NON EQ. QUANTUM MACROSCOPIC COMPLEX SYST

MANY ADJECTIVES

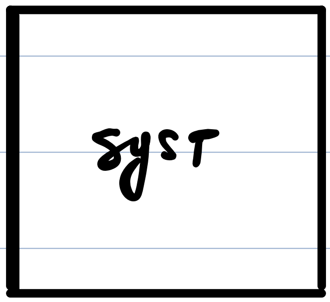
NON. EQ: WE'LL ALL TALK ABOUT OUT OF EQ
DYNAMICS

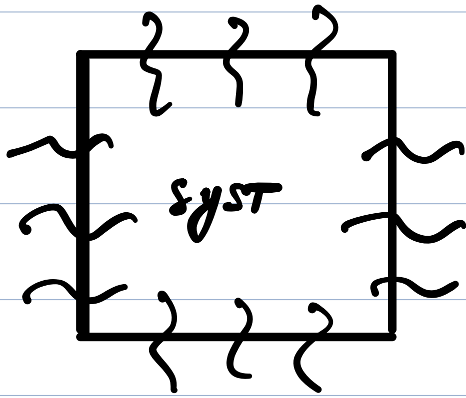
QUANTUM: WELL... I'LL GO FROM CLASSICAL TO
QUANTUM BACK & FORTH
NOT SO MANY DIFFERENCES, THOUGH
I'LL HIGHLIGHT THEM

MACROSC: VS. SINGLE PARTICLE. MANY-BODY
THERMODYN LIMIT IMPORTANT ⇒
LAW OF LARGE NUMBERS
STAT PHYS.

COMPLEX: HARD, NOT EASY ⇒ INTERESTING

CLOSED VS. OPEN





CLASS. NOTATION

SURROUNDINGS
ENVIRONMENT - BATH

$E = H(\{\vec{x}_i, \vec{p}_i\}) = \text{CST.}$

E FLUCTUATES

MAY $\exists$ OTHER CONSERVED QUANTITIES, EG MOMENTUM, ANG MOMENTUM, ETC.

EXCHANGES W/ BATH ALSO N OR SOMETHING ELSE

MICROCANONICAL

CANONICAL

GRAN. CANONICAL

EQUILIBRIUM MEASURES

MICROCANONICAL – CLOSED

$$\mathcal{P}(\{\vec{x}_i, \vec{p}_i\}) = \prod_{\mu=1}^{N_c} \delta\left(\mathcal{I}_\mu(0) - I_\mu(\{\vec{x}_i, \vec{p}_i\})\right)$$

N_c # CT. MOTION $I_\mu(\{\vec{x}_i, \vec{p}_i\})$ $\mu = 1, \dots, N_c$ $\quad \frac{dI_\mu}{dt} = 0$

$\mathcal{I}_\mu(0)$ THEIR INITIAL VALUES

CANONICAL – OPEN

$$\mathcal{P}(\{\vec{x}_i, \vec{p}_i\}) = \frac{1}{\mathcal{Z}(\{\gamma_\mu\})} e^{-\sum_{\mu=1}^{N_c} \gamma_\mu I_\mu(\{\vec{x}_i; \vec{p}_i\})}$$

e.g. $I_1 = H$, $\gamma_1 = \beta$ INVERSE TEMP

$I_2 = N$, $\gamma_2 = \mu$ CHEMICAL POT.

DYNAMICS

CLOSED

NEWTON - HAMILTON CLASS

HEISENBERG - SCHRÖDINGER QUANTUM

OPEN

LANGEVIN SINGLE NOISY TRAJECTORIES

FOKKER-PLANCK - MASTER EQ $P(\{\vec{x}_i, \vec{p}_i\}, t)$

MSR - JD GENERATING FUNCT TRAJ.

LINDBLAD EQ. GEN. MASTER EQ.

SCHWINGER - KELDYSH CLOSED TIME PATH WITH

INFLUENCE FUNCTIONALS IN GEN. FUNCT

GEN MSR - JD

15 min

JUSTIFICATION OF NOISE / DISSIP. IN LANGEVIN EQ.

CLASSICAL LANGEVIN. BROWNIAN MOTION, THEN GENER.

LANGEVIN ~ 1900

BETTER & MORE GENERAL:

MODEL OF SYST H_{syst} BUT ALSO

MODEL OF BATH & COUPLING

PHONONS, SIMPLEST IDEA

HARMONIC OSCILLATORS & LINEAR COUPLING

$$H_B = \sum_{a=1}^{N_0} \left(\frac{\pi_a^2}{2m_a} + \frac{m_a \omega_a^2}{2} q_a^2 \right)$$

d=1 TO SIMPLIFY THE NOTATION

$$H_I = \sum_{a=1}^{N_0} \frac{c_a q_a}{\sqrt{N_0}} \cdot f(x)$$

(COUNTERTERM LATER)

c_a COUPLING CONST. $O(1)$

$\{q_a, \pi_a\}$ OSCILLATOR'S PHASE SPACE VARIABLES

N_0 # OSCILLATORS (WILL GO TO ∞)

SOLVABLE NEWTON / HAM. DYN FOR THE OSC.

LINEAR EQS. FORCES BY THE COUPLING TO x

VIA $f(x)$ USUALLY $f(x) = x$ SIMPLEST CASE

GEN. LATER

SOME VERY SIMPLE CALCULATIONS & USING

THE IMPORTANT ASSUMPTION

$\mathcal{P}(\{q_a(0), \pi_a(0)\})$ EQUILIBRIUM BATH

CANONICAL BOLTZMANN

(COUPLING TO $x(0)$)

TRICKY ISSUE ABOUT A COUNTERTERM TO BE DISCUSSED

TAKING $N_0 \to \infty$ A BATH : ENORMOUS

GAUSSIAN RANDOM INITIAL CONDITIONS $\Longrightarrow$

INDUCE RANDOMNESS / STOCHASTICITY k_BT

FRICTION

THE DYN NO LONGER CONSERVES ENERGY OR

OTHER CONST MOTION OF THE SYST

$\Longrightarrow$

GENERALIZED LANGEVIN EQS

$$m\ddot{x}(t) + \int_0^t dt' \, \Gamma(t-t') \, \dot{x}(t') = F[x(t)] + \xi(t)$$

$$\xi(t) = -\sum_{a=1}^{N_0} c_a \left[\frac{\pi_a(0)}{m_a \omega_a} \sin \omega_a t + \left(q_a(0) + \frac{c_a \, x(0)}{m_a \omega_a^2} \right) \cos \omega_a t \right]$$

NOISE

STATISTICS OF $\{ q_a(0), \pi_a(0) \}$ $\Longrightarrow$ THE ONE OF ξ:

$$\langle \xi(t) \rangle = 0 \qquad \langle \xi(t) \, \xi(t') \rangle = k_B T \, \Gamma(t-t')$$

$\langle \cdots \rangle$ MEANS AVERAGE WITH RESPECT TO THE i.c. OF THE OSC.

WITH

$$\Gamma(t-t') = \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} \cos[\omega_a (t-t')]$$

SYMMETRIC FRICTION KERNEL

FUNCT FORM DEPENDS ON $\{ c_a, m_a, \omega_a^2 \}$

PROPS OF OSC. BATH

KAWASAKI, ZWANZIG 60s - 70s

USUALLY $$S(\omega) = \frac{1}{N_o} \sum_{a=1}^{N_o} \frac{C_a^2}{m_a \omega_a} \delta(\omega - \omega_a)$$

$$\Gamma(t-t') = \int_0^\infty d\omega \, \frac{S(\omega)}{\omega} \cos \omega (t-t')$$

THE CHARACTERISTICS OF THE OSCILLATORS

$\Rightarrow$ NOISE, FRICTION PROPS.

FROM CLASSICAL MECHANICS TO STOCHASTIC PROCESSES

REDUCED SYSTEM

NON-MARKOVIAN-MEMORY

SPECTRAL FUNCTION LEADING TO POWER-LAWS

$$\frac{S(\omega)}{\omega} = \frac{2\gamma_0}{\pi} \left(\frac{|\omega|}{\tilde{\omega}}\right)^{a-1} f_c\left(\frac{|\omega|}{\Lambda}\right)$$

USUAL CASES

$$\Gamma(t-t') = \begin{cases} 2\gamma_0 \, \delta(t-t') & \text{a=1 WHITE OHMIC} \\ \gamma_0 \exp\left(-\frac{|t-t'|}{\tau}\right) & \text{EXP. DECAY} \\ |t-t'|^{-a} & \text{POWER LAW SUB/SUPER. OHMIC} \end{cases}$$

ALL THE INFO ABOUT BATH IS IN $k_B T$, $\Gamma(t-t')$

OVERDAMPED LIMIT - USUAL IN STAT PHYS.

VALID FOR LARGE FRICTION/MASS $\gamma_0/m \gg 1$

OFTEN TIME SCALE OF MOMENTUM RELAX. ARE VERY SHORT, GAUSSIAN WITH

$\langle v(t) \rangle \rightarrow v_0$ $\qquad$ $\langle v^2(t) \rangle \rightarrow \frac{k_B T}{m}$

DROP $m\ddot{x}$ FROM EQ. MOTION

SEPARATION OF TIME- SCALES p AND x VERY DIFFERENT

$f(x)$ GENERIC $\qquad H_I = \sum_{a=1}^{N_0} c_a q_a f(x)$

$$m\ddot{x}(t) + f'(x(t)) \int_0^t dt' \, \Gamma(t-t') \, f'(x(t')) \, \dot{x}(t')$$

$$= F[x(t)] + \xi(t) \, f'(x(t))$$

MULTIPLICATIVE NOISE

MENTIONED COUNTERTERM.

