Quantum quenches in classical models

Leticia F. Cugliandolo

Sorbonne Universités, Université Pierre et Marie Curie
Laboratoire de Physique Théorique et Hautes Energies
Institut Universitaire de France
leticia@lpthe.jussieu.fr

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Dynamics of isolated disordered models: evolution, equilibration? the GGE & FDT effective temperatures

Leticia F. Cugliandolo

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Question

Does an isolated system reach equilibrium ?

Boosted by recent interest in

— the dynamics after quantum quenches of cold atomic systems

rôle of interactions (integrable *vs.* non-integrable)

— many-body localisation

novel effects of quenched disorder

And, an isolated classical systems ?

The (old) ergodicity question revisited

LFC, Lozano & Nessi 17. LFC, Lozano, Nessi, Picco & Tartaglia (in prep)

Quantum : Foini, Gambassi, Konik & LFC 17. de Nardis, Panfil *et al.* 17

Quantum quenches

Definition & questions

- Take an isolated quantum system with Hamiltonian $\hat{H}_0$
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $\hat{H}_0$ (or any $\hat{\rho}(t_0)$)
- Unitary time-evolution $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ with a Hamiltonian $\hat{H}$.

Does the system reach a steady state ?

Is it described by a thermal equilibrium density matrix $e^{-\beta\hat{H}}$?

Do at least some observables behave as thermal ones?

Does the evolution occur as in equilibrium ?

Other kinds of density matrices ?

Classical quenches

Definition & questions

- Take an isolated classical system with Hamiltonian H_0, evolve with H
- Initialize it in, say, ψ_0 a configuration, *e.g.* $\{\vec{q}_i, \vec{p}_i\}$ for a particle system

  ψ_0 could be drawn from a probability distribution, *e.g.* $Z^{-1} e^{-\beta_0 H_0(\psi_0)}$

Does the system reach a steady state ?

Is it described by a thermal equilibrium density matrix $e^{-\beta H}$?

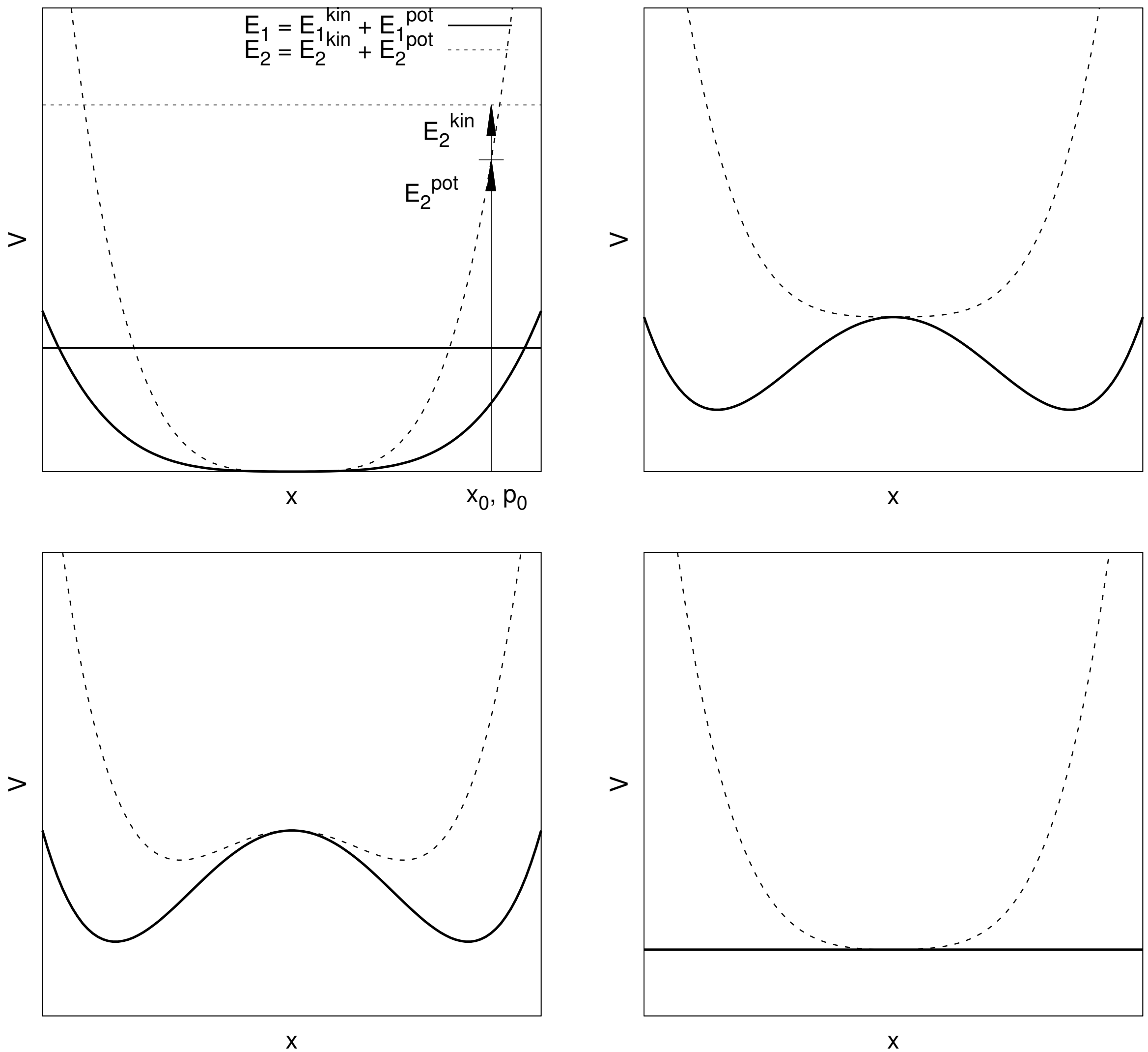
Do at least some observables behave as thermal ones?

Does the evolution occur as in equilibrium ?

Other kinds of probability distributions ?

Quenches

Simple examples (kind of building blocks)



At $t = 0$ change in V

Continuity in variables

$x(0^-) = x(0^+) = x_0$

$p(0^-) = p(0^+) = p_0$

Jump in (potential) energy

dashed to solid:

energy extraction

solid to dashed:

energy injection

Classical quenches

Models

We chose to study classical disordered models

isolated p spin spherical disordered models

Interesting & very well characterised

equilibrium phases & relaxational dissipative dynamics

rich free-energy landscapes with metastability, flat regions, large and small barriers, etc.

(also interesting in the context of many-body localisation studies)

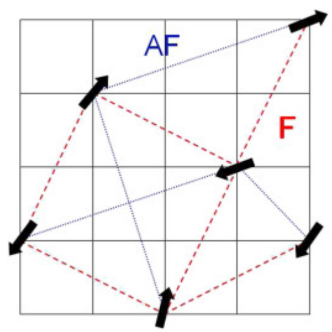
Quenched disorder

Spin Disordered Potential

$V = -\sum_{ij} J_{ij} s_i s_j - \sum_{ijk} J_{ijk} s_i s_j s_k + \dots$

the exchanges $J_{ij}, J_{ijk},$ etc. taken from

a probability distribution (details later)



Real variables $s_i \in \mathbb{R}$

Spherical constraint $\sum_{i=1}^{N} s_i^2 = N$

Connection with the following problem

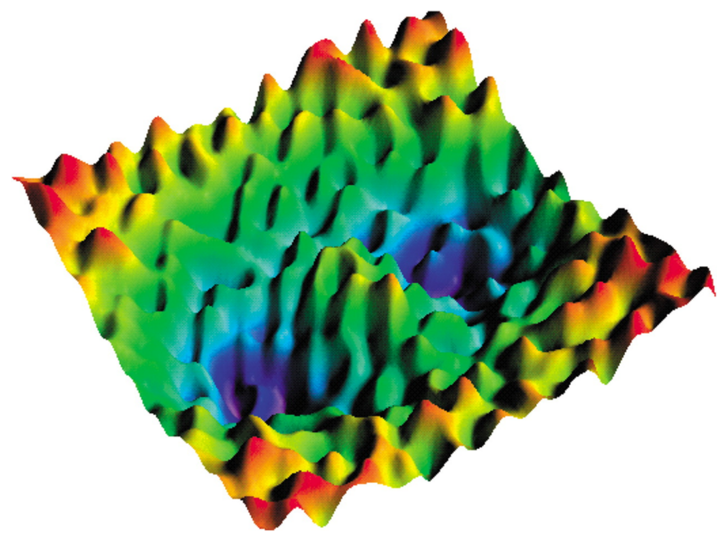
A particle

position $\vec{s} = (s_1, \dots, s_N)$

in a N dimensional space

under a random potential $V(\vec{s})$

Sketch for $N = 2$



but wrapped on the sphere

Classical dynamics

Coordinate-momenta pairs $\{\vec{s}, \vec{p}\}$ and Hamiltonian

$$H = K(\vec{p}) + V(\vec{s})$$

with the kinetic energy $K(\vec{p}) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2$

Newton-Hamilton equations

$$\boxed{\dot{s}_i = p_i/m \qquad \dot{p}_i = -dV(\vec{s})/ds_i}$$

The potential energy landscape makes the models behave differently

— N saddles (including min/max) for two body-interactions $V(\vec{s}) = \sum_{i \neq j} J_{ij} s_i s_j$

— x $\exp(N\Sigma)$ saddles for more than two body interactions $\sum_{i \neq j \neq k} J_{ijk} s_i s_j s_k$

(With dissipation used to model domain-growth & fragile glasses, respectively)

Classical dynamics

Coordinate-momenta pairs $\{\vec{s}, \vec{p}\}$ and Hamiltonian

$$H = K(\vec{p}) + V(\vec{s})$$

with the kinetic energy $K(\vec{p}) = \frac{1}{2m} \sum_{i=1}^{N} p_i^2$

Newton-Hamilton equations

$$\boxed{\dot{s}_i = p_i/m \qquad\qquad \dot{p}_i = -dV(\vec{s})/ds_i}$$

The potential energy landscape makes the models behave differently

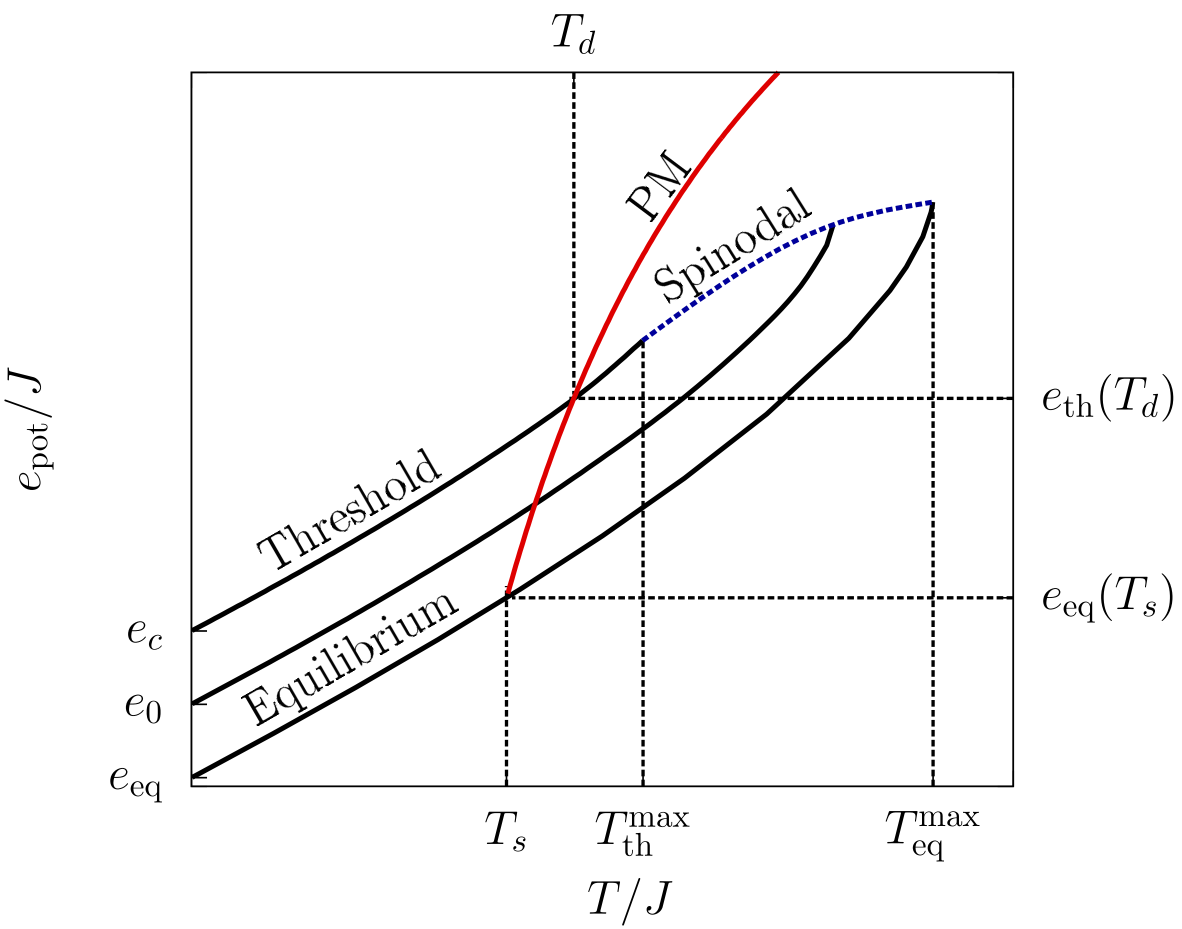
— Finite energy barriers for two body-interactions $V(\vec{s}) = \sum_{i \neq j} J_{ij} s_i s_j$

— Barriers scale with N for more than two body interactions $\sum_{i \neq j \neq k} J_{ijk} s_i s_j s_k$

(With dissipation used to model domain-growth & fragile glasses, respectively)

Three body interactions

Potential energy landscape



The initial conditions

- We chose initial states drawn from canonical equilibrium with Hamiltonian H_0 at inverse temperature β'
- The models have a phase transition at a finite β_c

  The high temperature phase is a disordered one, a paramagnet (PM)

  The low temperature phase is different in the two-body and more than two-body interaction models :

  – two ferromagnetic (FM)-like equilibrium states for two-body ($p = 2$)

  – $\mathcal{O}(e^{N\Sigma})$ metastable states, like in a glass, in the $p \geq 3$ case
- Initial conditions : disordered (PM) or confined (FM/metastable) TAP

