

to keep some memory of the initial conditions

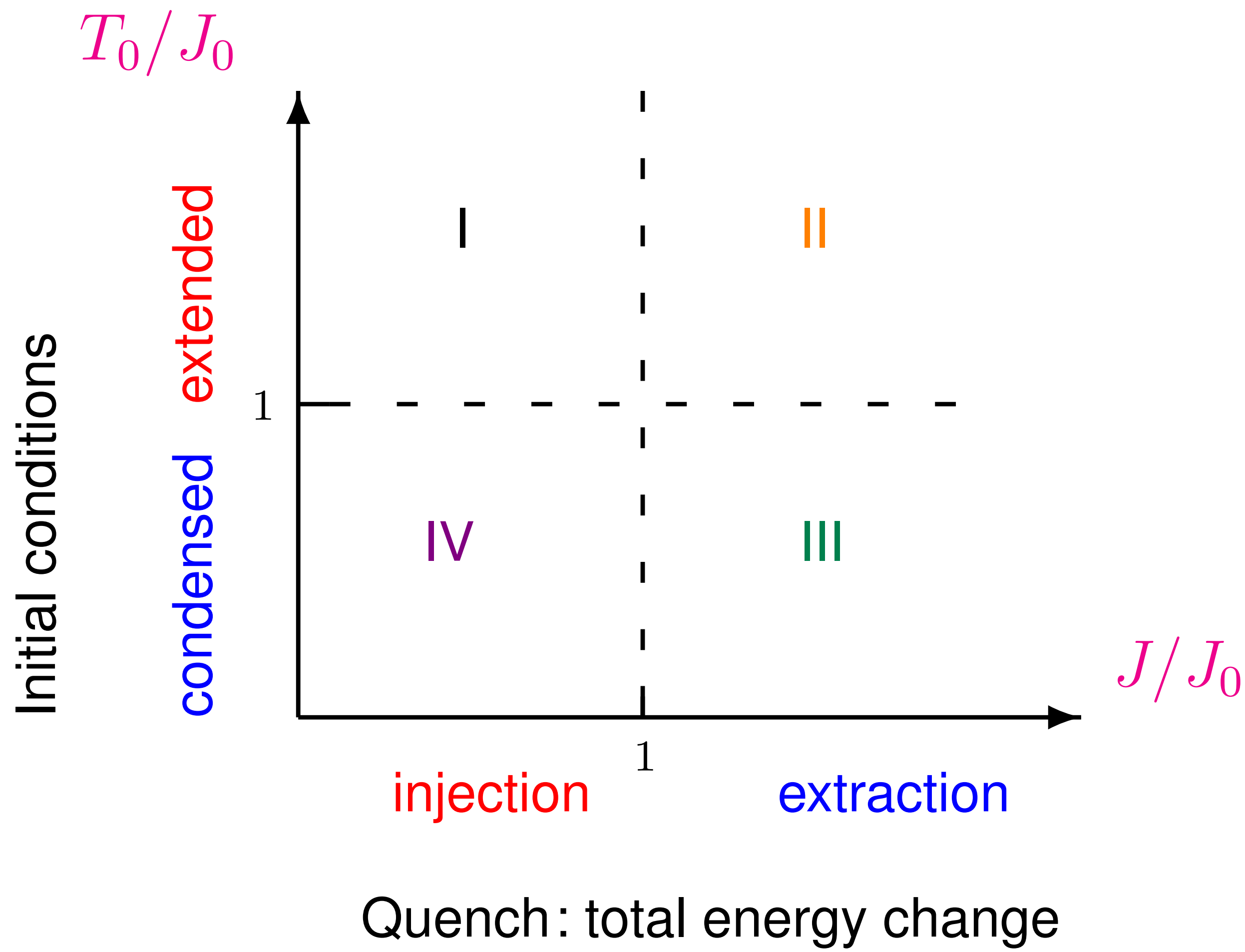
$$\boxed{\lambda_\mu^{(0)} \mapsto \lambda_\mu = \frac{J}{J_0}\lambda_\mu^{(0)}}$$

