Dynamics of Glassy Systems :

Lessons from a Family of Solvable Models

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General description

Plan

— Many-body systems out of equilibrium

Collective non-equilibrium relaxation

simple : *e.g.*, domain growth coarsening & the growing length

hard : glasses & spin-glasses, computer science, ecology, etc.

— Characterisation of the spontaneous and perturbed global relaxation

self-correlation and linear response

— Analytic description - dynamic mean-field theory

models and equations

separation of time scales & aging

effective temperatures

time-reparametrization invariance & fluctuations

Many-Body Systems Out of Equilibrium

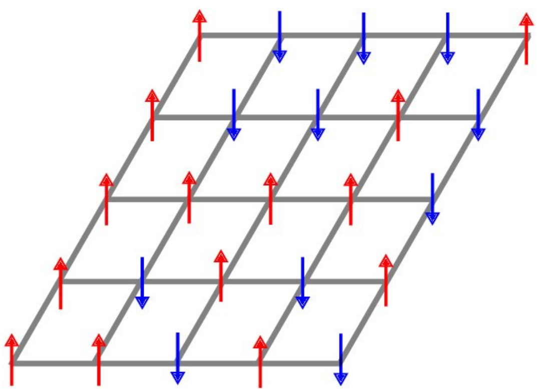
some examples

Many-body systems

Out of Equilibrium

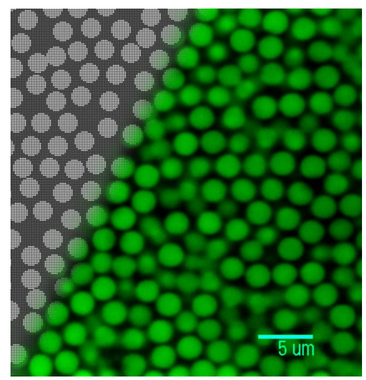
Ferromagnetic Ising Model

$$\mathcal{V} = -J \sum_{\langle ij \rangle} s_i s_j$$



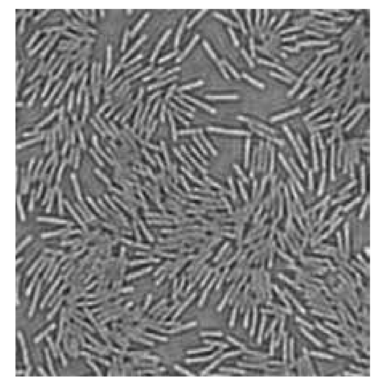
Particles in Interaction

$$\mathcal{V} = \sum_{i \neq j} V(r_{ij})$$



Active Matter

$$\vec{\mathcal{F}}_i \neq -\vec{\nabla}_i \mathcal{V}$$



in relaxation

driven

In physical systems the action-reaction principle is respected

beyond physics not necessarily, $\vec{\mathcal{F}}_{i \to j} \neq \vec{\mathcal{F}}_{j \to i}$ like in **ecosystems, markets**, etc.

The equilibration time of **macroscopic coarsening & glassy systems** in a wide range of parameters,

e.g. low temperatures,

goes beyond the experimental window t_{exp}

$$\lim_{N \gg 1} t_{\mathrm{eq}}(N) \gg t_{\mathrm{exp}}$$

t_{eq} grows with the system size N

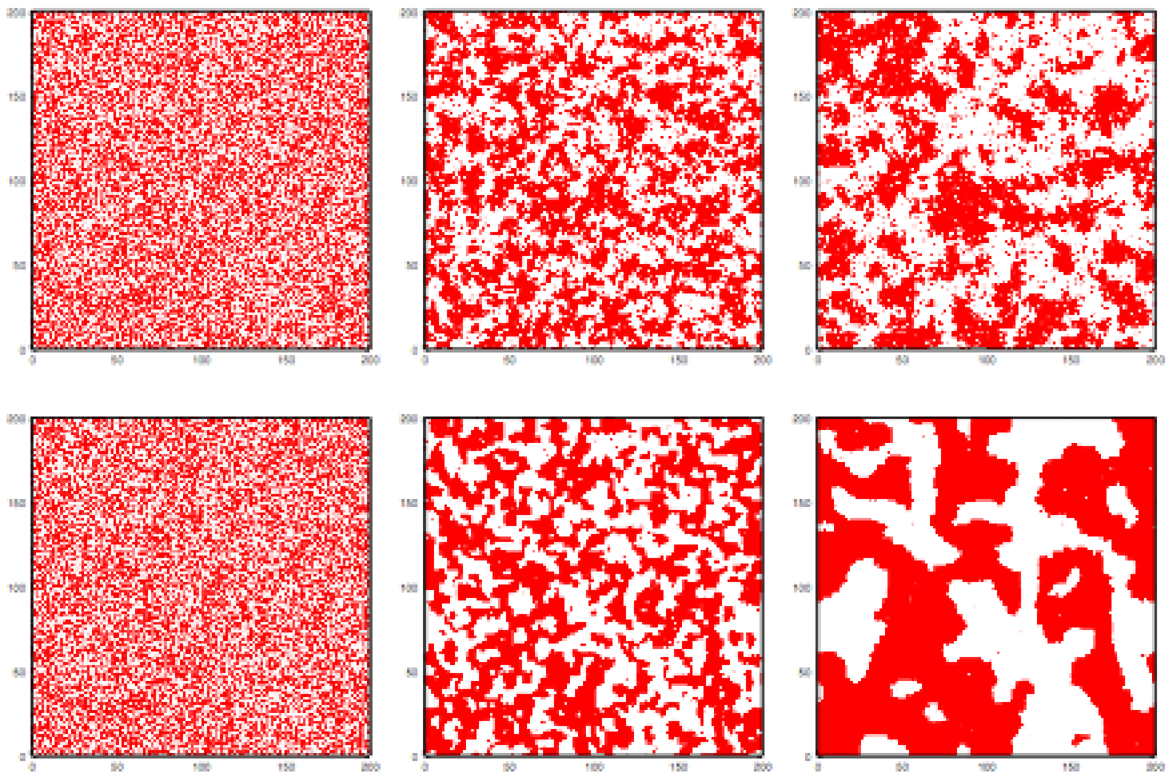
Collective Non-Equilibrium Relaxation

the simplest example, coarsening

$2d$ Ising model

Snapshots after an instantaneous quench from $T_0 \to \infty$ to $T \leq T_c$

$t = 0$ t_1 t_2



$T = T_c$

$T < T_c$

At $T = T_c$ **critical dynamics**

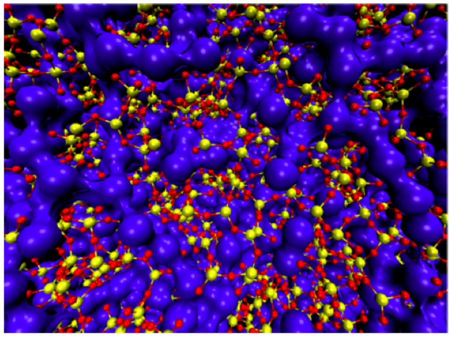
At $T < T_c$ **coarsening**

Dynamic scaling with a single characteristic growing length $\mathcal{R}(t) \ll L$