The quench

Spin Disordered Potential

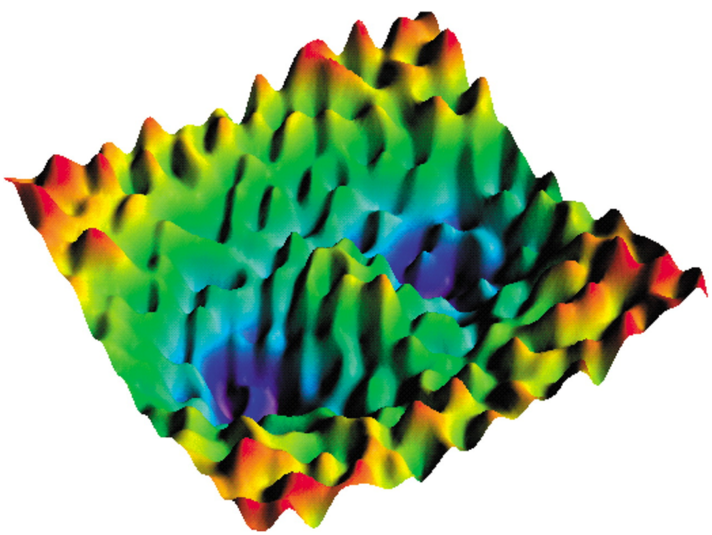
$V = -\sum_{ij} J_{ij} s_i s_j - \sum_{ijk} J_{ijk} s_i s_j s_k + \dots$

with exchanges J_{ij}, J_{ijk}, etc. taken from a Gaussian pdf

zero mean $[J_{ij}] = 0$ and

$[J^2_{i_1 \dots i_p}] = p! J_0^2 / (2N^{p-1})$

Initial energy scale J_0

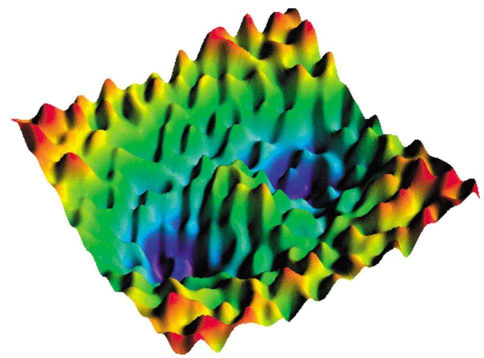


At time $t = 0$

Same configuration $\dot{s}_i(0), s_i(0)$

quench $\boxed{J^0_{i_1 \dots i_p} \mapsto J_{i_1 \dots i_p}}$

Final energy scale J



The rugged landscape is

stretched/contracted and pulled up/down

On the sphere

Dynamic equations

Conservative dynamics

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(m\partial_t^2 - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

$$\boxed{+\frac{\beta' J_0}{J}\, D(t,0)C(t_w,0)}$$

$$(m\partial_t^2 - z_t)R(t,t_w) = \int dt' \; \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

with the post-quench self-energy and vertex

$$D(t,t_w) = \frac{J^2 p}{2} C^{p-1}(t,t_w) \qquad \Sigma(t,t_w) = \frac{J^2 p(p-1)}{2} C^{p-2}(t,t_w)\, R(t,t_w)$$

and the Lagrange multiplier z_t fixed by $C(t,t) = 1$

Dynamic equations

Conservative dynamics

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(m\partial_t^2 - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

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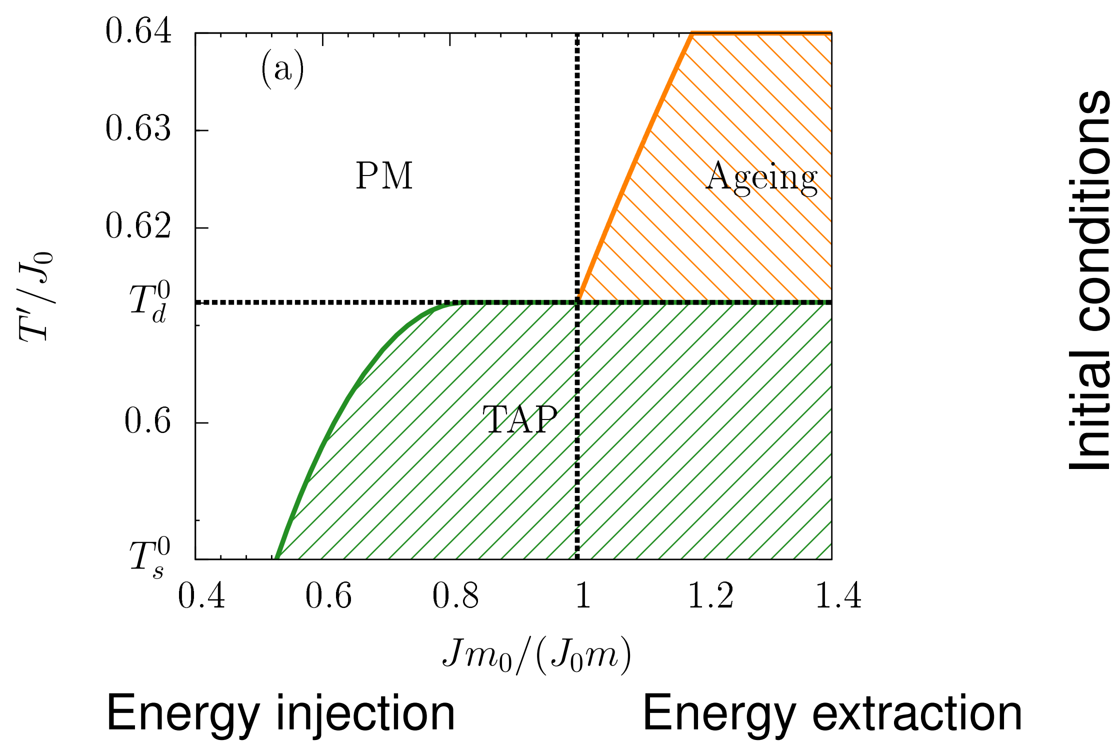
and the Lagrange multiplier z_t fixed by $C(t,t) = 1$

Solvable numerically & analytically for long times

Three body model

Dynamic phase diagram



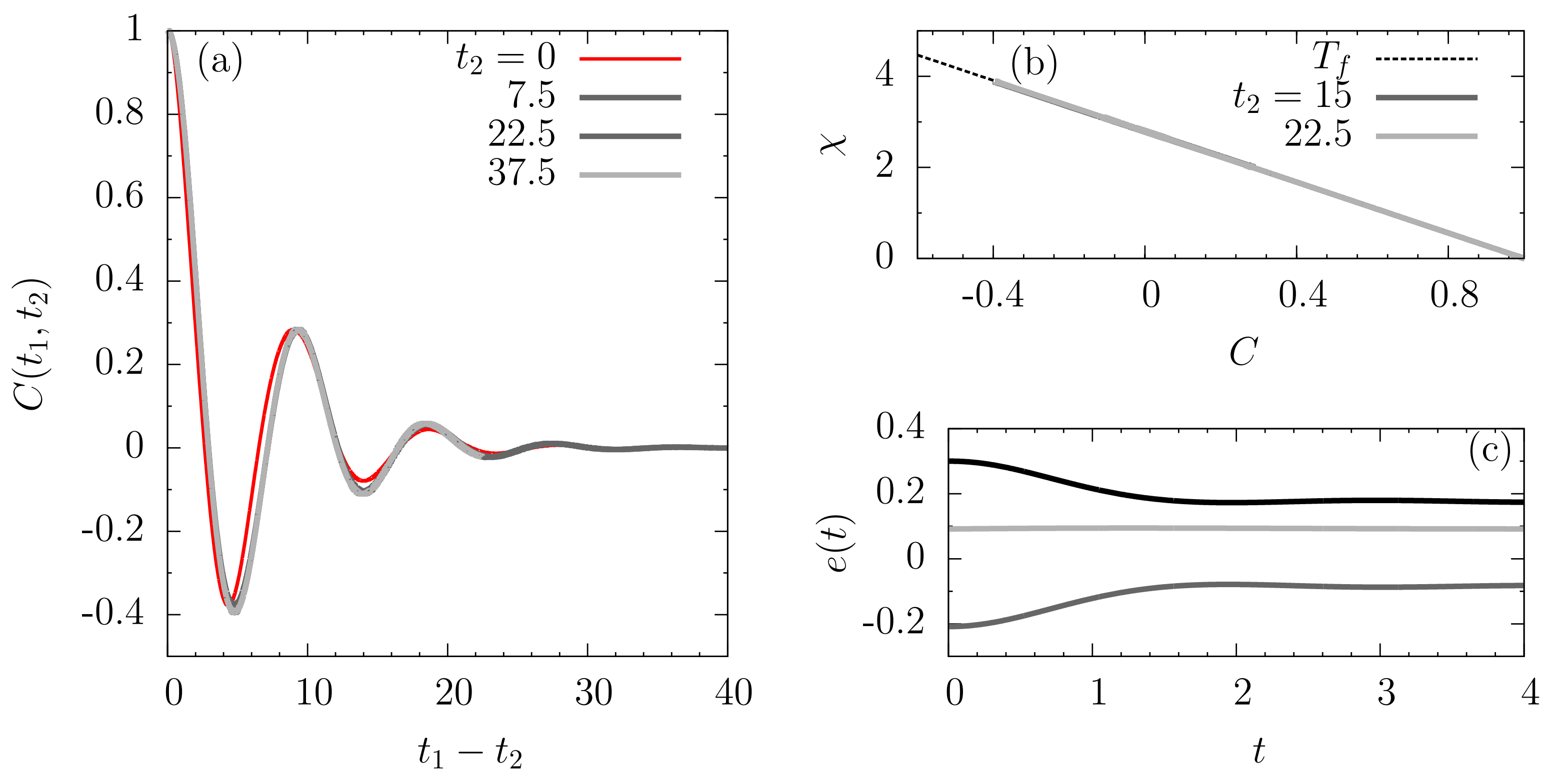
In PM, quenches go to GB equilibrium at $\beta_f(e_f)$ with e_f the final energy

Following metastable states, GB-like equilibration at β_f determined by e_f

Out of equilibrium relaxation with ageing effects when $e_f = e_{\rm th}$

Three body model

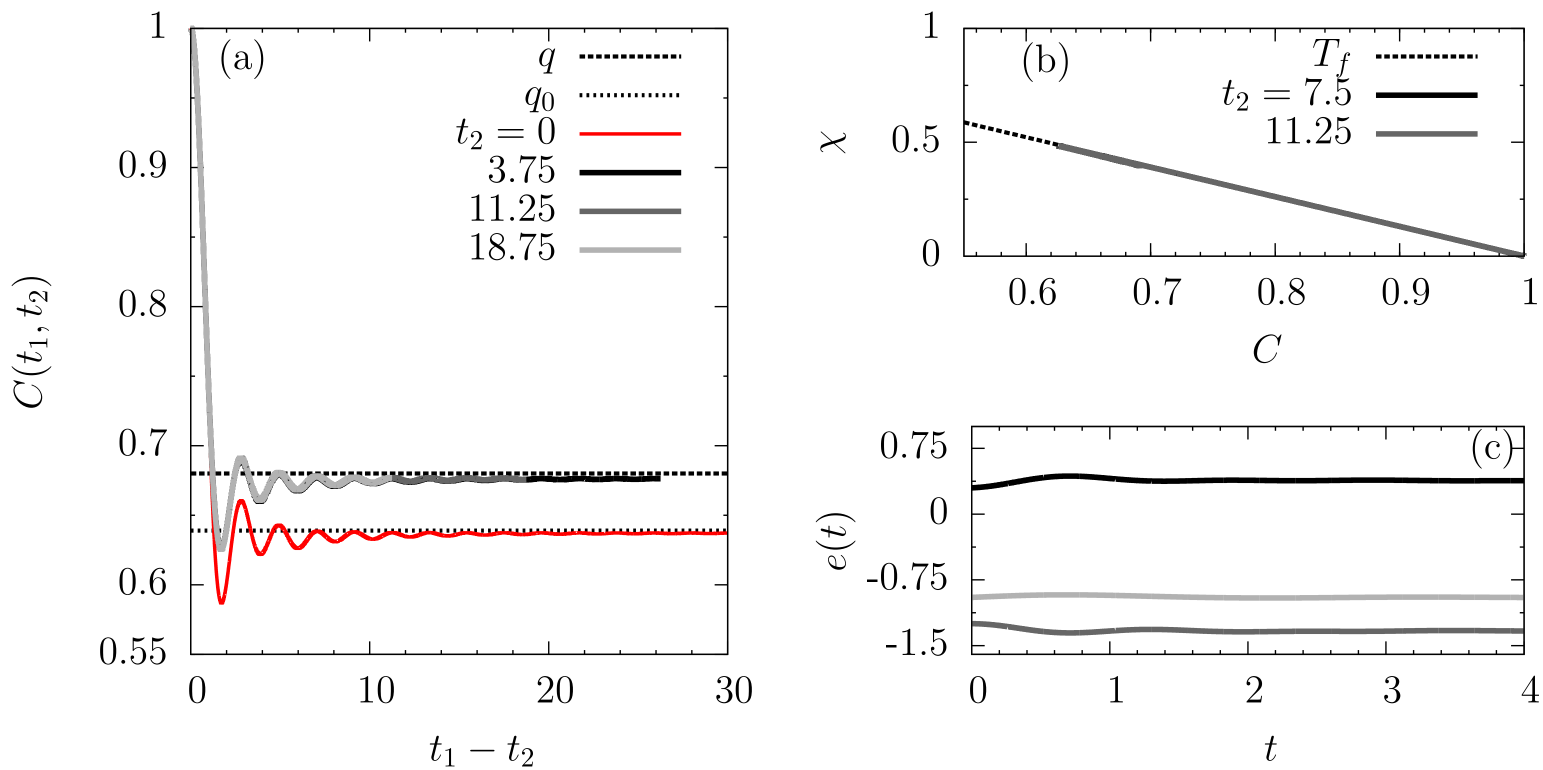
_e.g._, from equilibrium within a TAP state to the PM



GB equilibration at the temperature of a PM $\quad T_f = e_f + \sqrt{J^2 + e_f^2}$