No change in configuration $\{x_\mu(0^-) = x_\mu(0^+),\ p_\mu(0^-) = p_\mu(0^+)\}$ but macroscopic energy change

$$\Delta E = \begin{cases} > 0 \\ < 0 \end{cases} \quad \text{for} \quad \frac{J}{J_0} \begin{cases} < 1 & \textbf{Injection} \\ > 1 & \textbf{Extraction} \end{cases}$$

Control parameters

Total energy change & initial conditions



Classical quenches

Strategy

Choose a sufficiently simple classical *integrable interacting* model

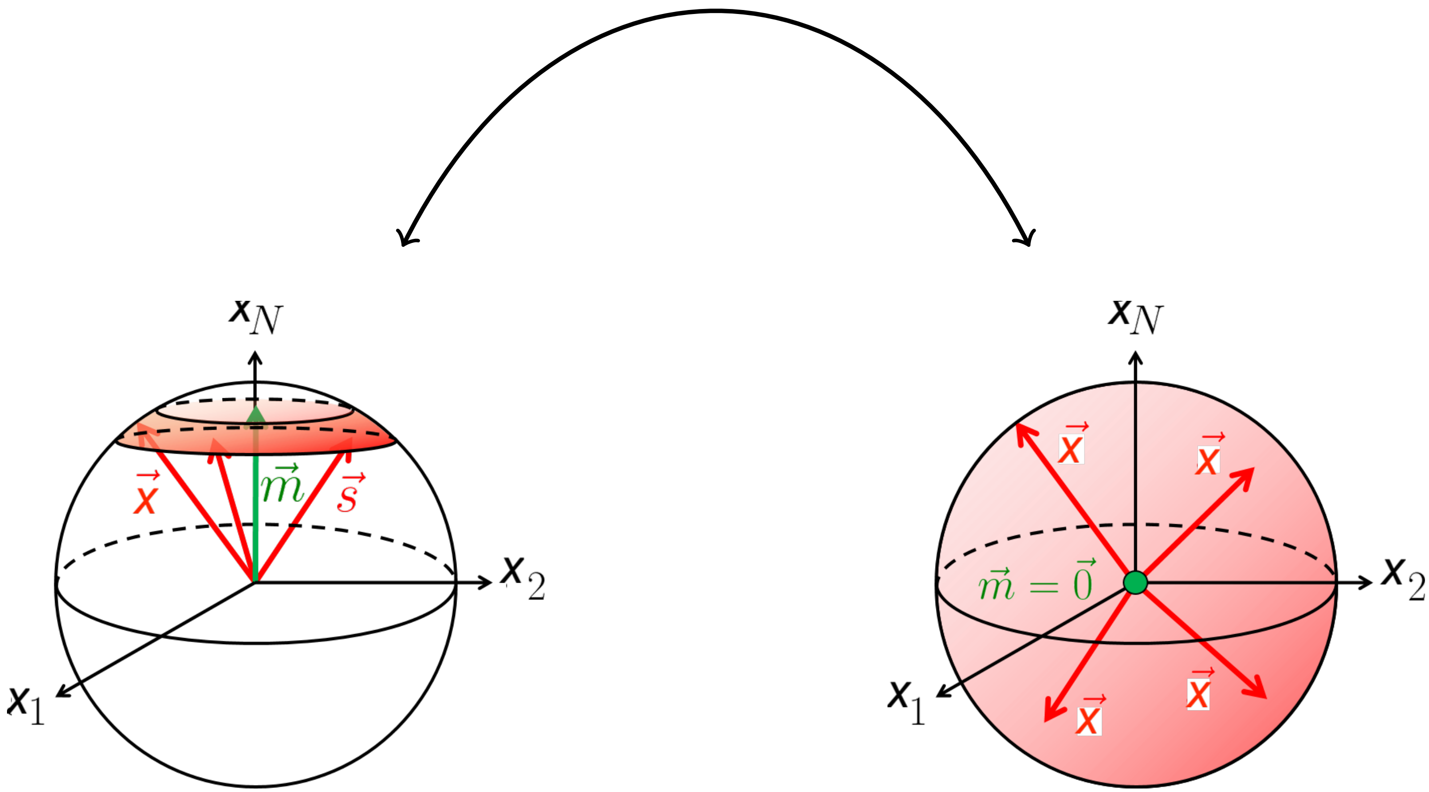
(not just harmonic oscillators)

with an interesting *phase diagram* to investigate different *initial conditions* and *quenches* across the *phase transition(s)*

Solve the dynamics after a quench
Build a Generalised Gibbs Ensemble
Prove that the asymptotic limit of local observables is captured by the GGE

Dynamics of the particle

Interpretation



Motion: Stay condensed Try to condense Remain or become extended

Energy: Extraction or little injection Extraction Injection or little extraction

Asymptotic measure

Is the Generalized Gibbs Ensemble the good one ?

