Phase ordering kinetics, aggregation and percolation in two dimensions

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In collaboration with

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STATPHYS26, July 2016

Quenches in statistical physics models

passing by critical percolation !

What is it about ?

Classical open systems

Statistical physics framework

Stochastic dissipative dynamics

Out of equilibrium

coarsening - phase ordering kinetics

percolation – fractality

Phenomenon

The talk focuses on a very well-known example

Dynamics following a change of a

control parameter

- If there is an equilibrium phase transition, the **equilibrium phases** are known on both sides of the transition.

  i.e. the asymptotic state is known.

- For a purely dynamic problem, the **absorbing states** are known.
- The **dynamic mechanism** towards equilibrium is understood

  the systems try to order locally in one of the few competing states.

Interests and goals

Practical & fundamental interest, *e.g.*

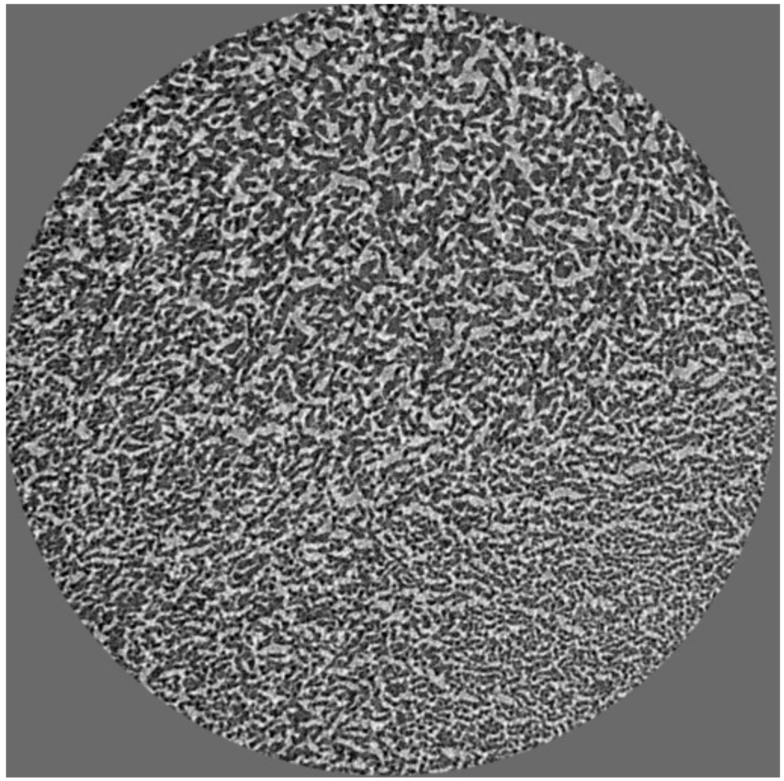
- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Some issues

- The role played by the initial conditions & short-time dynamics
- Full geometric characterisation of the structure
- When does the usual dynamic scaling regime set in ?
- The role played by the cooling rate

that are related to each other.

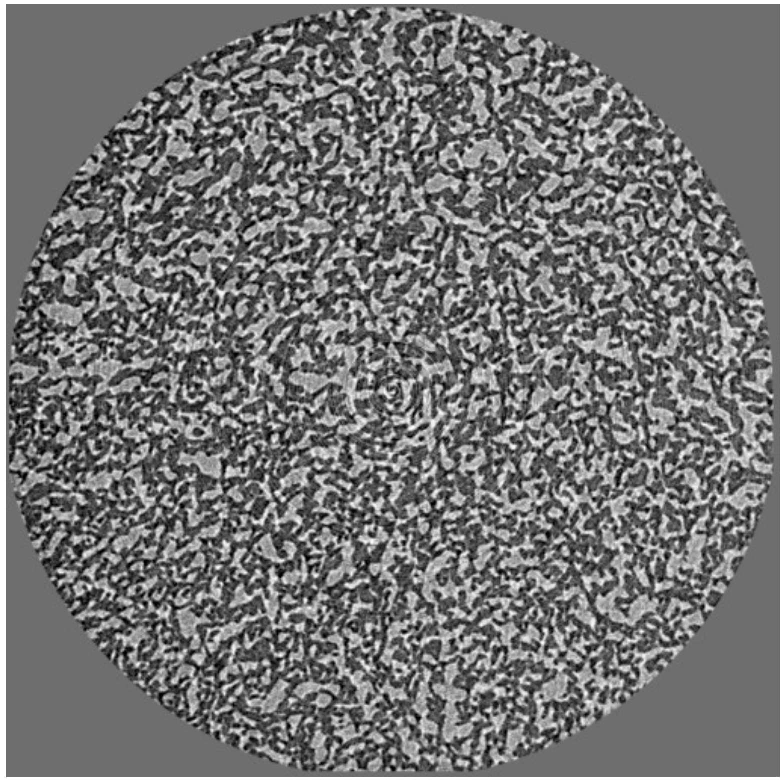
Phase separation in glasses



$t = 1$ min

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14

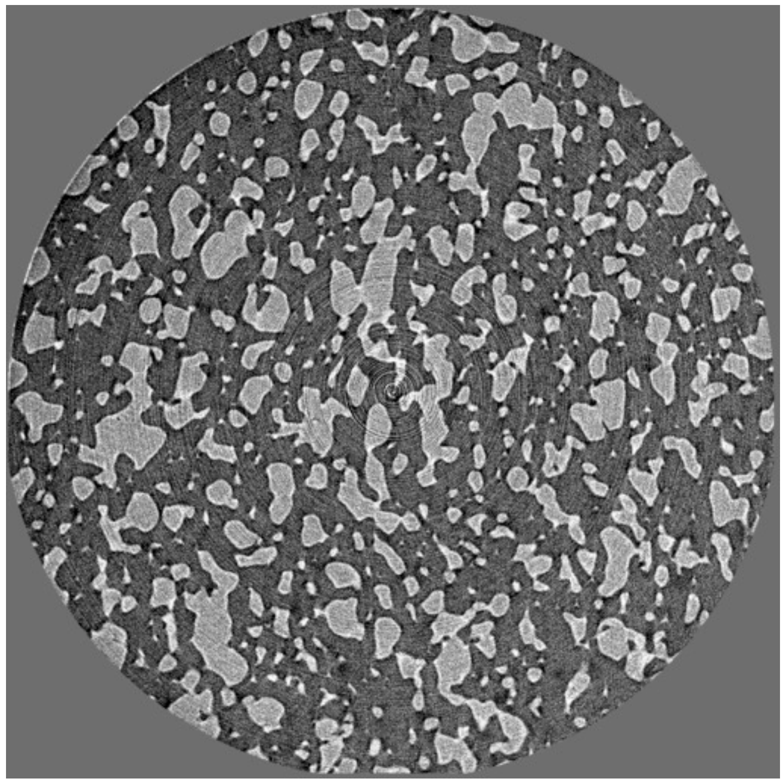
Phase separation in glasses



$t = 4$ min

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14

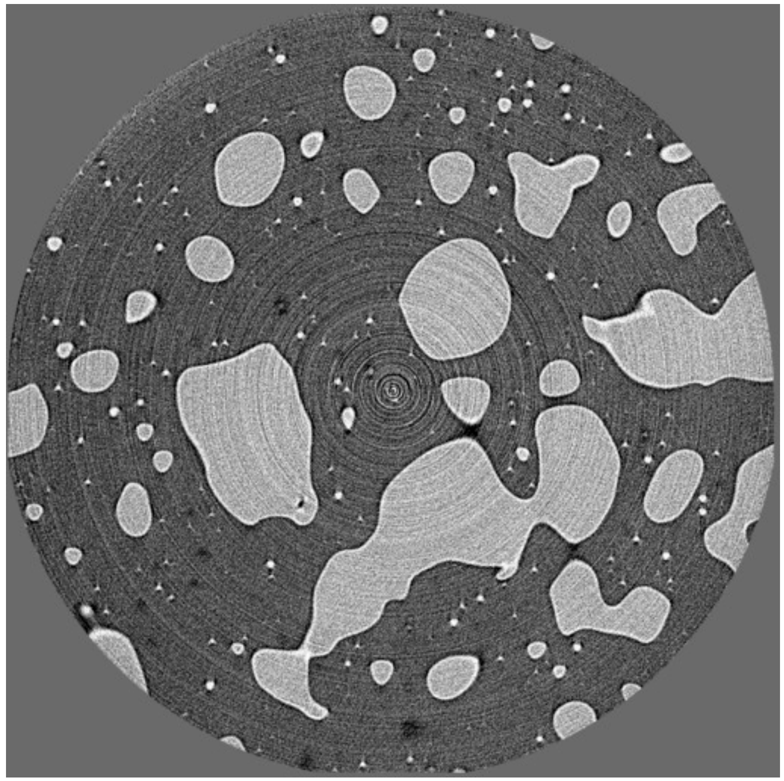
Phase separation in glasses



$t = 16$ min

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14

Phase separation in glasses



$t = 64$ min

Gouillart (Saint-Gobain), Bouttes & D. Vandembroucq (ESPCI) 11-14

Framework

The talk is on a very well-known problem

The stochastic dynamics of the $2d$ Ising model after an instantaneous quench from high to low temperature

- There is a 2nd order phase transition, and

  the **equilibrium phases** are the paramagnet at high T and the (degenerate) ferromagnet at low T.

- Standard knowledge :

  The **dynamic mechanism** is curvature-driven domain growth.

$2d$ Ising Model (IM)

Archetypical example for classical magnetic systems

$$H = -J \sum_{\langle ij \rangle} s_i s_j$$

$s_i = \pm 1$ Ising spins.

$\langle ij \rangle$ sum over nearest-neighbours on the lattice.

$J > 0$ ferromagnetic coupling constant.

critical temperature $T_c > 0$ for $d > 1$.

Monte Carlo rule $s_i \to -s_i$ accepted with

$p = 1$	if $\Delta E < 0$
$p = e^{-\beta \Delta E}$	if $\Delta E > 0$
$p = 1/2$	if $\Delta E = 0$

Non-conserved order parameter dynamics [$\uparrow\downarrow$ towards $\uparrow\uparrow$] etc. allowed.

[$m = 0$ to $m = 2$]

Phenomenon

Similar questions can be asked in very well-known problems in math, *e.g.*

Dynamics of a voter model starting from

a random initial condition

- Purely dynamic, violation of detailed balance, no phase transition
- Two absorbing states
- The **dynamic mechanism** towards absorption is understood

  domain growth is driven by interfacial noise

$2d$ Voter Model (VM)

Archetypical example of opinion dynamics

H does not exist - kinetic model

$s_i = \pm 1$ Ising spins that

sit on the vertices of a lattice.

Voter update rule

choose a spin at random, say s_i

choose one of its $2d$ neighbours at random, say s_j

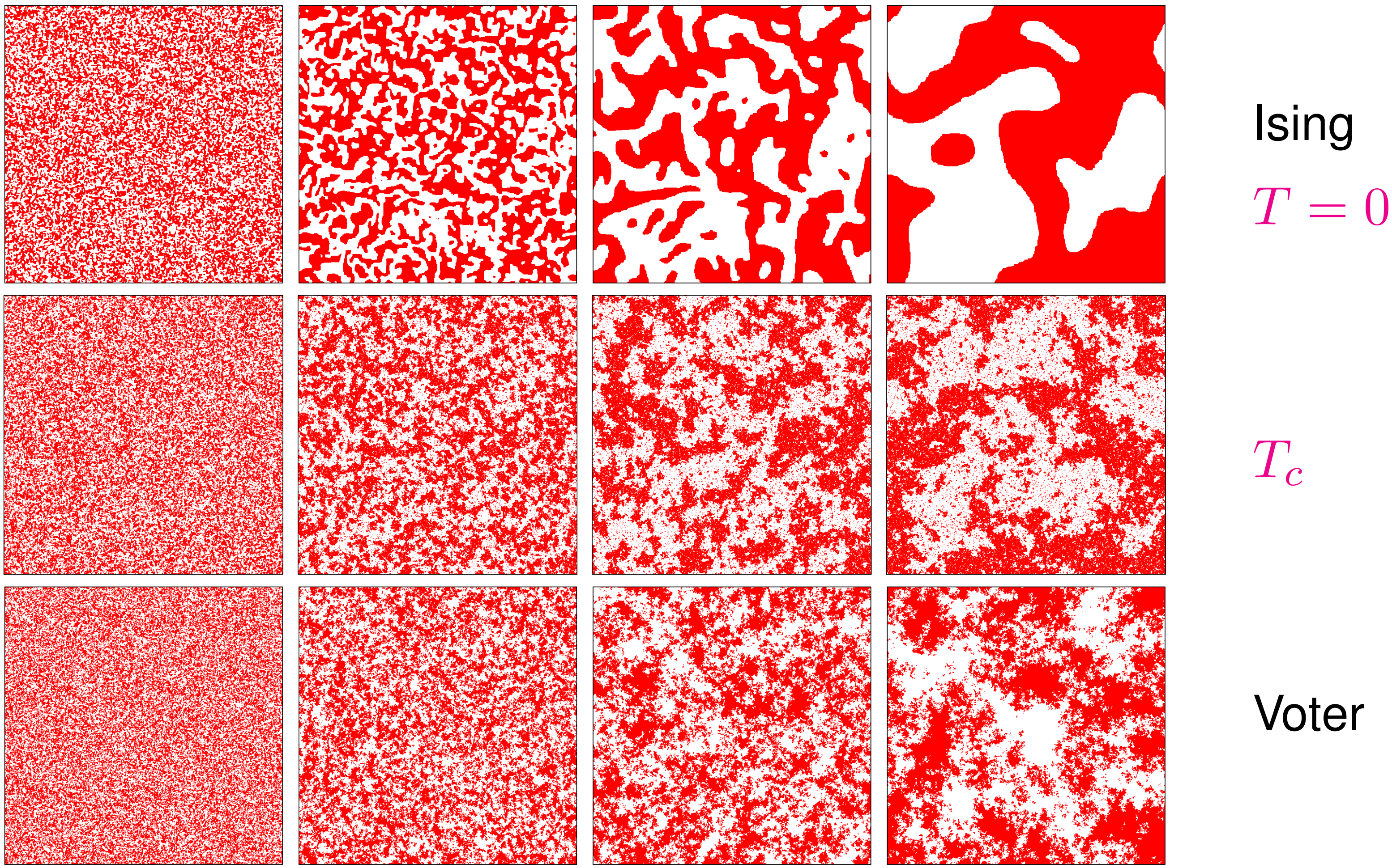
set $s_i = s_j$

In two dimensions full consensus, *i.e.* $m = L^{-d} \sum_{i=1}^{L^d} s_i = \pm 1$ is reached in a timescale $t_C \simeq L^2$ (with $\ln L$ corrections)

Clifford & Sudbury 73, Holley & Liggett 75, Cox & Griffeaths 86

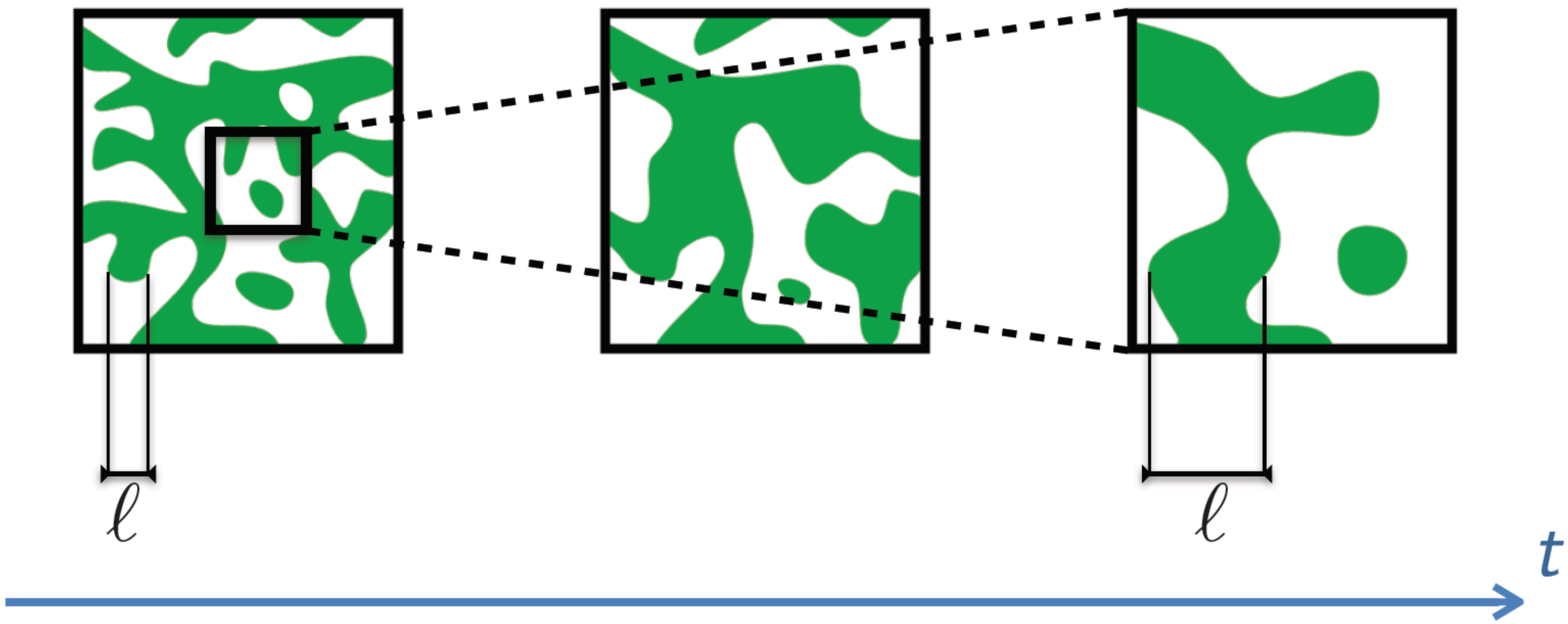
Phase ordering kinetics

$s_i = \pm 1$ **at** $t = 0$ **MCs, snapshots at** $t = 4,\ 64,\ 512, 4096$ **MCs**



Dynamic scaling

in phase ordering kinetics



Growing length $\ell \equiv \xi_d(t)$

Typically $\xi_d(t) \simeq t^{1/z_d}$

Excess energy w.r.t. the equilibrium one stored in the domain walls

Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains $\xi_d(t)$, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by $\xi_d(t)$, *e.g.*

$$C(r,t) \equiv \langle\, s_i(t) s_j(t)\,\rangle\big|_{|\vec{x}_i-\vec{x}_j|=r} \sim \langle\phi\rangle^2_{eq}\; f\left(\frac{r}{\xi_d(t)}\right)$$

$$C(t,t_w) \equiv \langle\, s_i(t) s_i(t_w)\,\rangle \sim \langle\phi\rangle^2_{eq}\; f_c\left(\frac{\xi_d(t)}{\xi_d(t_w)}\right)$$

etc. when $r \gg \xi_{eq}$, times such that $t, t_w \gg t_0$ and $C < \langle\phi\rangle^2_{eq}$.

Review Bray 94

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etc. when $r \gg \xi_{eq}$, times such that $t, t_w \gg t_0$ and $C < \langle\phi\rangle_{eq}^2$.

Review Bray 94

Is this really all there is ?