Three body model

Initial configuration in a metastable (TAP) state



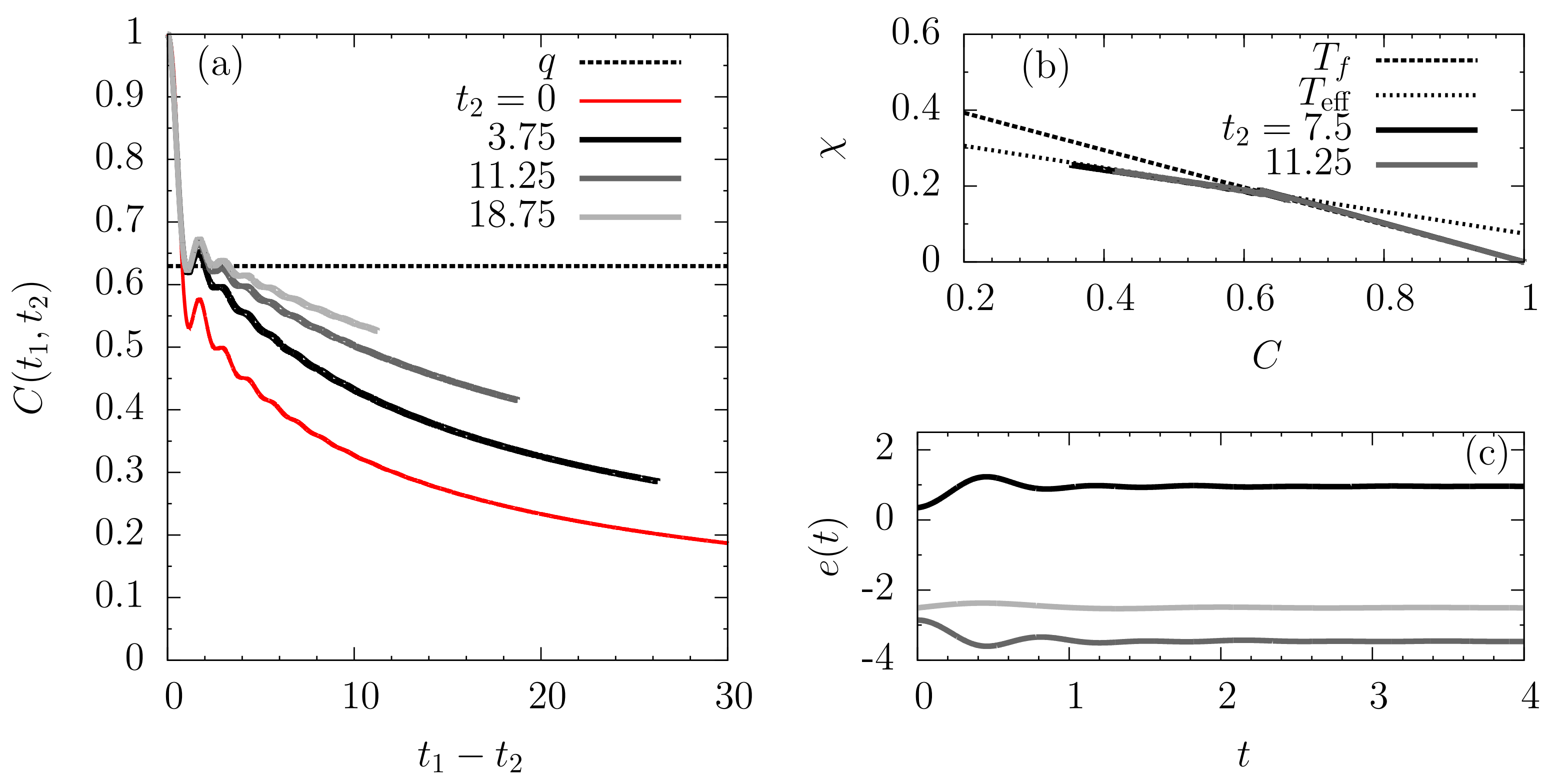
$C(t_1,0) \to q_0$ Fidelity

$\lim_{t_1-t_2\gg t_0} \lim_{t_2\gg t_0} C(t_1,t_2) = q$ Decorrelation

Following metastable states, equilibration at β_f fixed by $e_f = e_f^{\rm kin} + e_f^{\rm pot}$

Three body model

Energy extraction from PM to threshold

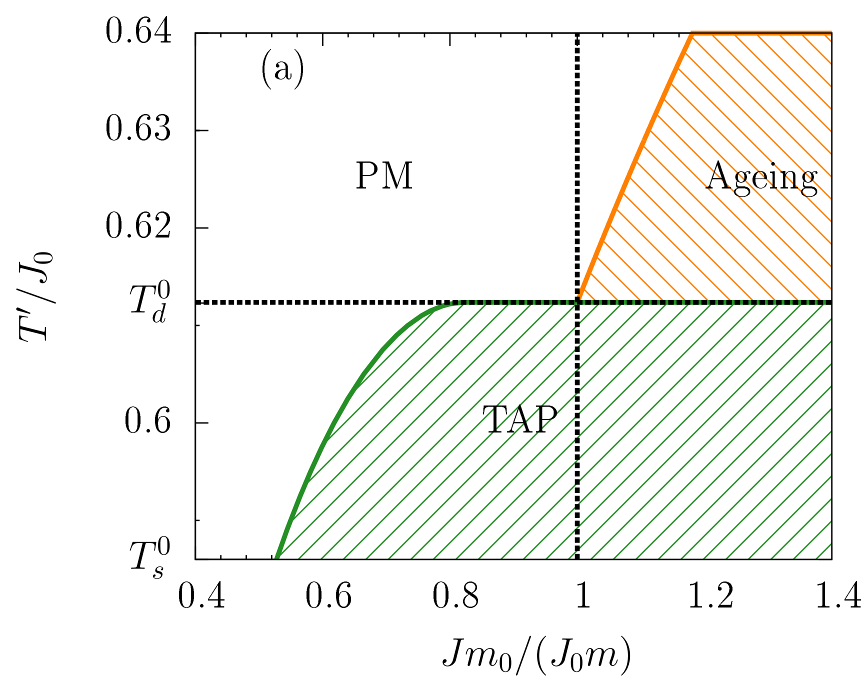


Similar to the relaxational case. Two temperature behaviour, fast and slow decay.

Out of equilibrium relaxation when quench parameters tuned so that $e_f = e_{\rm th}$

Three body model

Dynamic phase diagram - recap



Initial conditions

Energy injection

Energy extraction

In PM, quenches go to GB equilibrium at $\beta_f(e_f)$ with e_f the final energy

Following metastable states, GB-like equilibration at β_f determined by e_f

Out of equilibrium relaxation with ageing effects when $e_f = e_{\rm th}$

Two body model

Non-linear coupling through the Lagrange multiplier only

Diagonal in the basis of eigenvectors $\vec{v}_\mu$ of the interaction matrix J_{ij}

Projection of the coordinate (spin) vector on the eigenvectors $s_\mu = \vec{s} \cdot \vec{v}_\mu$ with $\mu = 1, \ldots, N$

Newton equations are almost quadratic

$$m\ddot{s}_\mu(t) = [z(t) - \lambda_\mu] s_\mu(t)$$

with $z(t)$ the Lagrange multiplier that enforces the spherical constraint and λ_μ the eigenvalues (semi-circle law, with support in $[-2J, 2J]$)

Two methods to solve :

— for $N \to \infty$, closed Schwinger-Dyson equations on $C(t, t_w)$ and $R(t, t_w)$, the global self-correlation and linear response (already shown for general p)

— for finite N, solve Newton equations under the spherical constraint

Dynamic equations

Conservative dynamics for $p = 2$

In the $N \to \infty$ limit exact causal Schwinger-Dyson equations

$$(m\partial_t^2 - z_t)C(t,t_w) = \int dt' \left[\Sigma(t,t')C(t',t_w) + D(t,t')R(t_w,t')\right]$$

$$+\frac{\beta' J_0}{J} D(t,0)C(t_w,0) + \text{Other Term}$$

$$(m\partial_t^2 - z_t)R(t,t_w) = \int dt' \, \Sigma(t,t')R(t',t_w) + \delta(t-t_w)$$

Other equation

with the post-quench self-energy and vertex

$$D(t,t_w) = J^2 \, C(t,t_w) \qquad \Sigma(t,t_w) = J^2 \, R(t,t_w)$$

and the Lagrange multiplier z_t fixed by $C(t,t) = 1$ (Technical)

Two body model

An implicit solution for finite N

The projection of the spin configuration on the eigenvector $\vec{v}_\mu$ reads ($m = 1$)

$$s_\mu(t) \quad = \quad s_\mu(0) \sqrt{\frac{\Omega_\mu(0)}{\Omega_\mu(t)}} \, \cos \int_0^t dt' \; \Omega_\mu(t') + \frac{\dot{s}_\mu(0)}{\Omega_\mu(0)\Omega_\mu(t)} \, \sin \int_0^t dt' \; \Omega_\mu(t')$$

The time-dependent frequency $\Omega_\mu(t)$ and Lagrange multiplier $z(t)$ are fixed by

$$\frac{1}{2} \frac{\ddot{\Omega}_\mu(t)}{\Omega_\mu(t)} - \frac{3}{4} \left(\frac{\dot{\Omega}_\mu(t)}{\omega_\mu(t)} \right)^2 + \Omega_\mu^2(t) = z(t) - \lambda_\mu$$

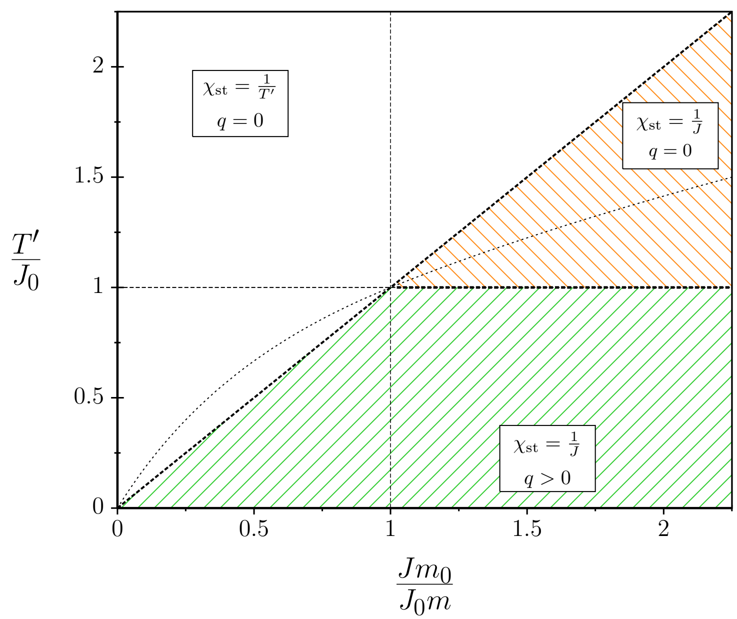
with initial conditions $\dot{\Omega}_\mu(0) = 0$, $\Omega_\mu^2(0) = \lambda_{\max} - \lambda_\mu$ and $z(t) = e_f + \frac{2}{N} \sum_\mu \lambda_\mu \langle s_\mu^2(t) \rangle$

Note that the initial conditions $\{s_\mu(0), \dot{s}_\mu(0)\}$ know about the pre-quench potential and the λ_μ about the post-quench one

Similar to **Sotiriadis & Cardy 10** for the quantum O(N) model

Two body model

Richer results !



Initial conditions

Three Sectors

I $\chi_{st} = 1/T'$ and $\lim_{t \gg t_w} C(t, t_w) = 0$

II $\chi_{st} = 1/J$ and $\lim_{t \gg t_w} C(t, t_w) = 0$

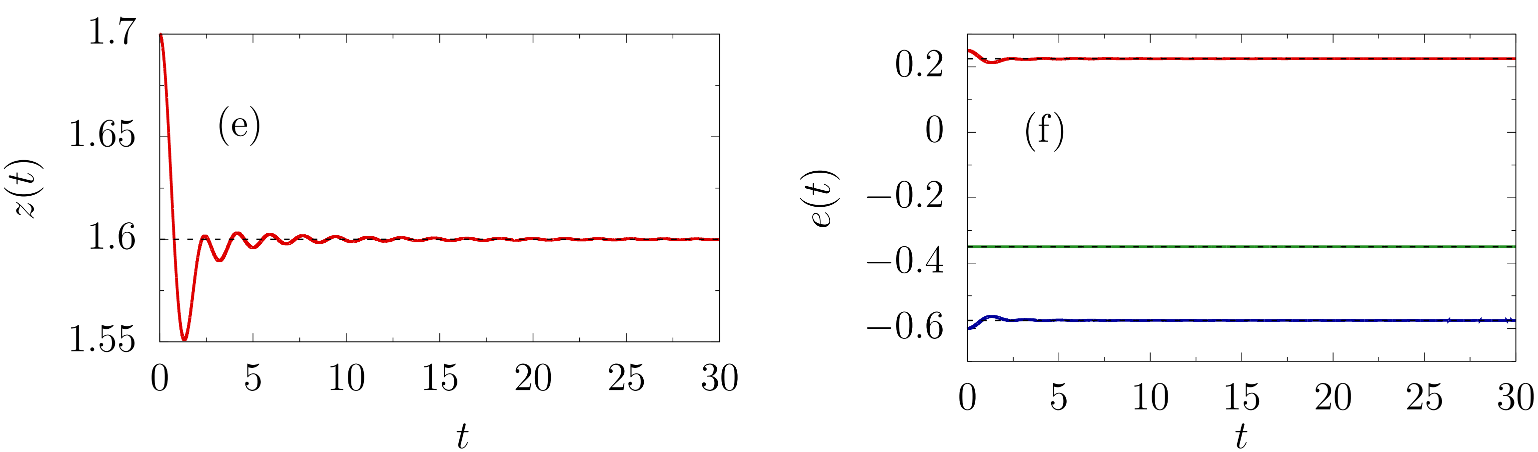
} **GGE ?**

III $\chi_{st} = 1/J$ and $\lim_{t \gg t_w} C(t, t_w) > 0$ **GB equilibrium ?**

Two body model

III Confined states global behaviour as in GB equilibrium at β_f

$$z_f = \lim_{t\to\infty} z(t) = \frac{1}{J}$$



$$e^f_{\text{pot}} = \lim_{t\to\infty} e_{\text{pot}}(t)$$

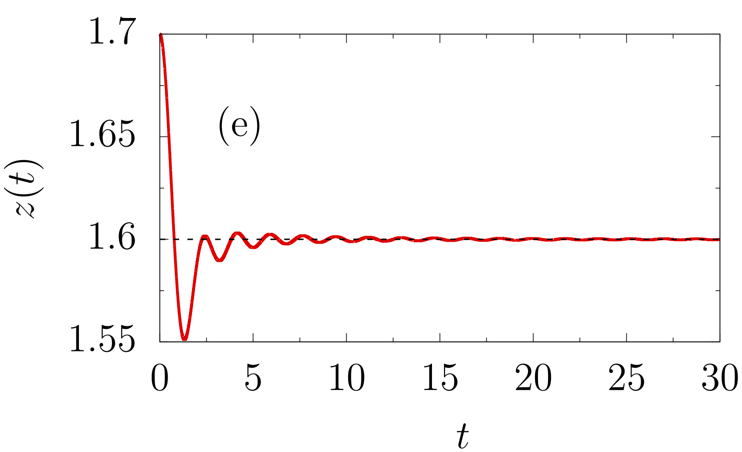
$$e^f_{\text{kin}} = \lim_{t\to\infty} e_{\text{kin}}(t)$$