Collective Non-Equilibrium Relaxation

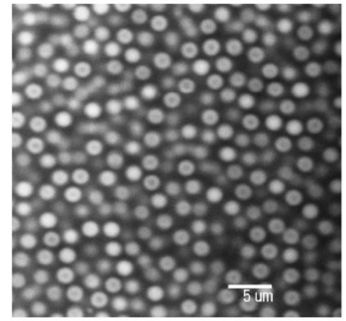
harder cases : glasses & spin glasses

Glasses & Spin Glasses



Simulations

Molecular (Sodium Silicate)

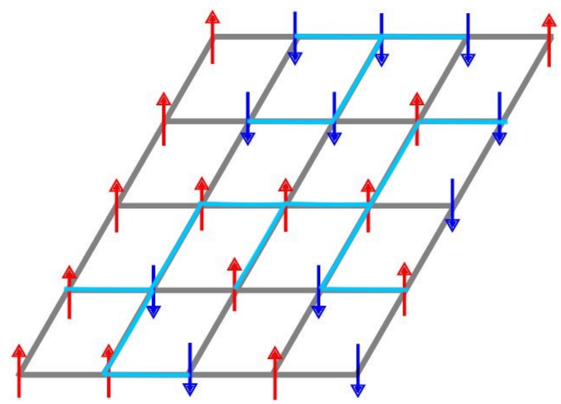


Confocal microscopy

Colloids (e.g. $d \sim 162$ nm in water)

Experiments

Granular matter



Exp & Simulations

Spin glasses

Characterisation of the Collective Relaxation

when there is no “visible” length

Global Observables

two-time correlations and linear responses

Two-time dependencies

Self displacement & correlation – integrated linear response

$$\Delta^2(t,t') \equiv \frac{1}{N}\sum_i [\langle (x_i(t) - x_i(t'))^2 \rangle] \qquad \text{Displacement}$$

$$C(t,t') \equiv \frac{1}{N}\sum_i [\langle x_i(t) x_i(t') \rangle] \qquad \text{Correlation}$$

Unperturbed

$$\sigma(t,t') \equiv \frac{1}{N}\sum_i \int_0^{t'} dt''\, R_i(t,t'') = \frac{1}{N}\sum_i \int_0^{t'} dt'' \; [\left. \frac{\delta \langle x_i(t) \rangle_h}{\delta h_i(t'')} \right|_{h=0}]$$

Extend the notion of order parameter

They are not related by FDT out of equilibrium

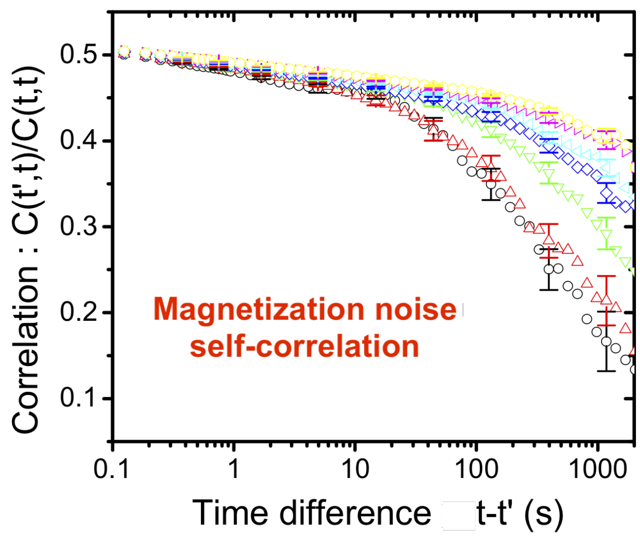
The averages are thermal (and over initial conditions) $\langle \ldots \rangle$

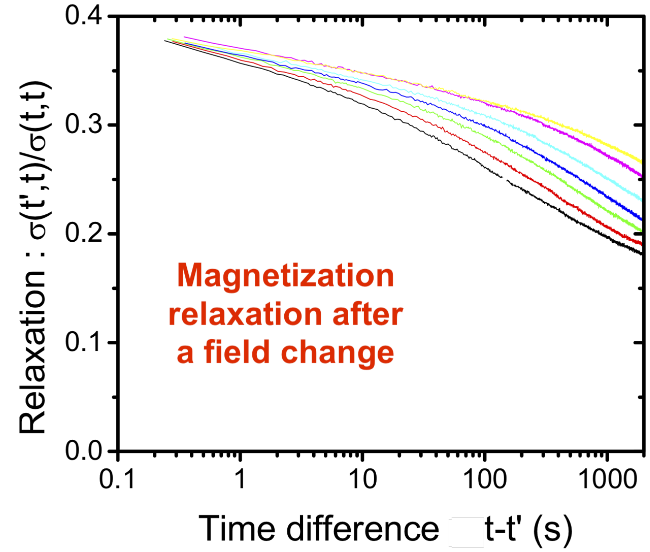
and over quenched randomness $[\ldots]$ (if present)

t' "waiting-time" and t "measuring-time" after preparation

Slow relaxation & Aging

Loss of Stationarity





Correlation **Integrated linear response**

0 t' t' t' t' time

Different curves are measured after log-spaced times t' after the quench

breakdown of stationarity $\Longrightarrow$ **aging**, far from equilibrium

No identifiable growing length $\mathcal{R}(t)$

microscopic mechanisms ?

Spin-glass experiments **Hérisson & Ocio 02-04**

Physical Aging

in words

Older systems (more time elapsed after the quench, longer t')

relax **more slowly** than younger ones

Breakdown of stationarity of correlation & integrated response

$$C(t,t') \neq C(t-t') \qquad \sigma(t,t') \neq \sigma(t-t')$$

In each regime – $\underbrace{\text{rapid}}_{\text{above plateau}}$ and $\underbrace{\text{slow}}_{\text{below plateau}}$ – there is scaling*

$$C(t,t') = C_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right) \qquad \sigma(t,t') = \sigma_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right)$$

* proven from general properties of temporal correlation functions

LFC & Kurchan 94

but no obvious interpretation of $\mathcal{R}(t)$ in glassy systems

Physical Aging & Memory

in words

Older systems (more time elapsed after the quench, longer t')

relax **more slowly** than younger ones

Breakdown of stationarity of correlation & integrated response

$C(t,t') \neq C(t-t') \qquad \sigma(t,t') \neq \sigma(t-t')$ **of the same magnitude**

In each regime, rapid and slow, there is scaling*

$$C(t,t') = C_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right) \qquad \sigma(t,t') = \sigma_{r,s}\left(\frac{\mathcal{R}_{r,s}(t)}{\mathcal{R}_{r,s}(t')}\right)$$

* proven from general properties of temporal correlation functions

LFC & Kurchan 94

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Mean-Field Modelling

Usual Curie-Weiss for PM-FM

Unusual for Glasses

Mean-Field Modelling

Classical p-spin Spherical Models

Potential energy

$$\mathcal{V} = -\sum_{i_1 \neq \ldots \neq i_p} J_{i_1 \ldots i_p} x_{i_1} \ldots x_{i_p} \qquad\qquad p \text{ integer}$$

quenched random couplings $J_{i_1 \ldots i_p}$ drawn from a Gaussian $P[\{J_{i_1 \ldots i_p}\}]$

