Phase ordering kinetics and percolation in two dimensions

Leticia F. Cugliandolo

Sorbonne Université
Laboratoire de Physique Théorique et Hautes Energies
Institut Universitaire de France

leticia@lpthe.jussieu.fr
www.lpthe.jussieu.fr/~leticia/seminars

In collaboration with

Jeferson Arenzon, Thibault Blanchard, Alan Bray, Federico Corberi, Ingo Dierking, Marco Esposito, Michika-zu Kobayashi, Ferdinando Insalata, Marcos-Paulo Loureiro, Marco Picco, Hugo Ricateau, Yoann Sarrazin, Alberto Sicilia and Alessandro Tartaglia.

Bangkok, January 2019
Quenches in statistical physics models

passing by critical percolation!

What is it about?

Classical open systems

Statistical physics framework

Stochastic dissipative dynamics

Out of equilibrium

coarsening – phase ordering kinetics

percolation – fractality (but static uncorrelated problem! So?)
The talk focuses on a very well-known example

Dynamics following a change of a control parameter

- If there is an equilibrium phase transition, the equilibrium phases are known on both sides of the transition. i.e. the asymptotic state is known.

- For a purely dynamic problem, the absorbing states are known.

- The dynamic mechanism towards equilibrium is understood the systems try to order locally in one of the few competing states.
Interests and goals

Practical & fundamental interest, *e.g.*

- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Some issues

- The role played by the *initial conditions & short-time dynamics*
- Full *geometric characterisation* of the structure
- When does the usual *dynamic scaling* regime set in?
- The role played by the cooling rate

that are related to each other.
Phase separation in glasses

$t = 1 \text{ min}$

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14
Phase separation in glasses

\[ t = 4 \text{ min} \]

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14
Phase separation in glasses

\[ t = 16 \text{ min} \]

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14
Phase separation in glasses

\[ t = 64 \text{ min} \]
The talk is on a very well-known problem

The stochastic dynamics of the $2d$ Ising model after an instantaneous quench from high to low temperature with non-conserved order parameter dynamics

- There is a 2nd order phase transition.
  The **equilibrium phases** are: the **paramagnet** at high $T$
  the (degenerate) **ferromagnet** at low $T$

- Standard knowledge:
  The **dynamic mechanism** is curvature-driven domain growth
**2d Ising Model (IM)**

Archetypical example for classical magnetic systems

\[
H = -J \sum_{\langle ij \rangle} s_i s_j
\]

\(s_i = \pm 1\) Ising spins.
\(\langle ij \rangle\) sum over nearest-neighbours
\(J > 0\) ferromagnetic coupling constant.
critical temperature \(T_c > 0\) for \(d > 1\)

**Monte Carlo rule** \(s_i \rightarrow -s_i\) accepted with

\[p = 1\] if \(\Delta E < 0\)
\[p = e^{-\beta \Delta E}\] if \(\Delta E > 0\)
\[p = 1/2\] if \(\Delta E = 0\)

**Non-conserved order parameter dynamics**

\[\text{[\(\uparrow\downarrow\) towards \(\uparrow\uparrow\)] etc. allowed.}\]
\[\text{[\(m = 0\) to \(m = 2\)]}\]
Similar questions can be asked in very well-known problems in math, e.g.

**Dynamics of a voter model starting from a random initial condition**

- Purely dynamic, violation of detailed balance, no phase transition
- Two absorbing states
- The *dynamic mechanism* towards absorption is understood
  
  domain growth is driven by interfacial noise
**2d Voter Model (VM)**

Archetypical example of opinion dynamics

$H$ does not exist - kinetic model

$s_i = \pm 1$ Ising spins that

sit on the vertices of a lattice.

**Voter update rule**

choose a spin at random, say $s_i$

choose one of its $2d$ neighbours at random, say $s_j$

set $s_i = s_j$

In two dimensions full consensus, i.e. $m = L^{-d} \sum_{i=1}^{L^d} s_i = \pm 1$ is reached in a timescale $t_C \simeq L^2$ (with $\ln L$ corrections)

Clifford & Sudbury 73, Holley & Ligget 75, Cox & Griffeath 86
Phase ordering kinetics

\[ s_i = \pm 1 \text{ at } t = 0 \text{ MCs, snapshots at } t = 4, 64, 512, 4096 \text{ MCs} \]

Ising
\[ T = 0 \]

\[ T_c \]

Voter
Dynamic scaling in phase ordering kinetics

Growing length $l(t)$

Typically $l(t) \sim t^{1/z_d}$

Excess energy w.r.t. the equilibrium one stored in the domain walls
Growing length