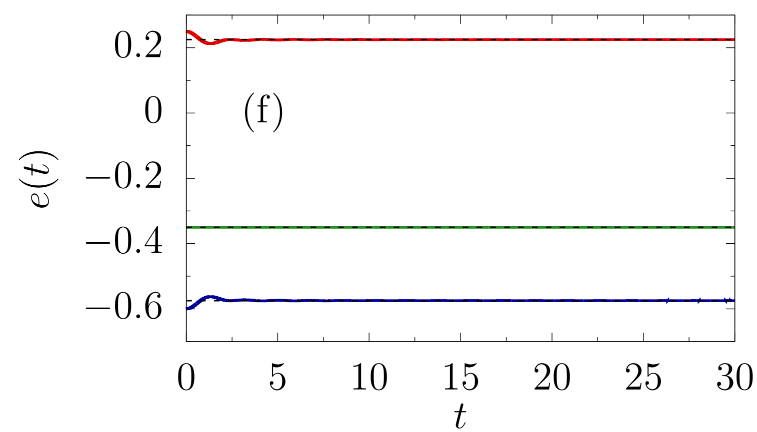
$$e_f = e^f_{\text{kin}} + e^f_{\text{pot}}$$

Two body model

III Confined states global behaviour as in GB equilibrium at β_f

$$z_f = \lim_{t\to\infty} z(t) = \frac{1}{J}$$





$$e^f_{\rm pot} = \lim_{t\to\infty} e_{\rm pot}(t)$$

$$T_f/2 = e^f_{\rm kin} = \lim_{t\to\infty} e_{\rm kin}(t)$$

$$e_f = e^f_{\rm kin} + e^f_{\rm pot}$$

Two body model

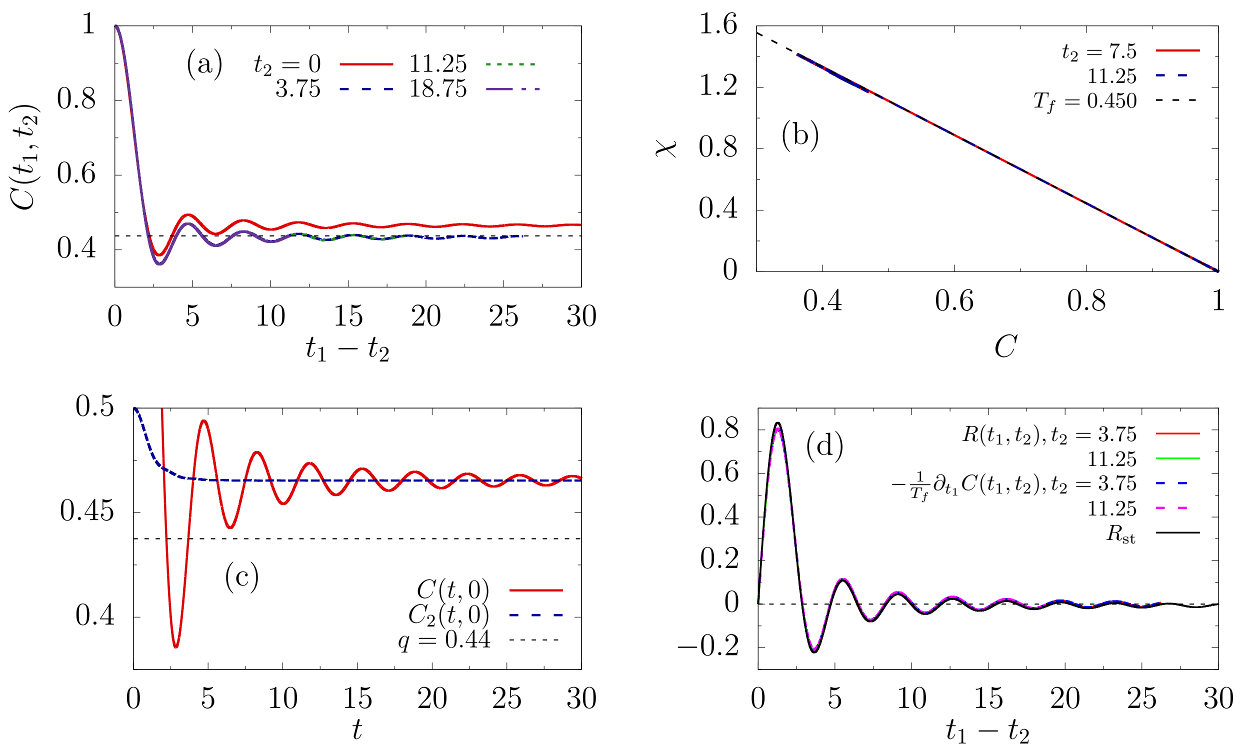
III Confined states global behaviour as in GB equilibrium at β_f

Fidelity

$C(t_1{,}0) \to q_0$

Integrated linear response

$\chi(t_1{,}t_2) = \int_{t_2}^{t_1} dt'\, R(t_1{,}t')$



$\chi = \frac{1}{T_f}(1{-}C)$

for

$C(t_1{,}t_2) \geq q$

q and T_f

as in GB equil.

in a confined state

Two body model

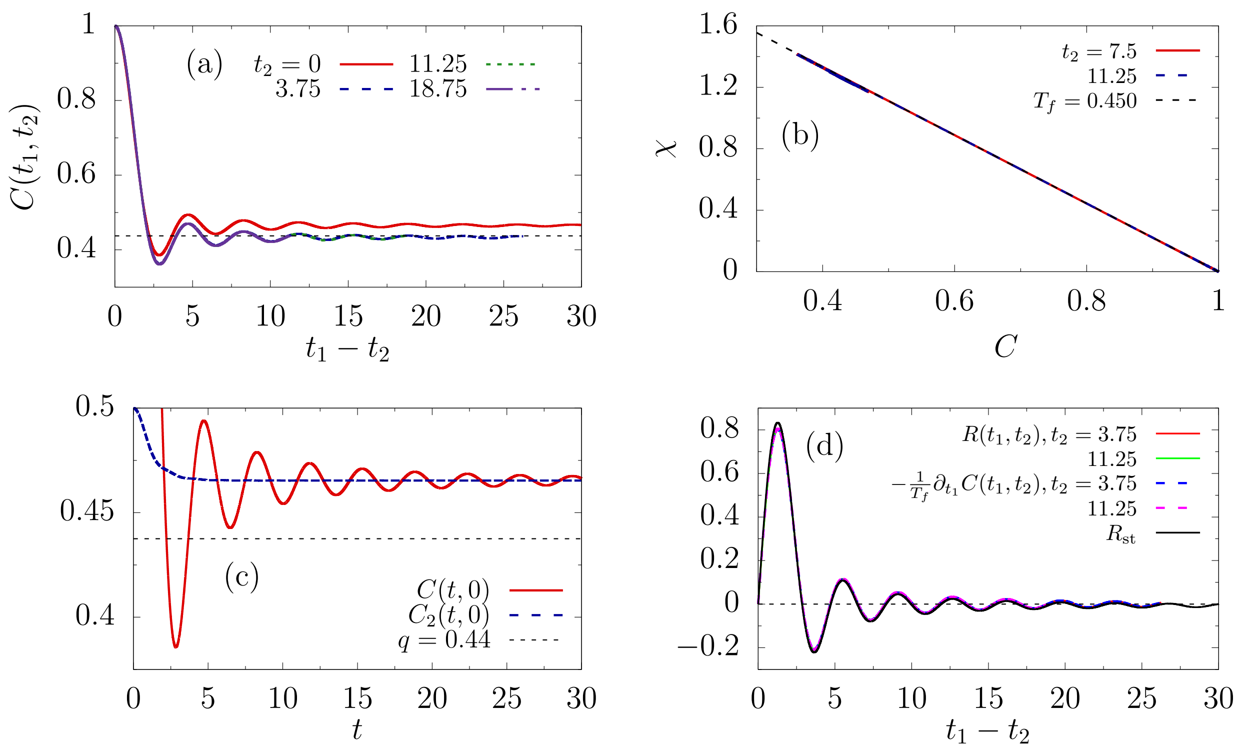
III Confined states global behaviour as in GB equilibrium at β_f

Fidelity

$C(t_1,0) \to q_0$

Integrated linear response

$\chi(t_1,t_2) = \int_{t_2}^{t_1} dt' \, R(t_1,t')$



$\chi = \frac{1}{T_f}(1-C)$

for

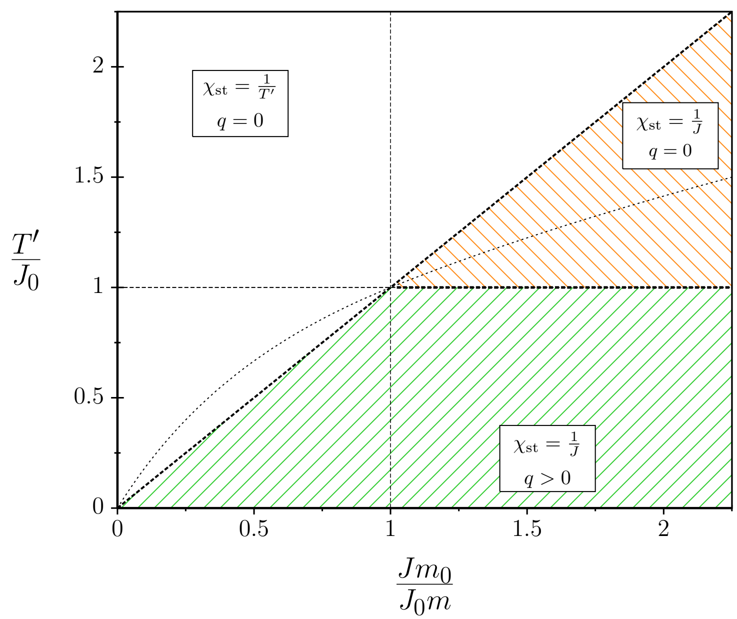
$C(t_1,t_2) \geq q$

$2e^f_{\rm kin} = T_f$

$2e^f_{\rm pot} = \frac{-1}{T_f}\,(1-q)$

Two body model

Richer results !



Initial conditions

Three Sectors

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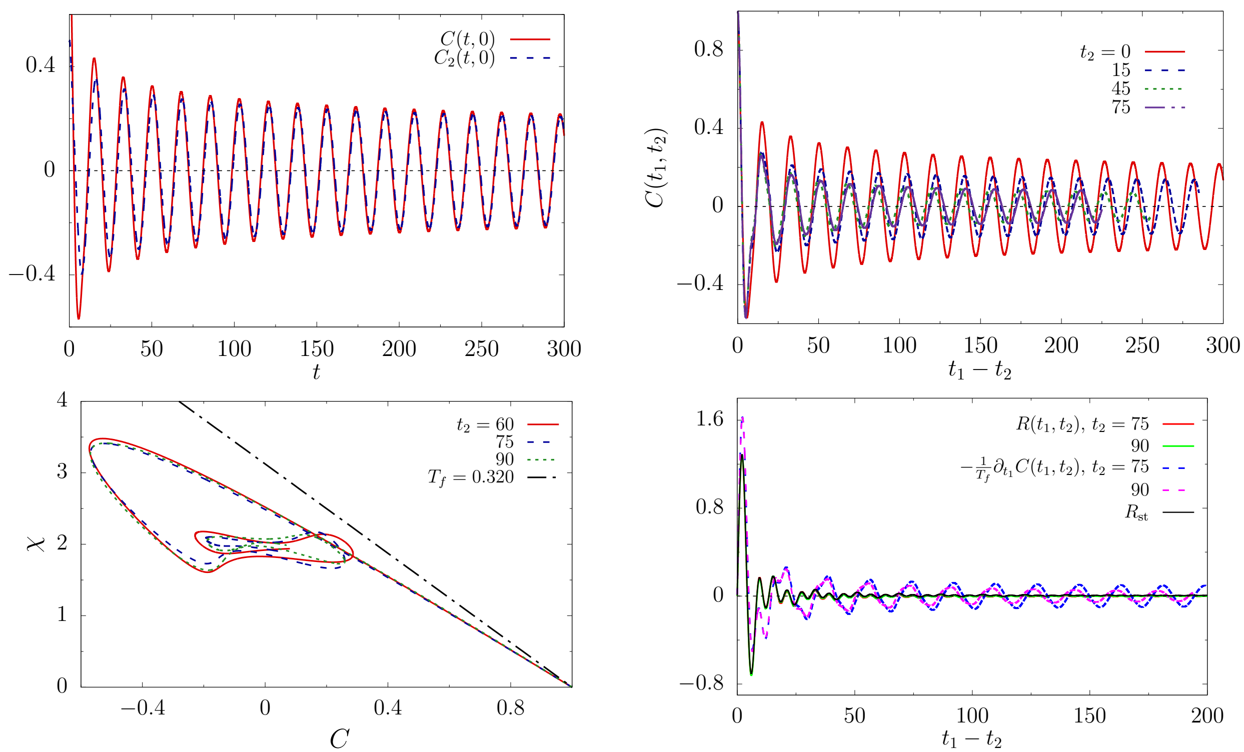
} **GGE ?**

III $\chi_{st} = 1/J$ and $\lim_{t \gg t_w} C(t, t_w) > 0$ **GB equilibrium ?**

Two body model

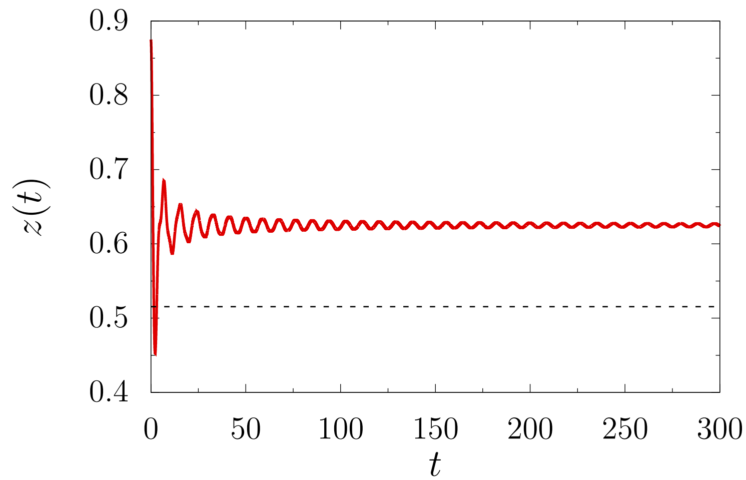
Large energy injection on a confined state