Interests and goals

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- Cooling rate effects on the density of topological defects in cosmology and condensed matter

Some issues

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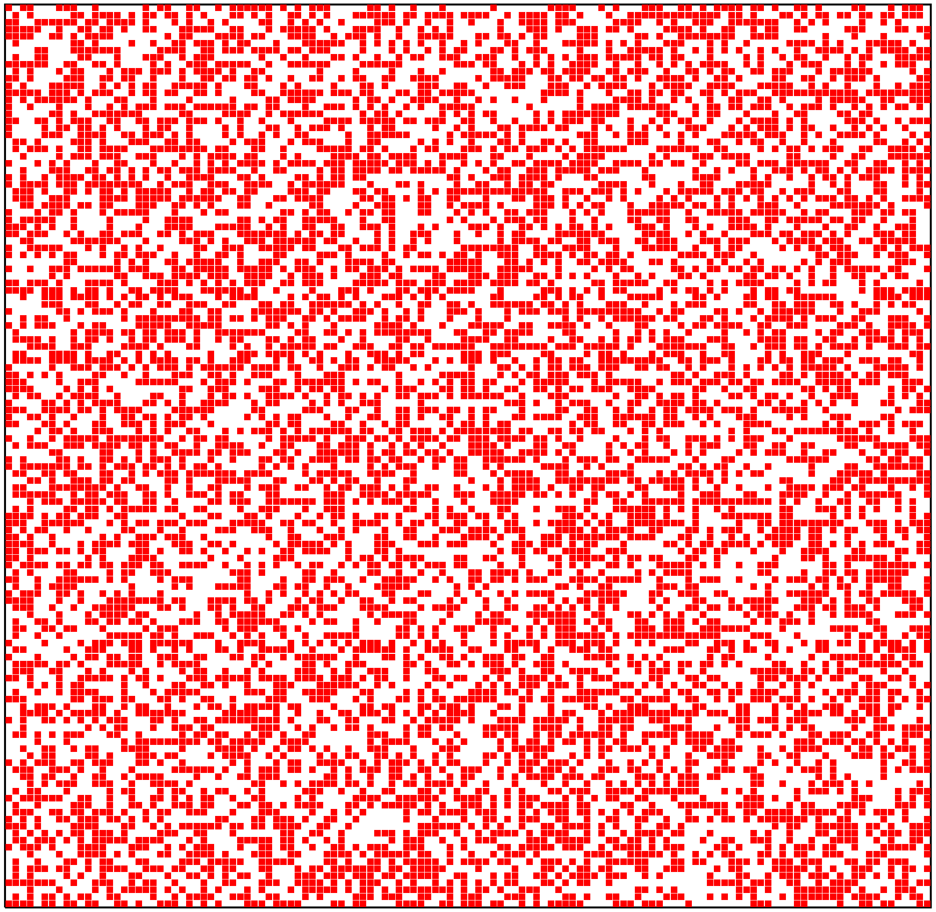
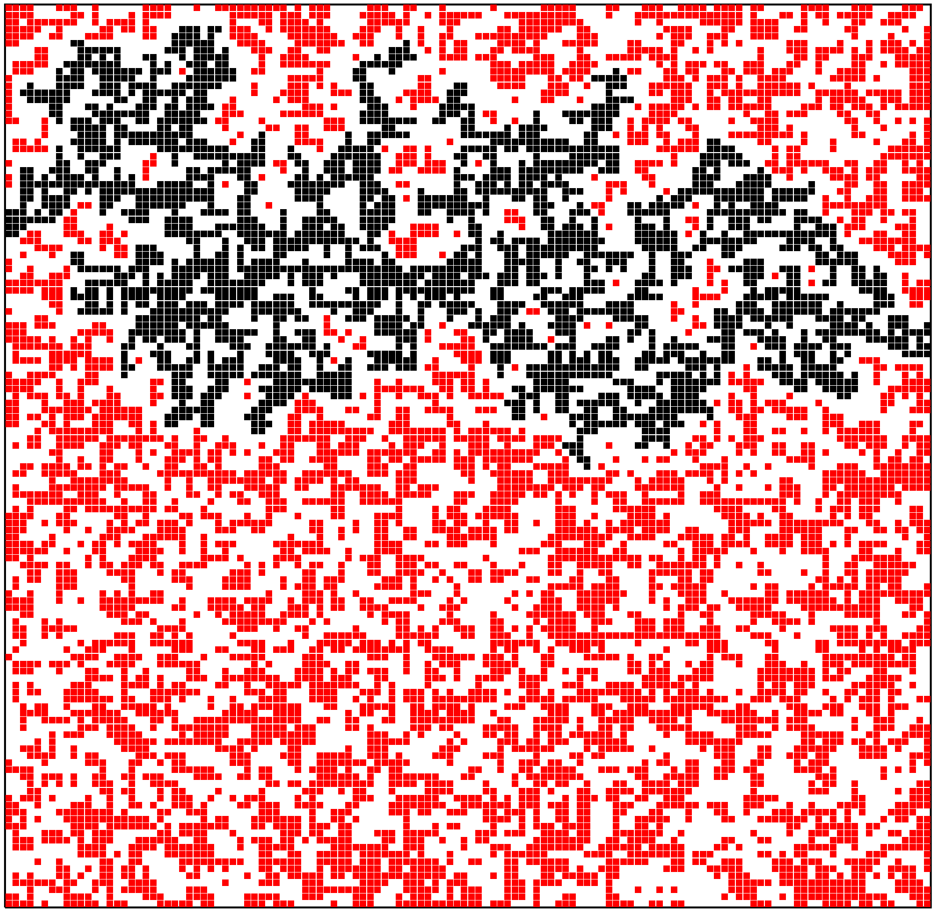
that are related to each other.

2d square IM at T=0



t=0.0

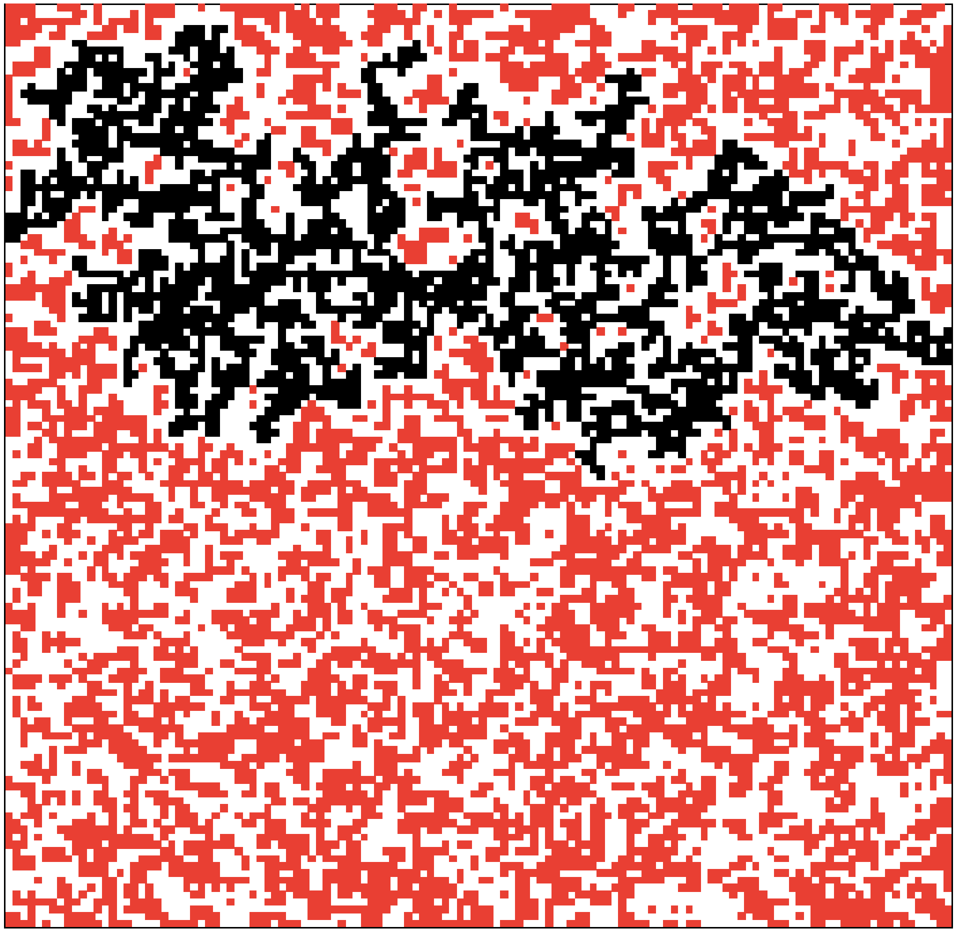
2d square IM at T=0



t=0.57533

Spanning cluster

Has this cluster something to do with (critical) percolation ?



Percolation

Purely geometric problem

Take a lattice Λ in d spatial dimensions.

Define a site occupation variable $n_i = 1, 0$ with probability $p, 1-p$

In the limit $L \to \infty$ there is a continuous phase transition at p_c such that the probability of there being a cluster of occupied nearest-neighbour sites that crosses a sample from one end to another in at least one Cartesian direction

$$\lim_{L\to\infty} P(p,L) \begin{cases} = 0 & \text{if } p \leq p_c \\ > 0 & \text{if } p > p_c \end{cases}$$

p_c depends on Λ and d.

At p_c the spanning cluster has fractal properties that are well characterised

Percolation

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p_c depends on Λ and d.

The distribution of finite size clusters is algebraic at p_c

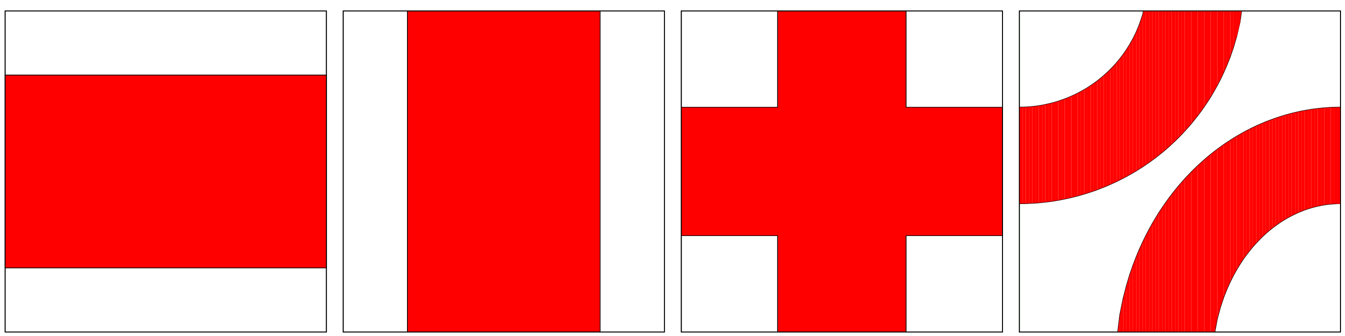
Percolation

Exact results at the critical threshold

Probability of percolation along the horizontal *or* vertical directions, π_1

Probability of percolation along the horizontal *and* vertical directions, π_{hv}

(On a torus) Probability of percolation along the diagonal direction, π_d



From SLE & CFT calculations.

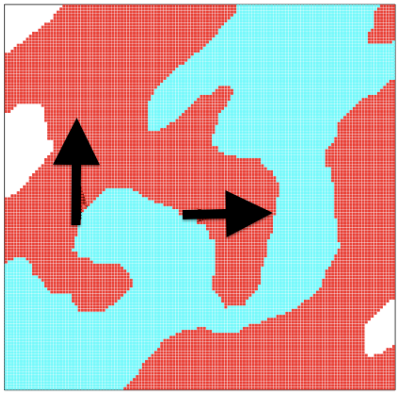
Percolation

Exact results at the critical threshold

Winding angle *vs.* curvilinear length of the wall

$$\langle \theta^2(x) \rangle = \text{ct} + \frac{4\kappa}{8+\kappa} \ln x$$

with $\kappa = 6$



From SLE & CFT calculations.

Coarsening

Is this really critical percolation ?

Back to the dynamic problem

2dIM : various lattices

Initial conditions for an instantaneous quench

Equilibrium at infinite temperature, $T_0 \to \infty$, initial condition.

The spins take ± 1 values with probability $1/2$.

Site occupation variable $n_i = (s_i + 1)/2 = 1, 0$ with $p = 1/2$

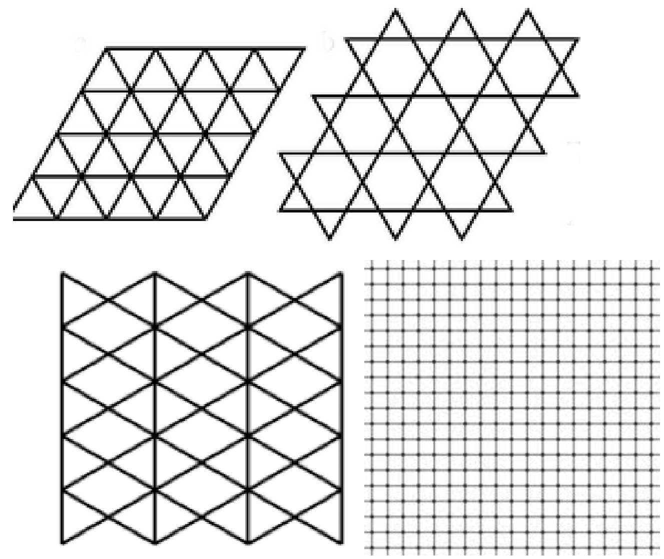
From a site percolation perspective :

$p_c = 0.65$ Kagome lattice.

$p_c = 0.59$ Square lattice.

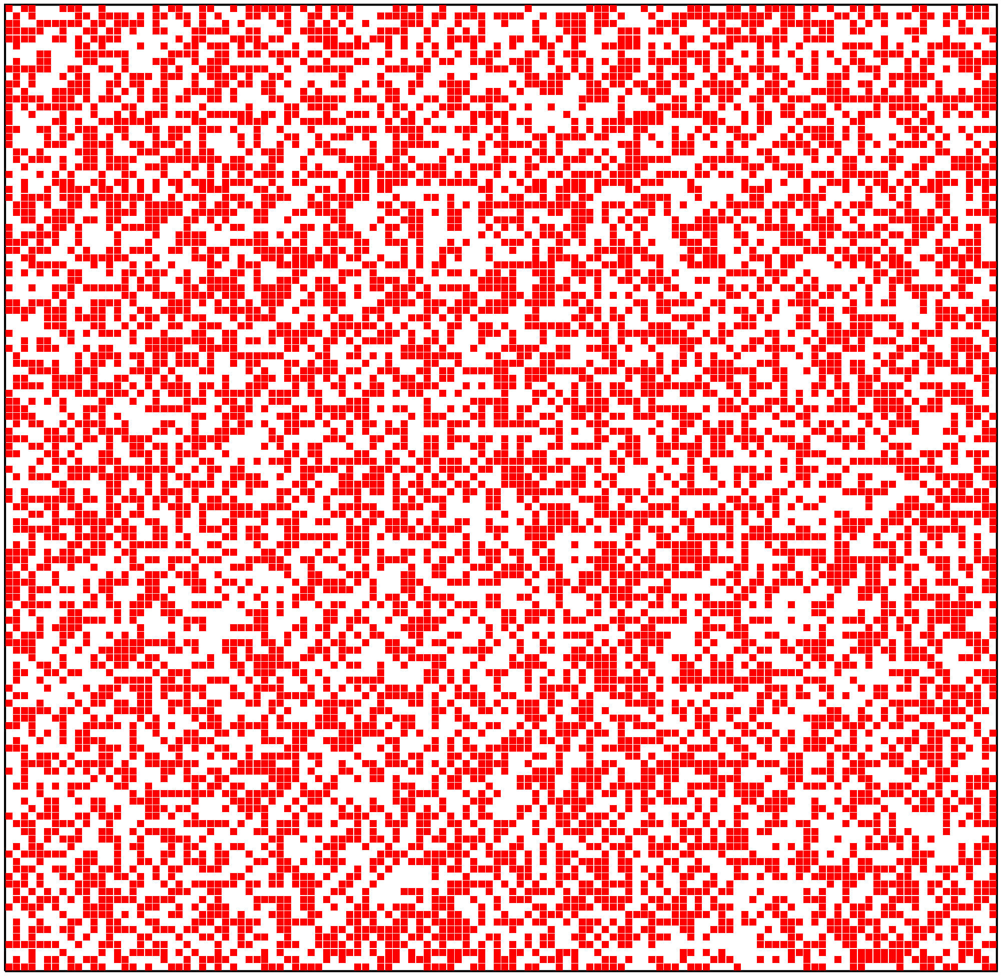
$p_c = 0.55$ Bow-tie lattice.

$p_c = 0.5$ Triangular lattice.



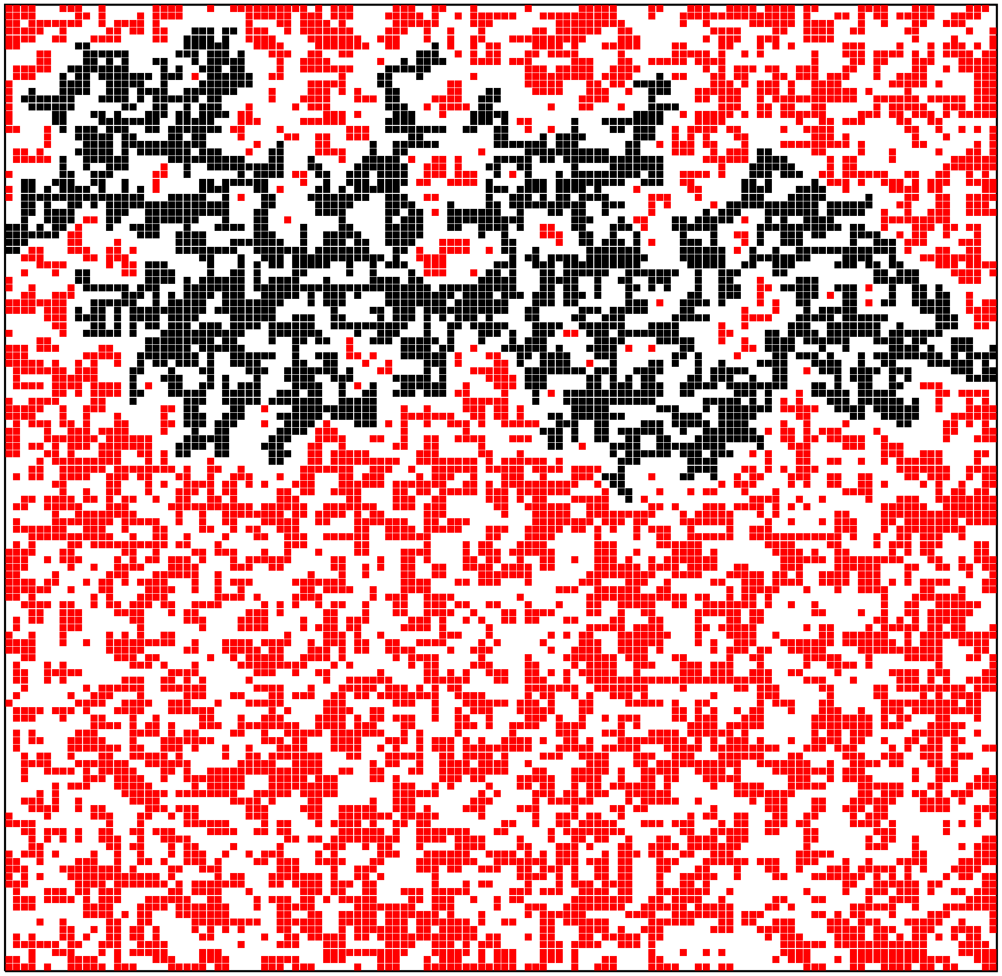
Initial condition at $p = 0.5$, below p_c