From excess energy

\[ \ell_G(t) = \frac{E_{eq}(T)}{E_{eq}(T) - E(t, T)} \sim t^{1/z_d} \quad \text{and} \quad z_d = 2 \]
Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains \( l(t) \), such that the domain structure is (in statistical sense) independent of time when lengths are scaled by \( l(t) \), e.g.

\[
C(r, t) \equiv \langle s_i(t) s_j(t) \rangle_{|\vec{x}_i - \vec{x}_j| = r} \sim \langle \phi \rangle_{eq}^2 f \left( \frac{r}{l(t)} \right)
\]

\[
C(t, t_w) \equiv \langle s_i(t) s_i(t_w) \rangle \sim \langle \phi \rangle_{eq}^2 f_c \left( \frac{l(t)}{l(t_w)} \right)
\]

etc. when \( r \gg \xi_{eq} \), times such that \( t, t_w \gg t_0 \) and \( C < \langle \phi \rangle_{eq}^2 \).

Review Bray 94
Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains $l(t)$, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by $l(t)$, e.g.

\[ C(r, t) \equiv \langle s_i(t)s_j(t) \rangle_{|x_i - x_j| = r} \sim \langle \phi \rangle_{eq}^2 f \left( \frac{r}{l(t)} \right) \]

\[ C(t, t_w) \equiv \langle s_i(t)s_i(t_w) \rangle \sim \langle \phi \rangle_{eq}^2 f_c \left( \frac{l(t)}{l(t_w)} \right) \]

etc. when $r \gg \xi_{eq}$, times such that $t, t_w \gg t_0$ and $C < \langle \phi \rangle_{eq}^2$. 

Review Bray 94
Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains $l(t)$, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by $l(t)$, e.g.

$$C(r, t) \equiv \langle s_i(t) s_j(t) \rangle_{|\vec{x}_i - \vec{x}_j| = r} \sim \langle \phi \rangle_{eq}^2 f \left( \frac{r}{l(t)} \right)$$

$$C(t, t_w) \equiv \langle s_i(t) s_i(t_w) \rangle \sim \langle \phi \rangle_{eq}^2 f_c \left( \frac{l(t)}{l(t_w)} \right)$$

etc. when $r \gg \xi_{eq}$, times such that $t, t_w \gg t_0$ and $C < \langle \phi \rangle_{eq}^2$.

Is this really all there is?
Interests and goals

Practical & fundamental interest, e.g.

- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Some issues

- The role played by the initial conditions & short-time dynamics
- Full geometric characterisation of the structure
- When does the usual dynamic scaling regime set in?
- The role played by the cooling rate

that are related to each other.
2d square IM at $T=0$

t=0.0
2d square IM at $T=0$

$t=0.57533$
Spanning cluster

Has this cluster something to do with (critical) percolation?
Percolation

Purely geometric problem

Take a lattice $\Lambda$ in $d$ spatial dimensions.

Define a site occupation variable $n_i = 1, 0$ with probability $p, 1 - p$

In the limit $L \to \infty$ there is a continuous phase transition at $p_c$ such that the probability of there being a cluster of occupied nearest-neighbour sites that crosses a sample from one end to another in at least one Cartesian direction

$$\lim_{L \to \infty} P(p, L) = \begin{cases} 
0 & \text{if } p \leq p_c \\
> 0 & \text{if } p > p_c 
\end{cases}$$

$p_c$ depends on $\Lambda$ and $d$.

At $p_c$ the spanning cluster has fractal properties that are well characterised
Percolation

Purely geometric problem

Probability of percolation along the horizontal or vertical directions, $\pi_{h,v}$

Probability of percolation along the horizontal and vertical directions, $\pi_{hv}$

(On a torus) Probability of percolation along the diagonal direction, $\pi_d$

Known exactly from SLE & CFT calculations.
Percolation

Purely geometric problem

Take a lattice $\Lambda$ in $d$ spatial dimensions.

Define a site occupation variable $n_i = 1, 0$ with probability $p, 1 - p$.

In the limit $L \to \infty$ there is a continuous phase transition at $p_c$ such that the probability of there being a cluster of occupied nearest-neighbour sites that crosses a sample from one end to another in at least one Cartesian direction

$$
\lim_{L \to \infty} P(p, L) \begin{cases} 
= 0 & \text{if } p \leq p_c \\
> 0 & \text{if } p > p_c
\end{cases}
$$

$p_c$ depends on $\Lambda$ and $d$.