TO GET THE EQ. I WROTE, ADD

$$H_c = \frac{1}{2} \sum_{a=1}^{N_0} \frac{c_a^2}{m_a \omega_a^2} x^2$$

AND CHOOSE INITIAL DIST OF $\{ \pi_a(0), q_a(0) \}$ WITH $x(0)$ WITHIN

NEEDED TO AVOID A "RENORM." OF H_{syst}

PHYSICAL CHOICE TO DO IT OR NOT

OUT OF EQ. BATHS

e.g. ACTIVE MATTER AOUP, OTHERS

BREAK BALANCE BETWEEN FRICTION AND THERMAL TERMS. e.g. USE $\Gamma_f(t-t') \neq \Gamma_{th}(t-t')$

e.g. USE COUPLING TO TWO BATHS, WITH DIFF Γs AND DIFF TEMPS. WHITE FAST AT T_f

COLOURED SLOW AT T_s

INVERSE PROBLEM

OBSERVE THE MOTION OF A TRACER

e.g. BROWNIAN MOTION

CONCLUDE ABOUT THE PROPS OF THE BATH

THE FORCES IT INDUCES OF TRACER

LANGEVIN DID IT THIS WAY, EARLY 1900s

EXPERIMENTS, THEORY

↓

TRACER IN A CELLULAR ENVIRONMENT

↓

~~HOW TO GO FROM~~ INFER FROM TRAJECTORIES

P. RONCERAY (MARSEILLE)

MIT GROUP EXPS.

MANY OTHERS

END 1st LECTURE

RECALL GENERALIZED LANGEVIN EQ.

CAUSALITY

$$m\ddot{x}(t) + f'(x(t))\int_0^{t} dt' \, \Gamma(t-t') \, f'(x(t')) \, \dot{x}(t')$$

$$= F[x(t), \lambda(t)] + \xi(t) \, f'(x(t))$$

POSSIBLE TIME-DEP PARAM.

ADD INDICES TO MAKE IT VALID FOR

$$i = 1, \dots, N$$

$$a = 1, \dots, d$$

LOOK AT THE NOTES FOR FULL GENERALITY

USUAL ADDITIVE NOISE CASE, ONE INDEPENDENT IDENTICAL BATH PER d.o.f.

$$m\ddot{x}_i^a + \int_0^t dt' \, \Gamma(t-t') \, \dot{x}_i^a =$$

$$= F_i^a[\vec{x}_i(t), \vec{\lambda}_i(t)] + \xi_i^a(t)$$

$$\langle \xi_i^a(t) \, \xi_j^b(t') \rangle = k_B T \, \Gamma(t-t') \, \delta_{ij} \, \delta^{ab}$$

CONNECTION WITH USUAL
MULT. NOISE WHITE NOISE LANG EQ

$$\dot{x} = \bar{f}(x) + g(x)\,\xi$$

TAKE $\Gamma(t-t') = 2\gamma_0\,\delta(t-t')$

DROP $m\ddot{x}$ OVERDAMPED LIMIT

SET $\gamma_0 = 1$ BY A REDEF. OF TIME

$$\left(f'(x(t))\right)^2 \dot{x}(t')$$

$$= F[x(t), \lambda(t)] + \xi(t)\, f'(x(t))$$

DIVIDE BY $\left(f'(x(t))\right)^2$ AND REDEFINE THE DETERM FORCE TERM $F/(f')^2 = \bar{f}$

AND

$$g = 1/f'$$

AND THAT'S IT.

REFS.

DERIVATION & DISCUSSION OF LANG EQ

http://www.lpthe.jussieu.fr/~leticia/SEMINARS/Langevin.pdf

SKETCHY SLIDES

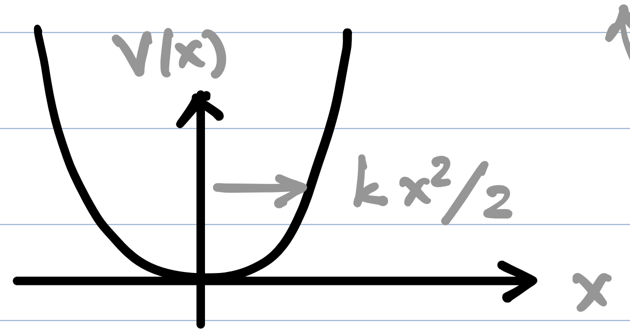
http://www.lpthe.jussieu.fr/~leticia/SEMINARS/ictp-non-Markovian-bis.pdf

YANG ET AL 03; MIN, LUO, CHERAYIL, KOU & XIE 05

$$C_x(t,t') = \langle x(t)\, x(t') \rangle$$

$$= \frac{1}{k}\, E_{\alpha,1}\left(- \frac{k\, |t-t'|^{\zeta}}{g} \right)$$

MITTAG LEFFLER - FCT.



$\zeta \simeq 0.5$

ex. SEPARATION OF TIME - SCALES

$$\dot{x} = p/m$$

WHITE MULT. NOISE

BROWNIAN MOTION

$$\dot{p} = -\gamma_0/m \, p + \xi$$

$$p(t) = e^{-\gamma_0/m \, t} p(0) + \int_0^t dt' e^{-\gamma_0/m (t-t')} \xi(t')$$

$$\langle p(t) \rangle = e^{-\gamma_0/m \cdot t} p(0) \xrightarrow[t \gg m/\gamma_0]{} 0$$

$$\langle p^2(t) \rangle = \underbrace{e^{-2\gamma_0/m \, t} p^2(0)}_{} \longrightarrow 0$$

$$+ 2\gamma_0 k_B T \int_0^t dt' e^{-2\gamma_0/m (t-t')}$$

$$= \cancel{2\gamma_0} k_B T \, e^{-2\gamma_0/m \, t} \frac{1}{\cancel{2\gamma_0}/m} \left(e^{+\frac{2\gamma_0}{m} t} - 1 \right)$$

$$= m k_B T$$

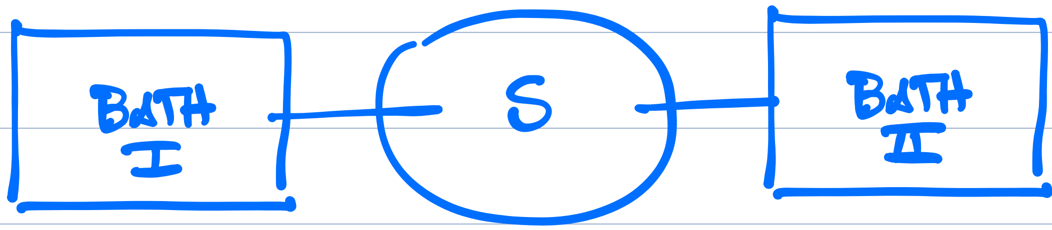
MAXWELL ok

$$\tau = m/\gamma_0 \ll 1$$

CALCULATE $\langle (x(t) - x(0))^2 \rangle \longrightarrow 2Dt$

5'

cfr L. PELITI'S SETTING



LANGEVIN EQ FORMULATION

FRICTION $\Gamma_1(t-t') + \Gamma_2(t-t')$

NOISE - NOISE CORR $k_B T_1 \, \Gamma_1(t-t') + k_B T_2 \, \Gamma_2(t-t')$

FOR THE SETTING IN WHICH EACH SYST VARIABLE COUPLED TO BOTH BATHS.

$F[x(t), \lambda(t)]$

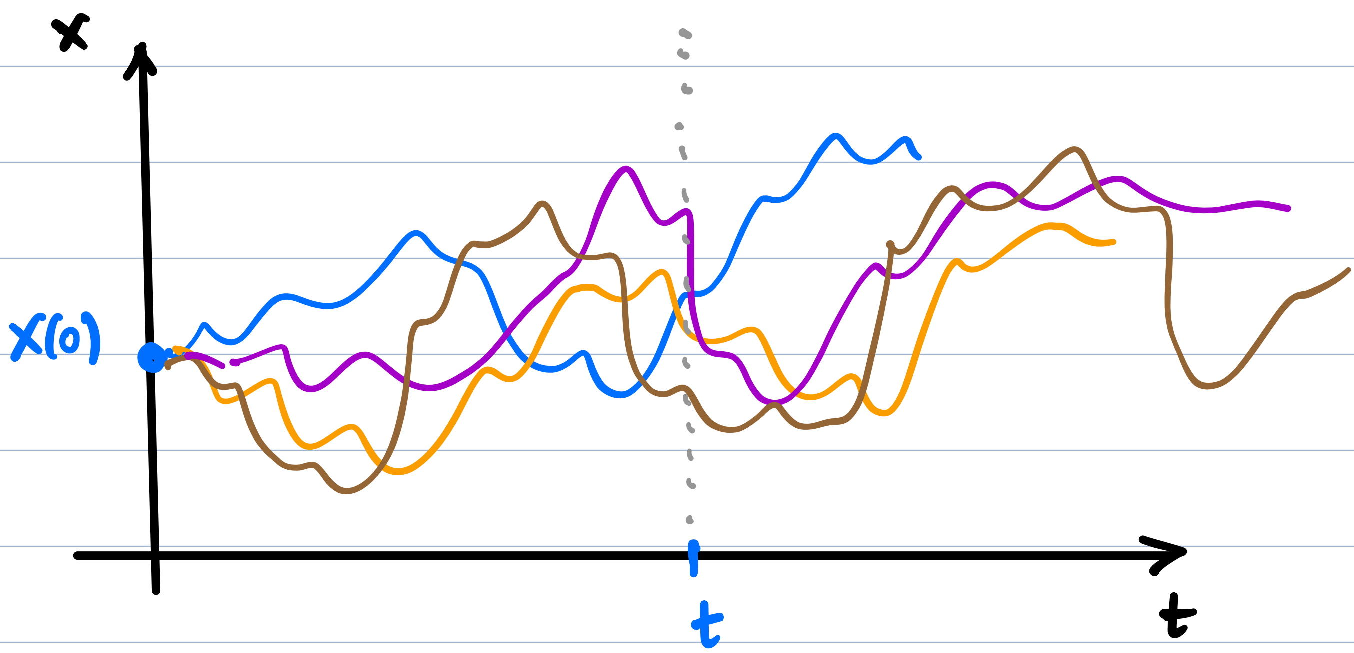
TIME DEP. PARAMETER

OTHER GENERALIZATIONS IN NOTES.