2d square IM at T=0



t=0.0

2d square IM at T=0



t=0.57533

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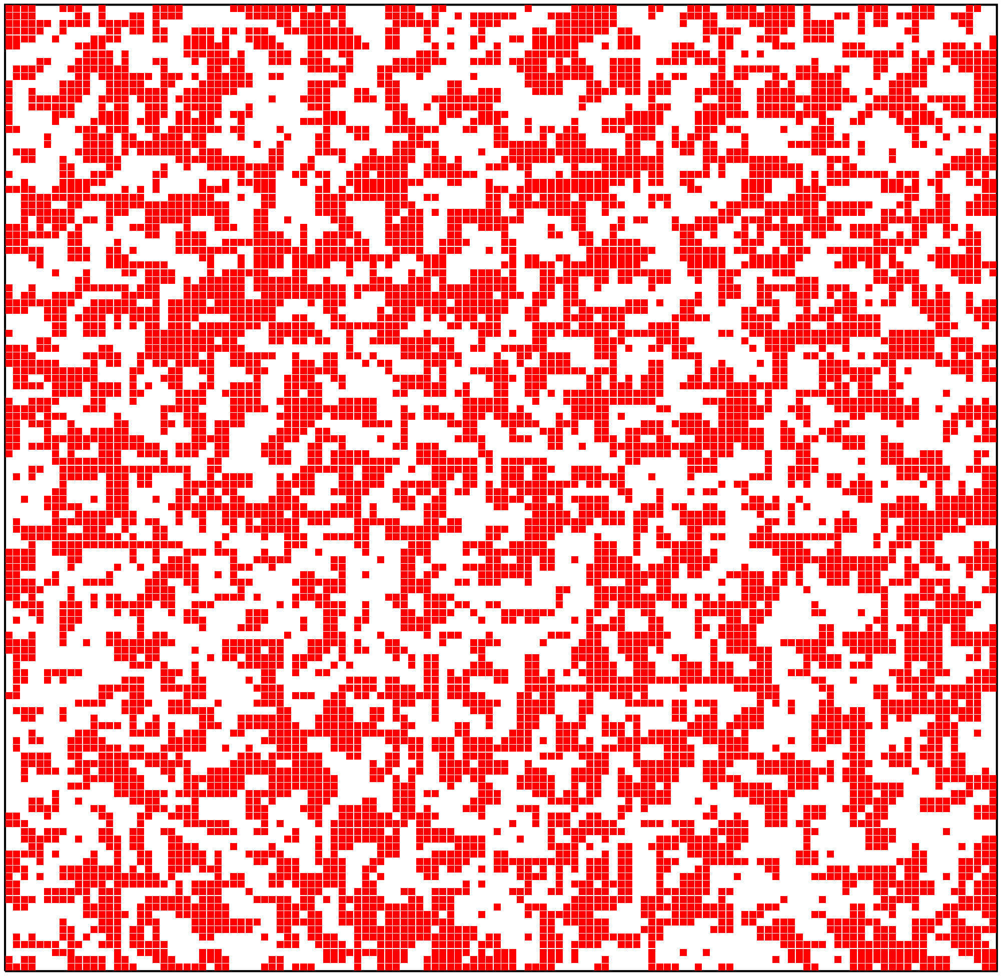
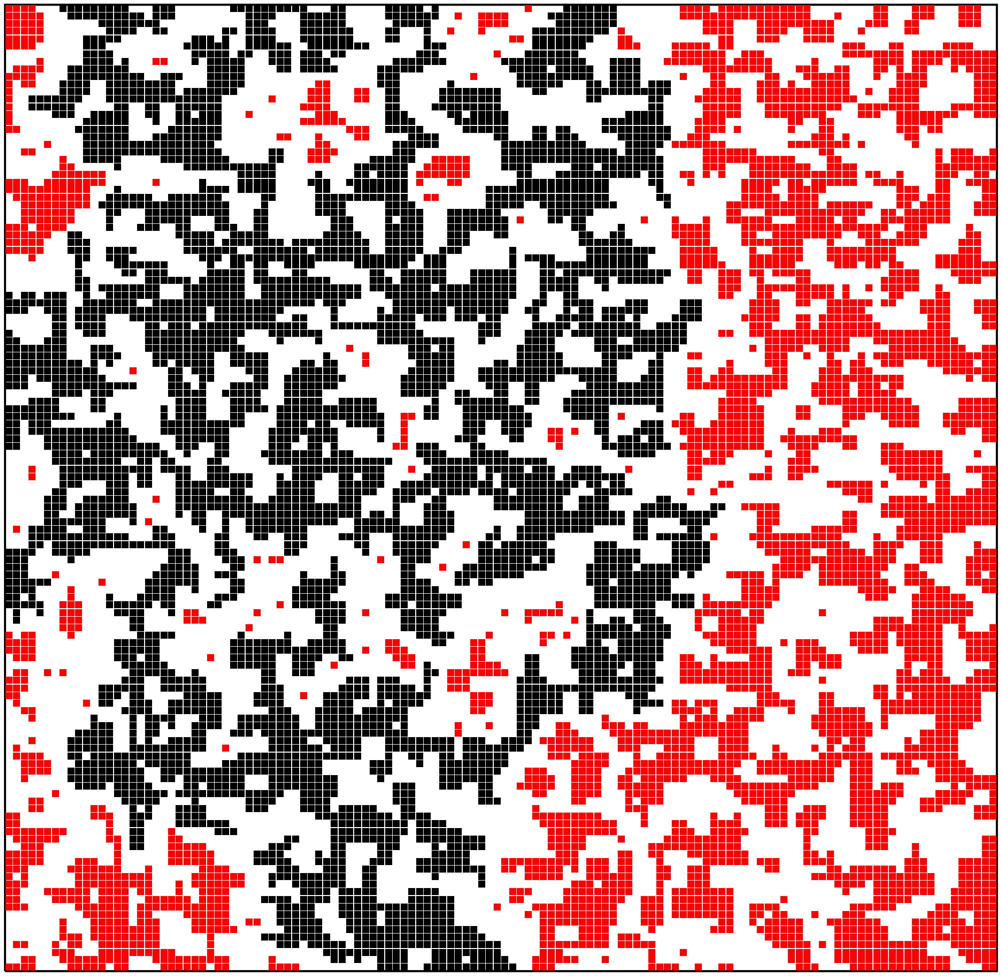


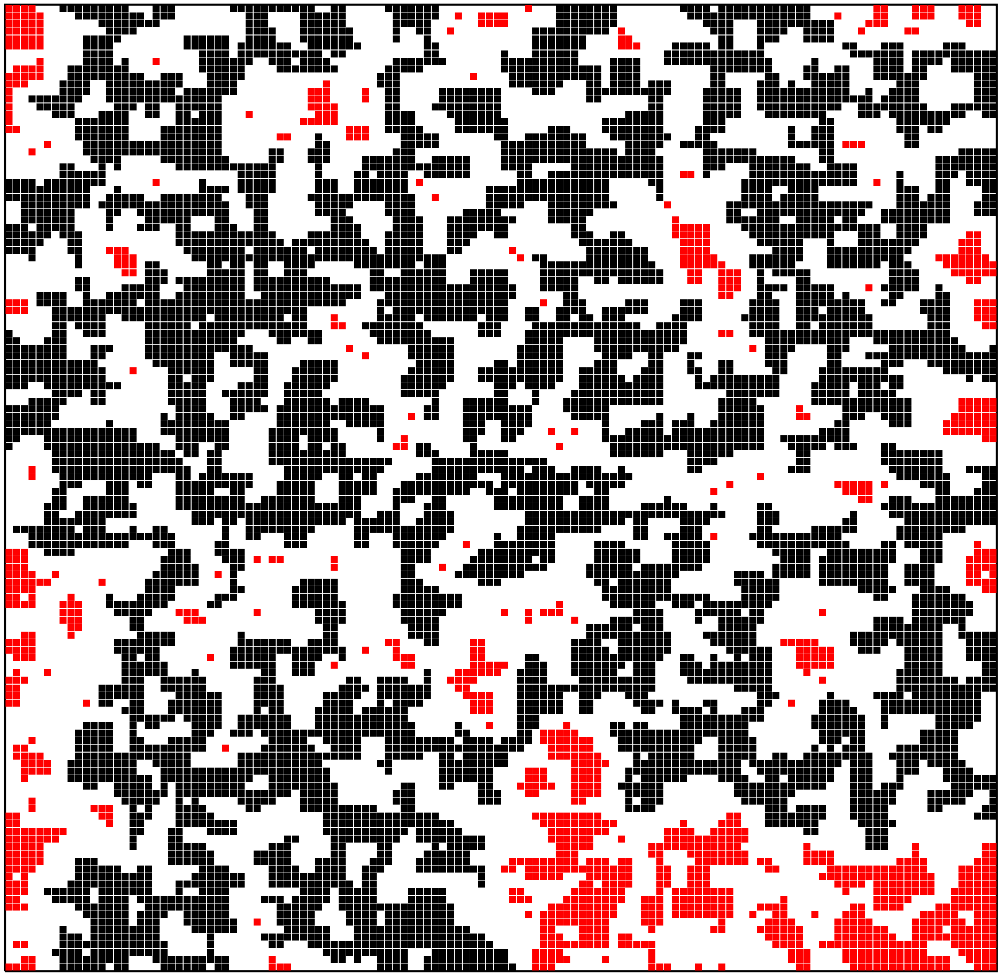
t=0.94844

2d square IM at T=0



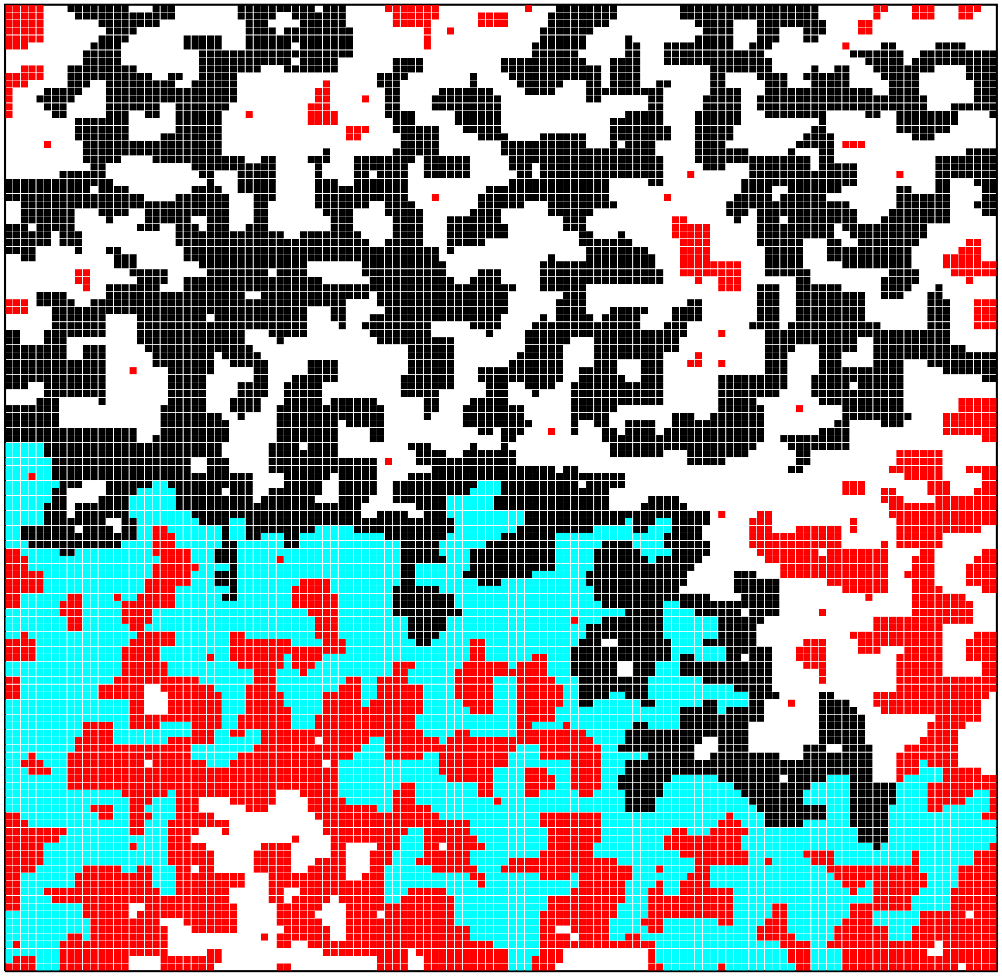
t=2.00847

2d square IM at T=0



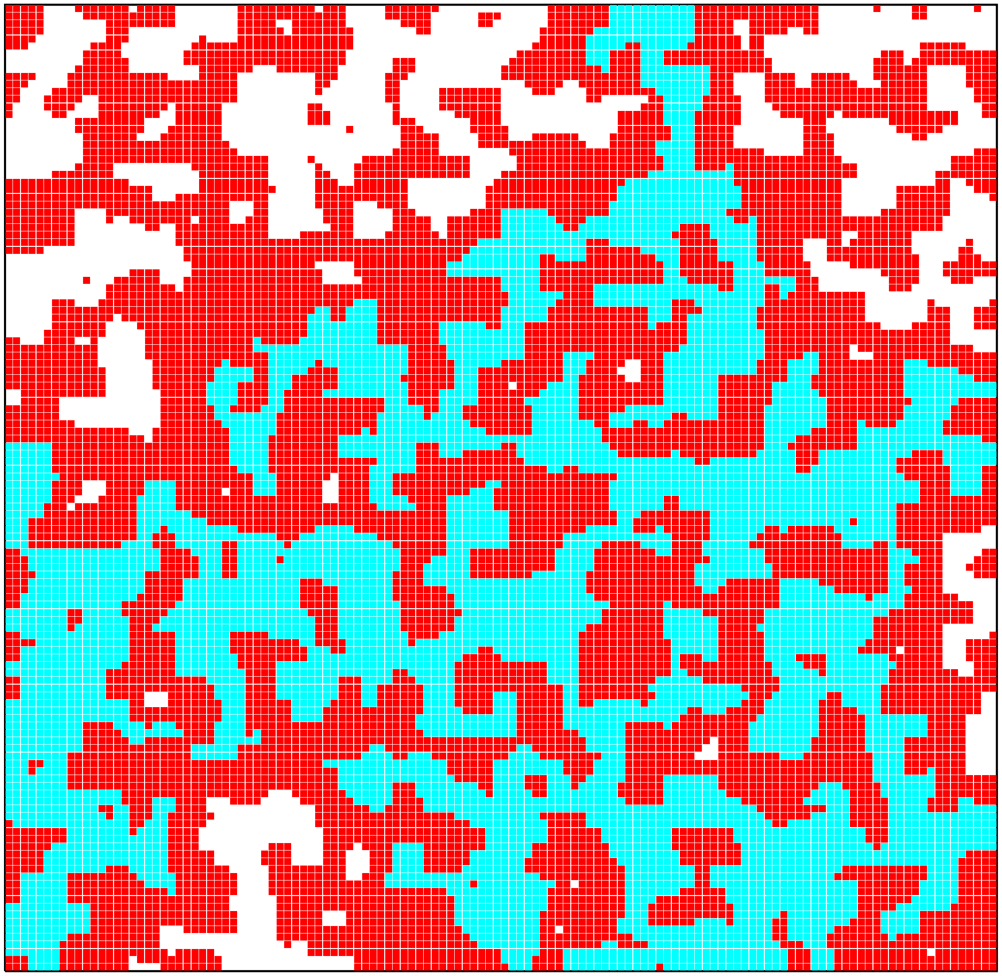
t=2.57898

2d square IM at T=0



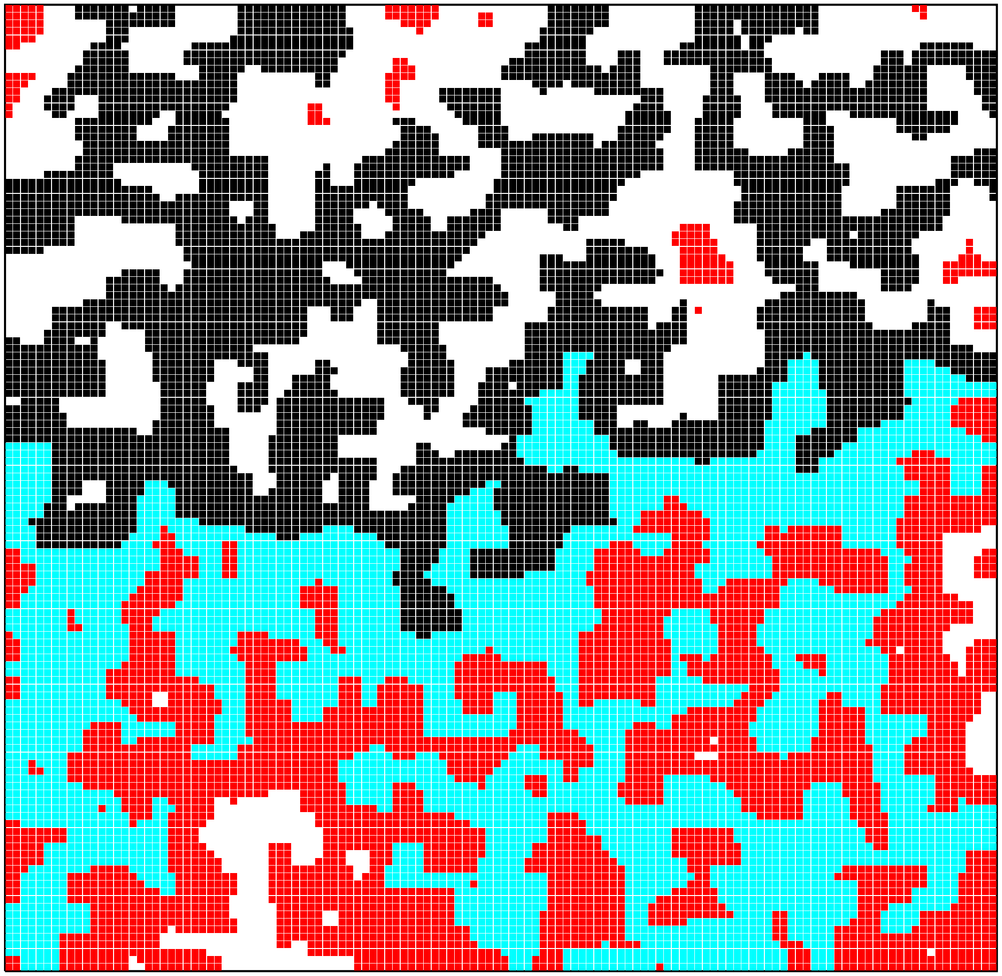
t=3.99211

2d square IM at T=0



t=6.58423

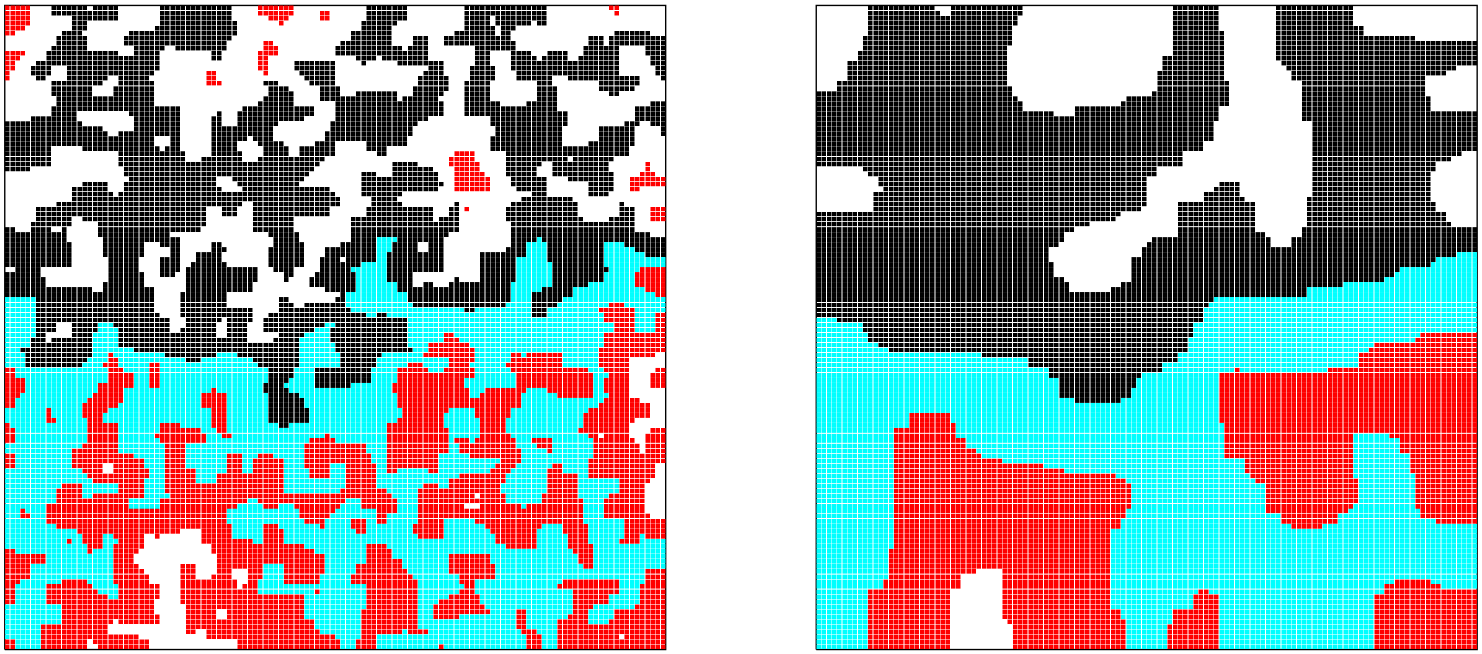
2d square IM at T=0



t=7.46144

2d square IM at T=0

The percolating structure was decided at $t_p \simeq 8$ MCs

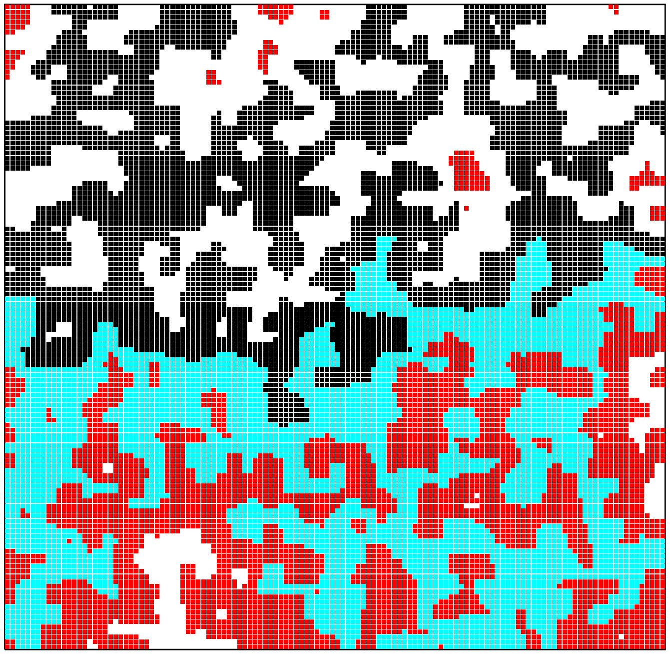


t=7.46144 t=128.0

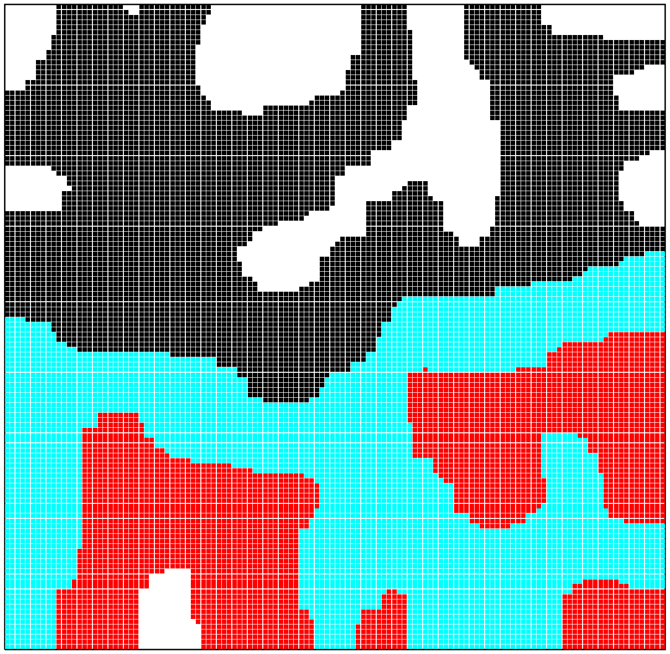
Arenzon, Bray, LFC & Sicilia 07 Blanchard, Corberi, LFC & Picco 14