(over-damped) **Langevin dynamics** for continuous spins $x_i \in \mathbb{R}$

coupled to a white bath $\langle \xi(t) \rangle = 0$ and $\langle \xi(t)\xi(t') \rangle = 2\gamma k_B T \delta(t-t')$

$$\gamma \frac{dx_i}{dt} = -\frac{\delta \mathcal{V}}{\delta x_i} + z_t x_i + \xi_i$$

z_t is a Lagrange multiplier that fixes the spherical constraint $\sum_{i=1}^{N} x_i^2 = N$

$p = 2$ mean-field **coarsening**

$p \geq 3$ RFOT modelling of **glasses**

Kirkpatrick, Thirumalai & Wolynes 87-89

Dynamic equations

Integro-differential eqs. on the correlation and linear response

In the $N \to \infty$ limit exact and closed causal Schwinger-Dyson equations

(Average over randomness, random initial conditions and thermal noise)

$$(\gamma\partial_t - z_t)C(t,t') = \int dt'' \left[\Sigma(t,t'')C(t'',t') + D(t,t'')R(t',t'')\right] + 2\gamma k_B T R(t',t)$$

$$(\gamma\partial_t - z_t)R(t,t') = \int dt'' \, \Sigma(t,t'')R(t'',t') + \delta(t-t')$$

where Σ and D are the self-energy and vertex, which for p spin models read

$$D(t,t') = \frac{p}{2} C^{p-1}(t,t') \qquad \Sigma(t,t') = \frac{p(p-1)}{2} C^{p-2}(t,t')\, R(t,t')$$

z_t is fixed by $C(t,t) = 1$

Sompolinsky & Zippelius 82, LFC & Kurchan 93

Similar to Mode-Coupling Theory for liquids **Götze et al 80s** or DMFT for quantum systems **Georges & Kotliar 90s**, but not necessarily in equilibrium

How to solve these equations ?

Input from numerical solutions $\Longrightarrow$

Asymptotic Ansatz

Weak ergodicity breaking

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} C(t,t') = q_{\rm EA} \qquad\qquad \lim_{t\gg t'} C(t,t') = 0$$

Bouchaud 92

Weak long-term memory

$$\lim_{t-t'\to\infty} \lim_{t'\to\infty} R(t,t') \simeq 0$$

but

$$\sigma(t,t') = \int_0^{t'} dt''\, R(t,t'') \longrightarrow f(C(t,t')) = \text{finite}$$

LFC & Kurchan 93

allow us to solve the integro-differential eqs. asymptotically

Weak ergodicity breaking

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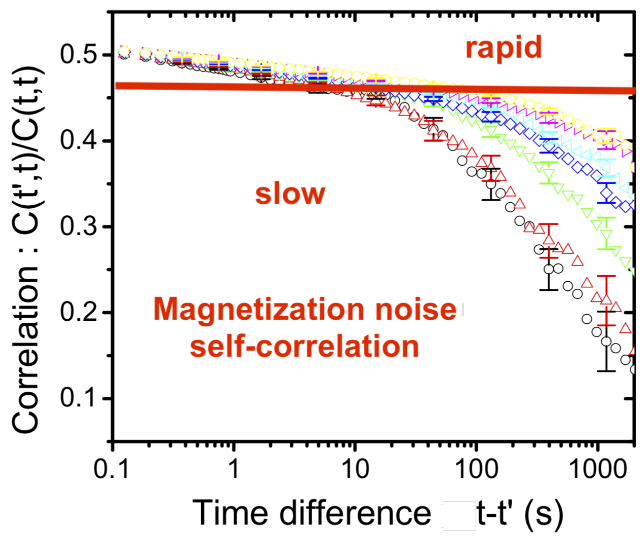
LFC & Kurchan 93

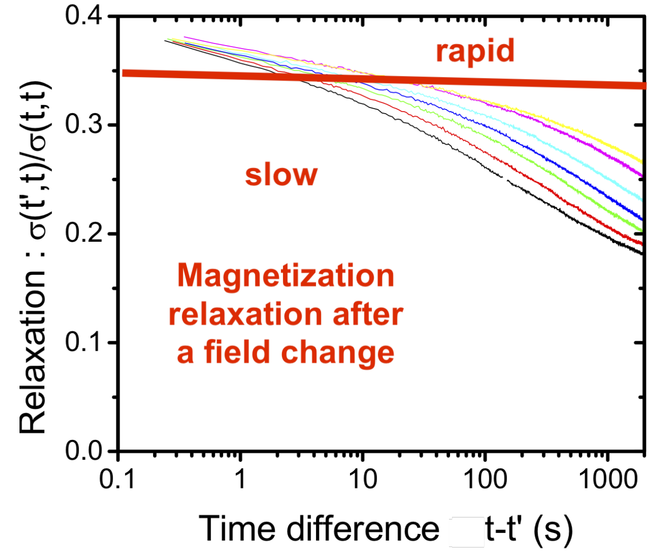
and capture aging

Slow relaxation & Aging

Loss of Stationarity



Correlation



Integrated linear response

$0 \quad t' \quad t' \quad t' \quad t' \quad$ time

Different curves are measured after log-spaced times t' after the quench

breakdown of stationarity $\Longrightarrow$ **aging**, far from equilibrium

microscopic mechanisms ?