The distribution of finite size clusters is algebraic at $p_c$. 
Percolation

Purely geometric problem

Domain area: sum of filled dots \( \mathcal{N}_d, \tau_d, D_A \)

External boundary or hull: red broken line \( \mathcal{N}_\ell, D_\ell \)

Hull-enclosed area: sum of lattice sites within the red boundary (including the two empty sites) \( \mathcal{N}_h, \tau, D \)
Percolation

Purely geometric problem

Winding angle vs. curvilinear length of the wall

$$\langle \theta^2(x) \rangle = ct + \frac{4\kappa}{8+\kappa} \ln x \quad \text{with} \quad \kappa = 6$$

Known exactly from SLE & CFT calculations.
Spanning cluster

Has this cluster something to do with (critical) percolation?

Back to the dynamic problem
Equilibrium at infinite temperature, $T_0 \rightarrow \infty$, initial condition.
The spins take $\pm 1$ values with probability $1/2$.
Site occupation variable $n_i = (s_i + 1)/2 = 1, 0$ with $p = 1/2$
From a site percolation perspective:

$p_c = 0.65$ Kagome lattice.
$p_c = 0.59$ Square lattice.
$p_c = 0.55$ Bow-tie lattice.
$p_c = 0.5$ Triangular lattice.

Initial condition at $p = 0.5$, below $p_c$
2d square IM at T=0

t=0.0
2d square IM at T=0

t=0.57533
2d square IM at T=0

t=0.94844
2d square IM at $T=0$

$t=2.00847$
2d square IM at T=0

t=2.57898
2d square IM at T=0

t=3.99211
2d square IM at $T=0$
2d square IM at $T=0$

t=7.46144
The percolating structure was decided at $t_p \simeq 8$ MCs
2d square IM at $T=0$

The final configuration will be one with two horizontal stripes

$t=7.46144$

$t=128.0$

Olejarz, Krapivsky & Redner 12, Blanchard & Picco 13
What is going on?

Rapid growth of the two largest clusters in $t \sim 10$ MCs

Further algebraic growth after $t \sim 10$ MCs

Data averaged over many runs. I’ll come back to these data later.
Cluster analysis

Counting and measuring

Domain area: sum of filled dots

External boundary or hull: red broken line

Hull-enclosed area: sum of lattice sites within the red boundary (including the two empty sites) $\mathcal{N}_h(A, t)$
Is it critical percolation?

Full distribution of areas after a quench from $T_0 \to \infty$

Three dynamic regimes.

– small areas $A \leq l^2(t)$: flat
– large but finite areas $l^2(t) \leq A \leq L^2$: power law decay
– even larger areas $A = \mathcal{O}(L^{DA})$: a bump
Is it critical percolation?

Tail and finite size scaling of the bump

Take $A$ to be the hull-enclosed area or the domain area.

At critical percolation, finite size scaling of the number density of areas

$$\mathcal{N}(A, L) = 2c A^{-\tau_A} + N_p(A/L^{D_A})$$

with $D_A = d/((\tau_A - 1)$ the fractal dimension of the percolating clusters.

Stauffer & Aharony 94

For hull-enclosed areas $\tau_A = 2$ and $D_A = 2$

For domain areas $\tau_A = 187/91 \simeq 2.05$ and $D_A \approx 1.9$

NB the corresponding exponents for critical Ising conditions are different but take values close to these.

The constants $2c$ are known, e.g. $2c_h \approx (16\pi\sqrt{3})^{-1}$ and $2c_d \approx 0.06$

Cardy & Ziff 03, Sicilia, Arenzono, Bray & LFC 07
Statistics of finite areas

Assuming percolation established at \( t = t_p \)

\[
n_h(A, t) \equiv \frac{(2)c_h}{(A + \lambda t)^2}
\]

\[
n_d(A, t) \approx \frac{(2)c_d (\lambda d t)^{\tau_A-2}}{(A + \lambda d t)^{\tau_A}}
\]

in the long time limit \( t \gg t_p \). Change in notation \( n = N^\tau \)

We derived the expected scaling forms, as \( l(t) = (\lambda t)^{1/2} \):

\[
l^4(t)n_h(A, t) = f_h \left( \frac{A}{l^2(t)} \right)
\]

\[
l^4(t)n_d(A, t) \approx f_d \left( \frac{A}{l^2(t)} \right)
\]

The new parameters are \( c_d = c_h + O(c_h^2) \) and \( \lambda_d = \lambda + O(c_h) \). Moreover, the sum rules, \( N_h(t) = N_d(t) \) and \( \int dA A n_d(A, t) = 1 \) relate \( c_h \) to \( \tau \) (or \( \tau' \))!