2d square IM at T=0

The final configuration will be one with two horizontal stripes



t=7.46144

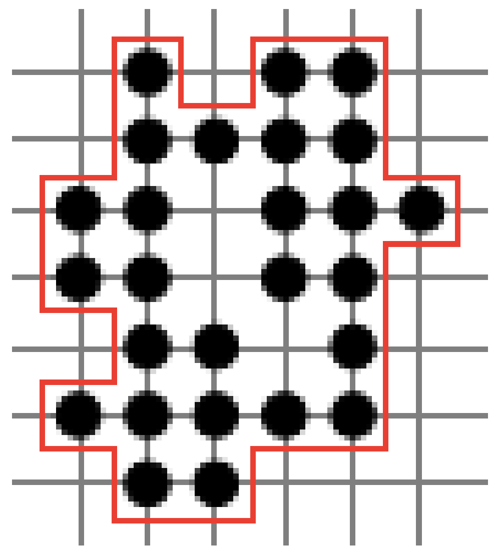


t=128.0

Olejarz, Krapivsky & Redner 12, Blanchard & Picco 13

Is it critical percolation ?

Analysis of clusters and boundaries



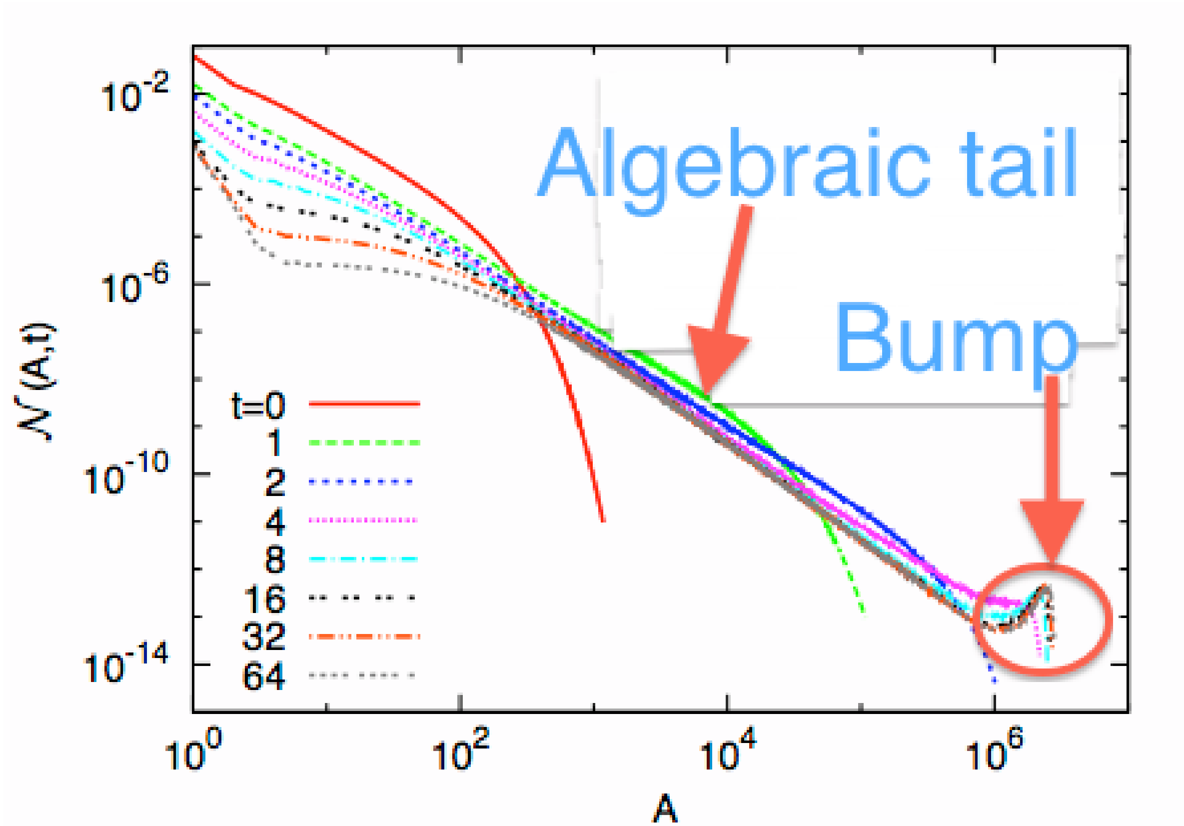
Domain area : sum of filled dots

External boundary or hull : red broken line

Hull-enclosed area : sum of lattice sites within the red boundary (including the two empty sites)

Is it critical percolation ?

Full distribution of hull-enclosed areas



Quench from $T_0 \to \infty$.

Is it critical percolation ?

Finite size scaling of the bump

Take A to be the hull-enclosed area or the domain area.

At critical percolation, finite size scaling of the number density of areas

$$\boxed{\mathcal{N}(A,L) = 2c\, A^{-\tau} + N_p(A/L^D)}$$

with $D = d/(\tau-1)$ the fractal dimension of the percolating clusters.

Stauffer & Aharony 94

For hull-enclosed areas $\tau = 2$ and $D = 2$

For domain areas $\tau = 187/91 \simeq 2.05$ and $D \simeq 1.9$

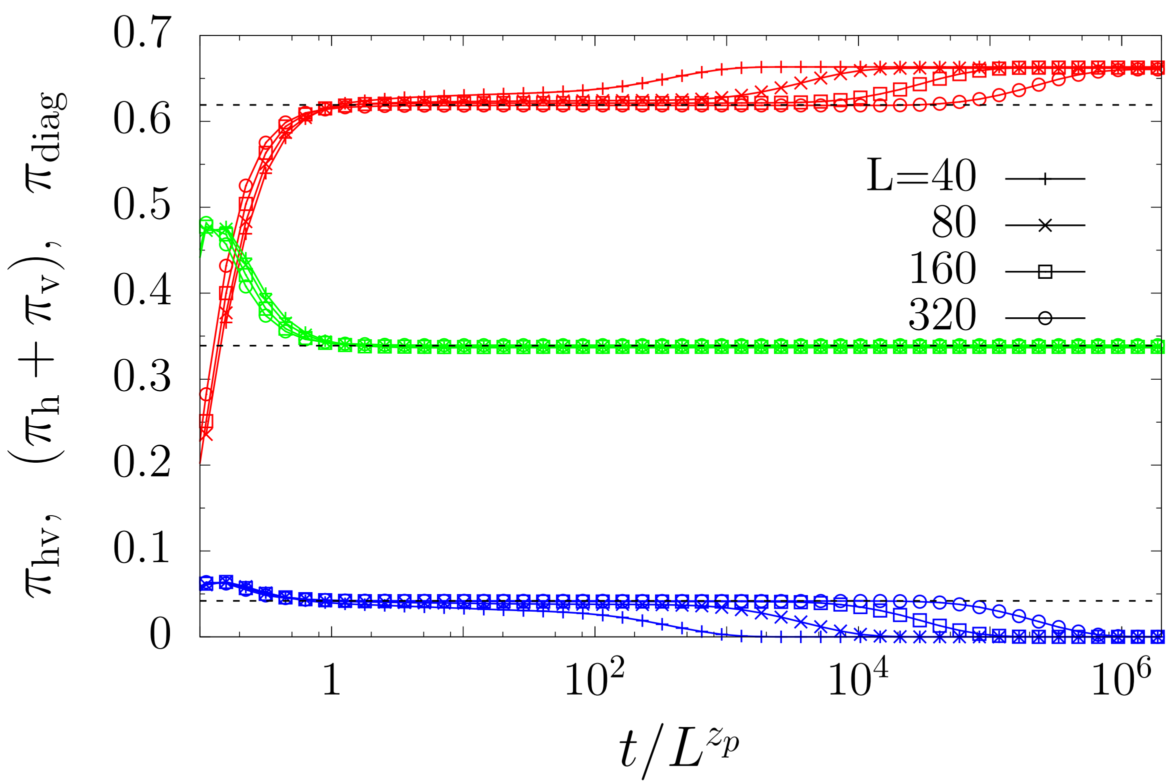
NB the corresponding exponents for critical Ising conditions are different but take values close to these.

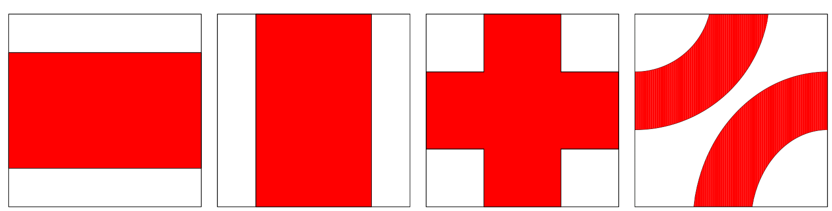
The constants $2c$ are known, *e.g.* $2c_d \approx 0.06$

Cardy & Ziff 03 Sicilia, Arenzon, Bray & LFC 07

Is it critical percolation ?

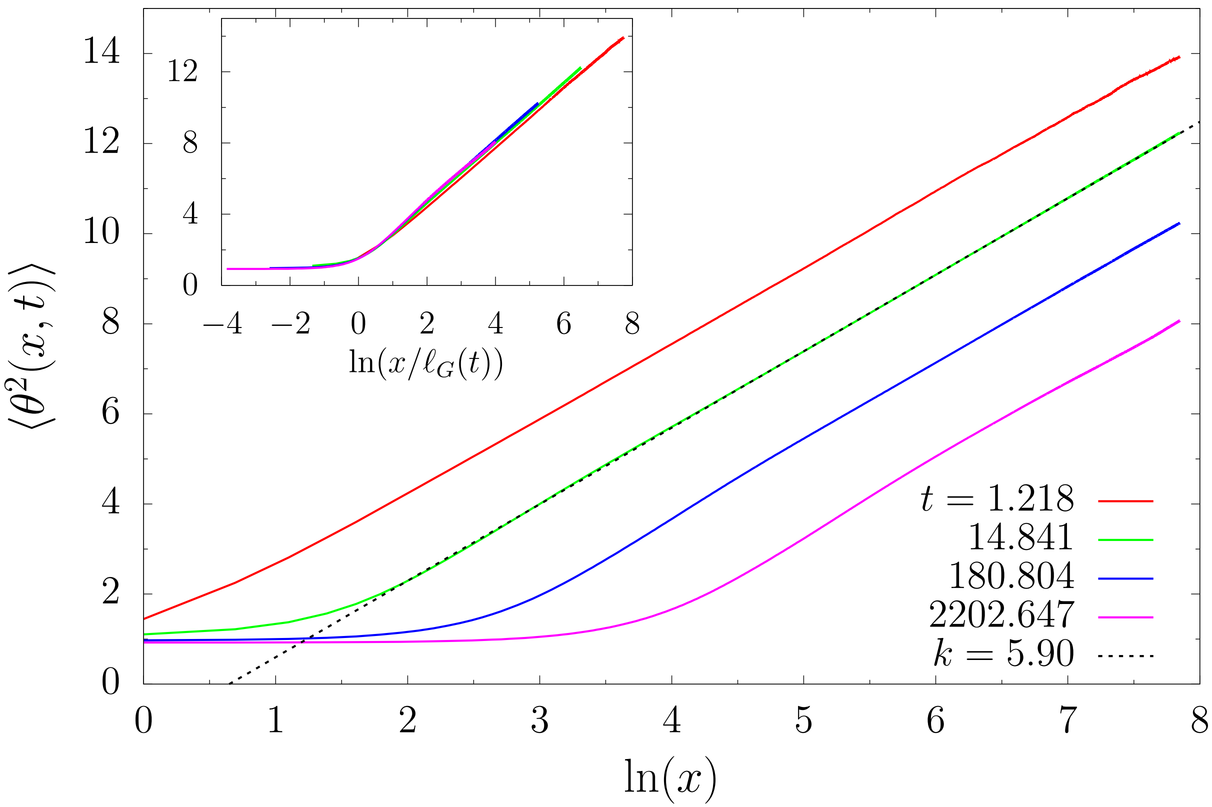
The probabilities of percolation in different directions





Is it critical percolation ?

The winding angle



$\kappa \simeq 5.9$

When ?

Let us call $t_p(L)$

the time needed to reach

the critical percolation state

Determination of $t_p(L)$

Cloning trick and measurement of the overlap

Quench a system from $T_0 \to \infty$ to $T = 0$ at $t = 0$.

Let it evolve at $T = 0$ until t_w.

Make a copy of the instantaneous configuration, $\sigma_i(t_w) = s_i(t_w)$.

Let the two clones evolve with different thermal noises.

Compute the time-dependent overlap

$$\boxed{q_{t_w}(t, L) = \frac{1}{L^d} \sum_{i=1}^{L^d} \langle s_i(t) \sigma_i(t) \rangle}$$

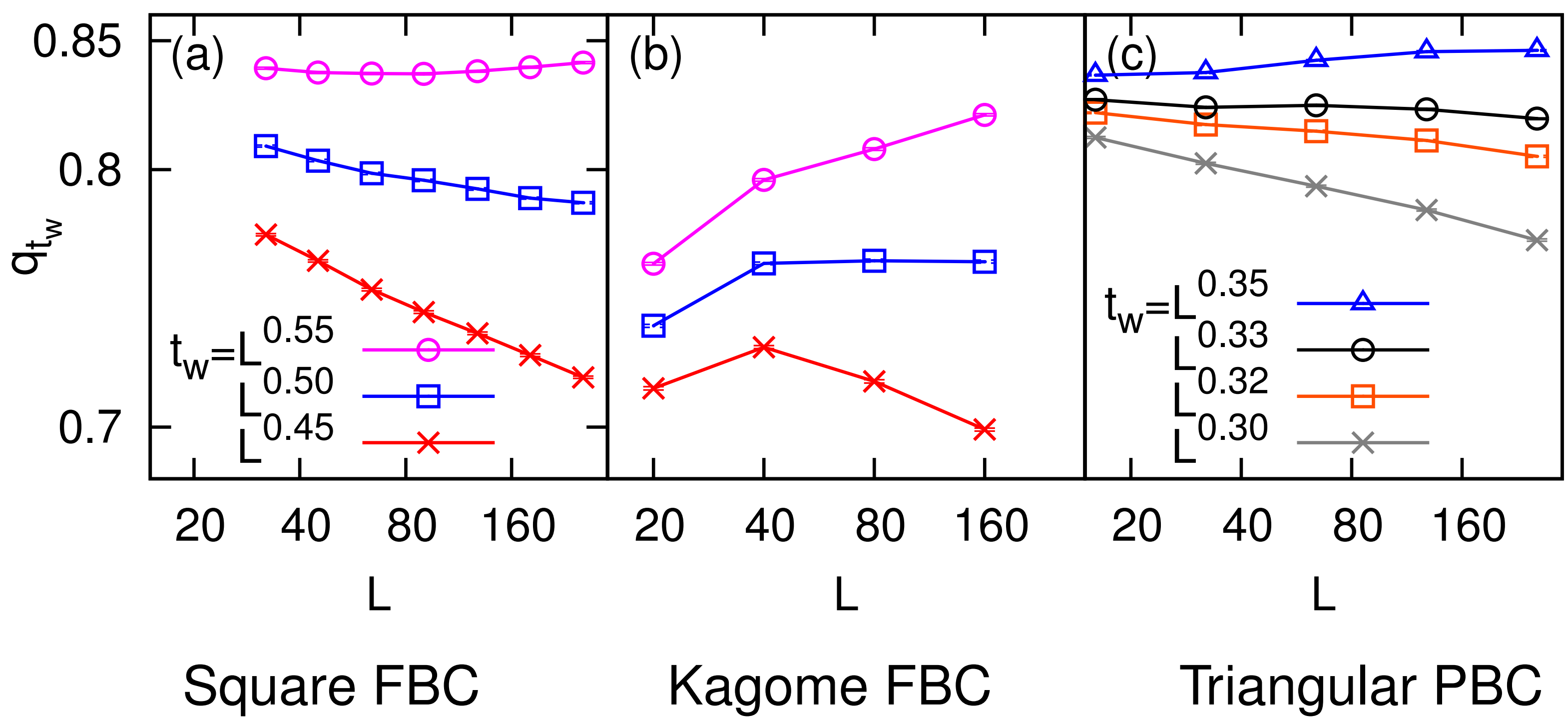
If $t_w < t_p(L)$ $\qquad \lim_{t \gg t_w} q_{t_w}(t, L) = 0$

If $t_w > t_p(L)$ $\qquad \lim_{t \gg t_w} q_{t_w}(t, L) > 0$

Determination of $t_p(L)$

The overlap

$\lim_{t\to\infty} q_{t_w(L)}(t, L)$ should reach a constant independent of L



$t_p(L) \simeq L^{0.5}$ $t_p(L) \simeq L^{0.33}$

First conclusion

Approach to critical percolation

The zero-temperature non-conserved order parameter dynamics of the $2d$ Ising model and the dynamics of the $2d$ Voter model, both starting from a totally uncorrelated $T_0 \to \infty$ paramagnetic initial state, approach uncorrelated critical percolation after a time $t_p \simeq L^{z_p}$.

The exponent z_p depends upon the *effective connectivity* of the lattice and the *microscopic dynamics*.

For instance, $z_p = 0.5$ for the $2d$ Ising model with non-conserved order parameter dynamics and $z_p = 1.667$ for the $2d$ Voter model, both on the square lattice.

Blanchard, Corberi, LFC & Picco 14 ; Tartaglia, LFC & Picco 15

First conclusion

Approach to critical percolation : why is this feature interesting ?

A mechanism that went unnoticed in this context so-far.

Seems to be universal.

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge : how can one prove this claim ?

Manifold consequences :

- metastability, blocked striped states at zero temperature ;
- corrections to dynamic scaling.

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Analytical challenge: perhaps in the voter model.