Spin-glass experiments **Hérisson & Ocio 02-04**

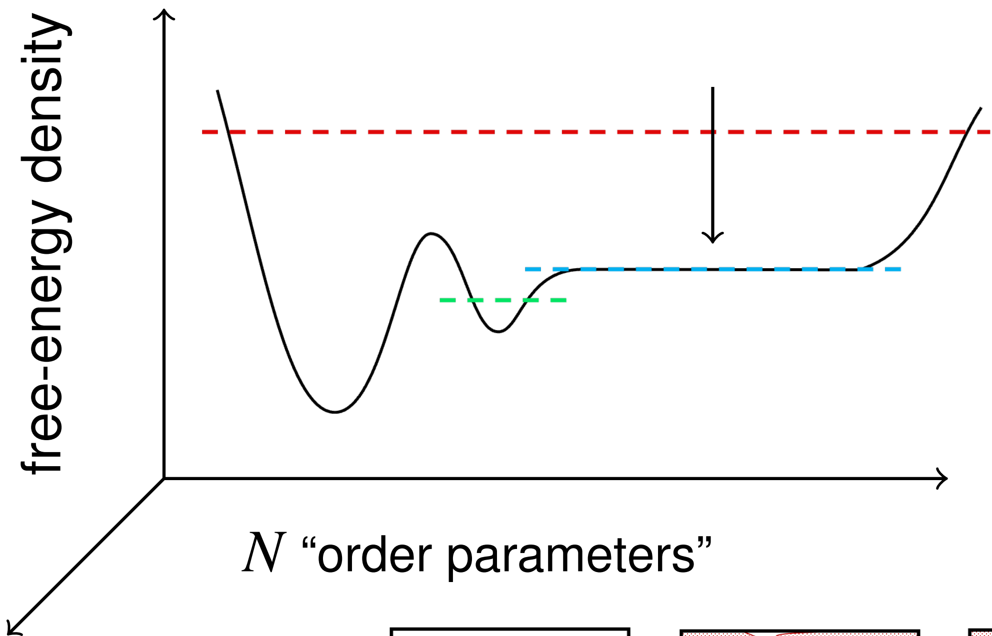
Interpretation

Complex Landscapes

Beyond Ginzburg-Landau

TAP Free-energy Landscape

$p \geq 3$ spin models

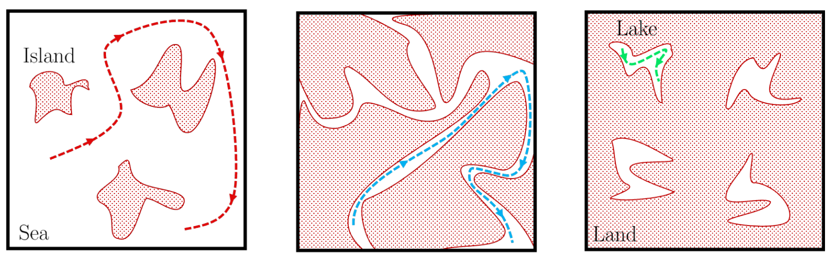


mostly maxima

threshold approached from quench

$$\lim_{t\to\infty} e(t) = e_{\rm th}$$

mostly minima

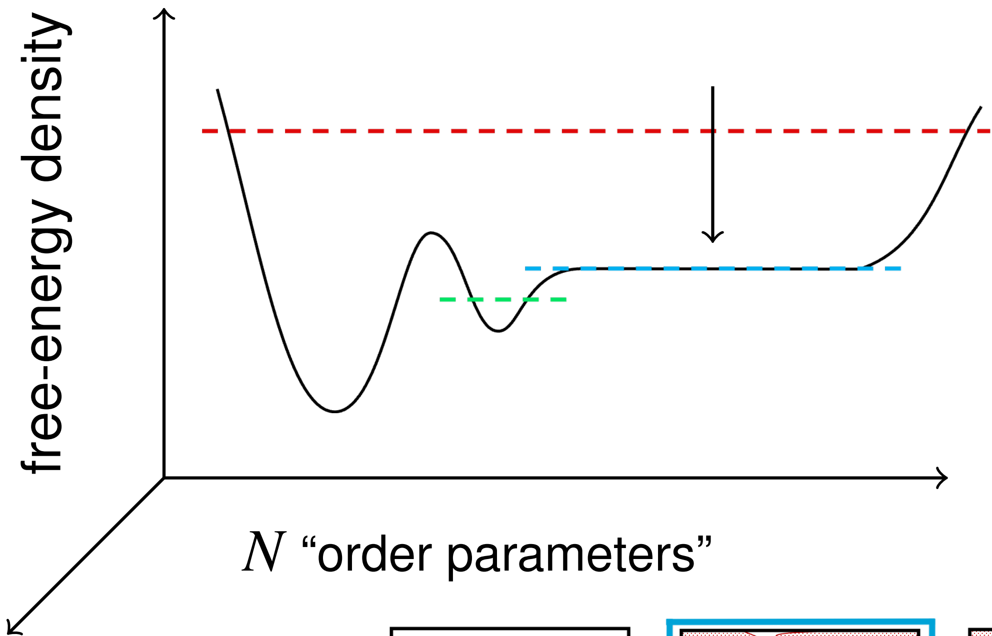


The dynamics is linked to the topography of the landscape

Both for **physical** and **algorithmic** dynamic rules

TAP Free-energy Landscape

$p \geq 3$ spin models

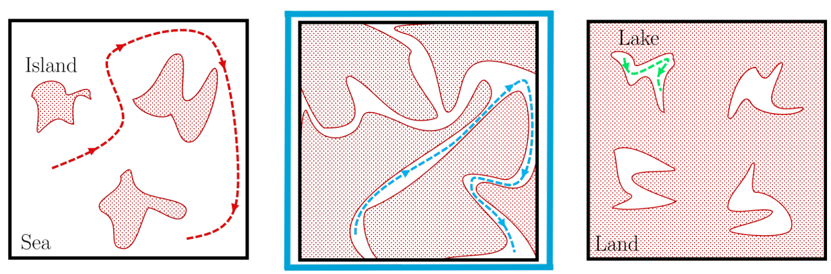


mostly maxima

threshold approached from quench

$$\lim_{t \to \infty} e(t) = e_{\text{th}}$$

mostly minima



Flat threshold as an attractor for the p-spin relaxation

Both for **physical** and **algorithmic** dynamic rules

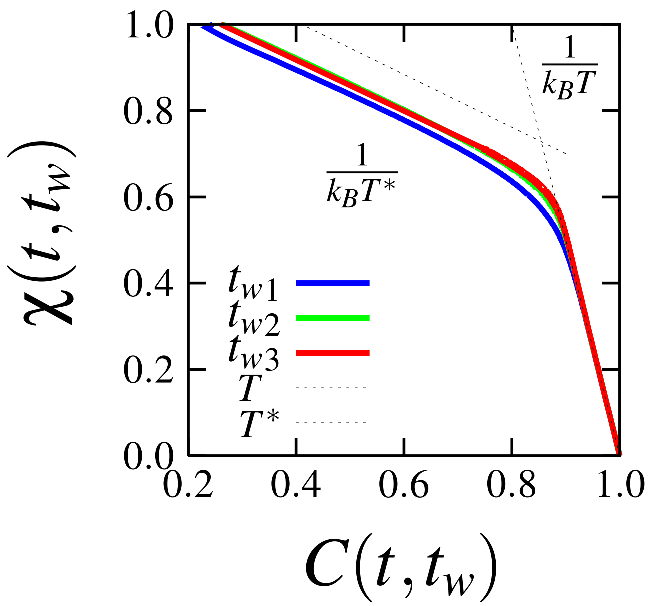
Some surprising predictions

with physical consequences

Fluctuation-dissipation

Induced vs. spontaneous fluctuations in glasses

A quench from $T_0 \to \infty$ to $T < T_c$



parametric construction

t_w fixed

$t_{w_1} < t_{w_2} < t_{w_3}$

$t - t_w$: $0 \to \infty$

used as a parameter

$T^* > T$

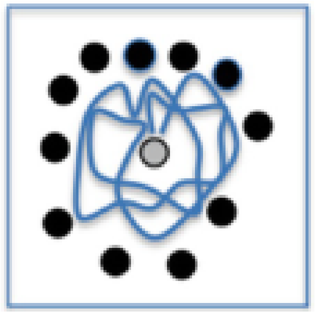
Breakdown of the equilibrium FDT $k_B T \chi = C$