The GGE "canonical" measure is

$$\boxed{\rho_{\rm GGE}(\boldsymbol{p},\boldsymbol{x}) = \mathcal{Z}^{-1}(\{\gamma_\mu\})\; e^{-\beta \sum_{\mu=1}^{N} \gamma_\mu I_\mu(\boldsymbol{p},\boldsymbol{x})}}$$

with

$$\boxed{I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}} \qquad \mu = 1, \dots, N$$

(quartic & non-local) and we fix the γ_μ **on average** by imposing

$$\boxed{\langle I_\mu \rangle_{\rm GGE} = \langle I_\mu \rangle_{i.c.}} \qquad \forall\, \mu$$

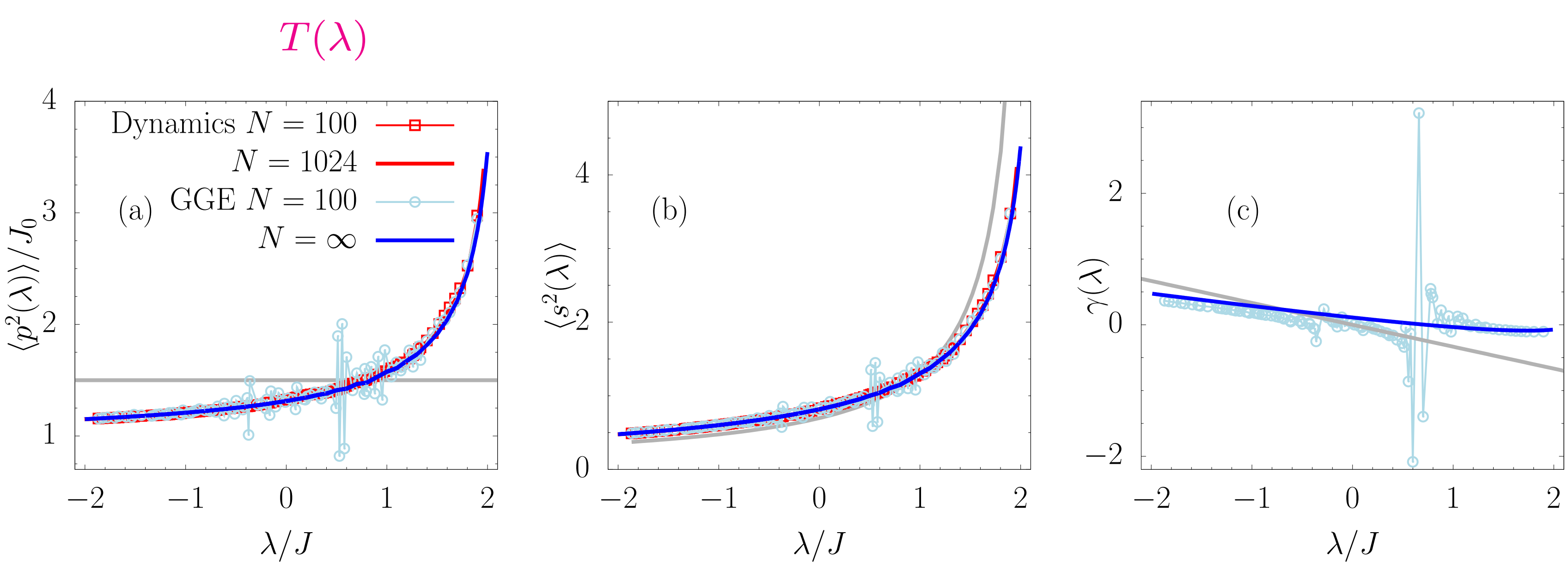
NB in interacting quantum integrable models the charges are usually not known. But we do know them for this model !