Similar master equation to the one of the $1d$ Glauber chain

Krapivsky et al. 90s

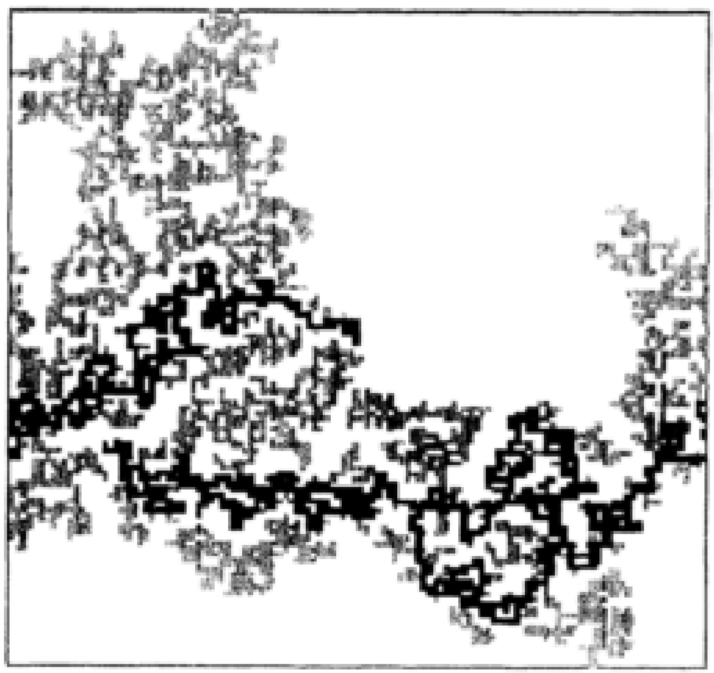
Mapping to random walks **Cox & Griffeaths 80s**

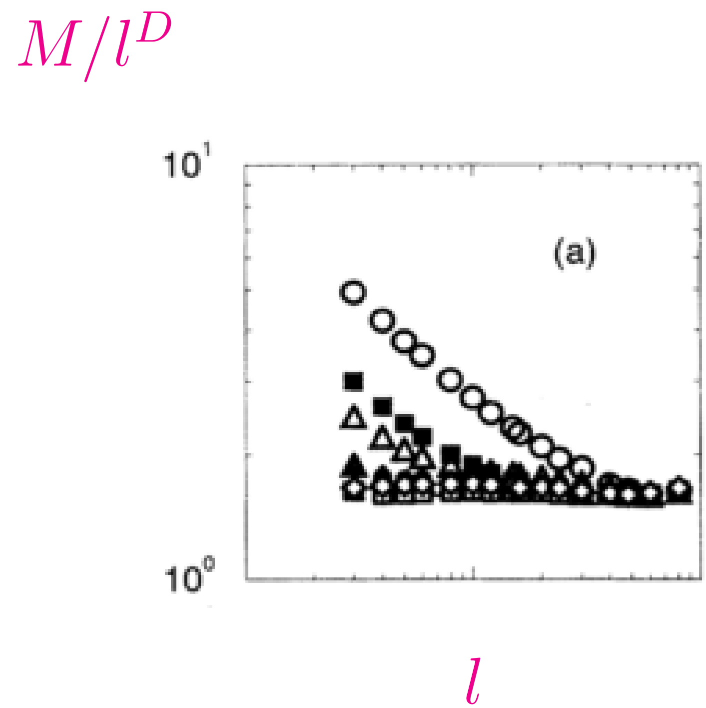
But... finite L effects searched

Aggregation

Approach to critical percolation : why is this feature interesting ?

Bidimensional diffusion-limited cluster-cluster aggregation





The gelation cluster at t_p

its scaled mass, with $D = 1.89$

Hasmy & Jullien 96

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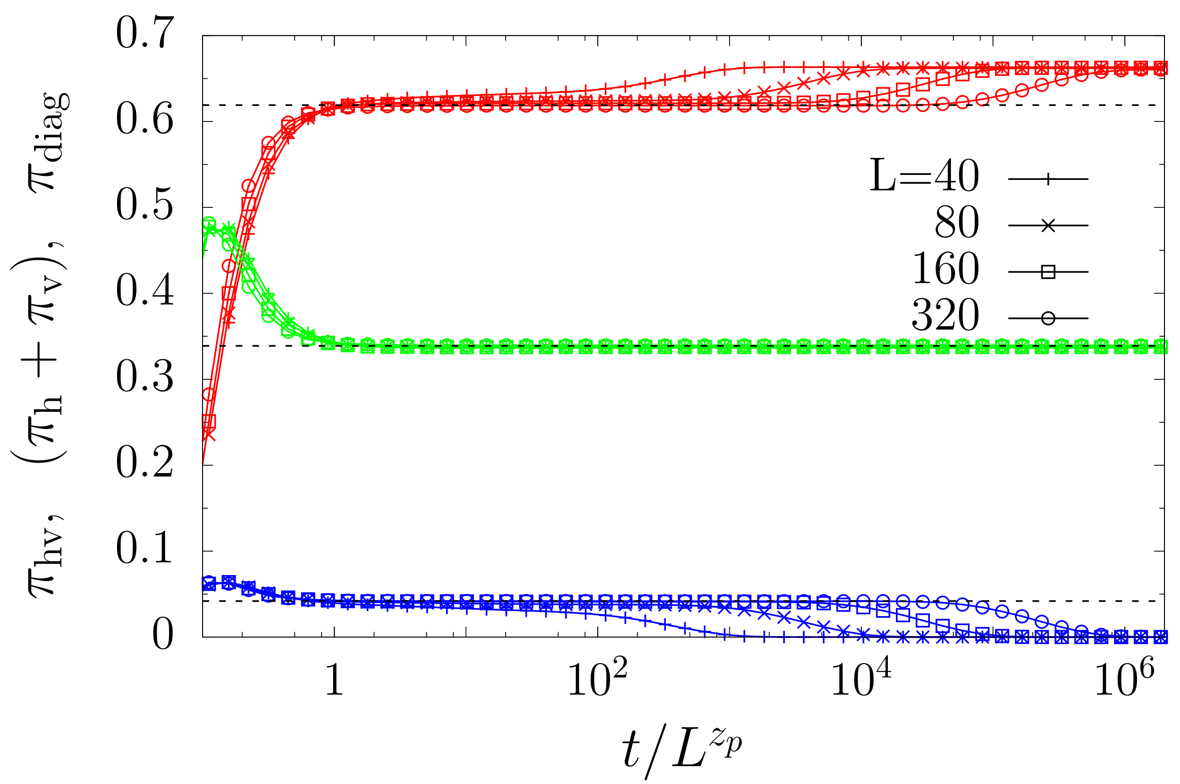
- metastability, blocked striped states at zero temperature ;
- corrections to dynamic scaling, $\xi_p(t) \simeq t^{1/z_p}, \xi_d(t) \simeq t^{1/z_d}$.

2d square IM at T=0

The final configuration was decided at t_p



$\pi_{hv} \approx 0.62$

$\pi_h + \pi_v \approx 0.34$

$\pi_d \approx 0.03$

stripe states with the probabilities of critical percolation

Barros, Krapivsky & Redner 09, Blanchard & Picco 13

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Approach to critical percolation: why is this feature interesting?

A mechanism that went unnoticed in this context so-far.

Seems to be universal.

In RG language it suggests the first approach to a fixed point that is not fully attractive (critical percolation) and the subsequent departure from it.

Analytical challenge : how can one prove this claim?

Manifold consequences:

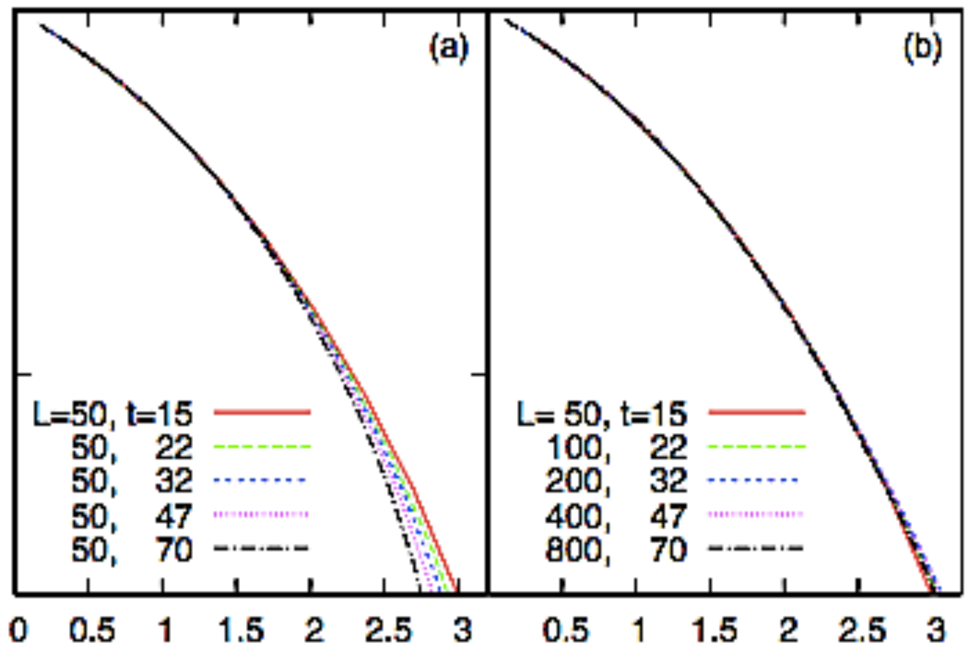
metastability, blocked striped states at zero temperature;

corrections to dynamic scaling, $\xi_p(t) \simeq t^{1/z_p}, \xi_d(t) \simeq t^{1/z_d}$.

Correction to scaling

Linear-Log scale, zoom over $C \lesssim 0.1$

$C(r,t,L)$



$\frac{r}{\xi_d(t)}$

$$f\left(\frac{r}{\xi_d(t)}\right) \qquad \boxed{g\left(\frac{r}{\xi_d(t)}, \frac{\xi_p(t)}{L}\right)}$$

with $\xi_p(t) \equiv t^{1/z_p}$ and $\xi_d(t) \equiv t^{1/z_d}$

$z_p \simeq 1/2$ for the square & Kagome, and $z_p \simeq 1/3$ for the triangular lattice.

Second conclusion

Early approach to percolation

$\xi_p(t) \simeq t^{1/z_p}$ is a new growing length-scale that brings about a new scaling variable to be taken into account in dynamic scaling.

Studies of the $2d$

Ising model with non-conserved order parameter dynamics

Voter model

Ising model with conserved (Kawasaki) order parameter dynamics

$\xi_p(t) \simeq t^{1/z_p}$ or $\xi_p(t) \simeq \xi_d^n(t)$

Blanchard, Corberi, LFC & Picco 14 ; Tartaglia, LFC & Picco 15

Statistics of finite areas

The predictions for $t > t_p$

$$n_h(A,t) \equiv \frac{(2)c_h}{(A+\lambda t)^2} \qquad\qquad n_d(A,t) \approx \frac{(2)c_d\,(\lambda_d t)^{\tau-2}}{(A+\lambda_d t)^{\tau}}$$

in the long time limit $t \gg t_p$.

We derived the expected scaling forms, as $\xi_d(t) = (\lambda t)^{1/2}$:

$$\xi_d^4(t) n_h(A,t) = f_h\left(\frac{A}{\xi_d^2(t)}\right) \qquad\qquad n_d(A,t) \approx (\lambda_d t)^{-2}\, f_d\left(\frac{A}{\lambda_d t}\right) \;.$$

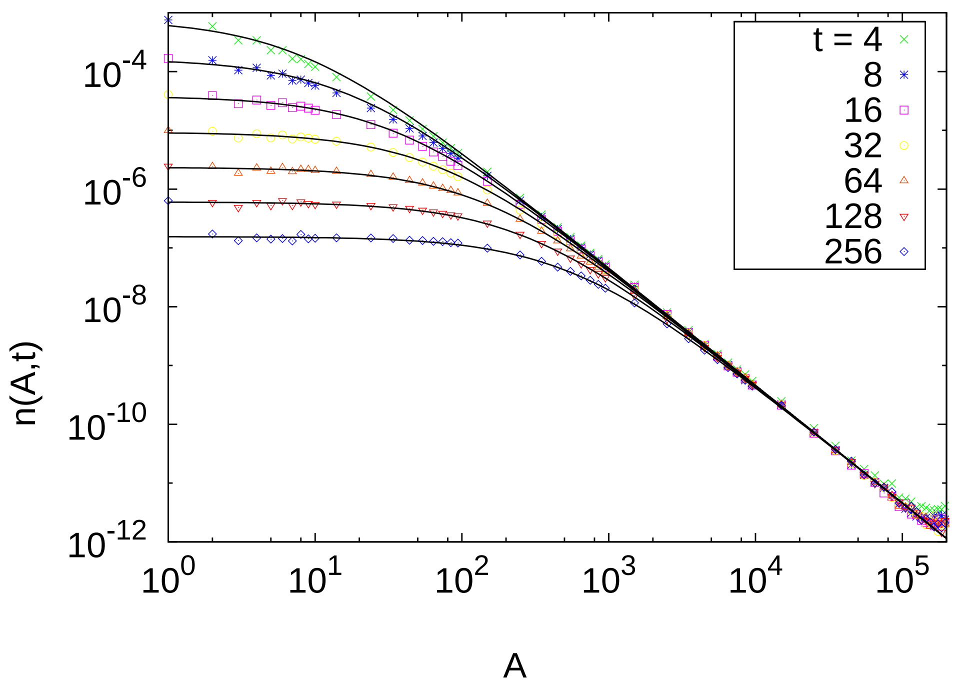
The new parameters are $c_d = c_h + O(c_h^2)$ and $\lambda_d = \lambda + O(c_h)$. Moreover, the sum rules, $N_h(t) = N_d(t)$ and $\int dA\; A\, n_d(A,t) = 1$ relate c_h to τ (or τ') !

Arenzon, Bray, LFC & Sicilia, PRL 07 & PRE 07.

Simulations vs. theory

Number density of (finite) hull-enclosed areas per unit area

$T_0 \to \infty$ **and** $T = 0$



Solid lines

analytical prediction :

$$n_h(A,t) \equiv \frac{(2)c_h}{(A+\lambda t)^2}$$

Summary

- Evidence for the approach to critical percolation at a time-scale that diverges with the system size as $t_p \simeq L^{z_p}$.
- The new growing length-scale, $\xi_p(t) \simeq t^{1/z_p}$ dominates at short times and is needed to improve the scaling of finite-size and finite-time data.

  This effect also exists at finite temperature. Metastability acquires a finite life-time.
- We derived the number density of hull & domain enclosed areas and interface length for $t > t_p$ and we showed that they satisfy dynamic scaling with respect to $\xi_d(t) \simeq t^{1/z_d}$ with $z_d = 2$ for times $t \gg t_p$.

Phase separation in glasses

— Sodium potassium borosilicate SiO_2 (70%) B_2O_3 (20%) BaO (10%) heterogeneous glass with special mechanical properties.

— Raw material mixed and melted at $T \simeq 1500$ C.

— Mixture cooled at ambient T : no dynamics (Paris $\to$ Grenoble).

— Mixture heated at $T = 1000$ C phase separates into

one glassy phase : 75% SiO_2 + 25% B_2O_3.

another glassy phase : 60% SiO_2 + 20% B_2O_3 + 20% BaO

roughly $c = 50\%$ each.

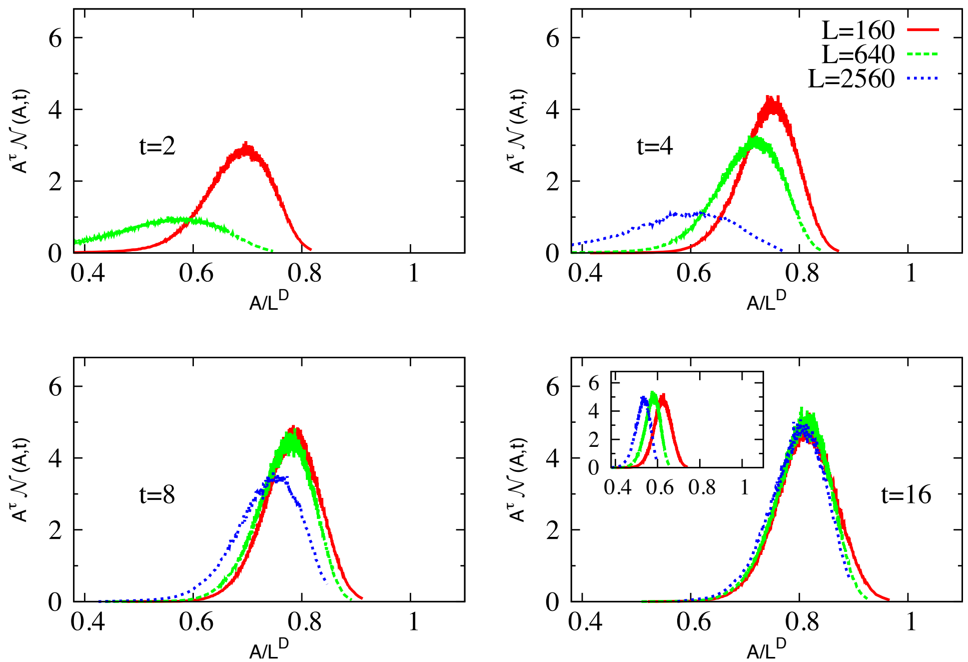
— 3d X tomography at ESRF (ID 19 line).

— Sample size 700 μml ; pixel size 0.7 μm.

D. Bouttes (ESPCI), E. Gouillart (Saint-Gobain) & D. Vandembroucq (ESPCI)

Is it critical percolation ?

Number density of the areas of percolated clusters



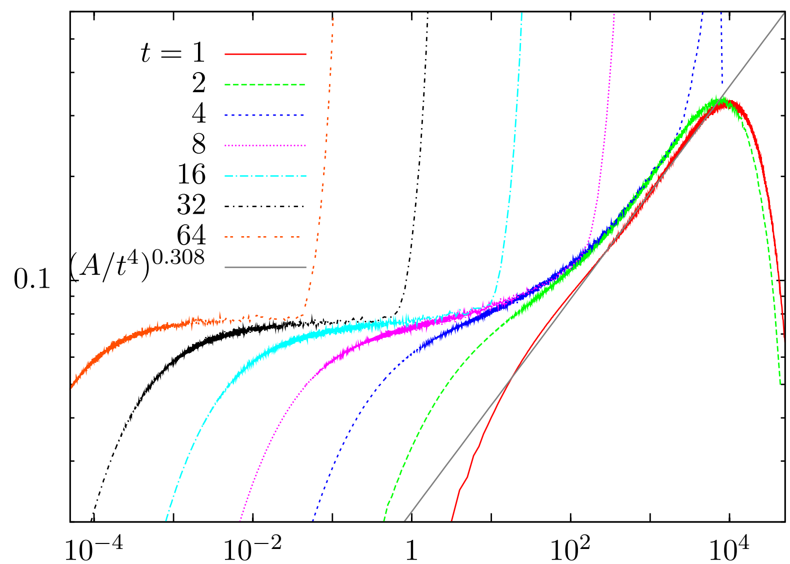
At $t = 16$ MCs the bump converged to a stationary form that satisfies the finite size scaling of critical site percolation with $\tau = 2.05$ and $D = 1.9$, and the scaling function is the one of critical percolation (not shown).

Insert: failure of collapse if critical Ising exponents are used.