Arenzon, Bray, LFC & Sicilia 07
Simulations vs. theory

Number density of (finite) hull-enclosed areas per unit area

\[ T_0 \to \infty \text{ and } T = 0 \]

Solid lines
analytical prediction:

\[ n_h(A, t) \equiv \frac{(2) c_h}{(A + \lambda t)^2} \]

\[ 2c_h = (16\pi \sqrt{3})^{-1} \]

Arenzon, Bray, LFC & Sicilia 07
Experiments vs. theory

Number density of (finite) domain areas per unit area

\[ T_0 \rightarrow \infty \text{ and } T = 0 \]
Is it critical percolation?

Number density of the domain areas of percolated clusters

At $t = 16$ MCs the bump converged to a stationary form that satisfies the finite size scaling of critical site percolation with $\tau_A = 2.05$ and $D_A = 1.9$

Insert: failure of collapse if critical Ising exponents are used.
Is it critical percolation?

The probabilities of percolation in different directions

\[ (\pi_h + \pi_v), \; \pi_{\text{diag}} \]

\[ t/L^{z_p} \]
Is it critical percolation?

Probabilities as in critical percolation

Scaling with $L^{z_p}$ and $z_p \approx 1/2$ in the approach to the constant values

$\pi_{hv} \approx 0.62$

$\pi_h + \pi_v \approx 0.34$

$\pi_d \approx 0.03$

Blanchard, Corberi, LFC, Picco & Tartaglia 14, 15
Is it critical percolation?

The final configuration was decided at $t_p \sim L^{z_p}$

Frozen stripe states with probability $\pi_h + \pi_v$

$\pi_{hv} \approx 0.62$

$\pi_h + \pi_v \approx 0.34$

$\pi_d \approx 0.03$

Barros, Krapivsky & Redner 09, Blanchard & Picco 13
Is it critical percolation? 

The final configuration was decided at $t_p$.

stripe states with the probabilities of critical percolation:

a spanning cluster along the two Cartesian directions $\equiv$ order $\pi_{hv} \approx 0.66$

spanning cluster along along only one of them $\equiv$ domain walls $1 - \pi_{hv} \approx 0.34$
Is it critical percolation?

The winding angle

\[ \theta^2 = ct + \frac{4\kappa}{8+\kappa} \ln x \]

\( x \) curvilinear coordinate

\[ \kappa \approx 5.9 \]

In the inset: scaling with \( \ell_G(t) \to l(t) \) estimated from the excess energy
The largest clusters

Algebraic growth after $t_p$ in MC simulations

In critical percolation: $\frac{A_{\text{max}}}{r_0^2} \simeq \left( \frac{L}{r_0} \right)^{D_A}$ with $r_0$ the lattice spacing.

In coarsening: $r_0 \mapsto l(t)$ the growing length.

After substitution $\frac{A_{\text{max}}}{L^2} \propto t^\alpha$ with $\alpha \equiv \frac{2 - D_A}{z_d} = 0.0521$
In critical percolation: \( \frac{A_{\text{max}}}{r_0^2} \sim \left( \frac{L}{r_0} \right)^{D_A} \) with \( r_0 \) the lattice spacing.

In coarsening: \( r_0 \mapsto l(t) \) the growing length. \[ \text{Almeida & Takeuchi 19} \]

After substitution \( \frac{A_{\text{max}}}{L^2} \propto t^\alpha \) with \( \alpha \equiv \frac{2 - D_A}{z} = 0.0521 \).
Let us call $t_p(L)$ the time needed to reach the critical percolation state.
Determination of $t_p(L)$

Cloning trick and measurement of the overlap

Quench a system from $T_0 \rightarrow \infty$ to $T = 0$ at $t = 0$.

Let it evolve at $T = 0$ until $t_w$.

Make a copy of the instantaneous configuration, $\sigma_i(t_w) = s_i(t_w)$.

Let the two clones evolve with different thermal noises.

Compute the time-dependent overlap

$$q_{t_w}(t, L) = \frac{1}{L^d} \sum_{i=1}^{L^d} \langle s_i(t) \sigma_i(t) \rangle$$

If $t_w < t_p(L)$

$$\lim_{t \gg t_w} q_{t_w}(t, L) = 0$$

If $t_w > t_p(L)$

$$\lim_{t \gg t_w} q_{t_w}(t, L) > 0$$
Determination of $t_p(L)$

The overlap

$$\lim_{t \to \infty} q_{tw}(L)(t, L)$$ should reach a constant independent of $L$

Square FBC

Kagome FBC

Triangular PBC

$t_p(L) \sim L^{0.5}$

$t_p(L) \sim L^{0.33}$
The zero-temperature non-conserved (also the conserved) order parameter dynamics of the $2d$ Ising model, (both) starting from a totally uncorrelated $T_0 \to \infty$ paramagnetic initial state, approach uncorrelated critical percolation after a time $t_p \sim L^{z_p}$.