Dynamics vs GGE

$$\langle x_\mu^2 \rangle_{\rm GGE} = \overline{\langle x_\mu^2(t) \rangle_{i.c.}} \quad \text{and} \quad \langle p_\mu^2 \rangle_{\rm GGE} = \overline{\langle p_\mu^2(t) \rangle_{i.c.}} \quad ?$$

Dynamics vs GGE

e.g., comparison for quenches in Phase I



In gray, the initial functions

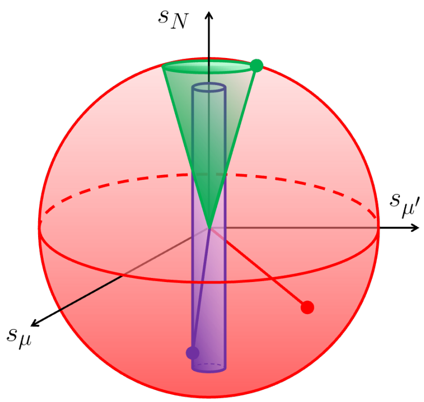
Similar coincidence in Phases II, III & IV

Interesting features linked to “fluctuations catastrophe” in Phase IV

Harmonic Ansatz $\equiv$ saddle-point evaluation of the GGE

Goals achieved

In the late times limit taken after the large N limit



We solved

– the *global dynamics* with Schwinger-Dyson/ DMFT eqs.

– the *mode dynamics* with parametric oscillator techniques of the *(soft) Neumann model*

With the *GGE measure*

$$\rho_{\rm GGE}(\boldsymbol{p}, \boldsymbol{x}) = \mathcal{Z}^{-1}(\{\beta\gamma_\mu\})\, e^{-\beta \sum_\mu \gamma_\mu I_\mu(\boldsymbol{p},\boldsymbol{x})}$$

– we calculated & proved

$$\langle x_\mu^2 \rangle_{\rm GGE} = \frac{T_\mu}{z_{\rm GGE} - \lambda_\mu} = \overline{\langle x_\mu^2(t)\rangle_{i.c.}}$$

$$\langle p_\mu^2 \rangle_{\rm GGE} = T_\mu = \overline{\langle p_\mu^2(t)\rangle_{i.c.}}$$

obtaining also $\{T_\mu, \beta\gamma_\mu\}$

The canonical GGE describes the asymptotic dynamics of this non-trivial integrable model

Barbier, LFC, Lozano, Nessi 22

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Langevin dynamics

of the $\mathcal{H}_{\rm GGE}$ model

$$m\dot{x}_\mu = \frac{\delta \mathcal{H}^z_{\rm GGE}}{\delta p_\mu} \qquad \dot{p}_\mu + \eta \dot{x}_\mu(t) = -\frac{\delta \mathcal{H}^z_{\rm GGE}}{\delta x_\mu(t)} + \xi_\mu(t)$$

with the white noise $\langle \xi_\mu(t) \rangle = 0$ and $\langle \xi_\mu(t) \xi_\nu(t') \rangle = 2\eta k_B T \delta_{\mu\nu} \delta(t-t')$

and the Hamiltonian $\mathcal{H}^z_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu + z(x^2 - N)$

where $I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$ note the weird form !

Check

$$\overline{x_\mu^2(t)} = \langle x_\mu^2 \rangle_{\rm GGE} \stackrel{?}{=} \langle x_\mu^2 \rangle_\xi$$