Approach to percolation

Scaling of finite-size area density with $\xi_p(t) \simeq t^{1/z_p}$

$A^\tau \mathcal{N}(A, L; t)$



$A/\xi_p^2(t)$

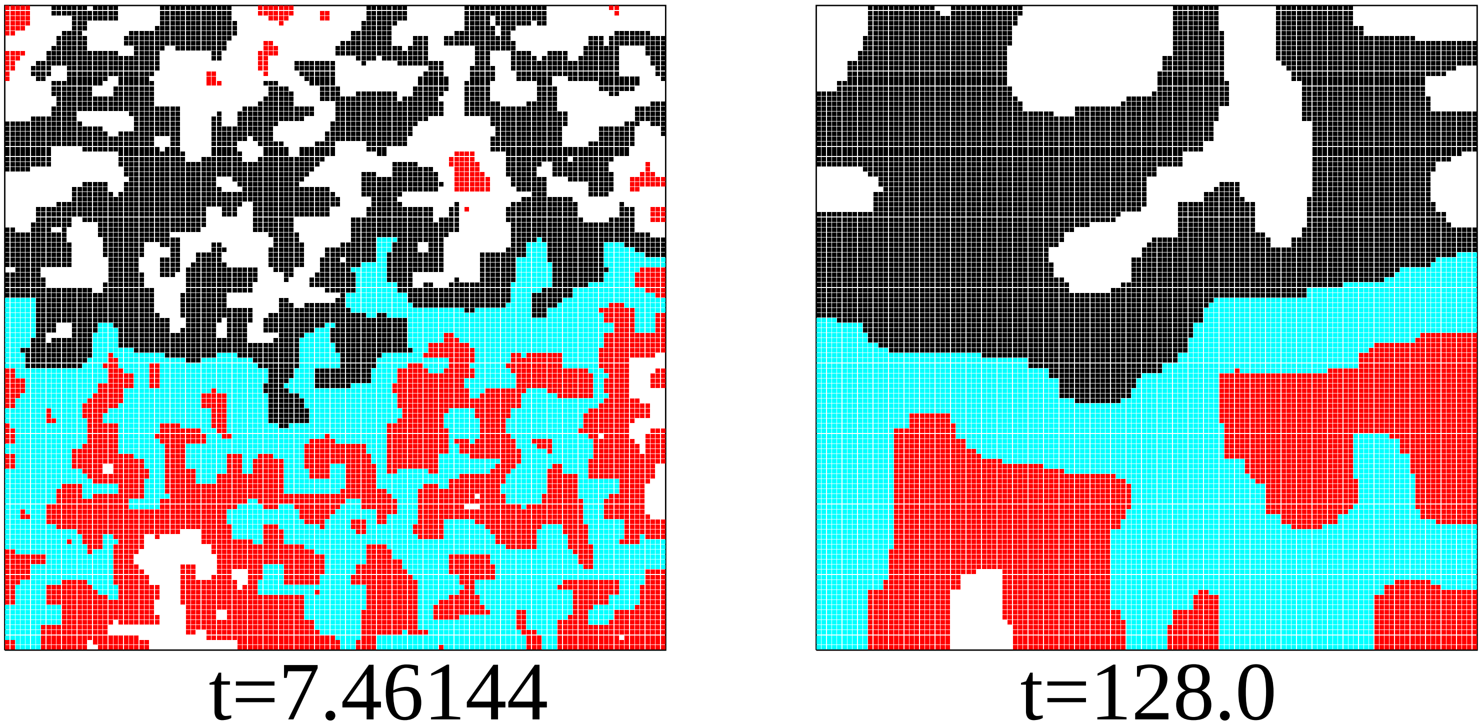
with $\xi_p(t) \equiv t^{1/z_p}$ and

conjecture: $z_p = z_d/n$ with n the lattice coordination and $z_d = 2$.

Blanchard, LFC, Picco & Tartaglia 16

2d square IM at T=0

The final configuration was decided at t_p



stripe states with the probabilities of critical percolation:

a spanning cluster along the two Cartesian directions $2\pi_{hv} \approx 0.64$

along only one of them $1 - 2\pi_{hv} \approx 0.36$

U(1) field theory in $3d$

Relativistic bosons ; also ^{4}He, type II superconductors, cosmology

$$\mathcal{L} = \frac{1}{c^2}|\dot\psi|^2 + i\mu\{\psi^*\dot\psi - \text{cc}\} - |\nabla\psi|^2 + g\rho|\psi|^2 - \frac{g}{2}|\psi|^4$$

$\mathcal{S} = \int d^dx \, \mathcal{L}$ action ; $\mathcal{L}$ Lagrangian density, $\psi(\vec{x}, t)$ complex field.

μ chemical potential, c velocity of light, ρ and g parameters in potential.

critical temperature $T_c > 0$ in $d = 3$.

Langevin dynamics $\quad -\gamma\dot\psi$ viscosity friction plus ξ noise

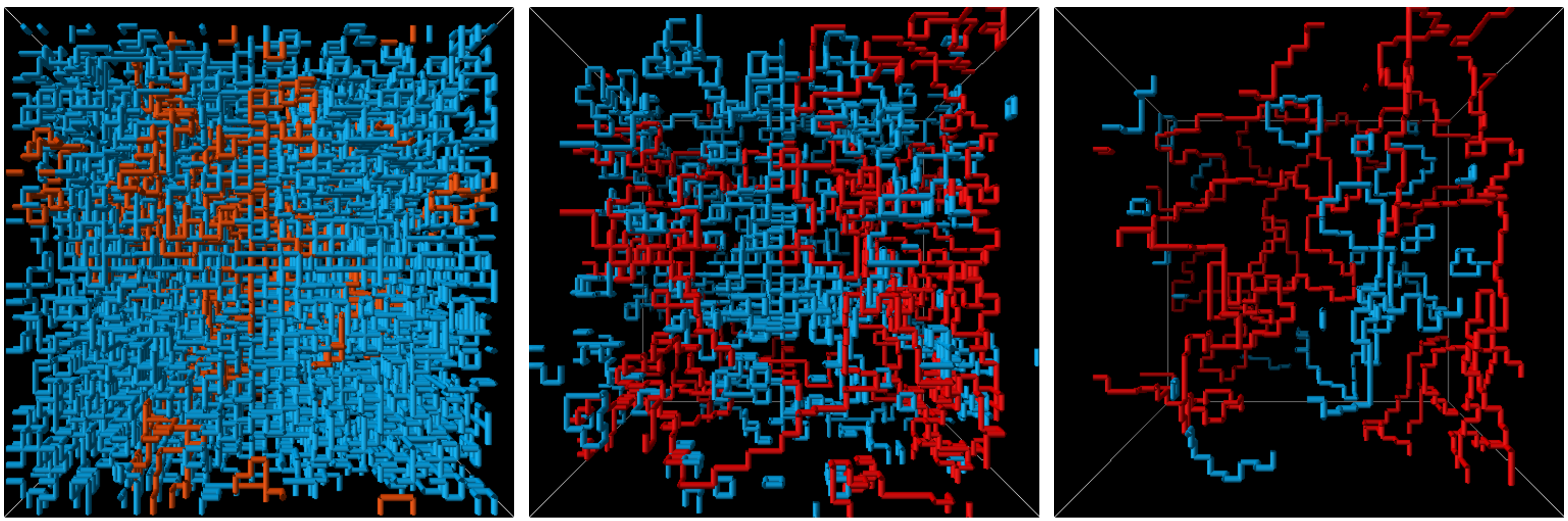
time-dependent complex Ginzburg-Landau, stochastic Goldstone ($\mu \to 0$) and Gross-Pitaevskii ($c \to \infty$) model for BECs close to the Mott insulator transition and in their gaseous phase, respectively.

Non-conserved order parameter dynamics

Gardiner et al 00s

U(1) field theory in $3d$

Time evolution of vortex configurations, tubes of $\psi = 0$

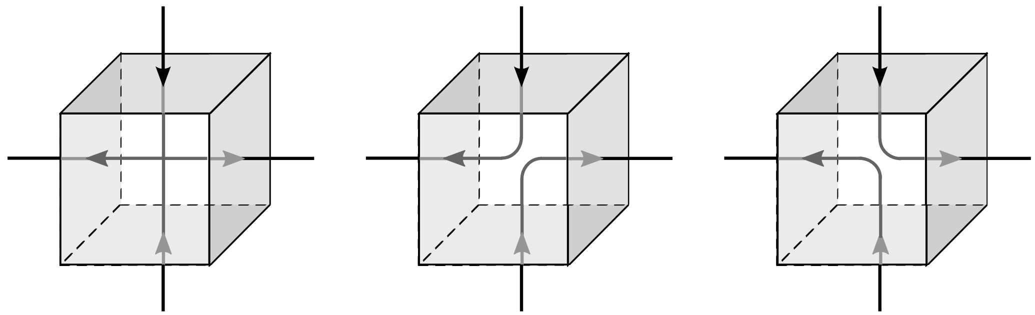


$t = 0$ $t = 3$ $t = 5$

U(1) field theory in $3d$

Vorticity & reconnection conventions

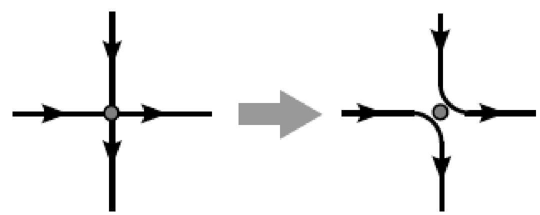
$2\pi v_x = \sum_{\text{plaq}} [\Delta\theta]_{2\pi} = 0, \pm 1, \ldots$



One field configuration with two possible line structures

Maximal & stochastic reconnection rules

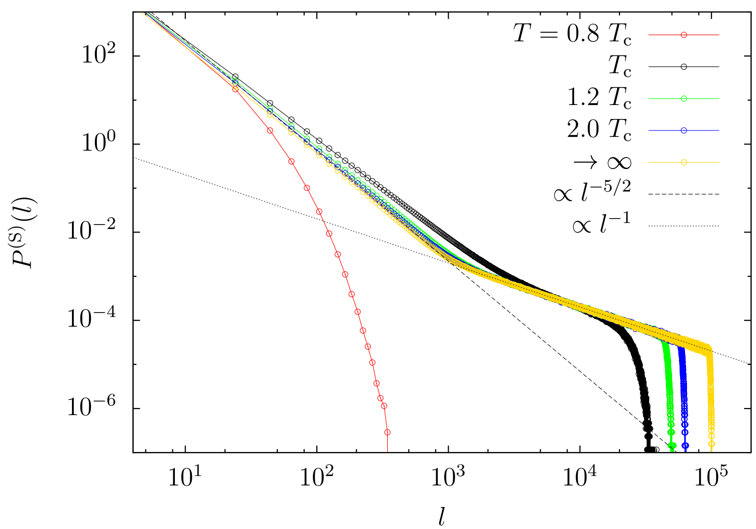
while just one choice in



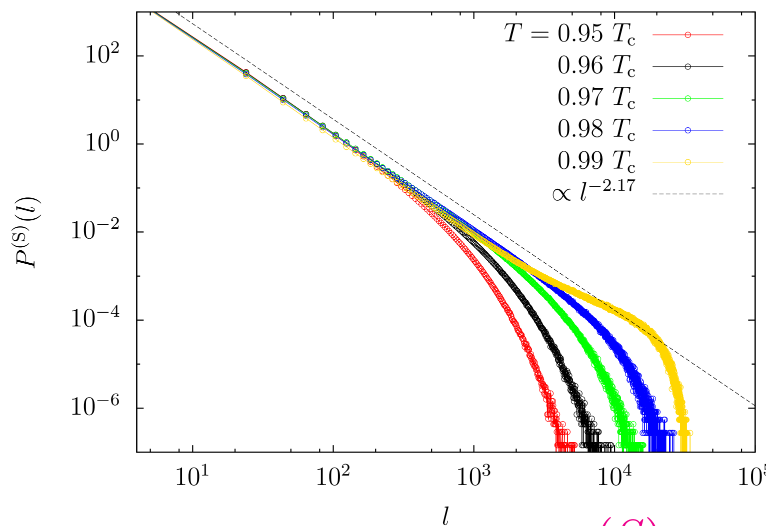
Kajantie et al. 00, Bittner, Krinner & Janke 05, Kobayashi & LFC 15

U(1) field theory in $3d$

Number density of vortex loops in equilibrium



High & low T



Close to $T_L^{(S)}$

Stochastic rule, three algebraic regimes :

At $T \gg T_c$, for $l \ll L^2$ Gaussian statistics $l^{-5/2}$

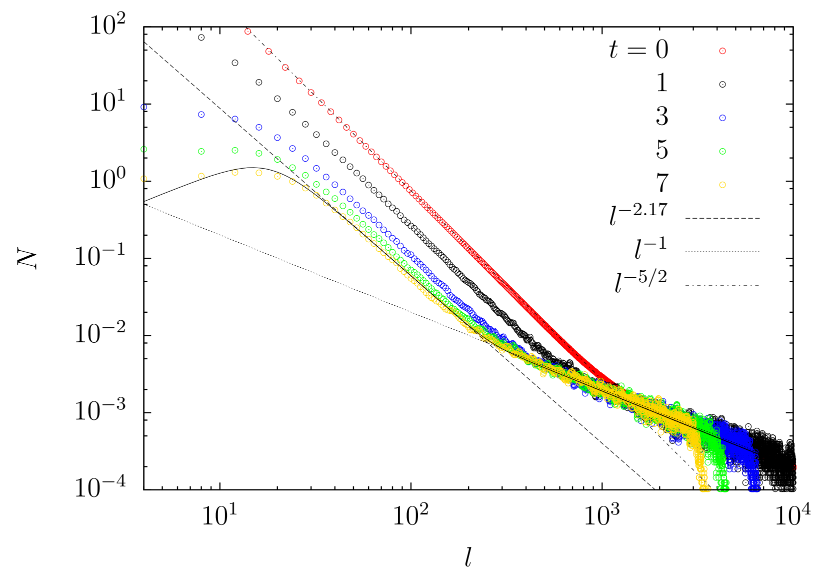
for $l \gg L^2$ fully-packed loops large-scale statistics l^{-1}

At $T_L^{(S)} < T_c$, percolation $l^{-2.17}$

Kobayashi & LFC in preparation

U(1) field theory in $3d$

Number density of vortex loops after a quench to $T = 0$



There is a $t_p \simeq 6$ such that for $t \gtrsim t_p$ three algebraic regimes :

$l \ll t^{1/2}$ dynamic scaling $l^2 P(l)$ against $l/t^{1/2}$

$t^{1/2} \ll l \ll L^2$ percolation-like $l^{-2.17}$

$L^2 \ll l$ fully-packed loops large-scale statistics l^{-1}

Kobayashi & LFC in preparation

Second conclusion

Early approach to percolation

A similar phenomenon observed in the $3d$ U(1) field theory or XY model.

$\xi_p(t) \simeq t^{1/z_p}$ is a new growing length-scale that brings about a new scaling variable to be taken into account in dynamic scaling.

In the $2d$ Ising model $\xi_p(t) \simeq t^{1/z_p}$ and $\xi_d(t) \simeq t^{1/z_d}$ are rather well separated as $z_p \simeq 0.5$ and $z_d = 2$.

In the $2d$ Voter model it is hard to disentangle the two since $z_p \simeq 1.667$ and $z_d = 2$.

In the $3d$ U(1) model we still do not know z_p.

Kobayashi & LFC in preparation

Interests and goals

Practical & fundamental interest, *e.g.*

- Mesoscopic structure effects on the opto-mechanical properties of phase separating glasses
- Cooling rate effects on the density of topological defects in cosmology and condensed matter

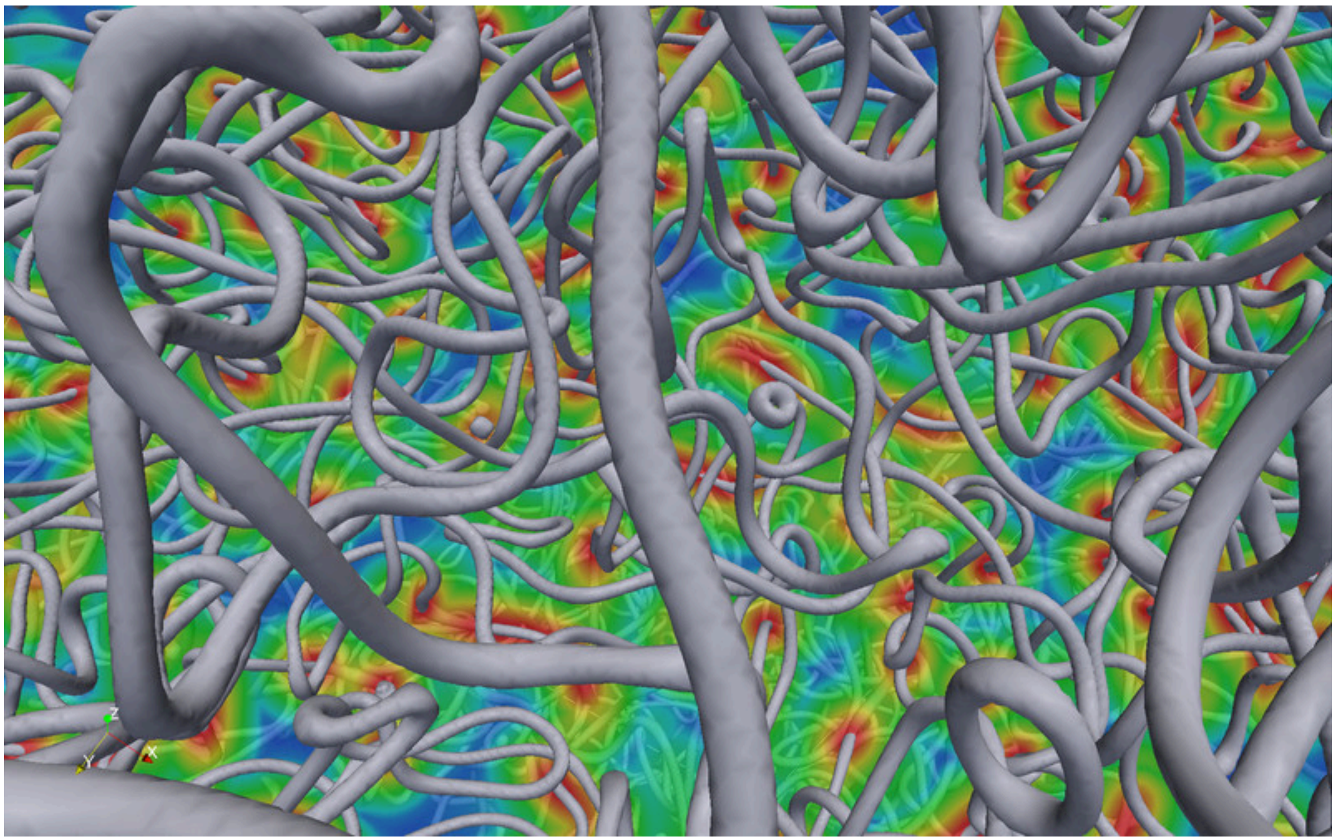
Some issues

- The role played by the initial conditions & short-time dynamics
- Full geometric characterisation of the structure
- When does the usual dynamic scaling regime set in ?
- The role played by the cooling rate

that are related to each other.

Theoretical motivation

Network of cosmic strings



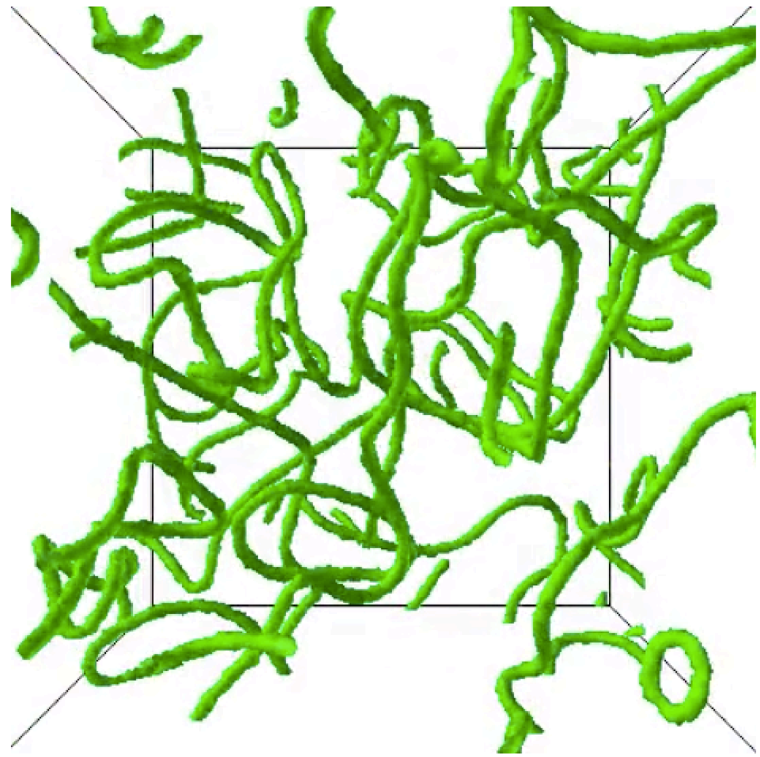
They should affect the Cosmic Microwave Background, double quasars, etc.