KRAMERS / FOKKER-PLANCK EQUATION

$\mathcal{P}(\{\vec{x}_i, \vec{p}_i\}; t)$ PROB. OF PARTICLES HAVING $\{\vec{x}_i, \vec{p}_i\}$ AT TIME t KNOWING INITIAL PROB.

eg. JUST x (OVERDAMPED) $d=1$ $N=1$



READ $\mathcal{P}(x; t)$ HERE

RESTRICTION

CLOSED DIFF EQ. ONLY FOR WHITE NOISE

KRAMERS EQ

BOTH MOMENTUM & POSITION

UNDERDAMPED

$$\frac{\partial P}{\partial t} + \frac{p}{m}\frac{\partial P}{\partial x} =$$

$$\gamma_0 \frac{\partial}{\partial p}(pP) + \frac{\partial}{\partial p}\left(-F P\right) + \gamma_0 k_B T \frac{\partial^2 P}{\partial p^2}$$

FOKKER-PLANCK

OVERDAMPED LIMIT $\Longrightarrow$

JUST POSITIONS

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}(F P) = \gamma_0 k_B T \frac{\partial^2 P}{\partial x^2}$$

THIS FORMALISM IS

USUALLY USED TO PROVE THAT IF

$$F = -V'(x) \Longrightarrow$$

cfr LUCA PELITI'S

$$P(\{x,p\}) = \frac{e^{-\beta H_{syst}(\{x,p\})}}{Z(\beta)}$$

STAT SOLUTION

BUT IF WE DON'T HAVE KRAMERS / FP ?

FUNCTIONAL FORMALISM

ANALYSIS CAN BE DONE AT THE LEVEL OF GEN FUNCT

$$Z_{dyn}[\lambda] = \int \prod_a dq_a \prod_a d\pi_a \int dx \int dp \; e^{\int_0^t dt' \lambda(t') x(t')}$$

$$\exp\{-S[\{x, p; q_a, \pi_a\}]\}$$

λ SOURCE

INTEGRATE AWAY THE $\{\pi_a, q_a\}$

$$Z_{dyn} = \int dx \int dp \; \exp\{-S[\{x,p\}; k_B T, \Gamma]\}$$

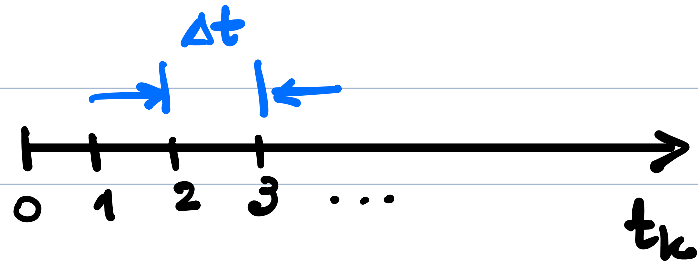
$$S = S_{syst}[\{x, p\}] + \underbrace{S_{INF}[\{x\}; k_B T, \Gamma]}_{\text{BATH EFFECT}}$$

THIS IS THE FORMALISM WHICH CAN BE GENERALIZED QUANTUM MECHANICALLY.

FEYNMAN - VERNON 60s

HINTS ON HOW TO OBTAIN THE ACTION S
AND SOME TRICKY ISSUES.

- DISCRETIZE TIME



- DISCRETIZE LANGEVIN EQ.

  RECALL PRESCRIPTION HAS TO BE CHOSEN

- IF INERTIA PRESENT OR COLOURED NOISE

  $\neq$ PRESCRIPTION ARE EQUIVALENT FOR

  $\Delta t \to 0$

- IF OVERDAMPED & WHITE NOISE :

  CAREFUL !

  SPECIALLY WHEN CONSTRUCTING THE GEN.

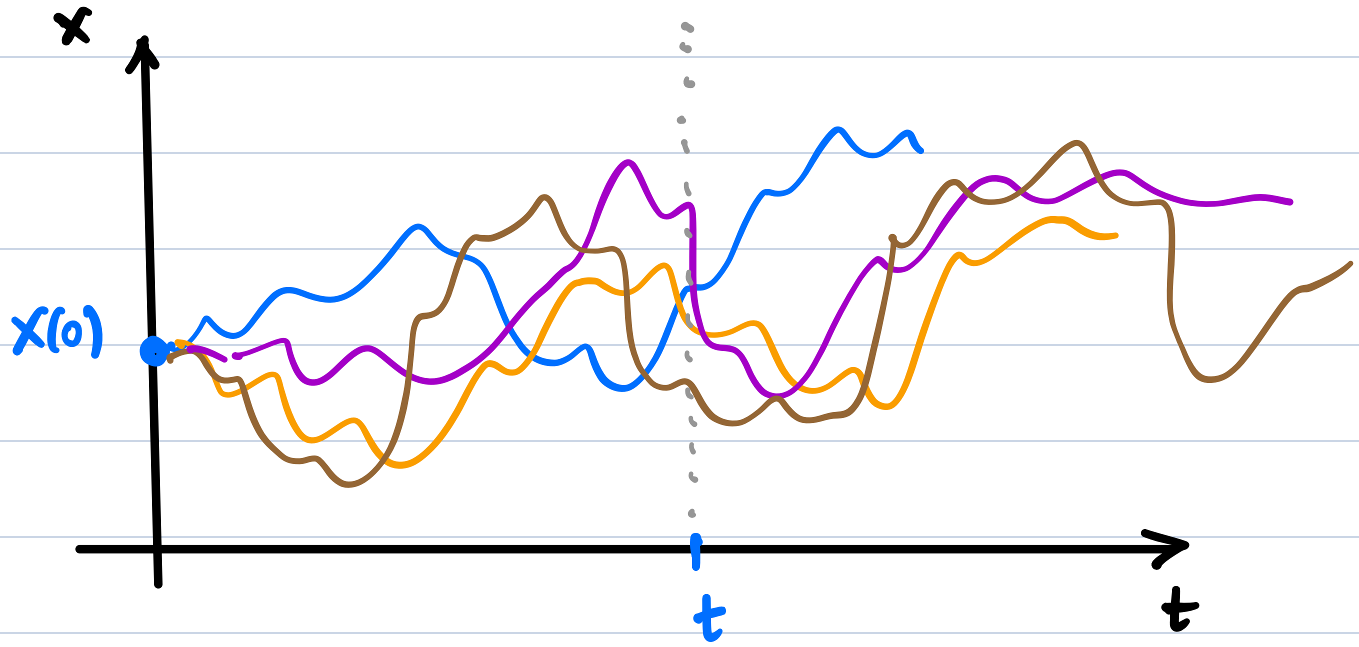
  FUNCT. ARNOULX DE PIREY, LFC,

  LECOMTE, VAN WIJLAND 22

SKETCH : SUM OVER ALL POSSIBLE TRAJ.

GENERATED BY THE LANGEVIN EQ.

UNDER ITS NOISE



$$Z_{dyn}(\lambda) = \left\langle e^{\int_0^t dt' \, \lambda(t') \, x_\xi(t')} \right\rangle_\xi$$

MEANS A SUM OVER ALL REAL'Z OF ξ WITH THEIR

$$P[\{\xi(t)\}]$$

AND $x_\xi(t)$ SOL. OF LANG EQ.

- IMPOSE THIS COND WITH A FUNCT δ-DIRAC
- EXPONENTIATE THE δ WITH AUX $\{i\hat{x}(t)\}$
- AVERAGE OVER $\{\xi(t)\}$

FOR EXAMPLE, FOR A SINGLE VARIABLE BILINEARLY COUPLED TO AN OSCILLATOR BATH

CLASSICALLY

$$S[x, i\hat{x}] = \underbrace{\ln P_0(x(0), \dot{x}(0))}_{\text{INITIAL COND.}}$$

$$- \int_0^t dt' \; i\hat{x}(t') \underbrace{\left\{ m\ddot{x}(t') + \int_0^{t'} dt'' \, \Gamma(t'-t'')\, \dot{x}(t'') - F[x(t')] \right\}}_{\text{LANG EQ. (APART FROM NOISE)}}$$

$$+ \underbrace{\frac{k_B T}{2} \int_0^t dt' \int_0^t dt'' \; i\hat{x}(t') \, \Gamma(t'-t'') \, i\hat{x}(t'')}_{\text{INTEGRATION OVER NOISE}}$$

$$+ \underbrace{\ln J[x]}$$

JACOBIAN – HAS TO DO WITH DISCRETIZATION USED TO DEFINE THE STOCH PROCESS AND BUILD THE PATH INTEGRAL

WHY IS THIS FORMALISM USEFUL?

- PERTURB. TH.
- SADDLE-POINT / MEAN-FIELD
- SYMMETRIES → WARD IDENTITIES, e.g. FLUCT THS.