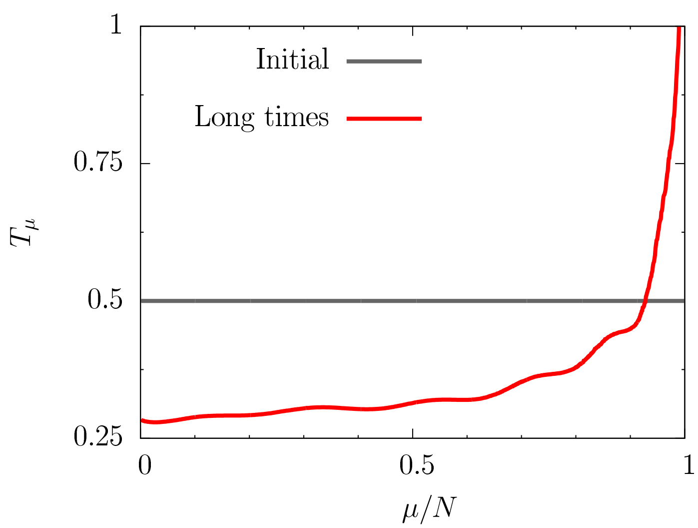
Stationary dynamics but no FDT at a single temperature: no GB equilibrium

Two body model

- **Large energy injection on a confined state: T_μ spectrum**



$T' = 0.5,\ J = 0.25,\ N = 1024$



$$z(t) \to z_f = T' + J^2/T'$$

The time-dependent frequencies too

$$\Omega^2_\mu(t) \to (z_f - \lambda_\mu)/m \equiv \omega^2_\mu$$

The μ modes $s_\mu(t)$ decouple and

become independent harmonic oscillators

with conserved energy

$$e_\mu = e^{\rm kin}_\mu(t) + e^{\rm pot}_\mu(t)$$

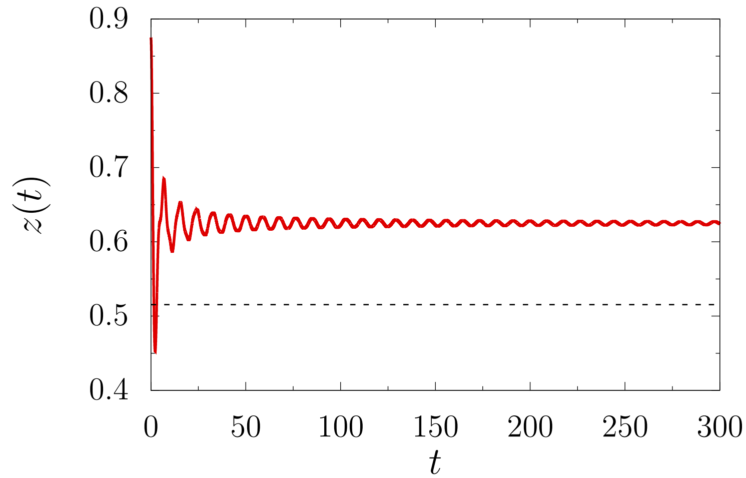
Mode temperatures

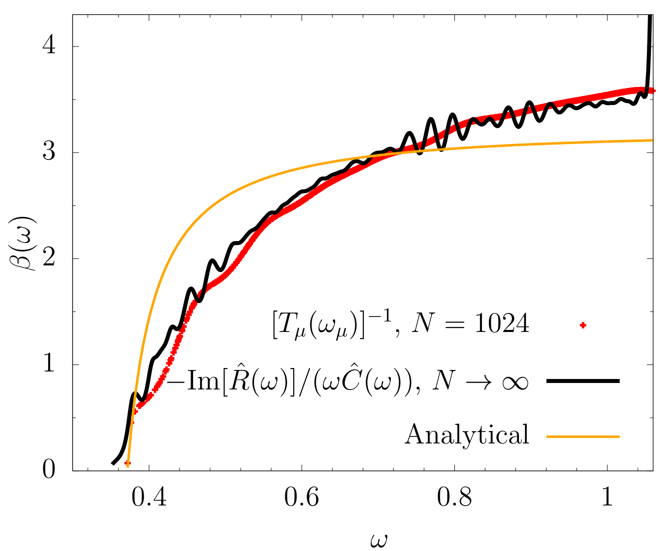
$$\overline{\langle H^{\rm kin}_\mu \rangle} = \overline{\langle H^{\rm pot}_\mu \rangle} = T_\mu$$

where $\overline{\ldots} = \lim_{\tau \gg 1} \frac{1}{\tau} \int_{t_{\rm st}}^{t_{\rm st}+\tau} dt' \ldots$

Two body model

I Large energy injection on a confined state: T_μ from the FDR





$z(t) \to z_f = T' + J^2/T'$

The time-dependent frequencies too

$\Omega_\mu^2(t) \to (z_f - \lambda_\mu)/m \equiv \omega_\mu^2$

The μ modes $s_\mu(t)$ decouple and

become independent harmonic oscillators

with conserved energy

$e_\mu = e_\mu^{\text{kin}}(t) + e_\mu^{\text{pot}}(t)$

Mode inverse temperatures *vs*

FDR inverse temperature

$-\text{Im}\hat{R}(\omega)/(\omega\hat{C}(\omega)) = \beta_{\text{eff}}(\omega)$

Two body model

An integrable model ? Yes, Neumann's model (1850)

Motion of a particle on S_{N-1}, enforced by $\sum_k x_k^2 = N$

The Hamiltonian is

$$H = \frac{1}{4N} \sum_{k \neq l} L_{kl}^2 + \frac{1}{2} \sum a_k x_k^2$$

with $L_{kl} = (x_k p_l - x_l p_k)/\sqrt{m}$

The integrals of motion are $I_k = x_k^2 + \sum_{l(\neq k)} \frac{L_{kl}^2}{a_k - a_l}$

K. Uhlenbeck 1982

Translation from Neumann to $p = 2$ spherical model

$a_k \mapsto -\lambda_\mu$ and $I_\mu = s_\mu^2 + \frac{1}{N} \sum_{\nu(\neq \mu)} \frac{s_\mu^2 p_\nu^2 + s_\nu^2 p_\mu^2 - 2 s_\mu p_\mu s_\nu p_\nu}{\lambda_\nu - \lambda_\mu}$

Two body model

Two (or more) possibilities : GB, GGE or none

- The system is able to act as a bath on itself and equilibrate to

$$\rho_{\mathrm{GB}} = Z^{-1}\, e^{-\beta_f H}$$

- The system is not able to act as a bath on itself as it is an integrable system.

  Does it approach a Generalised Gibbs Ensemble (GGE)

$$\boxed{\rho_{\mathrm{GGE}} = Z_{\mathrm{GGE}}^{-1}\, e^{-\sum_{\mu=1}^{N} \beta_\mu I_\mu}}$$

  with Uhlenbeck's constants of motion I_μ and β_μ fixed by

$$\boxed{\langle I_\mu \rangle_{\mathrm{GGE}} = I_\mu(t=0^+)}$$

It depends on the kind of quench, phases in the dynamic phase diagram, and on the observables !

Full classification in progress

Conclusions

Study of the quenched dynamics of classical isolated disordered models

We showed that they can

- equilibrate to GB measures
- undergo non-stationary (aging) dynamics
- or (most probably) approach a GGE

depending on the type of model (highly interacting or quasi quadratic) and the kind of quench performed.

Work on the extension of these studies to the quantum models and the better understanding of the approach to a GGE is under way

Fluctuation-dissipation relations

Classical setting

Measure

$\text{Im}\tilde{R}^{AB}(\omega)$ and $\omega\tilde{C}^{AB}(\omega)$

take the ratio and extract $\beta_{\text{eff}}^{AB}(\omega)$

In equilibrium all $\beta_{\text{eff}}^{AB}(\omega)$ should be equal to the same constant

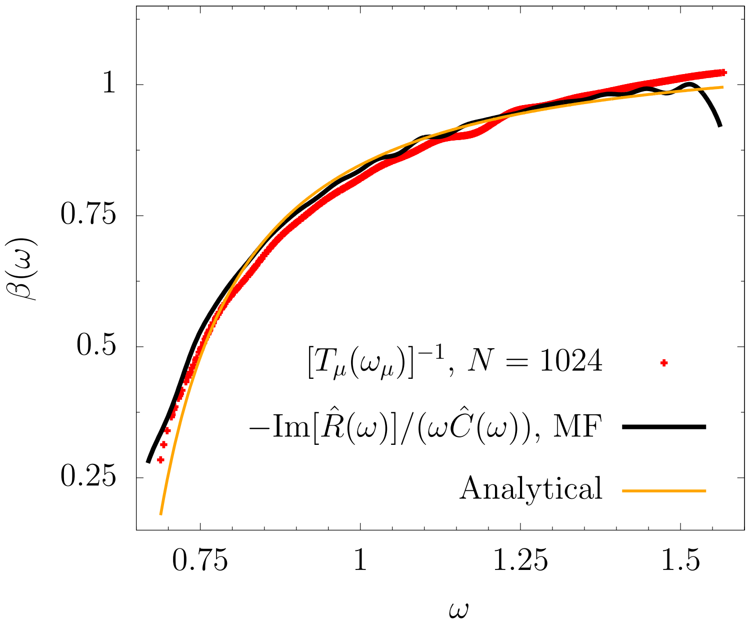
This is the fluctuation-dissipation theorem (FDT).

If there is a frequency or observable dependence, the system is not in Gibbs-Boltzmann equilibrium

Do these $\beta_{\text{eff}}(\omega)$ play a role in closed systems too ?

GGE and FDT temperatures

A generic method



FDR

$$\frac{2\mathrm{Im}\tilde{R}(\omega)}{\omega\tilde{C}(\omega)} = \beta(\omega)$$

The asympt mode freq

$$\omega_\mu^2 = [z_{\mathrm{as}} - \lambda_\mu]/m$$

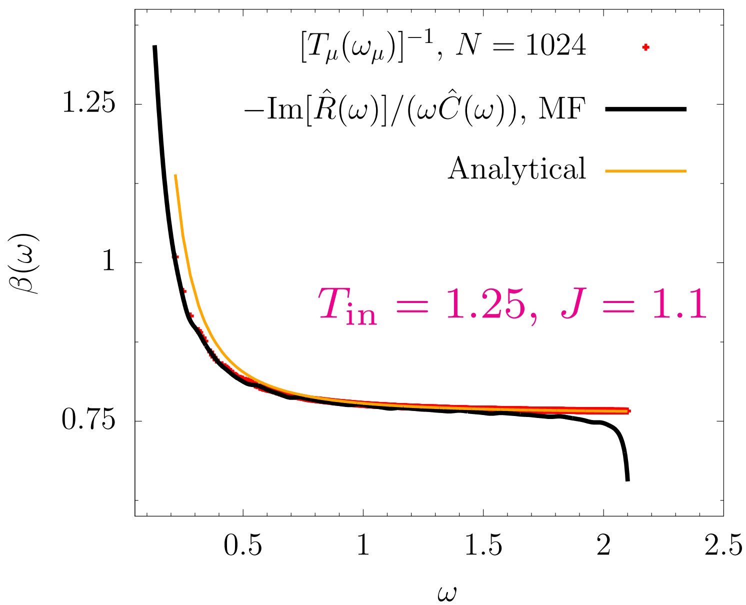
$$\boxed{\frac{2\mathrm{Im}\tilde{R}(\omega_\mu)}{\omega_\mu\tilde{C}(\omega_\mu)} = \beta_\mu}$$

$T_{\mathrm{in}} = 1.25, \ J = 0.5$

Using the idea in **Foini, Gambassi, Konik & LFC ; de Nardis, Panfil, ... 17** for **quantum integrable systems** now in a

GGE and FDT temperatures

A generic method



FDR

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The asympt mode freq

$$\omega_\mu^2 = [z_{\rm as} - \lambda_\mu]/m$$

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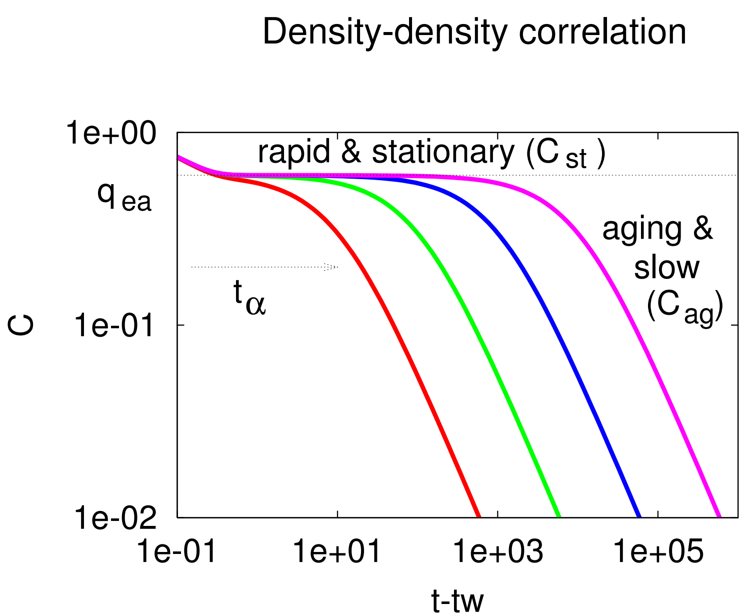
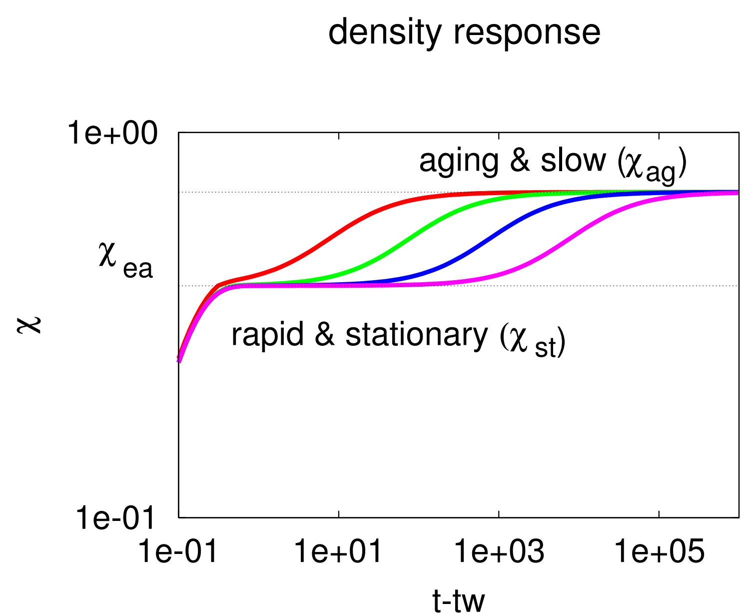
Using the idea in **Foini, Gambassi, Konik & LFC ; de Nardis, Panfil, ... 17**