Picture from M. Kunz's group (Université de Genève)

Topological defects

instantaneous configurations

Domain walls in the $2d$IM

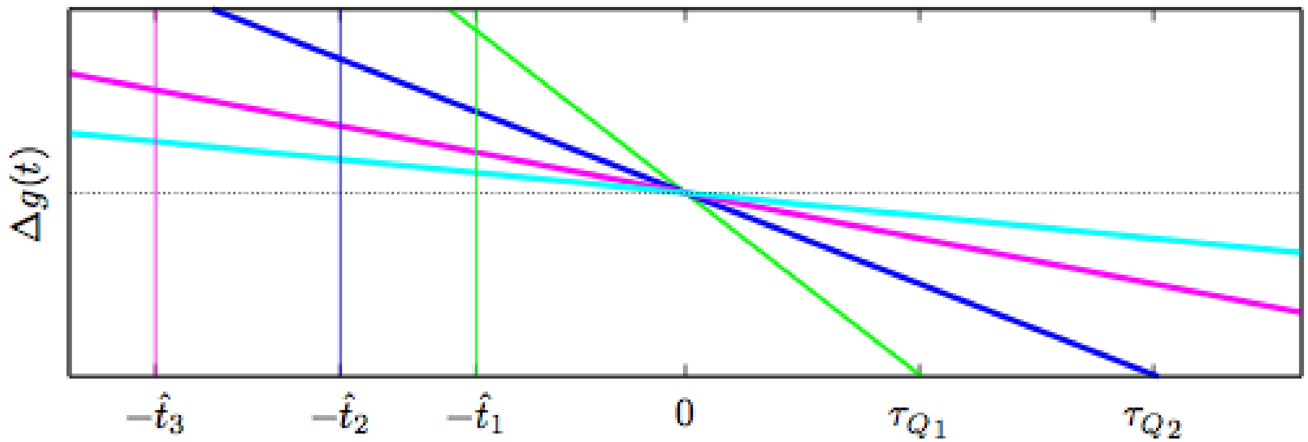


Vortices in the $3d$ xy model

One can give a precise mathematical definition but the visual one is enough

Finite rate quenching protocol

How is the scaling modified for a very slow quenching rate ?



$\Delta g \equiv g(t) - g_c = -t/\tau_Q$ with $\tau_{Q1} < \tau_{Q2} < \tau_{Q3} < \tau_{Q4}$

Standard time parametrization $g(t) = g_c - t/\tau_Q$

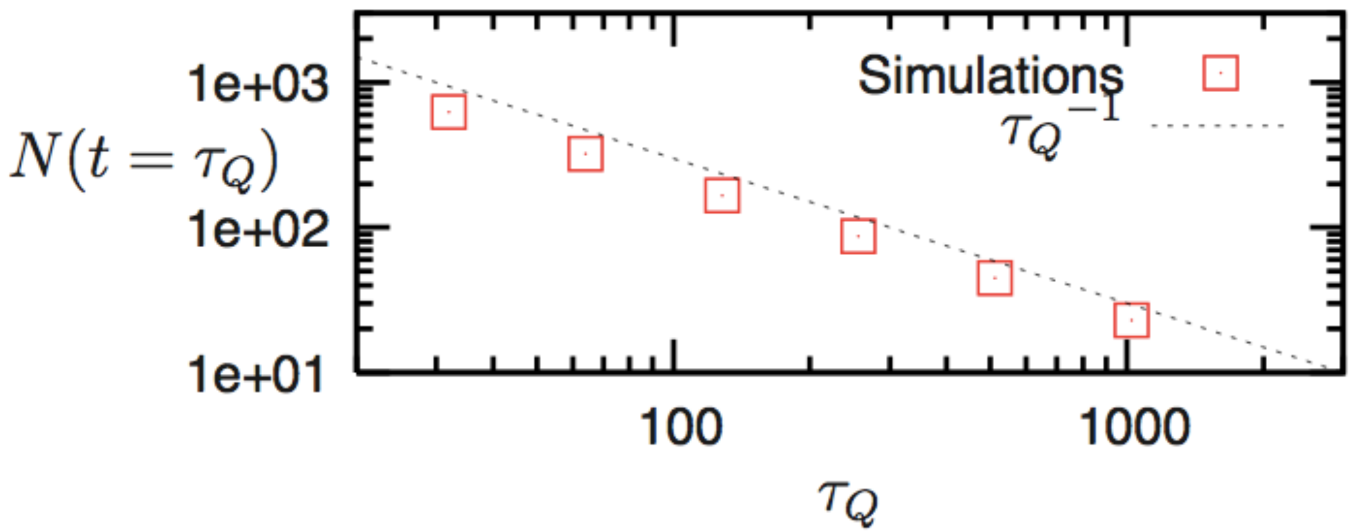
Simplicity argument: linear cooling could be thought of as an approximation of any cooling procedure close to g_c.

Density of domain walls

At $t \simeq \tau_Q$ in the 2dIM with NCOP dynamics

Coarsening in the low-T phase has to be taken into account

$$N(t \simeq \tau_Q, \tau_Q) = n(t \simeq \tau_Q, \tau_Q) L^2 \simeq \tau_Q^{-1}$$



while the KZ mechanism yields $N_{KZ} \simeq \tau_Q^{-\nu/(1+\nu z_c)} \simeq \tau_Q^{-0.31}$.

Biroli, LFC & Sicilia 10

Summary

Three parts

- Evidence for the approach to critical percolation at a time-scale that diverges with the system size as $t_p \simeq L^{z_p}$.
- The new growing length-scale, $\xi_p(t) \simeq t^{1/z_p}$ dominates at short times and is needed to improve the scaling of finite-size and finite-time data.
- In slow quenches, need to take coarsening into account at sufficiently long times, e.g., $t \simeq \tau_Q$ to describe the remnant density of topological defects ayer a finite rate quench correctly.

“Exact results for curvature driven coarsening in two dimensions”,
J. J. Arenzon, A. J. Bray, L. F. Cugliandolo, A. Sicilia. PRL **98**, 145701 (2007).

“Domain growth morphology in curvature driven two dimensional coarsening”,
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A. Tartaglia, L. F. Cugliandolo, M. Picco, arXiv :1506.05321

Zurek's argument

Slow quench from equilibrium well above g_c

The system follows the pace imposed by the changing conditions, $\Delta g(t) = -t/\tau_Q$, until a time $-\hat{t} < 0$ (or value of the control parameter $\hat{g} > g_c$) at which its dynamics are too slow to accommodate to the new rules. The system **falls out of equilibrium**.

$-\hat{t}$ is estimated as the moment when the relaxation time, τ_{eq}, is of the order of the typical time-scale over which the control parameter, g, changes :

$$\tau_{eq}(g) \simeq \left. \frac{\Delta g}{d_t \Delta g} \right|_{-\hat{t}} \simeq \hat{t} \qquad \Rightarrow \qquad \boxed{\hat{t} \simeq \tau_Q^{\nu z_c/(1+\nu z_c)}}$$

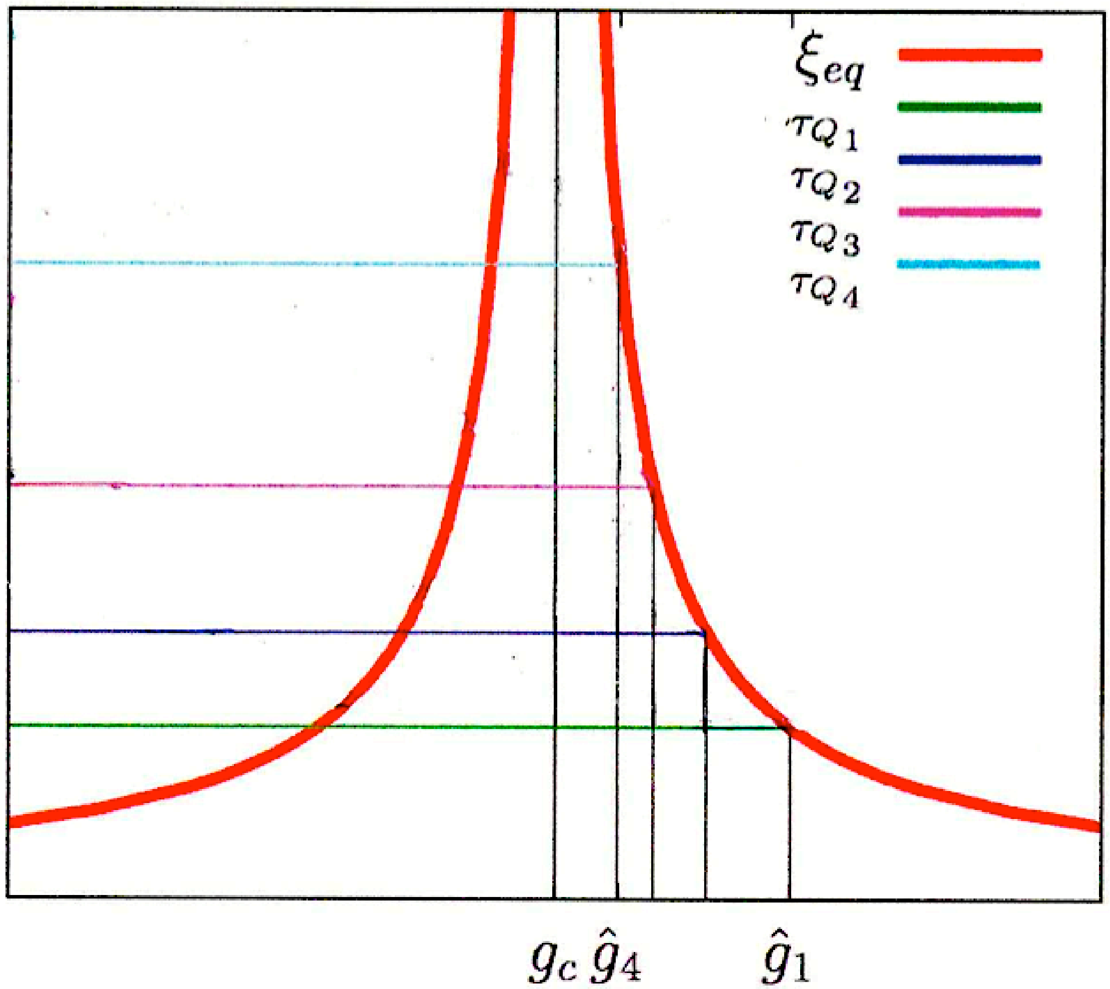
The density of defects is $\boxed{\hat{n}_{KZ} \simeq \xi_{eq}^{-d}(\hat{g}) \simeq (\Delta \hat{g})^{\nu d} \simeq \tau_Q^{-\nu d/(1+\nu z_c)}}$

and gets blocked at this value ever after

Zurek 85

Finite rate quench

Sketch of Zurek's proposal for R_{τ_Q}



Infinitely fast quench

Dynamic scaling

From a completely uncorrelated initial state either to the critical point or into the symmetry broken phase.

Infinite system size, thermodynamic limit.

$$\mathcal{R}_c(t) \simeq t^{1/z_c} \qquad \text{or} \qquad \mathcal{R}(t) \simeq t^{1/z_d}$$

and

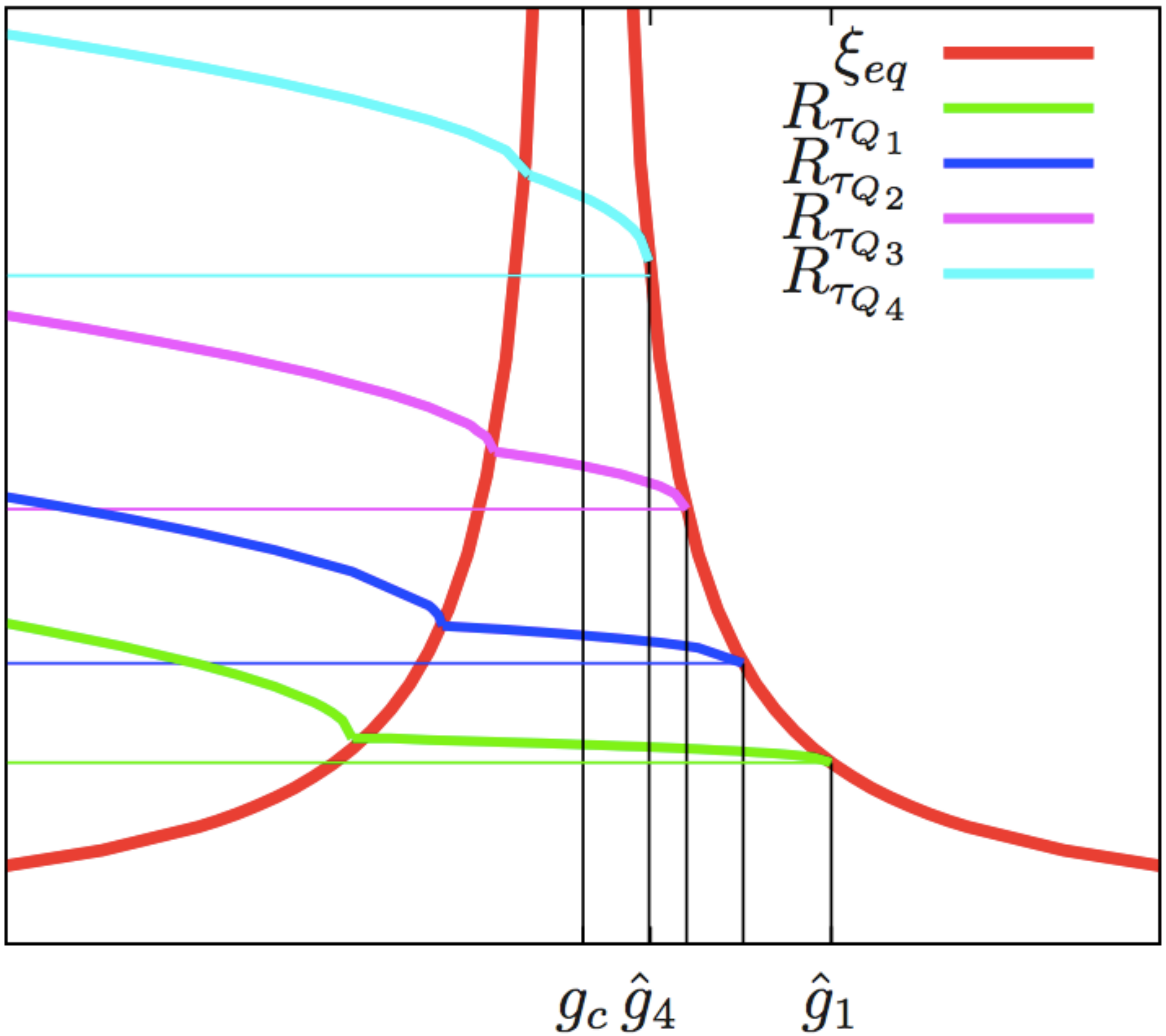
$$\boxed{n(t) \simeq \mathcal{R}^{-d}(t)}$$

These quantities relax.

Review in **Hohenberg & Halperin 77 ; Bray 94**

Finite rate quench

Sketch of the effect of τ_Q on $\mathcal{R}(t, g)$



cfr. constant thin lines, **Zurek 85**

Density of domain walls

Test of universal scaling in the 2dIM with NCOP dynamics

Dynamic scaling implies

$$n(t,\tau_Q) \simeq [\mathcal{R}(t,\tau_Q)]^{-d}$$

with d the dimension of space

Therefore

$$\boxed{n(t,\tau_Q) \simeq \tau_Q^{d\nu(z_c-z_d)/z_d}\; t^{-d[1+\nu(z_c-z_d)]/z_d}}$$

depends on *both* times t and τ_Q.

NB t can be much longer than t^* (time for starting sub-critical coarse-ning) ; in particular t can be of order τ_Q while t^* scales as τ_Q^α with $\alpha < 1$.

Since z_c is larger than z_d this quantity grows with τ_Q at fixed t.

Third conclusion

Need to take coarsening into account at sufficiently long times, e.g., $t \simeq \tau_Q$ to describe the remnant density of topological defects ayer a finite rate quench correctly.

Relevant for, *e.g.* recent study of vortices in Bose-Einstein condensates, *e.g.*

Su, Gou, Bradley, Fialko & Brand 13

Chomaz, Corman, Bienaimé, Desbuquois, Weitenberg, Nascimbène, Beugnon & Dalibard 15

Conclusion

A nice (non mysterious !) problem that, I think, will still surprise us.

- Amenable to nice analytic treatment and the use and develop-meant of new techniques.
- Easy to study numerically.
- Relevant to material science and experiments.
- Open to quantum and beyond physics extensions.

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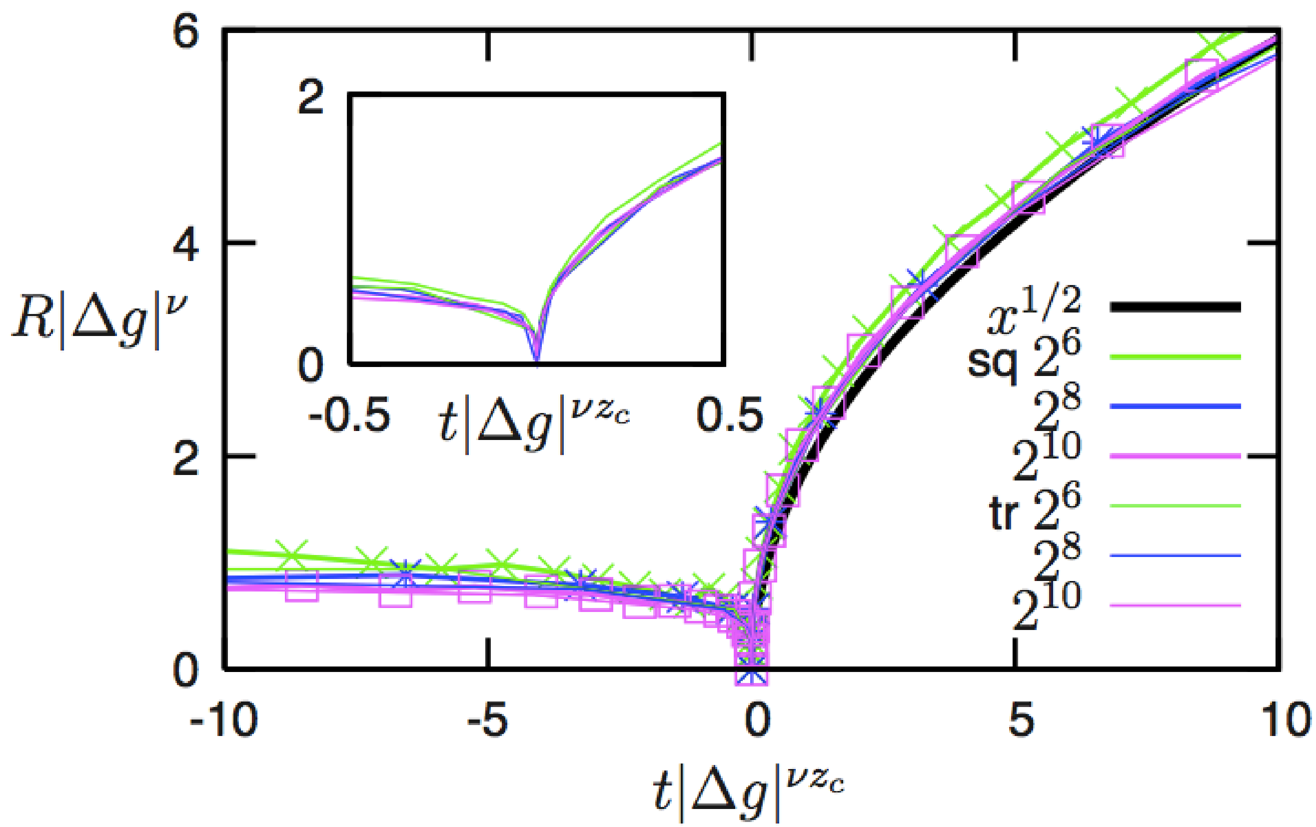
Simulations

Test of universal scaling in the 2dIM with NCOP dynamics

$\mathcal{R}\ |\Delta g|^{\nu}$ cst $(|\Delta g|^{\nu z_c} t)^{1/z_d}$



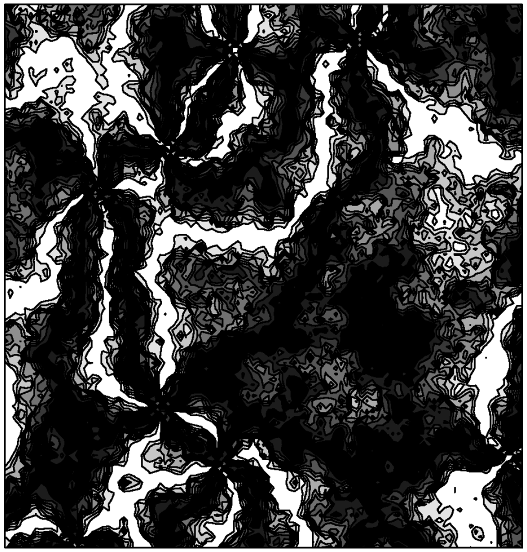
$z_c \simeq 2.17$ and $\nu \simeq 1$; the square root ($z_d = 2$) is in black

Also checked (analytically) in the $O(N)$ model in the large N limit.