20'

CORRELATIONS

$$\langle x(t)\, x(t')\rangle = \int \mathcal{D}x\, \mathcal{D}i\hat{x}\; e^{-S[\;]}\; x(t)\, x(t')$$

LINEAR RESPONSE

$$\left.\frac{\delta \langle x(t)\rangle_h}{\delta h(t')}\right|_{h=0} = \frac{\delta}{\delta h(t')} \int \mathcal{D}x\, \mathcal{D}i\hat{x}\; e^{-S_h[\;]}\; x(t) \Bigg|_{h=0}$$

in:

$$F_h[x(t)] = F[x(t)] + h(t)\, x(t)$$

RECALL $e^{\int_0^t dt''\; i\hat{x}(t'')\, F_h[x(t'')]}$ IN PATH INTEG.

$$\Longrightarrow \left.\frac{\delta\langle x(t)\rangle_h}{\delta h(t')}\right|_{h=0} = \int \mathcal{D}x\, \mathcal{D}i\hat{x}\; e^{-S[\;]}\; x(t)\, i\hat{x}(t')$$

$$= \langle x(t)\, i\hat{x}(t')\rangle$$

CAUSALITY, $\neq 0$ $t \geq t'$

RESPONSE FIELD

SYMMETRIES

PROVE FDT IF

$\mathcal{P}_0(x(0), i\hat{x}(0))$ MAXWELL-BOLTZM.

$F[x] = -V'[x]$ POTENTIAL FORCE WITH THE SAME $V[x]$ USED TO DRAW i.c.

$\Gamma(t-t')$ THE SAME IN FRICTION AND NOISE CORR.

PROOF. IT'S BASED ON THE INVAR. OF ACTION UNDER SIMULT TRANSF OF $\{x(t), i\hat{x}(t)\}$

$$S[x(t), i\hat{x}(t)] = S[\underbrace{x(-t)}_{\text{TIME REVERSAL}}, i\hat{x}(-t) + \underbrace{\beta \frac{d}{dt} x(-t)}_{\text{TRANSL.}}]$$

ALSO THE MEASURE $\mathcal{D}x\,\mathcal{D}i\hat{x}$ IS INVARIANT & INTEGRATION INTERVAL TOO

CONSEQUENCE ON CORRELATIONS & LINEAR RESP. ?

NEWTON INFLUENCE

NOTE THAT S_{syst} & S_{Diss} ARE SEPARATELY
\+ JAC
INVARIANT

$$\langle x(-t)\; i\hat{x}(-t')\rangle =$$

ALL THE $\langle \cdots \rangle$ OPERATION IS INV. SINCE
S & MEASURE ARE

$$\left\langle x(t)\left[i\hat{x}(t') - \beta \frac{d}{dt'} x(t') \right]\right\rangle \Longrightarrow$$

$$\langle x(t)\, i\hat{x}(t')\rangle = \langle x(-t)\; i\hat{x}(-t')\rangle + \beta \langle x(t)\; \dot{x}(t')\rangle$$

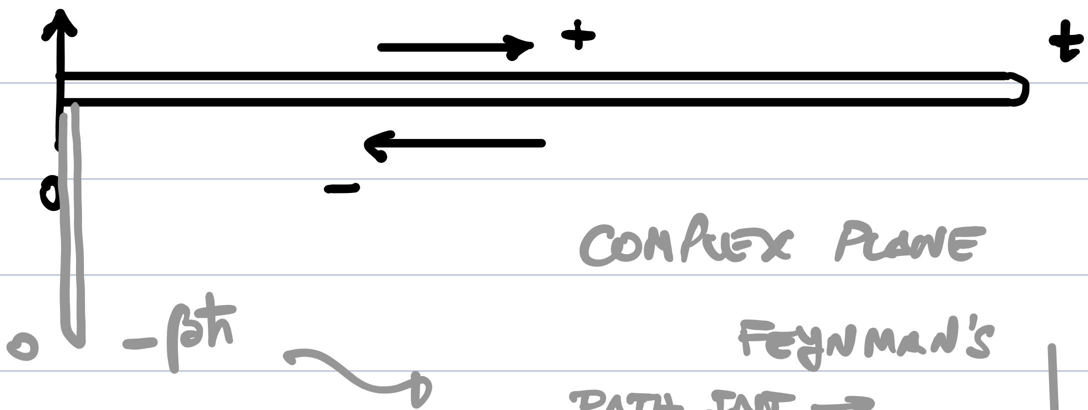
FDT

$$R(t,t') = R(-t,-t') + \beta \frac{\partial}{\partial t'} C(t,t')$$

THEREFORE, IF FDT $\Longrightarrow$ EQUIL. À LA
MAXWELL BOLTZMANN cfr J. KURCHAN'S
THEN THESE PROCESSES DO TAKE SYST
TO EQUIL. UNDER ABOVE CONDITIONS

QUANTUM

VERY SIMILAR !



TIME

COMPLEX PLANE

FEYNMAN'S PATH INT → STAT MECH.

$e^{-\beta \hat{H}/\hbar} = \hat{\rho}_{in.}$

$-it/\hbar \to -\beta$

$t = -i\tau$

$\tau = \beta$

IN ORDER TO CALCULATE CORR. LIKE

$$\langle \hat{A}(t)\, \hat{B}(t') + \hat{B}(t')\, \hat{A}(t) \rangle$$

NO TIME ORDERED, TRICK GO FORTH & BACK.

$$\frac{x^+(t) + x^-(t)}{2} \xrightarrow[\hbar \to 0]{} x(t)$$

$$\frac{x^+(t) - x^-(t)}{\hbar} \xrightarrow[\hbar \to 0]{} i\hat{x}(t)$$

SCHWINGER - KELDYSH

+ FEYNMAN VERNON

MSR-JD

COMPLETE FORMALISM THAT LET'S TREAT ANY KIND OF BATH - NOT ONLY OHMIC

SYMMETRIES

TRANSFORMATION OF PATHS IN THE COMPLEX PLANE

C. AR̲O̲N̲, G. BIROLI & LFC 18

LINDBLAD ?

WORKS ON $\hat{\rho}^{RED}_{syst}(t)$ DENSITY OPERATOR OR

DENSITY MATRIX

COUPLES SYST'S OPERATOR, SAY $\hat{x}$ TO
BATH OPERATORS, SAY $\{\hat{b}_a\}$

INTEGRATE AWAY THE $\{\hat{b}_a\}$ S $\Longrightarrow$ REDUCE

ASSUMPTIONS

RAPID DYNAMICS OF BATH COMPARED TO SYSTEM'S

NO INFLUENCE OF SYST ON BATH $N_b \gg 1$

MARKOVIAN STRONG !!!

GOOD IN QUANTUM OPTICS, NOT IN COND-MAT

LOCAL IN TIME EQ. $\approx$ SIMILAR TO

KRAMERS / FP.

$$\frac{d\hat{\rho}^{RED}_{syst}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_{syst}, \hat{\rho}^{RED}_{syst}\right] + \sum_{mn} h_{mn} \left(\hat{L}_n \hat{\rho}^{RED}_{syst} \hat{L}^{+}_m - \frac{1}{2}\left\{ \hat{L}^{+}_m \hat{L}_n, \hat{\rho}^{RED}_{syst} \right\} \right)$$

SUMMARY 1st & 2nd LECTURES

- I JUSTIFIED THE GENERALIZED LANGEVIN EQ. BEYOND BROWNIAN MOTION
  GENERIC MEMORY KERNEL & NOISE-CORR
  RELATED BY FDT IF EQ. BATH

- SKETCHED - ANDREA DID IT WELL - THE CLASSICAL GEN. FUNCT. CONSTRUCTION

- DISCUSSED ITS TIME-REV. SYMM. $\Rightarrow$ FDT & THEN EQUIL.

- SKETCHED THE CONSTRUCT OF QUANTUM GEN FCT - SCHWINGER-KELDYSH
  W/ GENERIC BATH - FEYNMAN- VERNON

WHAT ARE THE KERNELS FOR A QUANTUM OSC BATHS ?

$$\tilde{\Gamma}(t-t') = \int_0^\infty d\omega \; S(\omega) \; \coth\left(\frac{\beta\hbar\omega}{2}\right) \cos\omega(t-t')$$

$$\coth x : \lim_{x\to 0} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{2}{\cancel{1}+x-\cancel{1}+x} = \frac{1}{x}$$

$$\lim_{\beta\hbar\to 0} \Gamma = \int_0^\infty d\omega \; S(\omega) \; \frac{2}{\beta\hbar\omega} \; \cos\omega(t-t')$$

$$= \frac{k_B T}{\hbar} \underbrace{\int_{-\infty}^{\infty} d\omega \; \frac{S(\omega)}{\omega} \; \cos\omega(t-t')}_{\Gamma(t-t')}$$

THIS KERNEL APPEARS AS A TERM IN $e^{\frac{i}{\hbar} S_{diss}}$

NOISE

$$\frac{i S_{diss}}{\hbar} = \frac{(-1)}{\hbar} \int_0^t dt' \int_0^t dt'' \underbrace{\left(x^+(t') - x^-(t')\right)}_{i\hat{x}(t')\hbar} \tilde{\Gamma}(t'-t'') \underbrace{\left(x^+(t'') - x^-(t'')\right)}_{i\hat{x}(t'')\hbar}$$

$$\xrightarrow[\hbar\to 0]{} -\frac{\cancel{\hbar^2}}{\cancel{\hbar}} \frac{k_B T}{\cancel{\hbar}} \int_0^t dt' \int_0^t dt'' \; i\hat{x}(t) \, \Gamma(t-t') \, i\hat{x}(t'')$$

THERE IS ANOTHER TERM THAT DOES NOT DEP. ON TEMP. AND PLAYS THE RÔLE OF THE RETARDED FRICTION (RESPONSE OF BATH) RELATED BY QUANTUM FDT TO THIS ONE

$$\gamma'(\omega) = -\Theta(t-t')\int_0^\infty d\omega\, S(\omega)\, \sin\omega\,(t-t')$$

$$= \frac{1}{4\hbar}\lim_{\varepsilon\to 0^+}\int \frac{d\omega'}{\pi}\,\frac{1}{\omega-\omega'+i\varepsilon}\,\operatorname{th}\frac{\beta\hbar\omega'}{2}\,2\hbar\,\tilde{\Gamma}(\omega')$$

— IN LIST OF TOPICS I MENTIONED NOISE INDUCED PH. TRANS — HOT TOPIC QUANTUM

IMPORTANT

ALL THESE FORMALISMS SHOULD ADMIT (IF BATH IN EQ. $F=-V'$ AND) TO TAKE SYST FOR $t \gg t_{eq}$ (PARAM.) TO CANON. EQUIL

$$\frac{e^{-\beta H_{syst}}}{Z(\beta)}$$

NOT EASY IN GENERAL

→ OK FOR FOKKER PLANCK / KRAMERS

→ SYMM. IN Z_{dyn} FOR COLOURED NOISE

LINDBLAD ... UHM ...

t_{eq} ?? COULD BE VERY LONG AND SYST SIZE-DEPENDENT

⇒ LONG OUT OF EQ. RELAXATION STILL FOR EQ. BATH & $F=-V'$

LET'S DISCUSS SOME PHYSICAL PROBLEMS NOW.