in a **classical** system **LFC, Lozano, Nessi, Picco & Tartaglia (in prep)**

Glassy dynamics

Non stationary relaxation & separation of time-scales





$$C(t, t_w)$$

Correlation

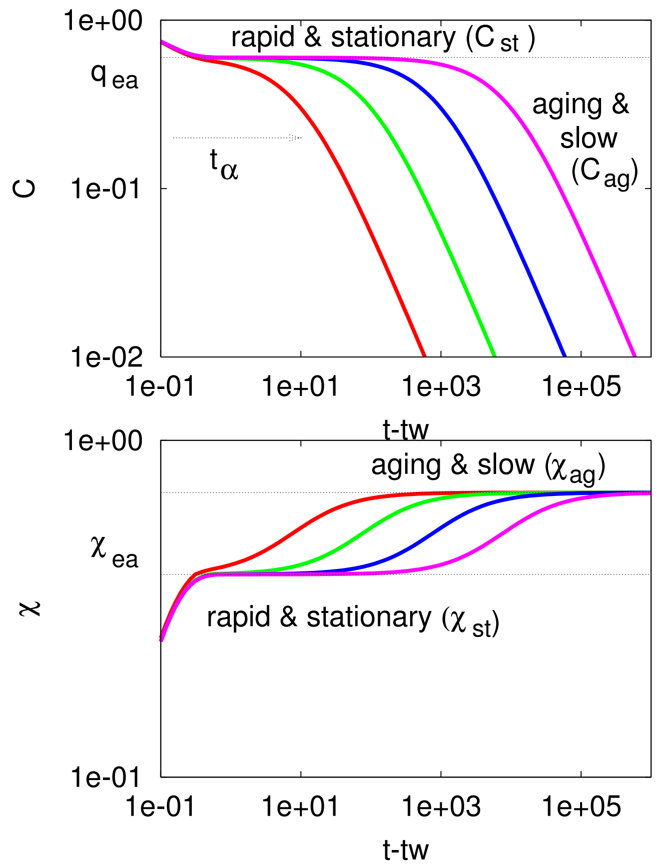
$$\chi(t, t_w) = \int_{t_w}^{t} dt' \; R(t, t')$$

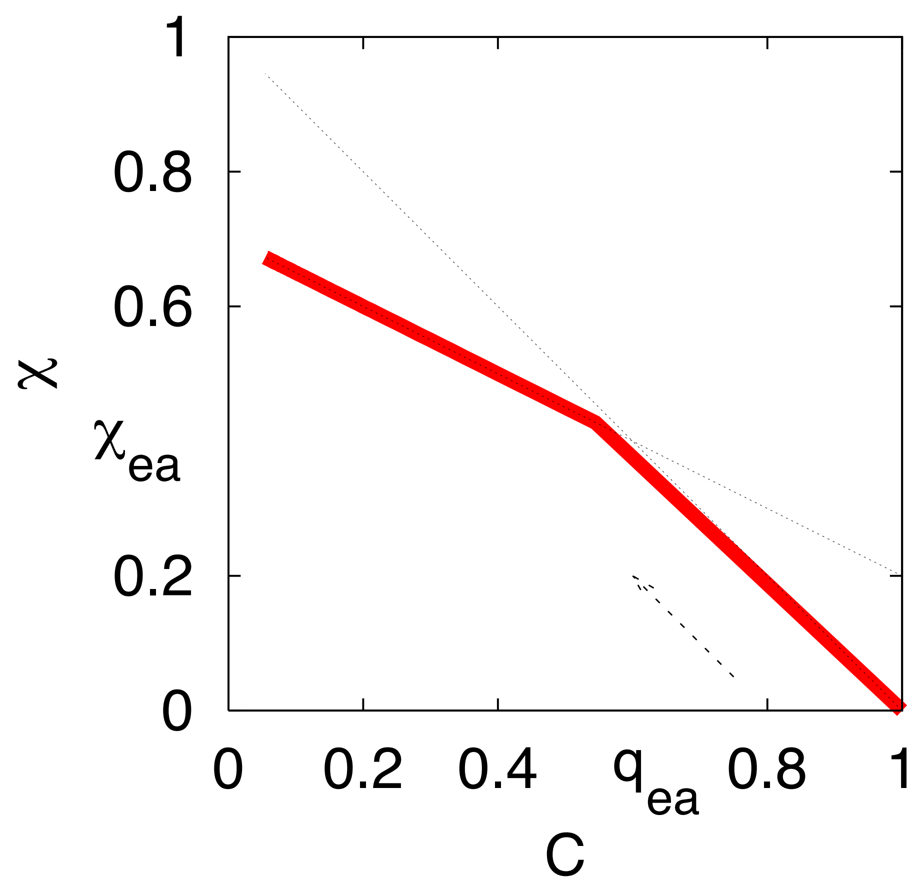
Time-integrated linear response

Analytic solution to a mean-field model **LFC & J. Kurchan 93**

Glassy dynamics

Fluctuation-dissipation relation : parametric plot

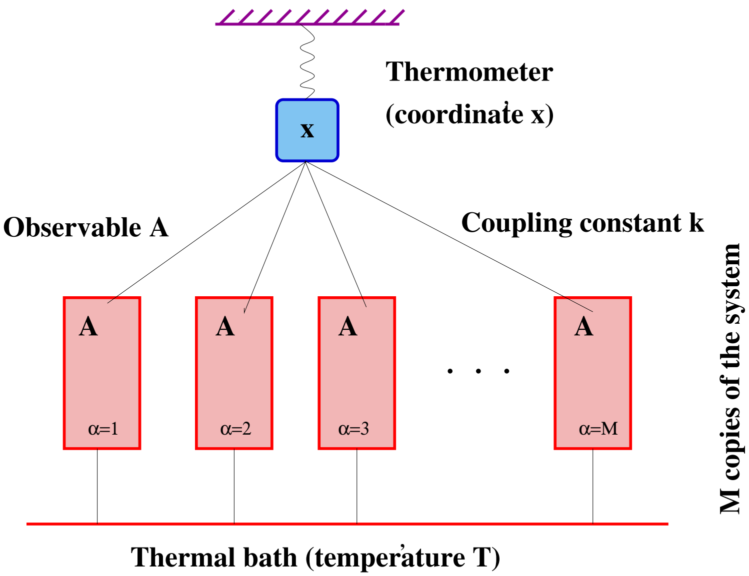


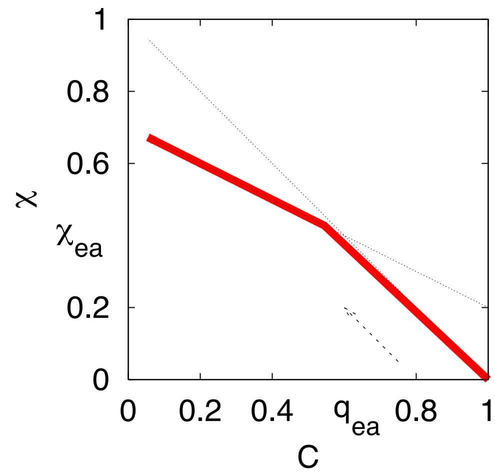


Analytic solution to a mean-field model **LFC & J. Kurchan 93**

FDR & effective temperatures

Can one interpret the slope as a temperature ?





(1) Measurement with a **thermometer** with

- Short internal time scale τ_0, fast dynamics is tested and T is recorded.
- Long internal time scale τ_0, slow dynamics is tested and T^* is recorded.

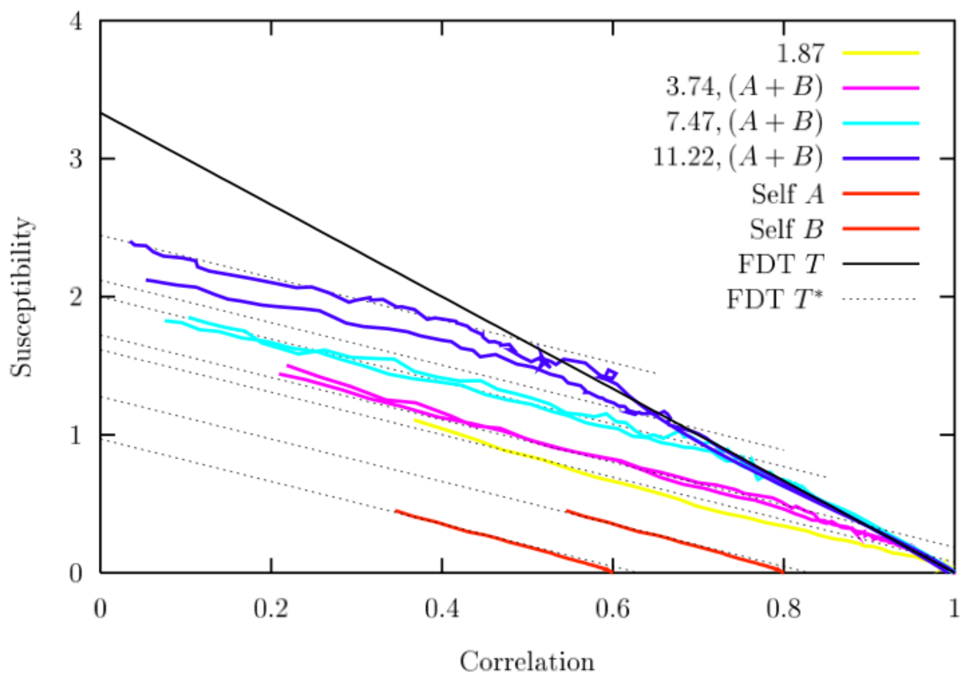
(2) **Partial equilibration**

(3) **Direction of heat-flow**

LFC, Kurchan & Peliti 97

FDT & effective temperatures

Sheared binary Lennard-Jones mixture



$\chi_k(C_k)$ plot for different wave-vectors k, partial equilibrations.

J-L Barrat & Berthier 00

Effective temperatures

Glasses, coarsening, driven systems

Different observables can depend differently (*e.g.* velocity vs. positions).

There is a separation of time-scales,

with a crossover at, roughly, ωt_w (or controlled by the drive)

The FDRs take a very special form:

- $\omega t_w \ll 1$ quasi-stationary relation and FDT with bath T
- $\omega t_w \gg 1$ non-stationary relation and FDR with another T^*.

$T_{\rm eff}(\omega, t_w)$ crosses over from T to T^* that depends upon

- the initial condition before the quench (disordered *vs.* ordered);
- weakly on other parameters of the systems.

Notion of interacting *vs.* non-interacting concerning partial equilibrations.

LFC 11, review

Fluctuation-dissipation relations

Quantum setting

Measure

$\text{Im}\tilde{R}^{AB}(\omega)$ and $\tilde{C}_{\pm}^{AB}(\omega)$

take the ratio and extract $\tanh(\beta_{\text{eff}}^{AB}(\omega)\hbar\omega/2)$

In equilibrium all $\beta_{\text{eff}}^{AB}(\omega)$ should be equal to the same constant

This is the fluctuation-dissipation theorem (FDT).

If there is a frequency or observable dependence, the system is not in Gibbs-Boltzmann equilibrium

Do these $\beta_{\text{eff}}(\omega)$ play a role in closed systems too ?

Plan

1. Introduction.
2. Fluctuation-dissipation relations
   — Measurements of effective temperatures and properties.
   — Relation to free-energy densities and entropy.
   — Fluctuation theorems.
3. Quantum quenches.
4. Integrable systems and Generalized Gibbs Ensembles.

LFC, Kurchan & Peliti 97 ; **Foini, LFC & Gambassi 11 & 12** ; **Foini, Gambassi, Konik & LFC 16** ; **de Nardis, Panfil, Gambassi, LFC, Konik & Foini 17**

Thanks to the joint autumn programs at KITP 2015

Plan

4 Integrable systems and Generalized Gibbs Ensembles.

- As a test of non-thermal equilibrium

Foini, LFC & Gambassi 11 & 12

- Integrable non-interacting systems

Foini, Gambassi, Konik & LFC 16

- Integrable interacting systems

de Nardis, Panfil, Gambassi, LFC, Konik & Foini 17

Thanks to the joint autumn programs at KITP 2015

Fluctuation-dissipation relations

Quantum Ising chain

The initial Hamiltonian $$\hat{H}_{\Gamma_0} = -\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \Gamma_0 \sum_i \hat{\sigma}_i^z$$

The initial state $|\psi_0\rangle$ ground state of $\hat{H}_{\Gamma_0}$

Instantaneous quench in the transverse field $\Gamma_0 \to \Gamma$

Evolution with $\hat{H}_{\Gamma}$.

Iglói & Rieger 00

Reviews: **Karevski 06 ; Polkovnikov et al. 10 ; Dziarmaga 10**

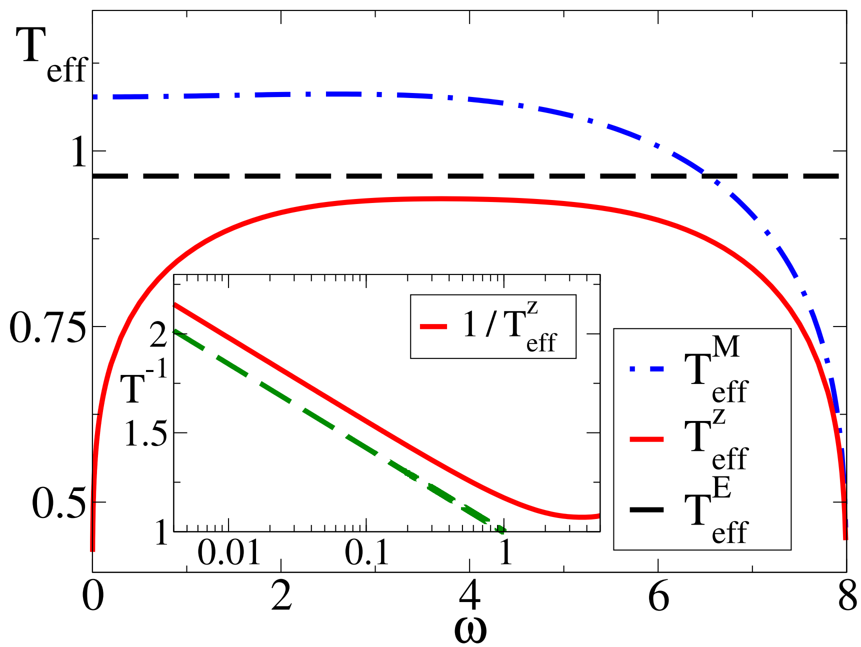
Observables : correlation and linear response of local longitudinal and transverse spin, etc.