The exponent $z_p$ depends upon the effective connectivity of the lattice and the microscopic dynamics.

For instance, $z_p \approx 0.5$ for the square lattice $2d$ Ising model with non-conserved order parameter dynamics.

Blanchard, Corberi, LFC & Picco 14; Tartaglia, LFC & Picco 15
Conclusions

Time scales & length scales

- Pre-perc.
- Dynamical scaling with $\ell_d(t)$
- Critical perc. for $r > \ell_d(t)$
- Post equilibration

$\ell_p(t_p) \sim L$

$t_{eq} \sim L^{z_d}$

$\ell_p(t) \gg l(t)$
Conclusions

Approach to critical percolation: why is this feature interesting?

A mechanism that went unnoticed in this context so-far.

Seems to be universal (NCOP, LCOP, NLCOP, Voter)

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge: how can one prove this claim?

Manifold consequences:

- metastability, blocked striped states at zero temperature;
- corrections to dynamic scaling.
Conclusions

Approach to critical percolation: why is this feature interesting?

A mechanism that went unnoticed in this context so-far.

Seems to be universal (NCOP, LCOP, NLCOP, Voter).

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge: perhaps in the voter model.

Similar master equation to the one of the $1d$ Glauber chain

Krapivsky et al. 90s

Mapping to random walks

Cox & Griffeaths 80s

But... finite $L$ effects searched
Conclusions

Approach to critical percolation: why is this feature interesting?

(a) $t = 0$

(b) $t = 4$

(c) $t = 16$

(d) $t = 64$

(e) $t = 256$

(f) $t = 1024$
Conclusions

Approach to critical percolation: why is this feature interesting?
Conclusions

Approach to critical percolation: why is this feature interesting?

A mechanism that went unnoticed in this context so-far.

Seems to be universal.

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge: how can one prove this claim?

Manifold consequences:

- metastability, blocked striped states at zero temperature;
- corrections to dynamic scaling, $\ell_p(t) \sim t^{1/z_p}, l(t) \sim t^{1/z_d}$. 

Conclusions

Approach to critical percolation: why is this feature interesting?

A mechanism that went unnoticed in this context so-far.

Seems to be universal.

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge: how can one prove this claim?

Manifold consequences:

- metastability, blocked striped states at zero temperature;
- corrections to dynamic scaling, $\ell_p(t) \sim t^{1/z_p}, l(t) \sim t^{1/z_d}$. 
Correction to scaling

Linear-Log scale, zoom over $C \lesssim 0.1$

$C(r,t,L)$

\[
\frac{r}{l(t)}
\]

\[
f \left( \frac{r}{l(t)} \right) \quad g \left( \frac{r}{l(t)}, \frac{\ell_p(t)}{L} \right)
\]

with $\ell_p(t) \equiv t^{1/z_p}$ and $l(t) \equiv t^{1/z_d}$

$z_p \simeq 1/2$ for the square & Kagome, and $z_p \simeq 1/3$ for the triangular lattice.
Conclusions

Early approach to percolation

\[ \ell_p(t) \sim t^{1/z_P} \] is a new growing length-scale that brings about a new scaling variable to be taken into account in dynamic scaling.

Studies of the 2d
- Ising model with non-conserved order parameter dynamics
- Voter model
- Ising model with conserved (Kawasaki) order parameter dynamics

\[ \ell_p(t) \sim t^{1/z_P} \text{ or } \ell_p(t) \sim \xi^n_d(t) \]

Blanchard, Corberi, LFC & Picco 14; Tartaglia, LFC & Picco 15
Summary

- Evidence for the approach to critical percolation at a time-scale that diverges with the system size as $t_p \sim L^{z_p}$.

- The new growing length-scale, $\ell_p(t) \sim t^{1/z_p}$ dominates at short times and is needed to improve the scaling of finite-size and finite-time data.

  This effect also exists at finite temperature. Metastability acquires a finite life-time.

- We derived the number density of hull & domain enclosed areas and interface length for $t > t_p$ and we showed that they satisfy dynamic scaling with respect to $l(t) \sim t^{1/z_d}$ with $z_d = 2$ for times $t \gg t_p$. 