1- SEP. OF TIME SCALES p VS. x $\Longrightarrow$ EX.

2- DROP INERTIA, JUST x. SEPARATION OF TIME SCALES IN ITS OWN BEHAVIOUR.
NOT SO SURPRISING. THINK ABOUT RW AGAIN, FOR

$t < t_{eq}^{(p)}$ $\Longrightarrow$ BALLISTIC $x(t) \simeq \frac{p}{m} t$

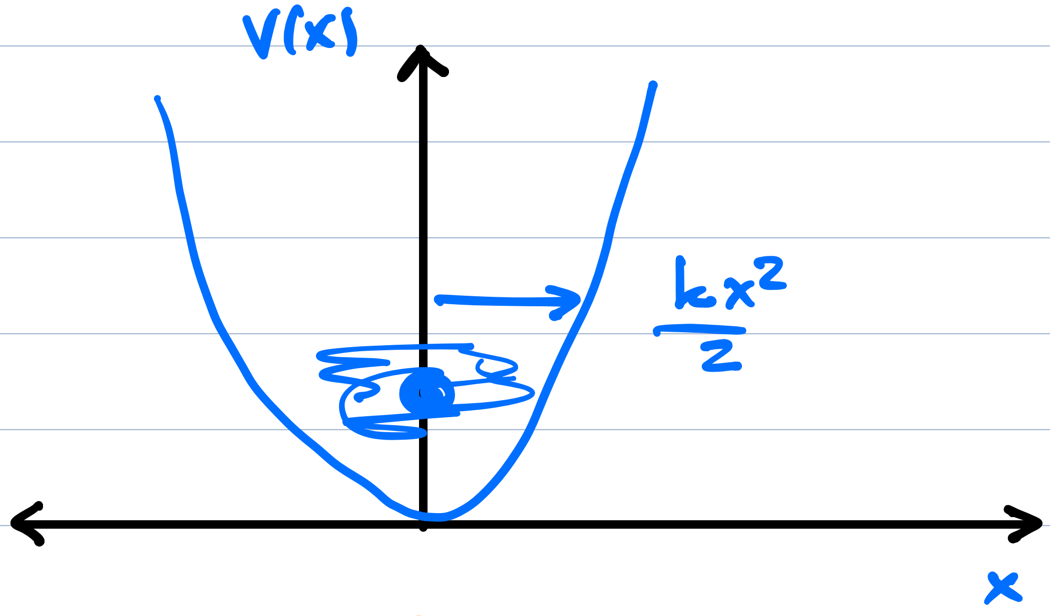
$t > t_{eq}^{(p)}$ $\Longrightarrow$ DIFFUSIVE WHEN NOISE CONTROLS

LET'S LOOK AT ANOTHER SIMPLE EX IN THE OVERDAMPED LIMIT

$$\gamma_0 \dot{x} = -kx + \xi + h$$

TO MEASURE LINEAR RESP.

A PARTICLE IN A HARM WELL UNDER ADD. WHITE NOISE



$$x(t) = \cancel{x_0}^{\,0}\, e^{-k/\gamma_0\, t} + \qquad x_0 = 0$$

$$+ \frac{1}{\gamma_0} \int_0^t dt'' \, e^{-k/\gamma_0 (t-t'')} \left[\xi(t'') + h(t'') \right]$$

$$\langle x(t) \rangle_h = \frac{1}{\gamma_0} \int_0^t dt'' e^{-\frac{k}{\gamma_0}(t-t'')} h(t'')$$

$$R(t,t') = \frac{\delta \langle x(t) \rangle_h}{\delta h(t')} = \frac{1}{\gamma_0} e^{-\frac{k}{\gamma_0}(t-t')} \theta(t-t')$$

<u>STATIONARY</u>

<u>CAUSAL</u>

$$\boxed{\tau \equiv \frac{\gamma_0}{k}}$$ TIME SCALE FOR RELAX.

NB $\tau \xrightarrow[k \to 0]{} \infty$ $\quad \tau \xrightarrow[\gamma_0 \to 0]{} 0$

$$\langle x(t)\, x(t')\rangle_{h=0} = \frac{k_BT}{k}\left[e^{-\frac{k}{\gamma_0}|t-t'|} - \underbrace{e^{-\frac{k}{\gamma_0}(t+t')}}_{} \right]$$

STATIONARY

SAME $\tau = \frac{\gamma_0}{k}$

$\to 0$

$t+t' \to 0$

iff $k \neq 0$

$$\frac{\partial \langle x(t)\, x(t')\rangle}{\partial t'} \xrightarrow[t>t']{} \frac{k_BT}{\cancel{k}} \, \frac{\cancel{k}}{\gamma_0} \, e^{-\frac{k}{\gamma_0}(t-t')}$$

$$\partial_{t'} C(t,t') = \frac{k_BT}{\gamma_0} e^{-\frac{k}{\gamma_0}(t-t')}$$

cfr

$$\boxed{R(t,t') = \frac{1}{k_BT} \partial_{t'} C(t,t') \qquad \text{FDT}}$$

ex. CHECK WHERE THIS FAILS IF $k=0$

$$\chi(t,t') = \int_{t'}^{t} dt'' \, R(t,t'') = \frac{1}{k_BT}\left[C(t,t) - C(t,t')\right]$$

WHAT HAPPENS IF YOU COUPLE THE SAME PART TO TWO BATHS ?

$$\Gamma_f(t-t') = 2\gamma_f\,\delta(t-t') \qquad k_BT_f$$

$$\wedge$$

$$\Gamma_s(t-t') = \frac{\gamma_s}{\tau_s}\, e^{-|t-t'|/\tau_s} \qquad k_BT_s$$

WE HAD $\tau = \dfrac{\gamma_0}{k}$ BUT NOW IT'S CLEAR THAT IT'LL BE MORE COMPLEX

$$\int_0^t dt' \left[\Gamma_s(t-t') + 2\gamma_0\,\delta(t-t')\right] \dot{x}(t') =$$

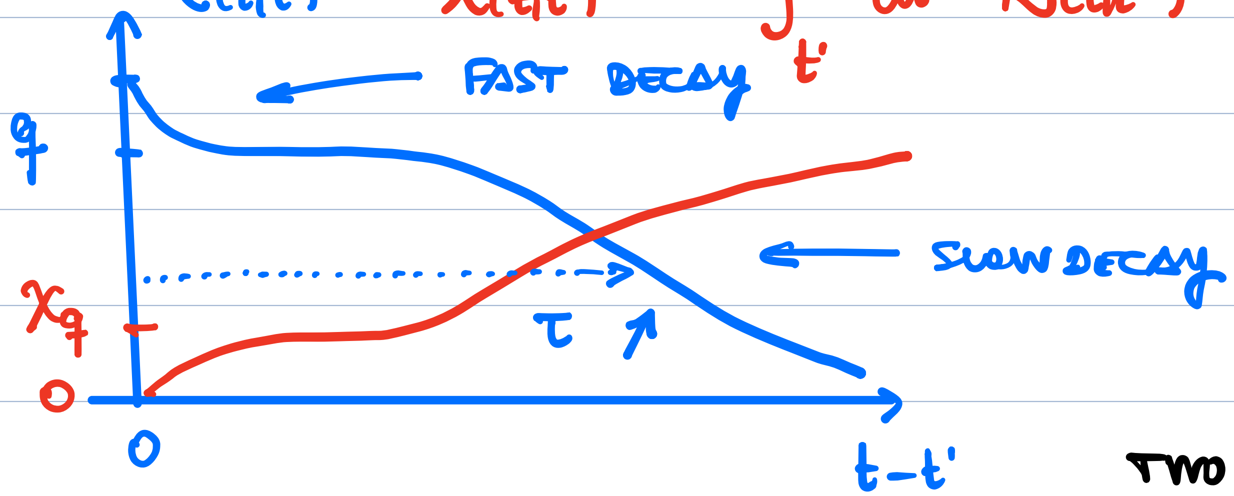
$$= -k\,x(t) + \xi(t) + h(t)$$

STILL A LINEAR 1st ORDER DIFF EQ.
SOLVABLE ANALYTICALLY.