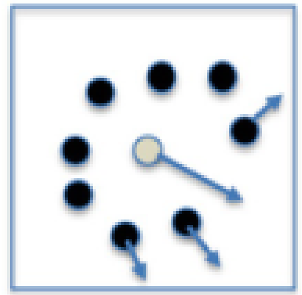
Convergence to $k_B T \chi(C)$, two linear relations for $C \lessgtr q_{ea}$

Mean-field models **LFC & Kurchan 93** & effective temperature interpretation **LFC, Kurchan & Peliti 97**

Fluctuation-dissipation

Interpretation





Short-scale re-arrangements ruled by the **equilibrium external bath** & **local properties of landscape**

The fluctuation-dissipation relation holds with the bath temperature T

Large-scale re-arrangements follow the systems' **internal dynamics** & **large scale props of landscape**

The fluctuation-disspation relation $\underbrace{\text{holds with another temperature } T^*}$

After **cooling** from equilibrium at $T_0 > T_d$, hotter $T^* > T$

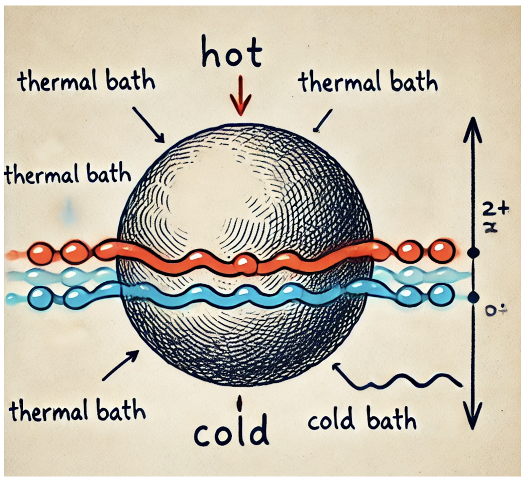
After **heating** from equilibrium at $T_0 < T_d$, colder $T^* < T$

Support for this interpretation :

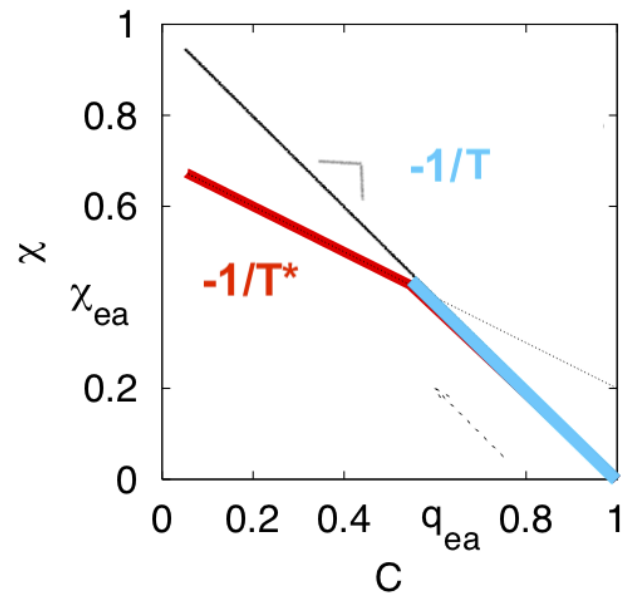
Effective temperatures

Induced by one (or more) baths

Exercise : motion in contact with a complex bath



Sketch created by ChatGPT



$\Gamma = \Gamma_{\rm cold}$ + $\Gamma_{\rm hot}$

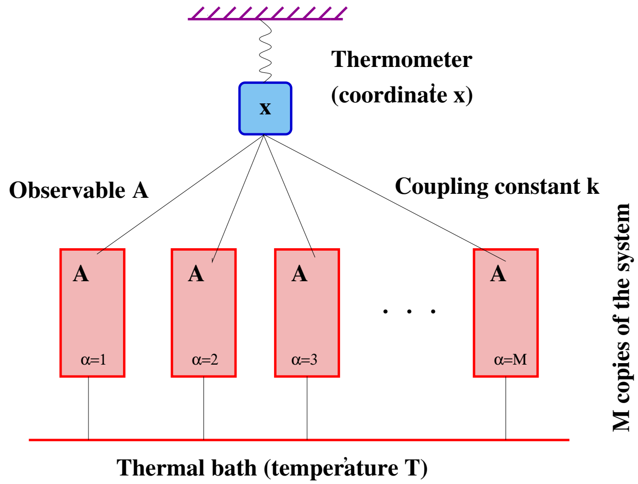
$\Gamma_{\rm cold}(t-t') = 2\gamma\delta(t-t')$ and T $\qquad$ $\Gamma_{\rm hot}(t-t') = \gamma_{\rm hot}\, e^{-(t-t')/\tau}$ and T^*

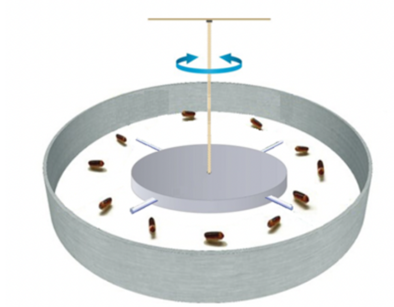
LFC & Kurchan 00, Ilg & Barrat 07, etc., cfr. tracer in pasive & active bath

Effective temperatures

Measurement with thermometers

LFC, Kurchan & Peliti 97





Grigera & Israeloff 99 - glassy

D'Anna, Mayor, Barrat, Loreto & Nori 03 - granular

Boudet, Jagielka, Guerin, Barois, Pistolesi & Kellay 24

artificial active matter - robots

- Short internal time scale fast dynamics is tested and T is recorded.
- Long internal time scale slow dynamics is tested and T^* is recorded.

Related to the phenomenological *fictive temperatures* of **Tool 46, Gardon & Narayanaswamy 70, Moynihan et al 76, etc.** but measurable & with a thermodynamic interpretation

Also appearing in stochastic thermodynamic relations

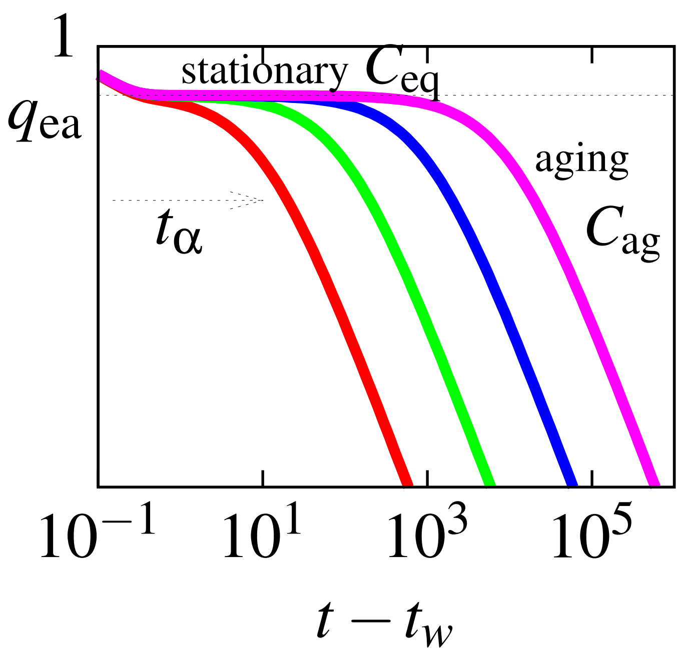
Time reparametrization invariance

and fluctuations

Time reparametrization invariance

In the long t_w limit

Fast $t - t_w \ll t_w$



The aging part is **slow**

$$\mathcal{R}(t)/\mathcal{R}(t_w) = O(1)$$

$$C_{ag}(t,t_w) \sim f_{ag}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$