Specially interesting case $\Gamma_c = 1$ the critical point. **Rossini et al. 09**

Quantum quench

$T_{\rm eff}$ from the FDR (quench to $\Gamma_c = 1$)

$$\hbar \, \mathrm{Im}\tilde{R}(\omega) = \tanh(\beta_{\rm eff}(\omega)\omega\hbar/2) \;\; \tilde{C}_+(\omega)$$



$$\boxed{\beta^z_{\rm eff}(\omega) \neq \beta^M_{\rm eff}(\omega) \neq \text{ct}}$$

Foini, LFC & Gambassi 11 & 12

Similar ideas in **Bortolin & Iucci 15** (hard core bosons)

Chiocchetta, Gambassi & Carusotto 15 (photon/polariton condensates)

Summary

Fluctuation-dissipation relations

Use of fluctuation-dissipation relations to check for deviations from Gibbs-Boltzmann equilibrium in the dynamics of closed quantum systems

Fluctuation-dissipation relations

Can they be used to infer the steady state density operator?

From the FDR to the GGE

Lehman representation

The correlation and linear response are

$$C(t_2,t_1) = \frac{1}{2}\langle[\hat{A}(t_2),\hat{A}^\dagger(t_1)]_+\rangle$$

$$R(t_2,t_1) = i\langle[\hat{A}(t_2),\hat{A}^\dagger(t_1)]_-\rangle\;\theta(t_2-t_1)$$

The expectation value $\langle\cdots\rangle$ is calculated over a generic density matrix $\hat{\rho}$

(units such that $\hbar=1$ and $[\hat{X},\hat{Y}]_\pm \equiv \hat{X}\hat{Y}\pm\hat{Y}\hat{X}$)

Taking a Fourier transform wrt to $t_2 - t_1$

$$\tilde{C}(\omega) = \pi\sum_{m,n\geq 0}\delta(\omega+E_n-E_m)|A_{nm}|^2(\rho_{nn}+\rho_{mm})$$

$$\text{Im}\,\tilde{R}(\omega) = \pi\sum_{m,n\geq 0}\delta(\omega+E_n-E_m)|A_{nm}|^2(\rho_{nn}-\rho_{mm})$$

where the sums run over a complete basis of eigenstates $\{|n\rangle\}_{n\geq 0}$ of the Hamiltonian $\hat{H}$ with increasing eigenvalues E_n, and $X_{mn} = \langle m|\hat{X}|n\rangle$.

From the FDR to the GGE

Lehman representation

Note that $\tilde{C}(\omega)$ and $\text{Im}\,\tilde{R}(\omega)$ are non-zero only if ω takes values within the discrete set $\{E_m - E_n\}_{m,n\geq 0}$ (due to the delta functions) and $A_{nm} \neq 0$.

In Gibbs-Boltzmann equilibrium $\rho_{nn} \propto \exp(-\beta E_n)$ since there is a single charge, $\hat{Q}_1 = \hat{H}$, and for any bosonic $\hat{A}$, the FDT holds $\forall\ \omega$

$$\text{Im}\,\tilde{R}(\omega) = \tanh(\beta\omega/2)\ \tilde{C}(\omega)$$

In contrast, for the GGE, $\rho_{nn} \propto \exp(-\sum_k \lambda_k Q_{kn})$ with $Q_{kn} \equiv \langle n|\hat{Q}_k|n\rangle$. By properly choosing $\hat{A}$ we can extract the λ_k's from the corresponding FDR.

From the FDR to the GGE

Example: a non-interacting integrable model

Take a non-interacting Hamiltonian in its diagonal form

$$\hat{H} = \sum_k \epsilon_k \, \hat{\eta}_k^\dagger \hat{\eta}_k \, ,$$

$\hat{\eta}_k$'s are creation operators for excitations of energy ϵ_k

The number operators $\hat{Q}_k = \hat{\eta}_k^\dagger \hat{\eta}_k$ are the (commuting) conserved charges.

The GGE density matrix is $\hat{\rho} \propto e^{\sum_k \lambda_k \hat{Q}_k}$ with $\hat{Q}_k = \hat{\eta}_k^\dagger \hat{\eta}_k$,

while $\beta_k \equiv \lambda_k / \epsilon_k$ defines a mode-dependent inverse "effective temperature".

For $\boxed{\hat{A} = \sum_k (\alpha_k \hat{\eta}_k + \alpha_k^* \hat{\eta}_k^\dagger) \qquad \Rightarrow \qquad \frac{\mathrm{Im}\, \tilde{R}(\omega_k)}{\tilde{C}(\omega_k)} = \tanh(\lambda_k/2)}$

with $\omega = \omega_k \equiv \epsilon_k$ (in the absence of degeneracies with respect to k and $\alpha_k \in \mathbb{C}$)

Quantum quench

Hard-core bosons in one dimension

Consider the Lieb-Liniger model with density ϱ

$$\hat{H}_c = \int dx \left[\partial_x \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) + c \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}(x) \right]$$

initialized in the ground state of $\hat{H}_{c=0}$ and evolved with $\hat{H}_{c\to\infty}$

Mapping to hard-core bosons, that after a Jordan-Wigner transformation become free fermions,

$$\hat{H}_{c\to\infty} = \sum_k \epsilon_k \hat{f}_k^\dagger \hat{f}_k \text{ with } \epsilon_k = k^2$$

The conserved charges are $\langle \hat{Q}_k \rangle = \langle \hat{f}_k^\dagger \hat{f}_k \rangle = 4\varrho^2 / (\epsilon_k^2 + 4\varrho^2)$

and the Lagrange multipliers $\lambda_k = \ln[\epsilon_k^2 / (4\varrho^2)]$

Quantum quench

Hard-core bosons in one dimension

Consider again the $c = 0$ to $c \to \infty$ quench of the Lieb-Liniger model

In the stationary limit (and for $q \neq 0, \pi$)

$$C(q,t) = \sum_k e^{-i(\epsilon_k - \epsilon_{k-q})|t|} (\frac{n_{k-q} + n_k}{2} - n_{k-q} n_k)$$

$$R(q,t) = i\theta(t) \sum_k e^{-i(\epsilon_k - \epsilon_{k-q})t} \, (n_{k-q} - n_k)$$

The Fourier transform picks $\omega = \epsilon_k - \epsilon_{k-q}$ with two solutions $k_{1,2}(q,\omega)$

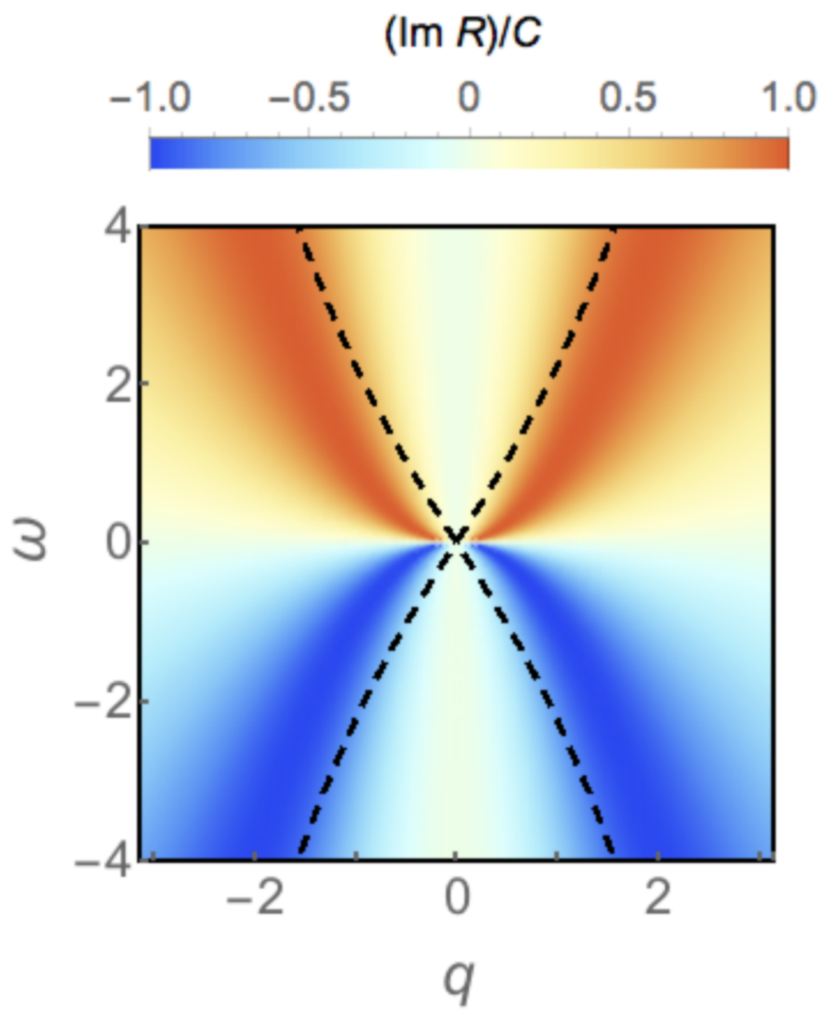
Measuring at frequency ω and wave-vector q related by $\omega = 2\epsilon_{(q+\pi)/2}$,

a single mode is selected, $k_1 = k_2 = (q+\pi)/2$, and the FDR becomes

$$\text{Im}\, R(q,\omega_q)/C(q,\omega_q) = \tanh \lambda_q \quad \text{with} \quad \lambda_q = \ln[\epsilon_q^2/(4\varrho^2)]$$

Quantum quench

Hard-core bosons in one dimension



Quantum quench

The one dimensional Bose gas

Consider the Lieb-Liniger model with density ϱ

$$\hat{H}_c = \int dx \left[\partial_x \hat{\phi}^\dagger(x) \partial_x \hat{\phi}(x) + c \hat{\phi}^\dagger(x) \hat{\phi}^\dagger(x) \hat{\phi}(x) \hat{\phi}(x) \right]$$

initialized in the ground state of $\hat{H}_{c=0}$ and evolved with $\hat{H}_{c<+\infty}$, now an interacting problem.

Bethe *Ansatz* solution:

$$\lim_{t\to\infty} \langle O \rangle = \langle \vartheta_{\mathrm{GGE}} | O | \vartheta_{\mathrm{GGE}} \rangle$$

with the eigenstate $|\vartheta_{\mathrm{GGE}}\rangle$ characterised by a “mode occupation” $\vartheta_{\mathrm{GGE}}(\lambda)$ computed, for this problem, in

De Nardis, Wouters, Brockmann & Caux 14

Quantum quench

The one dimensional Bose gas

Let us parametrize $\vartheta_{\rm GGE}(\lambda)$ as

$$\vartheta_{\rm GGE}(\lambda) = \frac{1}{1+e^{\epsilon(\lambda)}}$$

and $\epsilon(\lambda_F)=0$.

One particle-hole kinematics at slow momentum $k \ll k_F = \pi\varrho$

The FDR of the density-density correlation and linear response is

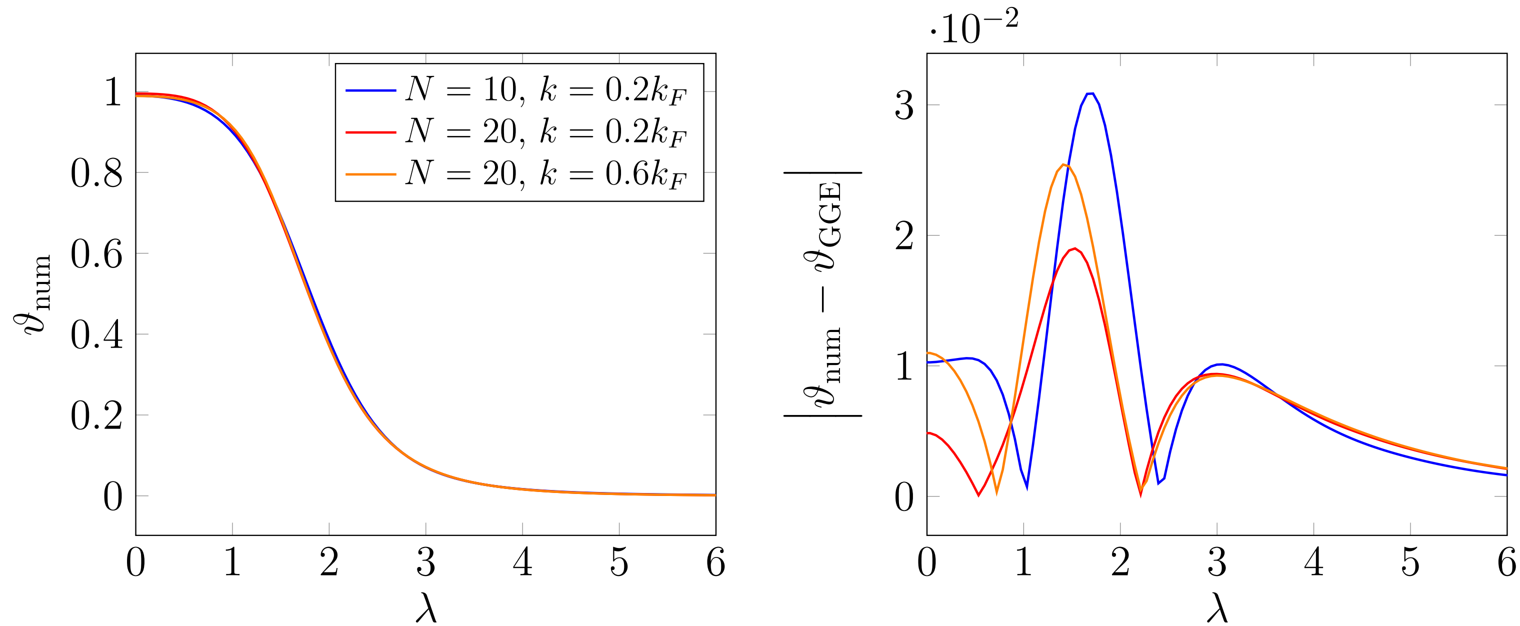
$$\boxed{\text{Im}\ \tilde{R}(k,\omega)/\tilde{C}(k,\omega) = \tanh(k\partial_\lambda\epsilon(\lambda)/(2\pi\rho_t(\lambda))|_{\lambda(k,\omega)}}$$

Without entering the technical details, $\epsilon(\lambda)$, $\rho_t(\lambda)$ and $\lambda(k,\omega)$ depend on $\vartheta_{\rm GGE}(\lambda)$.