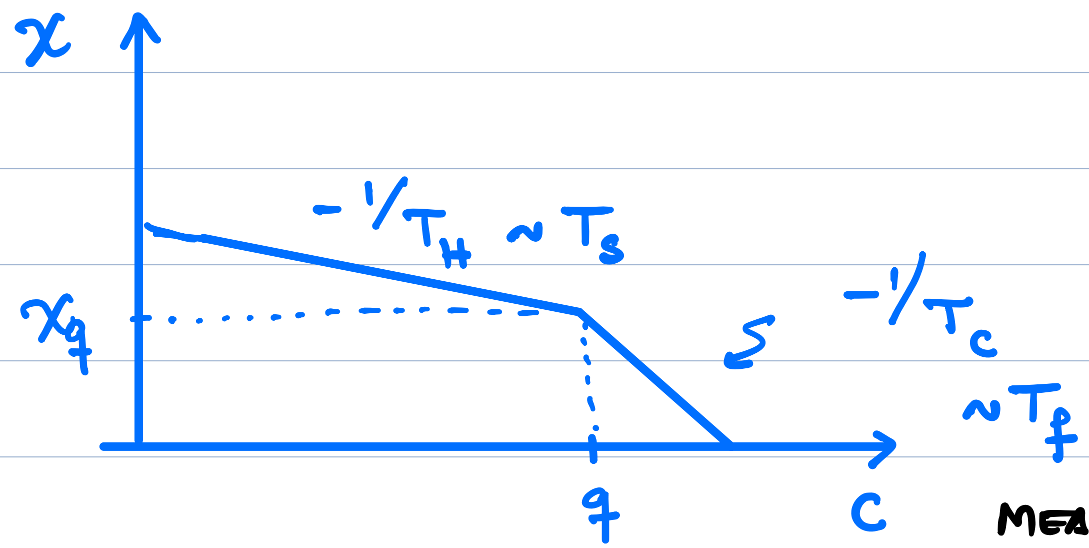
TRICK : SET $t_0 = 0$ TO $t_0 = -\infty$, $\gamma = \theta(t-t')\,\Gamma(t-t')$ IN FRICTION AND USE FOURIER TRANSF.

DEPENDING ON PARAMETERS

$$X(t,t') = \int_{t'}^{t} dt'' \, R(t,t'')$$



TWO STEP RELAX.



WITH TWO TEMP OBS IN FDT MEASUREM.

$$\frac{1}{T_{eff}(\omega)} = \frac{\gamma_f\,(1+\omega^2\tau_s^2) + \gamma_s}{T_f\,\gamma_f\,(1+\omega^2\tau_s^2) + T_s\,\gamma_s}$$

FINALLY COMPLEX SYSTEMS, BEYOND SINGLE VARIABLE.

DMFM WHAT IS IT ?

METHODS TO DEAL WITH DYNAMICS IN MEAN-FIELD.

GROSSO MODO. SINGLE OUT ONE VARIABLE OF YOUR WHOLE SYSTEM AND IDENTIFY SELF-CONSISTENTLY WHICH IS THE FRICTION & NOISE-NOISE KERNEL THAT SHOULD GO IN THE GEN. LANG. EQ.

$$m \ddot{x} + \gamma_0 \dot{x} = \int_0^t dt' \, \Sigma(t,t') \, x(t') + \xi + \eta$$

$$\langle \eta(t) \, \eta(t') \rangle = D(t,t')$$

NB Σ IS LIKE $\dfrac{d\Gamma(t-t')}{dt'}$ A RESPONSE

D IS LIKE $\Gamma(t-t')$ A CORR

WHO ARE Σ AND D ?

IN "MEAN-FIELD" MODELS THEY ARE FCTS OF $\{C, R\}$

$$C = \langle x(t)\, x(t')\rangle$$

$$R = \frac{\delta \langle x(t)\rangle}{\delta h(t')}$$

OF COURSE, THIS IS A MUCH HARDER PROBLEM TO SOLVE!

- SIMILAR STRUCT. CLASSICALLY & QUANTUM

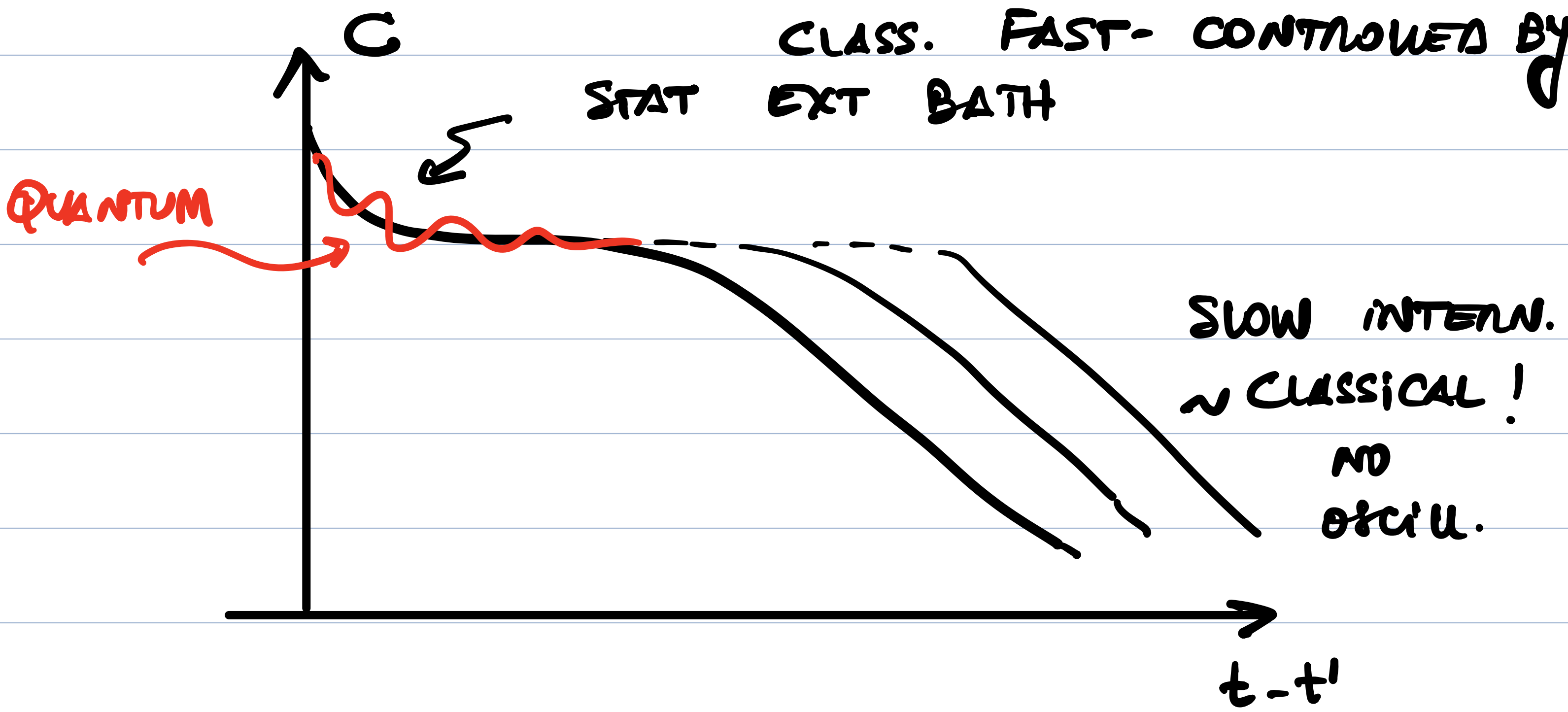
- CAN BE PUT IN A COMPUTER

- CAN BE USED TO WRITE CLOSED EQS. FOR $\{C, R\}$ $\longrightarrow$ SCHWINGER-DYSON EQS. BBGKY HIERARCHY CLOSED FOR MF MODELS

PHYSICS ? GLASSY MODELS.

TWO (OR MORE) SCALE RELAX. $\Rightarrow$

SELF INDUCED.



$$\chi(C) = \lim_{\substack{t' \to \infty \\ C(t,t') \text{ FIXED}}} \chi(t,t')$$

CLASSICAL

$$-\frac{1}{T_{eff}(C)} = \frac{d\chi(C)}{dC} \quad \begin{cases} C > q & = -\frac{1}{T} \\ C < q & = -\frac{1}{T^*} \end{cases}$$

QUANTUM $C > q \Rightarrow$ QUANTUM FDT

$C < q \Rightarrow$ CLASSICAL ! $-\frac{1}{T^*}$

4th LECTURE $-\frac{1}{k_B T_{eff}(C)} = X'(C)$

WE DISCUSSED OPEN GLASSY DYN T_{eff}

PIECEWISE CONST TEMP. INTERP. LFC, JK, PELITI

$$\lim_{N\to\infty} t_{eq}(N; T_f) \longrightarrow \infty$$ MODELS

IN PRACTICE BECOMES TOO LONG BELOW $T_f = T_d$

AGING EFFECTS AND FDT VIOLATIONS T^*

A SINGLE $T^* > T_f$

FOR HIGH T_0 TO LOW T_f QUENCHES

BUT AT THE BEGINING OF THE LECTURES I TALKED ABOUT CLOSED SYSTEMS. WHAT ABOUT THEM ? cf JK's & VULPIANI's LECT

QUANTUM QUENCHES

$|\psi_0\rangle$ EVOLVE WITH $\hat{H}$ $(\neq \hat{H}_0)$

$|\psi(t)\rangle$ WHAT HAPPENS ?

DOES THE REST ACT AS A BATH ON S ?

IN WHICH CASES ?

NON INTEGRABLE vs. INTEGRABLE

$\Downarrow$ $\Downarrow$

SHOULD ?

RESTRICTED BOLTZMANN GGE

$$\frac{e^{-\beta \hat{H}_{syst}}}{Z} \qquad \frac{e^{-\sum_\mu \gamma_\mu \hat{I}_\mu}}{Z(\{\gamma_\mu\})}$$

WHY NOT ASK THE SAME QUESTION CLASSICALLY ?