$$\partial_t C_{ag}(t,t_w) \propto \frac{\dot{\mathcal{R}}(t)}{\mathcal{R}(t)} \xrightarrow[t\to\infty]{} 0$$

$$\boxed{\partial_t C_{ag}(t,t_w) \ll C_{ag}(t,t_w)}$$

Eqs. for the slow relaxation $C_{ag} < q_{ea}$ are invariant under

$$\boxed{t \to h(t) \quad C(t,t_w) \to C(h(t),h(t_w)) \quad R(t,t_w) \to \dot{h}(t')/h(t_w)\, R(h(t),h(t_w))}$$

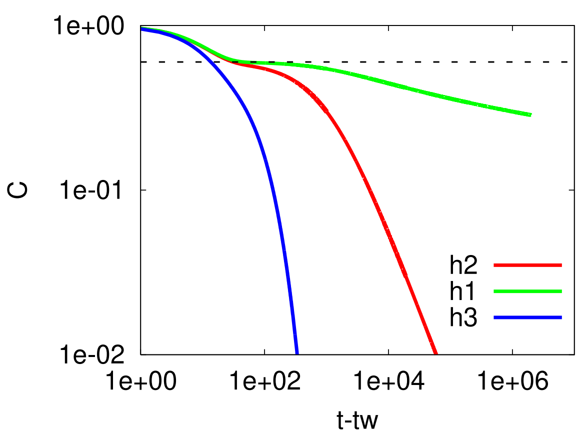
Leading fluctuations

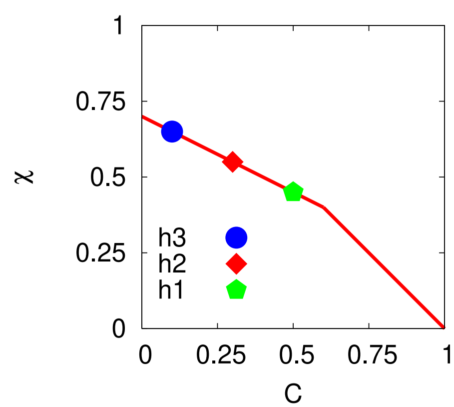
Global to local correlations & linear responses

$$C_{\text{ag}}(t,t_w) \approx f_{\text{ag}}\left(\frac{\mathcal{R}(t)}{\mathcal{R}(t_w)}\right)$$ **global correlation**

Global time-reparametrization invariance $\Rightarrow\ C^{\text{ag}}_{\vec{r}}(t,t_w) \sim f_{\text{ag}}\left(\frac{h_{\vec{r}}(t)}{h_{\vec{r}}(t_w)}\right)$

Ex. $h_{\vec{r}_1} = \frac{t}{t_0}$, $h_{\vec{r}_2} = \ln\left(\frac{t}{t_0}\right)$, $h_{\vec{r}_3} = e^{\ln^{a>1}\left(\frac{t}{t_0}\right)}$ in different spatial regions





Castillo, Chamon, LFC, Iguain & Kennett 02, 03

Chamon, Charbonneau, LFC, Reichman & Sellitto 04

Jaubert, Chamon, LFC & Picco 07

More recent perspective : time-reparametrization invariance in SYK models

Kitaev 15, Maldacena & Stanford 16, more in J. Kurchan's talk

Each problem

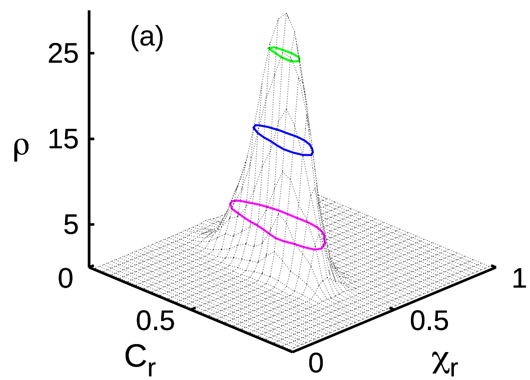
with its own peculiarities

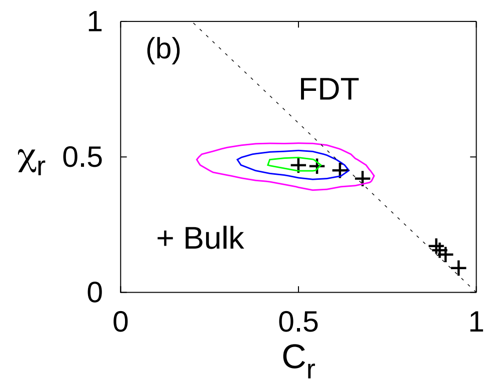
& much more to say !

Local correlations & responses

$3d$ **Edwards-Anderson spin-glass**

$$C_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} s_i(t) s_i(t_w) \; , \quad \chi_{\vec{r}}(t,t_w) \equiv \frac{1}{V_{\vec{r}}} \sum_{i \in V_{\vec{r}}} \int_{t_w}^{t} dt' \; \left. \frac{\delta s_i(t)}{\delta h_i(t')} \right|_{h=0}$$





+ Bulk : Parametric plot $\chi(t,t_w)$ vs $C(t,t_w)$ for t_w fixed and 7 $t \; (> t_w)$

ρ corresponds to the maximum t yielding the smallest C (left-most +)

Castillo, Chamon, LFC, Iguain, Kennett 02

Kinetically constrained models + Charbonneau, Reichman & Sellitto 04

Sigma Model

Conditions & expression

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad\qquad C_{ag}(\vec{r},t,t_w) = f_{ag}\left(e^{-\int_{t_w}^{t} dt'\,\partial_{t'}\varphi(\vec{r},t')}\right)$$

i. The action must be invariant under a global time reparametrization $t \to h(t)$.

ii. If our interest is in short-ranged problems, the action must be written using local terms. The action can thus contain products evaluated at a single time and point in space of terms such as $\varphi(\vec{r},t)$, $\partial_t\varphi(\vec{r},t)$, $\nabla\varphi(\vec{r},t)$, $\nabla\partial_t\varphi(\vec{r},t)$, and similar derivatives.

iii. The scaling form in eq. (29) is invariant under $\varphi(\vec{r},t) \to \varphi(\vec{r},t) + \Phi(\vec{r})$, with $\Phi(\vec{r})$ independent of time. Thus, the action must also have this symmetry.

iv. The action must be positive definite.