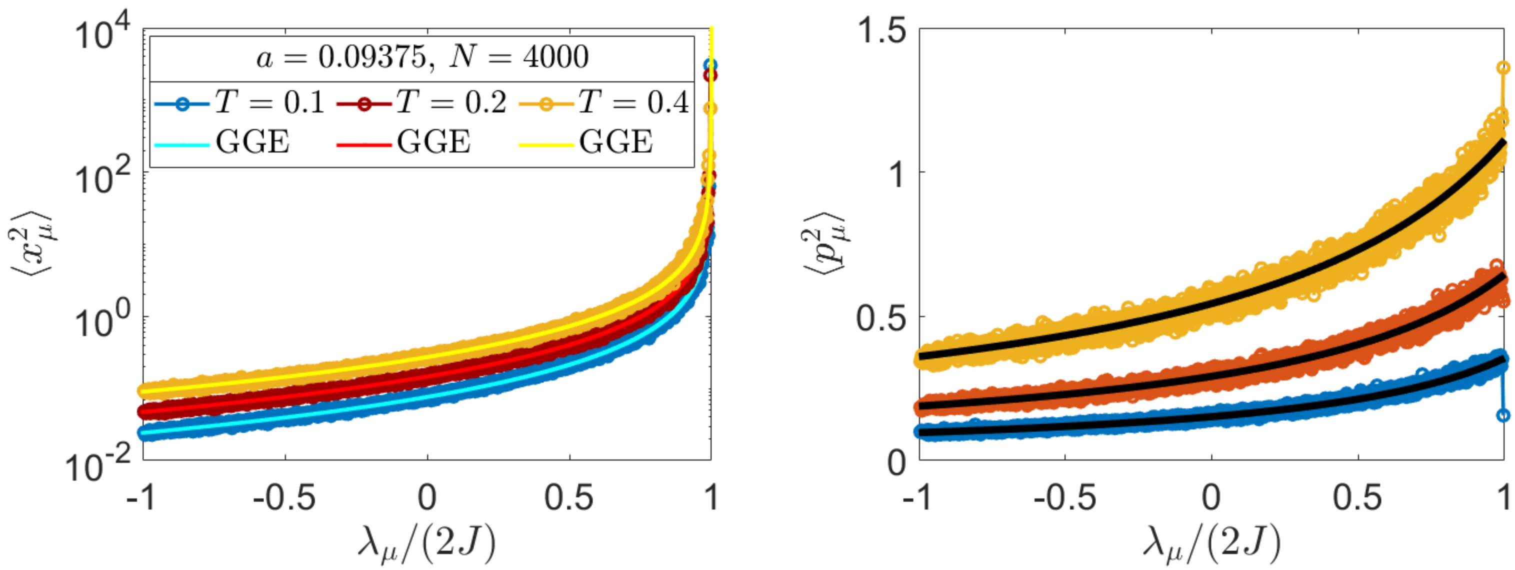
$$\overline{p_\mu^2(t)} = \langle p_\mu^2 \rangle_{\rm GGE} \stackrel{?}{=} \langle p_\mu^2 \rangle_\xi$$

Tests

High temperatures, $\beta < \beta_c$

$$\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$$

Choice of parameters $\gamma_\mu = {}^{a}/_{J}\, \lambda_\mu^2 - {}^{1}/_{2}\, \lambda_\mu$



Data points $\langle \dots \rangle_\xi$ and solid lines $\langle \dots \rangle^{\mathcal{H}}_{\rm GGE} = \langle \dots \rangle^{\mathcal{H}_{\rm GGE}}_{\rm GB}$