WE CAN CONSIDER THE SAME KIND OF MODELS THAT YIELD GLASSY DYN WHEN OPEN, NON CLOSED.

$$H_J[\{x_\mu\}] = -\sum_{\mu\neq\nu\neq\rho} J_{\mu\nu\rho}\, x_\mu x_\nu x_\rho + z\left(\sum_\mu x_\mu^2 - N\right)$$

$[J_{\mu\nu\rho}] = 0$

$[J_{\mu\nu\rho}^2] \propto \dfrac{J^2}{N^2}$

WE KNOW THAT WHEN COUPLED TO WHITE NOISE BATH

$\downarrow$

GLASSY BELOW T_d

GIVE THEM NEWTON DYN EXPECTED NON INTEGRABLE

$P_0(\{x_\mu(0), p_\mu(0)\})$?

- DRAW i.c. FROM P_0

eg. MAXWELL - BOLTZMANN AT T_0

- EVOLVE W/ H ($\neq H_0$) : A QUENCH

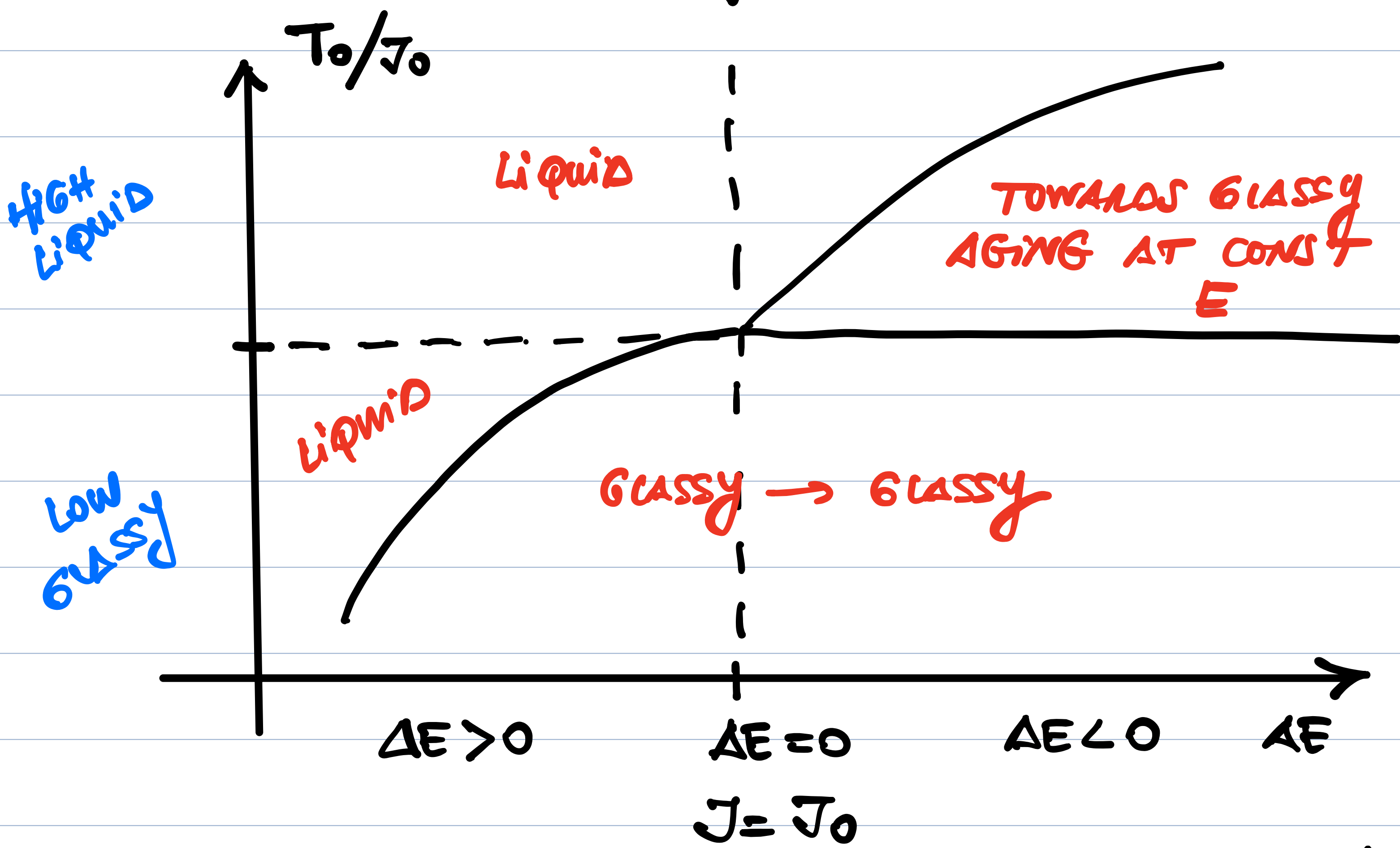
eg

$$J_0 \to J$$

IN INTERNAL PARAM.
IT'S WHAT QUANTUM PEOPLE DO.

- WHAT HAPPENS ?

NORMAL EQUIL. MAXWELL BOLTZMANN EQ.



CLASSICAL INTEGRABLE MODEL $\{I_\mu, \mu = 1 \dots N\}$

$$H = \sum_a^{N_0} \frac{m\omega_a^2}{2} x_a^2 + \sum_a^{N_0} \frac{p_a^2}{2m} \qquad \left(-h\sum_a x_a\right)$$

UNCOUPLED HARM OSC. $\Longrightarrow$ INTEGRABLE !

N_0 CONST OF MOTION, THE N_0 ENERGIES OF EACH OSCILLATOR

$$e_a(0) = \frac{m\omega_a^2}{2} x_a^2(0) + \frac{p_a^2(0)}{2m} = \text{CONST.}$$

SOLUTION TO NEWTON EQS.

$$m\ddot{x}_a = -m\omega_a^2 x_a + h_a$$

FORCING TO CALCULATE LINEAR RESPONSE

$$x_a(t) = \text{i.c. DEP}_a + \int_0^t dt' \underbrace{G_a(t-t')}\, h_a(t')$$

$$R_a(t,t') = G_a(t-t') \qquad \frac{\sin\omega_a(t-t')}{m\omega_a}$$

$$\overline{C_a}(t,t') = \frac{1}{2m\omega_0^2} \underbrace{\left(m\omega_0^2 x_0 + \frac{p_0^2}{m\omega_0^2}\right)}_{2e_a(0)} \cos\omega_a(t-t')$$

AFTER AVERAGING OVER A TIME WINDOW AROUND t' TO GET RID OF NON-STAT TERMS

CLASSICAL FDT IN FOURIER

$$\underbrace{\mathrm{Im}\,\tilde{R}(\omega)} = \frac{1}{2k_BT}\underbrace{\omega\,\tilde{C}(\omega)}$$

WHAT DO WE GET HERE IF WE COMPARE ?

$$\mathrm{Im}\,\tilde{R}(\omega) = \frac{\pi}{2m\omega_a}\,\delta(\omega-\omega_a)$$

$$\omega\,\tilde{C}(\omega) = \frac{\pi\omega}{2m\omega_a^2}\,2e_a(0)\,\delta(\omega-\omega_a)$$

AT $\omega=\omega_a$

$$\boxed{\frac{2\,\mathrm{Im}\,\tilde{R}(\omega)}{\omega\,\tilde{C}(\omega)} = \frac{1}{e_a(0)} = \frac{1}{k_BT_{eff}}}$$

AND IF WE CALCULATE β_{GGE} ?