These requirements largely restrict the possible actions. The one with the smallest number of spatial derivatives (most relevant terms) is

$$\mathcal{S}[\varphi] = \int d^d r \int dt \left[K \, \frac{(\nabla\partial_t\varphi(\vec{r},t))^2}{\partial_t\varphi(\vec{r},t)} \right] , \qquad (30)$$

Chamon & LFC 07

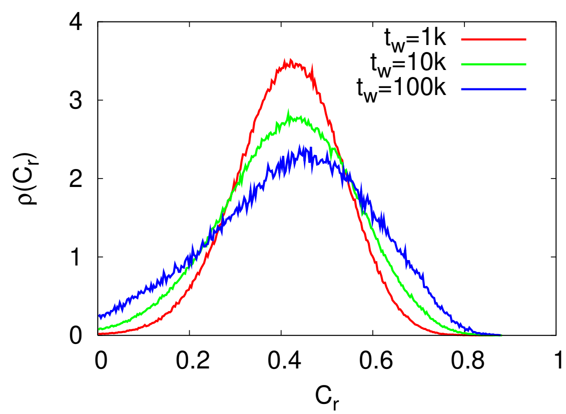
Sigma Model

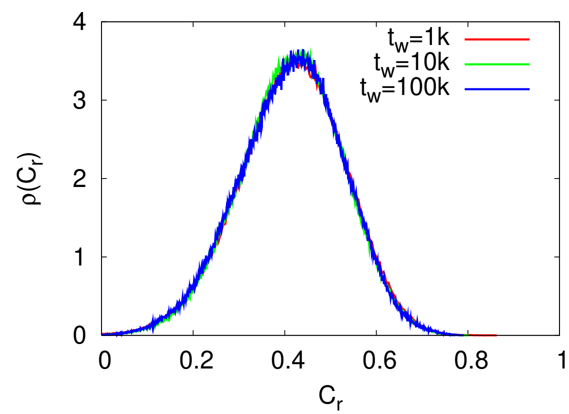
Some consequences - $3d$ Edwards Anderson model

$$h(\vec{r},t) = e^{-\varphi(\vec{r},t)} \qquad C_{ag}(\vec{r},t,t_w) = f_{ag}(e^{-\int_{t_w}^{t} dt'\, \partial_{t'}\varphi(\vec{r},t')})$$

Distribution of local correlations depends on times t, t_w only through C, ξ

$$\rho(C_{\vec{r}};\, t, t_w, \ell, \xi(t,t_w)) \to \rho(C_{\vec{r}};\, C_{ag}(t,t_w), \ell/\xi(t,t_w))$$





t, t_w such that $C_{ag}(t,t_w) = C$ $\quad$ ℓ such that $\ell/\xi = \text{cst}$ **Jaubert, Chamon, LFC, Picco 07**

predictions on the form of ρ derived from $S[\varphi]$ too

Tests in Lennard-Jones systems **Avila, Castillo, Mavimbela, Parsaeian 06-12**

How general is this ?

Coarsening & domain growth

e.g. the d-dimensional $O(N)$ model in the large N limit (continuous space limit of the Heisenberg ferro with $N \to \infty$)

N component field $\vec{\phi} = (\phi_1, \ldots, \phi_N)$ with Langevin dynamics

$$\partial_t \phi_\alpha(\vec{r},t) = \nabla^2 \phi_\alpha(\vec{r},t) + \lambda |N^{-1} \phi^2(\vec{r},t) - 1| \phi_\alpha(\vec{r},t) + \xi_\alpha(\vec{r},t)$$

$\phi_\alpha(\vec{k},0)$ Gaussian distributed with variance Δ^2

Time reparametrization invariance is reduced to time rescalings

$$t \to h(t) \qquad \Rightarrow \qquad t \to \lambda t$$

Same in the $p = 2$ spherical model

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Coarsening & domain growth

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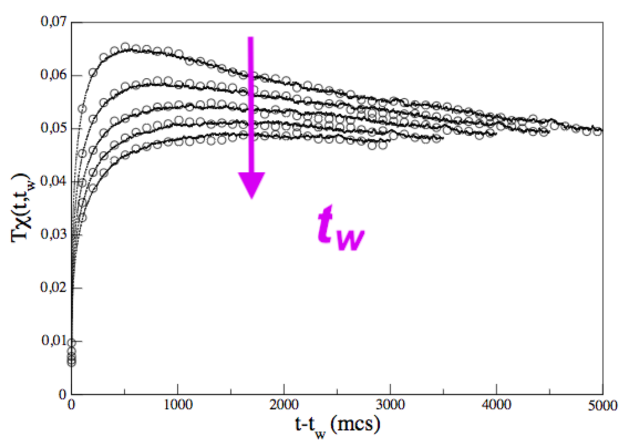
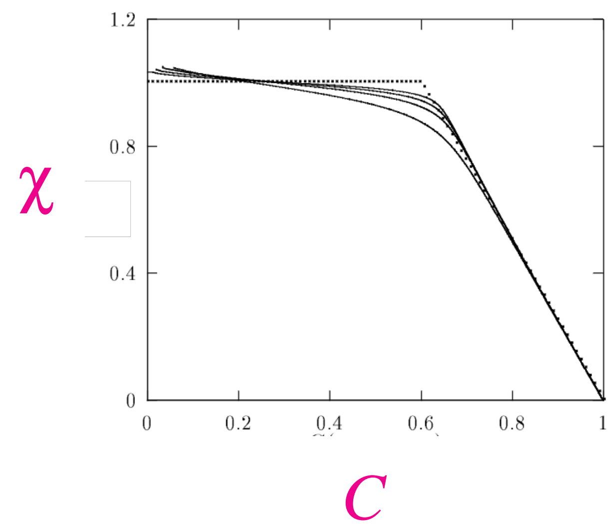




Ising FM, $O(N)$ field theory, or $p = 2$ spherical model

Related to $T^* \to \infty$ and simplicity of free-energy landscape

Triangular relations

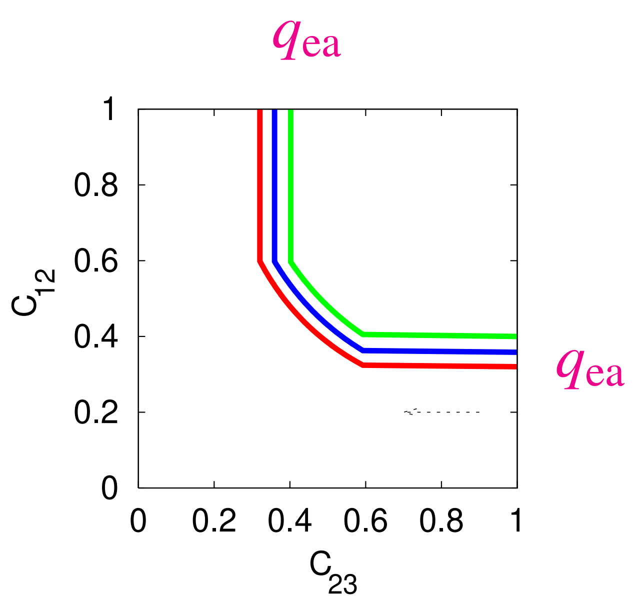
Scaling of the aging global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three global correlations