Computing the left-hand-side one can reconstruct $\vartheta_{\rm GGE}(\lambda)$ and compare it to the exact form derived by **De Nardis et al 14**

Quantum quench

Hard-core bosons in one dimension



$\vartheta_{\rm num}$ from FDR & $\vartheta_{\rm GGE}$ from direct calculation.

Error due to the low k expansion present in both evaluations.

de Nardis, Panfil, Gambassi, LFC, Konik, Foini 17

Summary

Fluctuation-dissipation relations

The FDRs of carefully chosen, but quite natural, observables "contain" the GGE effective temperatures.

They can be used to measure them or, even more generally, to infer the steady state density matrix

References

4 Integrable systems and Generalized Gibbs Ensembles.

- As a test of non-thermal equilibrium

Foini, LFC & Gambassi 11 & 12

- Integrable non-interacting systems

Foini, Gambassi, Konik & LFC 16

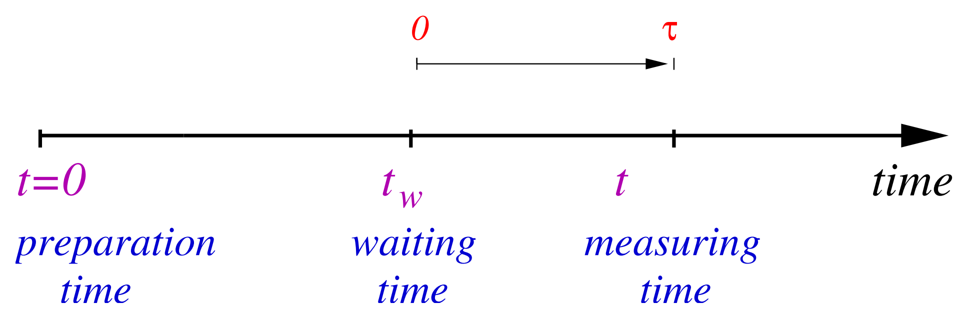
- Integrable interacting systems

de Nardis, Panfil, Gambassi, LFC, Konik & Foini 17

Thanks to the joint autumn programs at KITP 2015

Two-time observables

Correlations



The two-time correlation between two observables $\hat{A}(t)$ and $\hat{B}(t_w)$ is

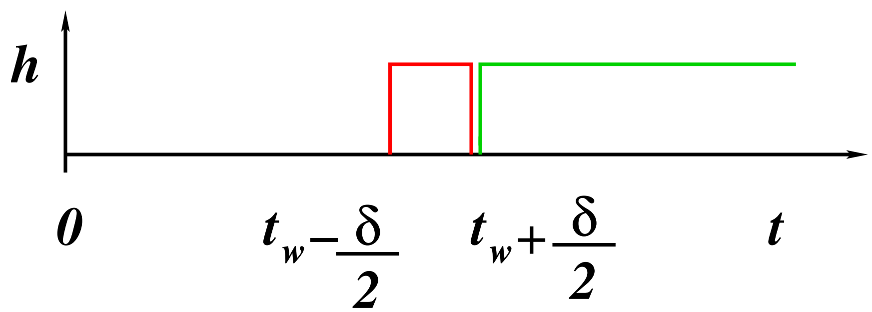
$$C_{AB}(t, t_w) \equiv \langle \hat{A}(t)\hat{B}(t_w) \rangle$$

expectation value in a quantum system, $\langle \ldots \rangle = \text{Tr} \ldots \hat{\rho}/\text{Tr}\hat{\rho}$

or the average over realizations of the dynamics (initial conditions, random numbers in a MC simulation, thermal noise, etc.) in a classical system.

Two-time observables

Linear response



The perturbation couples linearly to the observable $\hat{B}$ at time t_w

$$\hat{H} \quad \to \quad \hat{H} - h(t_w)\hat{B}$$

The linear instantaneous response of another observable $\hat{A}(t)$ is

$$R_{AB}(t, t_w) \equiv \left. \frac{\delta \langle \hat{A}(t) \rangle_h}{\delta h(t_w)} \right|_{h=0}$$

Similarly in a classical system

Fluctuation-dissipation theorem

Gibbs-Boltzmann density operator $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}}$

$$\tilde{C}_{BA}(-\omega) = e^{\beta\omega} \tilde{C}_{AB}(\omega)$$

and then

$$\boxed{\text{Im}\tilde{R}^{AB}(\omega) = [\hbar^{-1} \tanh(\beta\hbar\omega/2)]^{\pm 1} \, \tilde{C}_{\pm}^{AB}(\omega)}$$

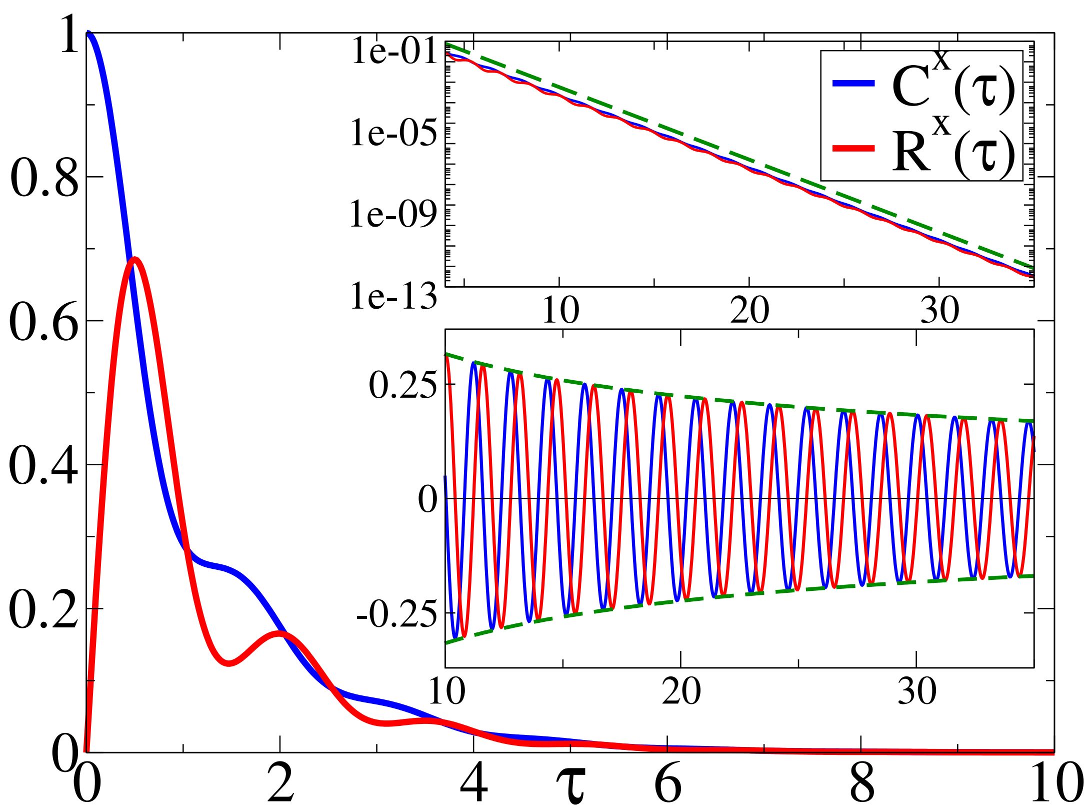
Bosons

Fermions

Classical limit : $\text{Im}\tilde{R}^{AB}(\omega) = \beta\omega \, \tilde{C}^{AB}(\omega)$

Quantum quench

$T_{\rm eff}$ from the longitudinal spin FDR



Insets

$e^{-\tau/\tau_{\rm c}}$

$\tau^{-2}\sin(4\tau+\phi)$

$$C^x(\tau) \simeq A_{\rm c} e^{-\tau/\tau_{\rm c}}[1 - a_{\rm c}\tau^{-2}\sin(4\tau+\phi_{\rm c})]$$

$$R^x(\tau) \simeq A_R e^{-\tau/\tau_{\rm c}}[1 - a_R\tau^{-2}\sin(4\tau+\phi_R)]$$

Foini, LFC & Gambassi 11

Quantum quench

$T_{\rm eff}$ from FDT ?

For sufficiently long-times such that one drops the power-law correction

$$-\beta^x_{\rm eff} \simeq \frac{R^x(\tau)}{d_\tau C^x_+(\tau)} \simeq -\frac{\tau_{\rm c} A_R}{A_{\rm c}}$$

A constant consistent with a classical limit but

$$T^x_{\rm eff}(\Gamma_0) \neq T_e(\Gamma_0)$$

Morever, a complete study in the full time and frequency domains confirms that $T^x_{\rm eff}(\Gamma_0,\omega) \neq T^z_{\rm eff}(\Gamma_0,\omega) \neq T_e(\Gamma_0)$ (though the values are close).

Fluctuation-dissipation relations as a probe to test thermal equilibration

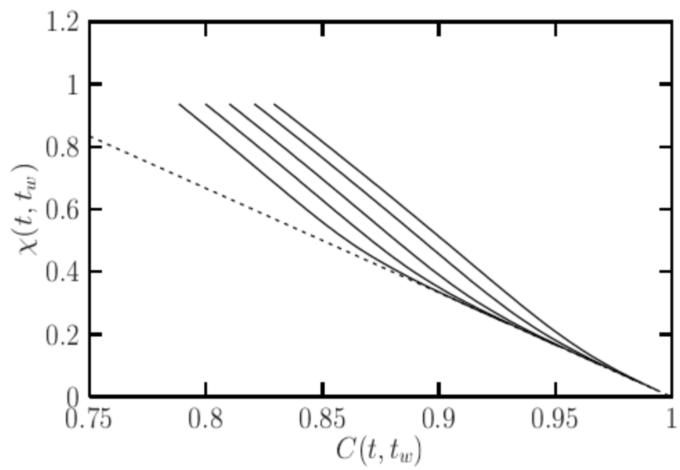
No equilibration for generic Γ_0 in the quantum Ising chain

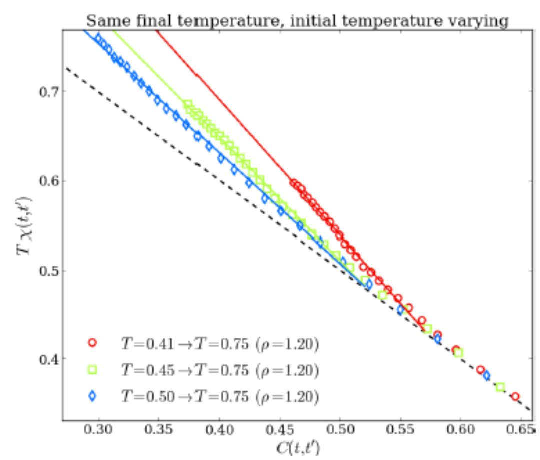
FDT & effective temperatures

Role of initial conditions

$T^* > T$ found for quenches from the disordered into the glassy phase

(Inverse) quench from an ordered initial state, $\boxed{T^* < T}$





2d XY model or O(2) field theory

Binary Lennard-Jones mixture

Berthier, Holdsworth & Sellitto 01

Gnan, Maggi, Parisi & Sciortino 13

Quantum quenches

Expectations

Non-integrable systems expected to eventually thermalise.

Integrable systems ?

role of initial states ; non critical *vs.* critical quenches, *etc.*

- Definition of T_e from $\langle\psi_0|\hat{H}|\psi_0\rangle = \langle\hat{H}\rangle_{T_e} = Z_{\beta_e}^{-1}$ Tr $\hat{H}e^{-\beta_e\hat{H}}$

Just one number, it can always be done

- Comparison of dynamic and thermal correlation functions, *e. g.*

$$C(r,t) \equiv \langle\psi_0|\hat{\phi}(\vec{x},t)\hat{\phi}(\vec{y},t)|\psi_0\rangle \textit{ vs. } C(r) \equiv \langle\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})\rangle_{T_e}.$$

Calabrese & Cardy ; Rigol et al ; Cazalilla & Iucci ; Silva et al, etc.

But the functional form of correlation functions can be misleading !

Quantum quenches

Questions

Non-integrable systems expected to thermalise.

Integrable systems ?

role of initial states ; non critical *vs.* critical quenches, *etc.*

- Definition of T_e from $\langle\psi_0|\hat{H}|\psi_0\rangle = \langle\hat{H}\rangle_{T_e} = Z_{\beta_e}^{-1}\ \text{Tr}\ \hat{H}e^{-\beta_e\hat{H}}$

Just one number, it can always be done

- Comparison of dynamic and thermal correlation functions, *e. g.*

$C(r,t) \equiv \langle\psi_0|\hat{\phi}(\vec{x},t)\hat{\phi}(\vec{y},t)|\psi_0\rangle$ *vs.* $C(r) \equiv \langle\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})\rangle_{T_e}$.

Calabrese & Cardy ; Rigol et al ; Cazalilla & Iucci ; Silva et al, etc.

Proposal : put qFDT to the test to check whether $T_{\text{eff}} = T_e$ exists

Foini, LFC & Gambassi 11 & 12

Two-body model

Integrable system ?

Model with N constants of motion: (extensions of) the Lewis invariants

$$2I_\mu = \rho_\mu^{-2}(t)s_\mu^2(t) + m(\rho_\mu(t)\dot{s}_\mu(t) - \dot{\rho}_\mu(t)s_\mu(t))^2$$

with $\rho_\mu(t)$ given by the Ermakov equation

$$m\ddot{\rho}_\mu(t) + [z(t) - \lambda_\mu] - \rho_\mu^{-1/3}(t) = 0$$

Such system should approach a Generalized Gibbs Ensemble

$$\boxed{\rho_{\mathrm{GGE}} = Z_{\mathrm{GGE}}^{-1}(\beta_\mu)\, e^{-\sum_{\mu=1}^{N}\beta_\mu I_\mu}}$$

How to fix β_μ after a quench ? $\boxed{\langle I_\mu\rangle_{\mathrm{GGE}} = I_\mu(t=0^+)}$

H.R. Lewis 67, Ermakov Univ. Izv. (Kiev) 20, 1 (1880)

Two-body model

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Such system should approach a Generalized Gibbs Ensemble ?

$$\rho_{\mathrm{GGE}} = Z_{\mathrm{GGE}}^{-1}(\beta_\mu, \rho_\mu(t))\ e^{-\sum_{\mu=1}^{N} \beta_\mu I_\mu(s_\mu, p_\mu, \rho_\mu(t))}$$

How to fix β_μ after a quench ? $\langle I_\mu \rangle_{\mathrm{GGE}} = I_\mu(t = 0^+)$

H.R. Lewis 67, Ermakov Univ. Izv. (Kiev) 20, 1 (1880)