$$P_{GGE}(\{x,p\}) = \frac{e^{-\beta_{GGE} H(\{x,p\})}}{Z}$$

CONDITION

$$e(0) = \int dx \int dp \; P_{GGE}(\{x,p\}) \, H(\{x,p\})$$

$$\Longrightarrow \boxed{e(0) = \beta_{GGE}^{-1} = k_B T_{GGE}}$$

SO WE GET $\boxed{T_{eff} = T_{GGE}}$

IF WE REPEAT FOR AN ENSEMBLE OF OSC w/ $\neq \omega$ WE'LL GET

$$\boxed{T_{eff}(\omega) = T_{GGE}(\omega)}$$

AND ENSEMBLE OF OSC IS INTEGRABLE

NB ω "LABELS" THE CONS QUANT $\Sigma_0(\omega)$

OBSERVED QUANTUM FOR TRANSV ISING CHAIN $\Rightarrow$ MAPPABLE TO FREE FERMIONS

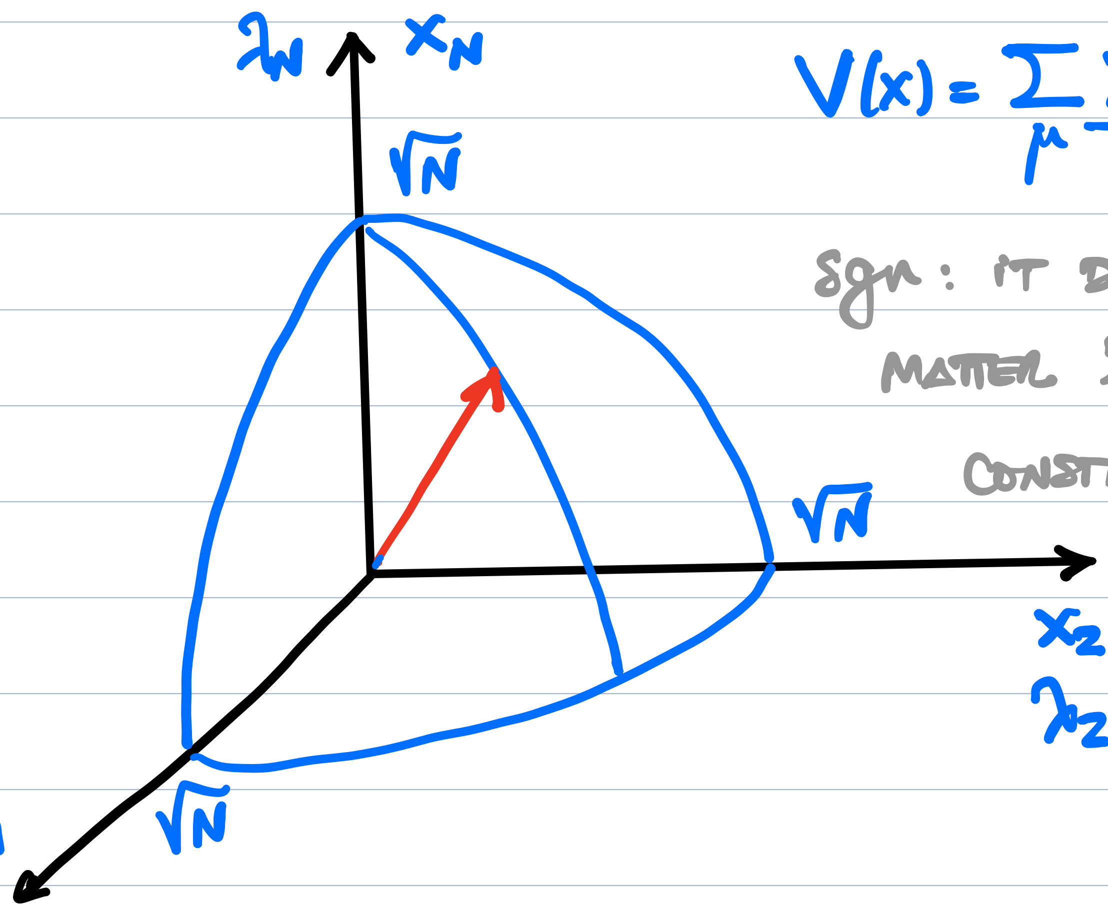
A NON TRIVIAL CLASSICAL INTEGRABLE MODEL WITH A GGE DESCRIPTION

$$\dot{x}_\mu = p_\mu / m$$

NEWTON / HAMILTON

$$\dot{p}_\mu = \lambda_\mu x_\mu - z(\vec{x}, \vec{p})\, x_\mu$$

WITH $\sum_\mu x_\mu^2 = N$ SAY $\lambda_1, \lambda_2 \dots \lambda_N$



$$V(x) = \sum_\mu \frac{\lambda_\mu}{2} x_\mu^2$$

sgn: IT DOESN'T MATTER $\{\lambda_\mu \geq 0\}$

CONSTRAINT BOUNDS MOTION

$$\mathcal{Z}(\vec{x},\vec{p}) = \frac{p^2}{2m} - V(x) = \frac{p^2}{2m} + \sum_\mu \frac{\lambda_\mu}{2} x_\mu^2$$

THIS IS NOT A LINEAR SET OF EQ

IT IS NOT A SET OF INDEP. HARM. OSC.

INTEGRABLE MODEL

$$I_\mu = x_\mu^2 + \frac{1}{mN} \sum_{\nu(\neq\mu)} \frac{(x_\mu p_\nu - x_\nu p_\mu)^2}{\lambda_\nu - \lambda_\mu}$$

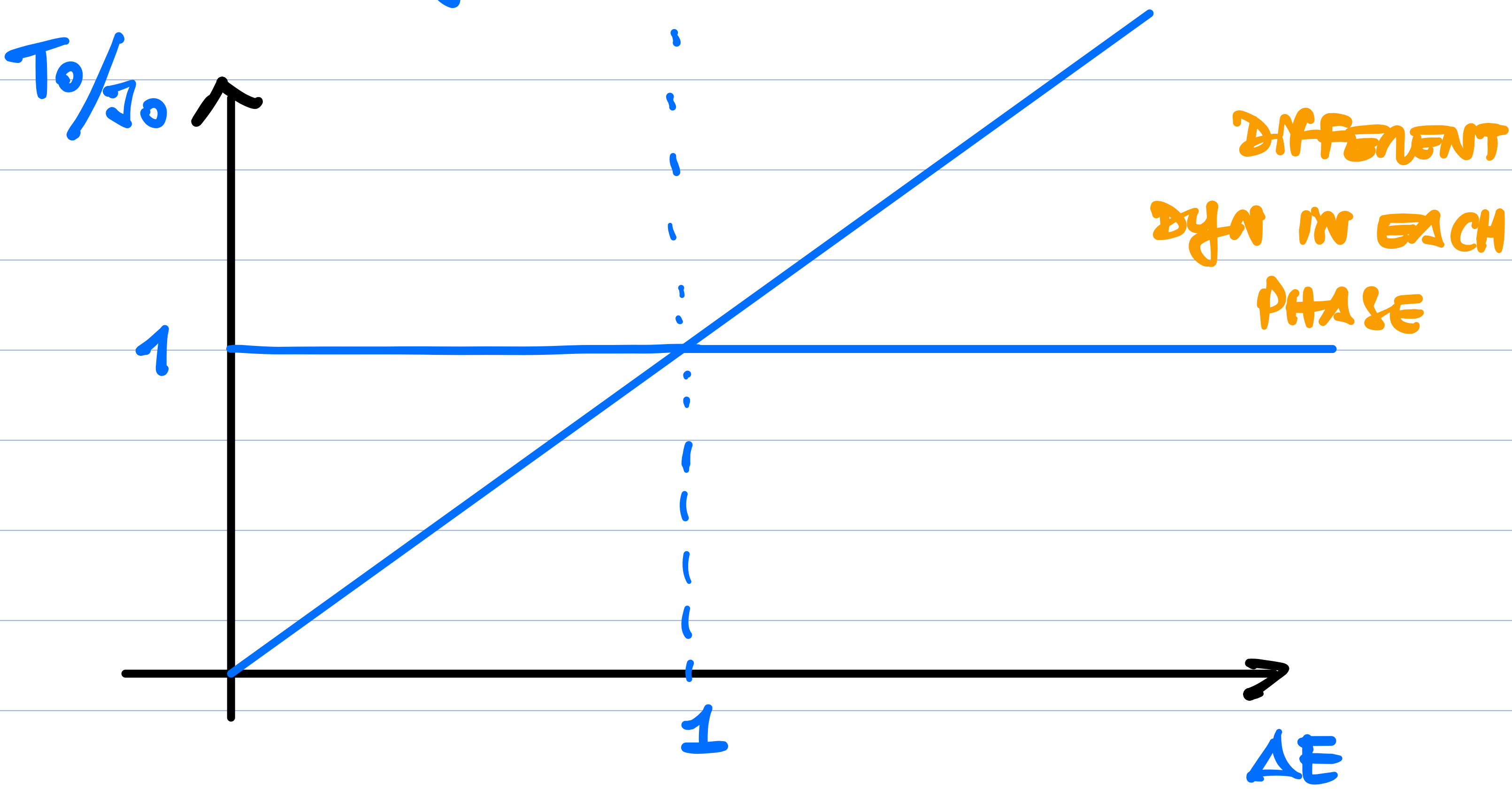
$$\frac{dI_\mu}{dt} = 0$$

WHAT HAPPENS UNDER QUENCHES?

$$P_0(\{\vec{x},\vec{p}\}) = \frac{\exp\{-\beta_0 H_0(\{\vec{x},\vec{p}\})\}}{Z}$$

EVOLVE WITH H $\quad \{\lambda_\mu^{(0)} \to \lambda_\mu\}$

SOLVE DYN



J_0 SCALE OF $\{\lambda_\mu^{(0)}\}$

$\Delta E = J/J_0$ $\quad$ J SCALE OF $\{\lambda_\mu\}$

$$\lim_{t\to\infty} \overline{x_\mu^2(t)}$$ ← TIME AVERAGE

$$= \langle x_\mu^2 \rangle_{GGE}$$ ← GGE AVERAGE

$\forall \mu$ ALSO p_μ !

$$\mathcal{P}_{GGE}(\vec{x},\vec{p}) = \frac{e^{-\sum_\mu \gamma_\mu I_\mu}}{\mathcal{Z}}$$

γ_μ FIXED FROM

$$\langle I_\mu \rangle_{GGE} = I_\mu(0) \qquad \forall \mu$$

SO, THE LONG TIMES LIMIT APPROACHES
THE GGE

WHAT ABOUT GEN. FDT ?

$$\frac{2\,\mathrm{Im}\,\tilde{R}(\omega)}{\omega\,\tilde{C}(\omega)} = \beta_{eff}(\omega) = \frac{1}{k_B T_{eff}(\omega)}$$

CALCULATE IT DURING
DYNAMICS

AND EVALUATE IT AT

$$m\,\omega_\mu^2 = \left[\lim_{t\to\infty} z(t) - \lambda_\mu\right] \geq 0 \quad \checkmark$$

EITHER FOR $\mu = 1 \ldots, N$ FINITE

OR FOR $N \to \infty$

COMPARE TO

$$m \langle p_\mu^2 \rangle_{GGE} = k_B T_\mu^{GGE}$$

IT WORKS ! IN ALL PHASES.

IN THE END THE MODES GET UNCOUPLED, EACH W/ ITS OWN T_μ BUT FIXED BY ALL THE OTHERS.

SOLVABLE - NON TRIVIAL EX FOR WHICH GGE DESCRIPTION WORKS

PHASE TRANSITIONS & NON-TRIVIAL DYN

$T_{eff}(\omega)$ → ALSO HAS A MEANING