$$C(t_1, t_2),\ C(t_2, t_3),\ C(t_1, t_3)$$

If, in the aging regime $C^{ij}_{\rm ag} \equiv C_{\rm ag}(t_i, t_j) = f_{\rm ag}\left(\frac{h(t_i)}{h(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C^{12}_{\rm ag} = f_{\rm ag}\left(\frac{h(t_1)}{h(t_3)}\frac{h(t_3)}{h(t_2)}\right) = f_{\rm ag}\left(\frac{f^{-1}_{\rm ag}(C^{13}_{\rm ag})}{f^{-1}_{\rm ag}(C^{23}_{\rm ag})}\right)$$



choose t_3 and t_1 so that $C^{13} = 0.3$

the arrow shows the t_2 ‘flow’ from t_3 to t_1

e.g. $C^{12} = q_{\rm ea} C^{13}/C^{23}$

Triangular relations

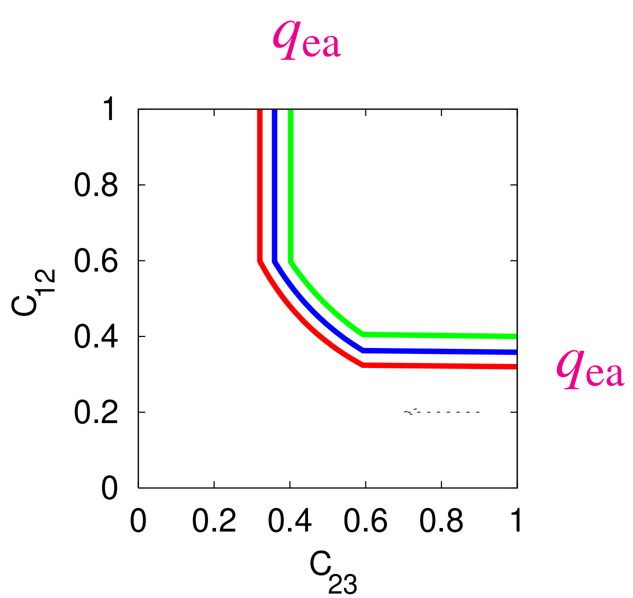
Scaling of the slow part of the global correlation

Take three times $t_1 \geq t_2 \geq t_3$ and compute the three local correlations

$$C_{\vec{r}}(t_1,t_2),\ C_{\vec{r}}(t_2,t_3),\ C_{\vec{r}}(t_1,t_3)$$

If, in the aging regime $C_{\vec{r}}^{ij} \equiv C_{\vec{r}}(t_i,t_j) = f_{\rm ag}\left(\frac{h_{\vec{r}}(t_i)}{h_{\vec{r}}(t_j)}\right)$ with $t_i \geq t_j \Rightarrow$

$$C_{\vec{r}}^{12} = f_{\rm ag}\left(\frac{f_{\rm ag}^{-1}(C_{\vec{r}}^{13})}{f_{\rm ag}^{-1}(C_{\vec{r}}^{23})}\right)$$



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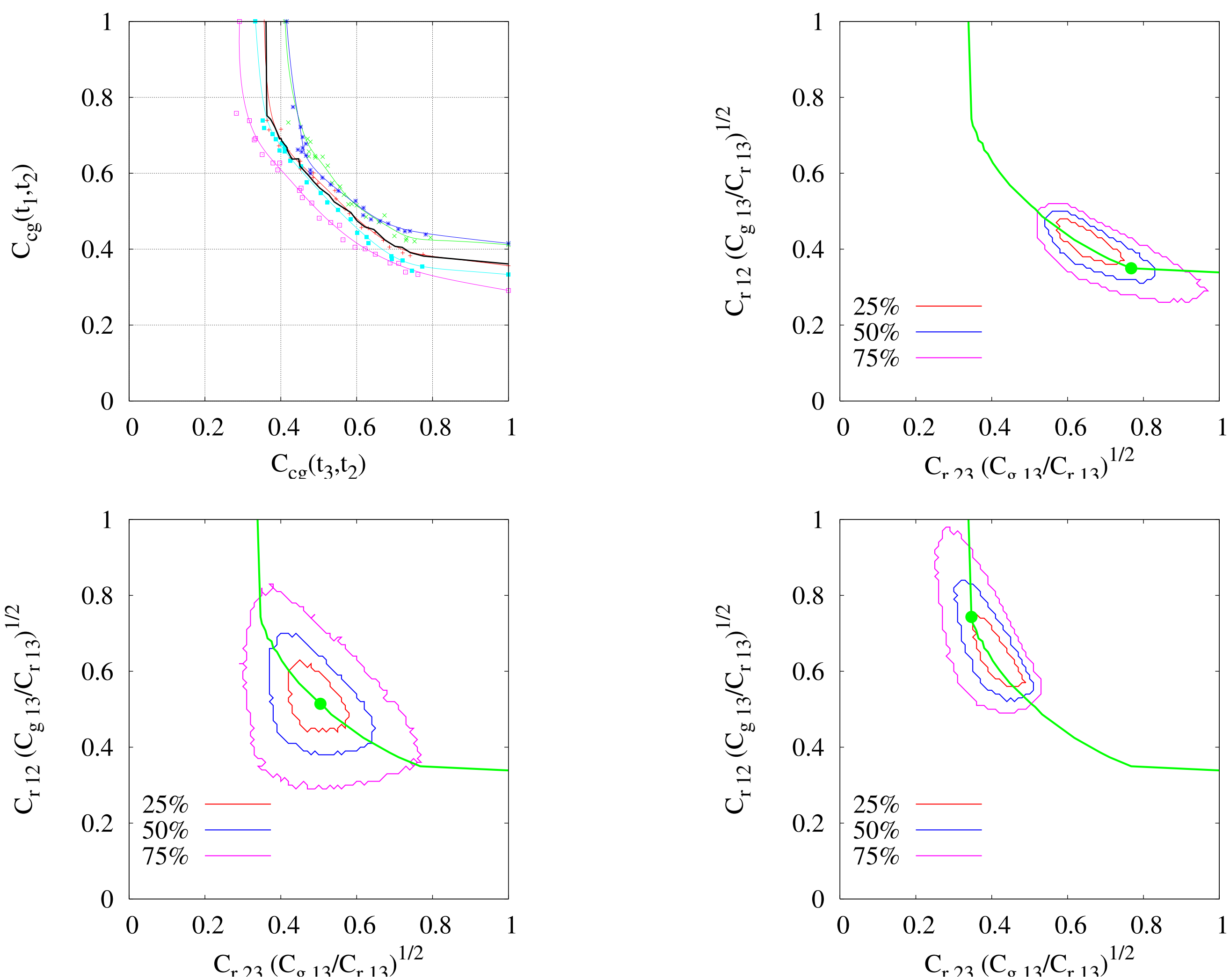
e.g. $C_{\vec{r}}^{12} = q_{\rm ea}\, C_{\vec{r}}^{13} / C_{\vec{r}}^{23}$.

Triangular relations

$3d$ Edwards-Anderson model



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