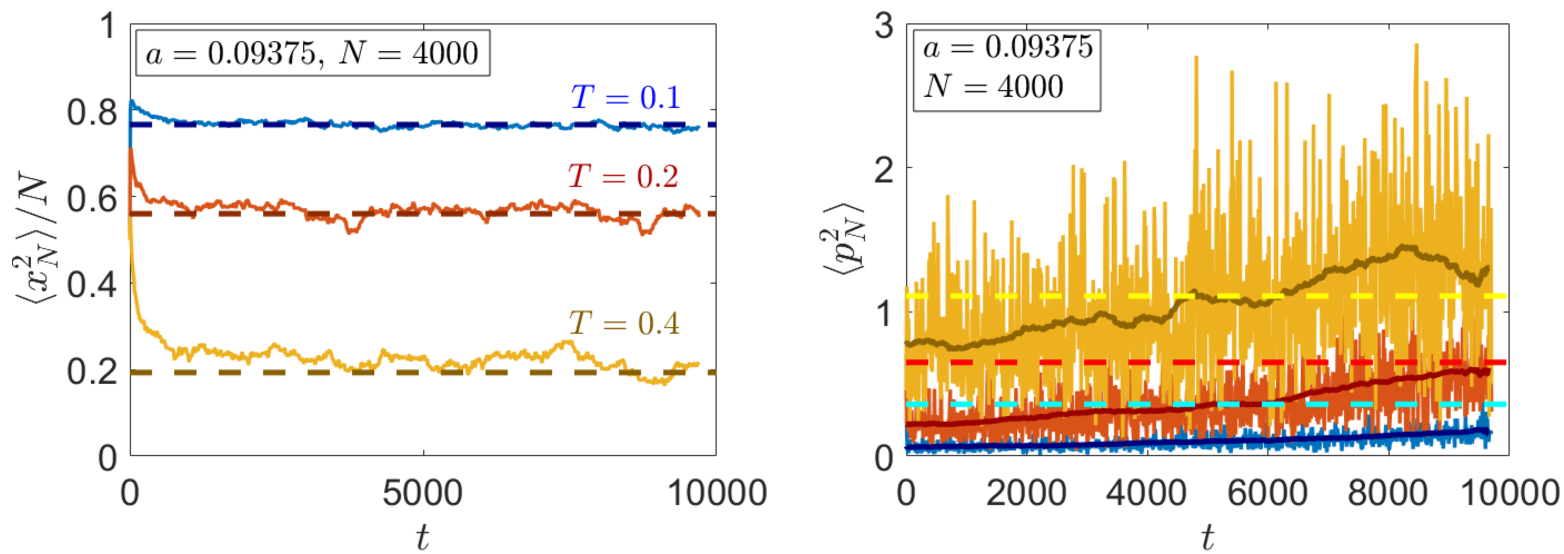
LFC, González-Albaladejo, Lozano & Stariolo 25

Tests

Low temperatures, $\beta > \beta_c$, condensation close to the North pole

$$\mathcal{H}_{\rm GGE} = \sum_\mu \gamma_\mu I_\mu$$

Choice of parameters $\gamma_\mu = {}^a\!/_J\, \lambda_\mu^2 - {}^1\!/_2\, \lambda_\mu$



Data points $\langle \dots \rangle_\xi$ and dashed lines $\langle \dots \rangle^{\mathcal{H}}_{\rm GGE} = \langle \dots \rangle^{\mathcal{H}_{\rm GGE}}_{\rm GB}$

LFC, González-Albaladejo, Lozano & Stariolo 25

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Newton dynamics after a quench of $\mathcal{H}_N$ the integrable model	“Ergodicity” & the **GGE ensemble** $P_{\rm GGE} \propto e^{-\beta \mathcal{H}_{\rm GGE}}$ also the GB measure of $\mathcal{H}_{\rm GGE}$	**Langevin relaxation** at β^{-1} of $\mathcal{H}_{\rm GGE}(\{\gamma_\mu\})$ $\leftarrow$ approaches the GB measure
$\overline{\langle I_\nu(t>0)\rangle_{i.c.}}$	$\langle I_\nu\rangle_{\rm GGE}$	$\lim_{t\to\infty}\langle I_\nu(t)\rangle$
$\mathcal{I}_\nu\left(\frac{T_0}{J_0},\frac{J}{J_0}\right)$	$= I_\nu^{\rm GGE}\left(\{\gamma_\mu\},\frac{T_0}{J_0},\frac{J}{J_0}\right)$ $\{\gamma_\mu\}$ fixed from $\mathcal{I}_\nu = I_\nu^{\rm GGE}$ to represent a quench of $\mathcal{H}_{\rm N}$	$= I_\nu^\infty\left(\{\beta\gamma_\mu\}\right)$
$\overline{\langle p_\mu^2(t>0)\rangle_{i.c.}}$	$\langle p_\mu^2\rangle_{\rm GGE}$	$\lim_{t\to\infty}\langle p_\mu^2(t)\rangle_{\xi,i.c.}$
$\overline{\langle x_\mu^2(t>0)\rangle_{i.c.}}$	$\langle x_\mu^2\rangle_{\rm GGE}$	$\lim_{t\to\infty}\langle x_\mu^2(t)\rangle_{\xi,i.c.}$

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