Dynamics in the $2d$ XY model

Vortices : planar spins turn around points

Schrielen pattern : gray scale according to $\sin^2 2\theta_i(t)$



After a quench vortices annihilate and tend to bind in pairs

$$\mathcal{R}(t,g) \simeq \lambda(g)\{t/\ln[t/t_0(g)]\}^{1/2}$$

Pargellis *et al* **92, Yurke** *et al* **93, Bray & Rutenberg 94**

Dynamics in the $2d$ XY model

KT phase transition & coarsening

- The high T phase is plagued with vortices. These should bind in pairs (with finite density) in the low T quasi long-range ordered phase.
- Exponential divergence of the equilibrium correlation length above T_{KT}

$$\xi_{eq} \simeq a_\xi e^{b_\xi [(T-T_{KT})/T_{KT}]^{-\nu}} \qquad \text{with} \qquad \nu = 1/2.$$

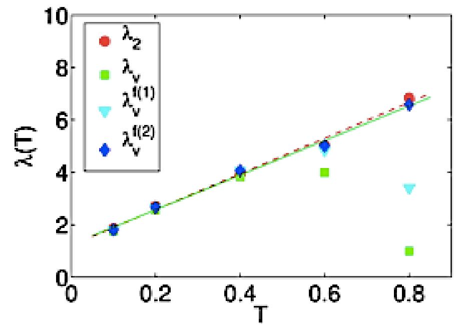
- Zurek's argument for falling out of equilibrium in the disordered phase

$$\hat{\xi}_{eq} \simeq (\tau_Q / \ln^3(\tau_Q/t_0))^{1/z_c} \qquad \text{with } z_c = 2 \text{ for NCOP.}$$

- Logarithmic corrections to the sub-critical growing length

$$\mathcal{R}(t,T) \simeq \lambda(T) \left[\frac{t}{\ln(t/t_0)} \right]^{1/z_d}$$

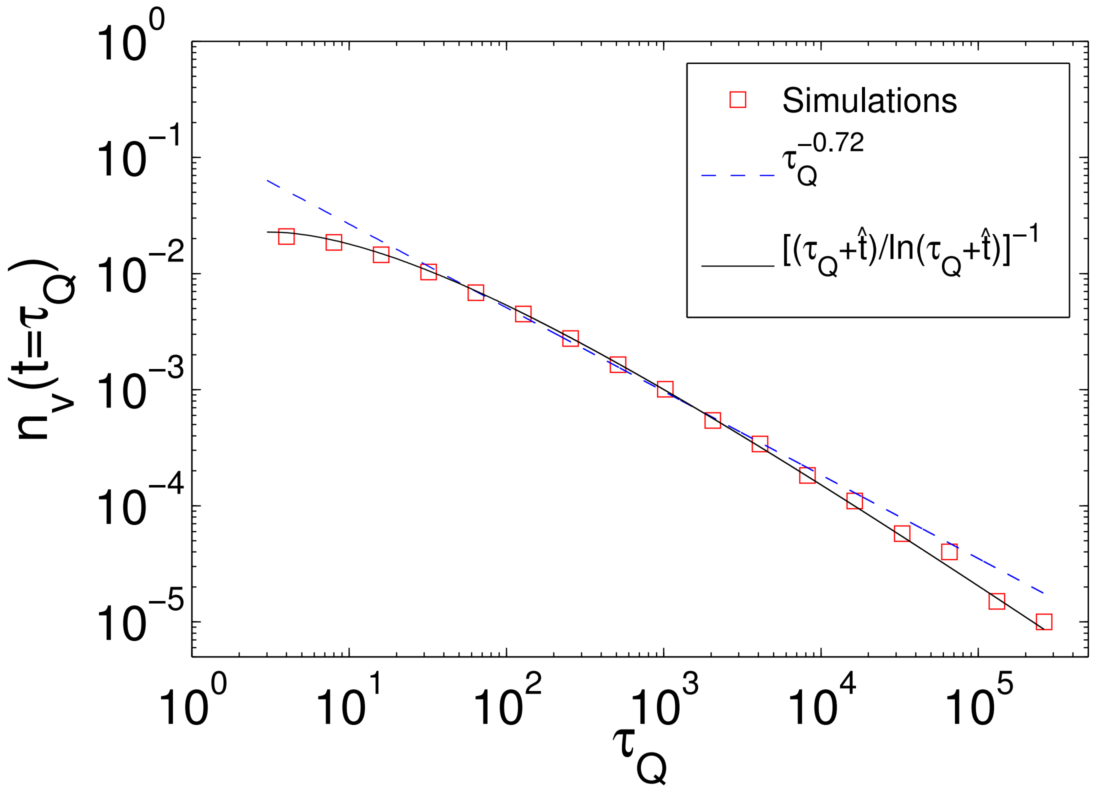
with $z_d = 2$ for NCOP



Dynamics in the $2d$ XY model

KT phase transition & coarsening

$$n_v(t \simeq \tau_Q, \tau_Q) \simeq \ln[\tau_Q / \ln^2 \tau_Q + \tau_Q]/(\tau_Q/\ln^2 \tau_Q + \tau_Q)$$



Large τ_Q

$$n_v \simeq \frac{\ln \tau_Q}{\tau_Q}$$

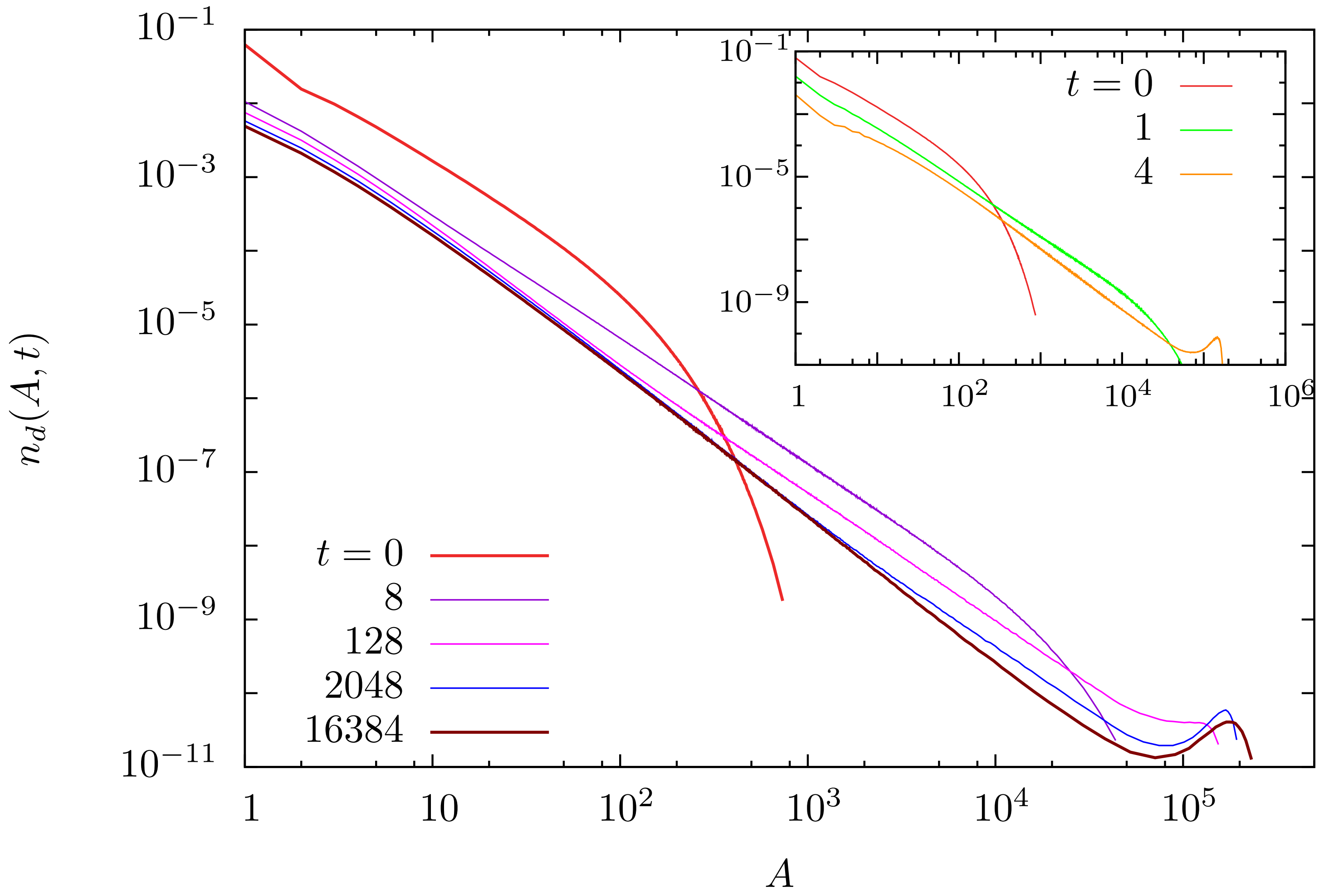
while

$$n_{KZ} \simeq \frac{\ln^3 \tau_Q}{\tau_Q}$$

Jelić and LFC 11

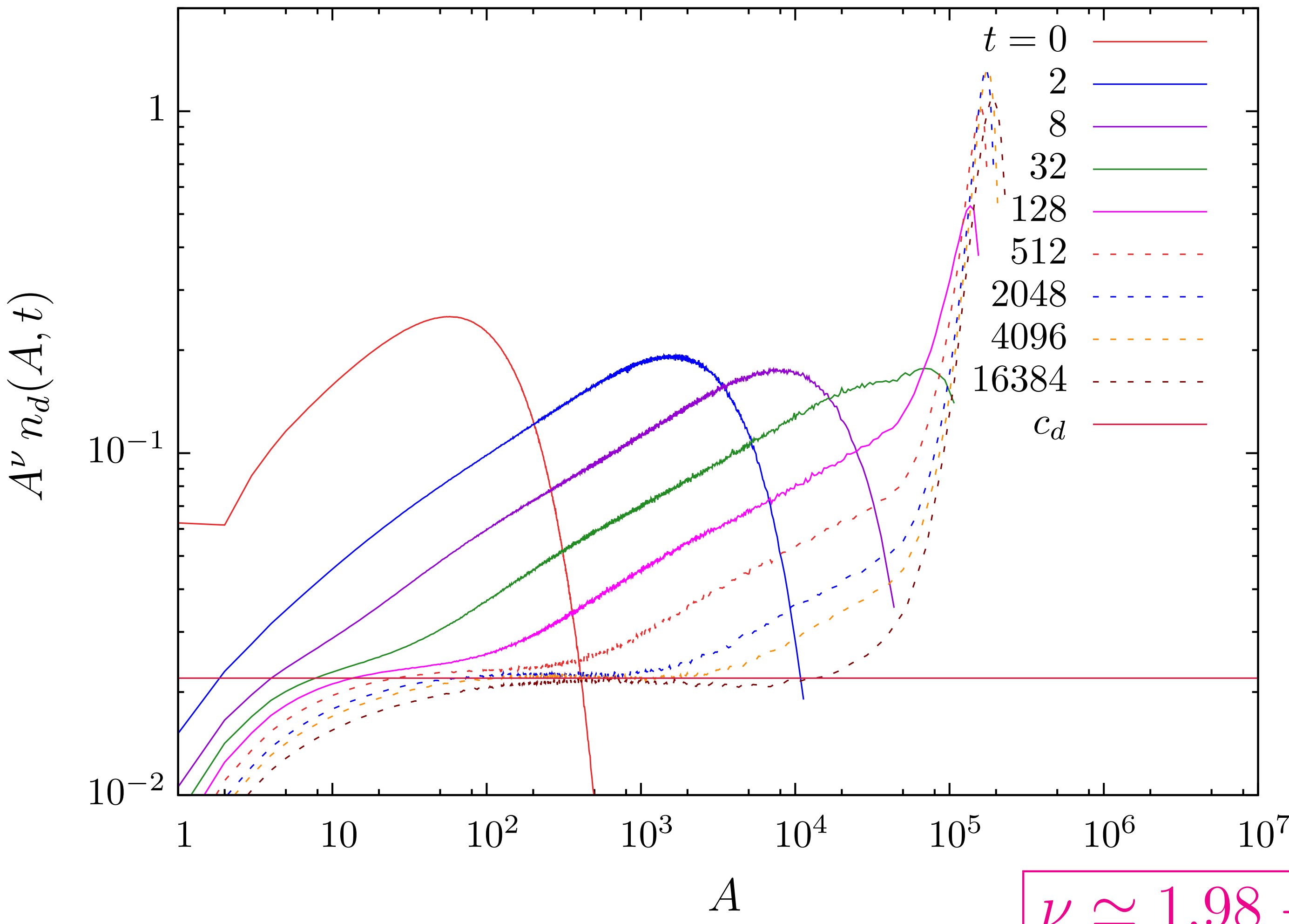
Square lattice 2dVM

Number density of cluster areas



Square lattice 2dVM

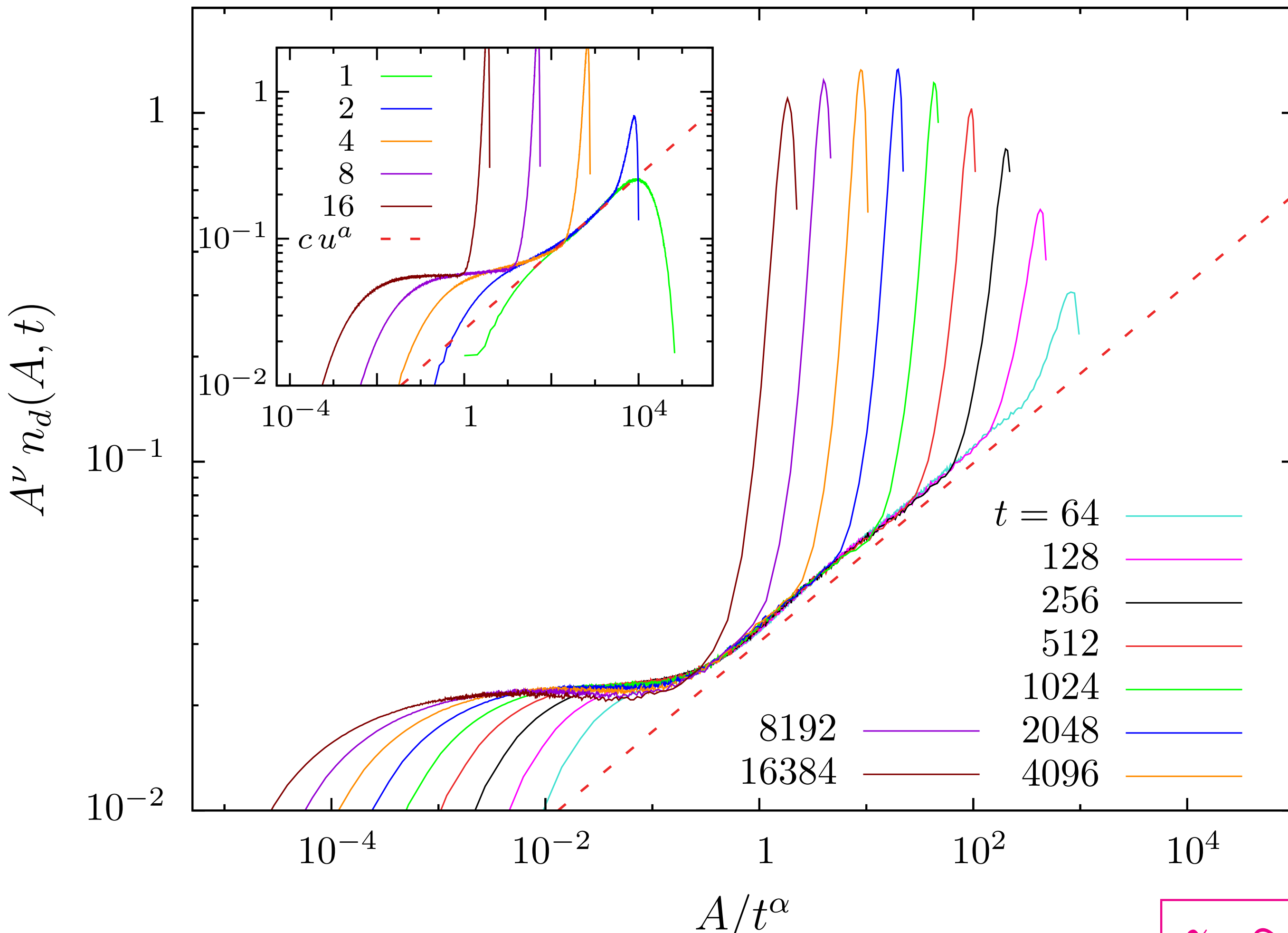
Scaled number density of cluster areas



$\nu \simeq 1.98 \to > 2$

Square lattice 2dVM

Scaled number density of cluster areas



$z_p \simeq 1.667$

Microscopic dynamics ?

The voter model : opinion dynamics

H does not exist - kinetic model

$s_i = \pm 1$ Ising spins that

sit on the vertices of a lattice.

Voter update rule

choose a spin at random, say s_i

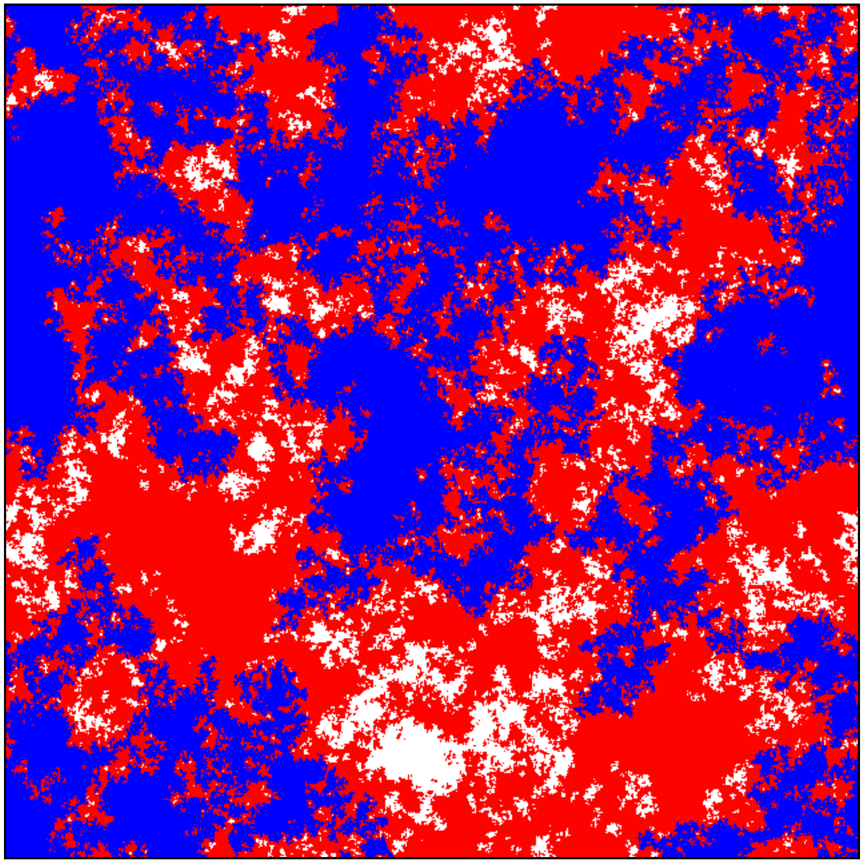
choose one of its $2d$ neighbours at random, say s_j

set $s_i = s_j$

Non-physical dynamics, no detailed balance.

Clifford & Sudbury 73, Holley & Liggett 75, Cox & Griffeaths 86

Square lattice 2dVM



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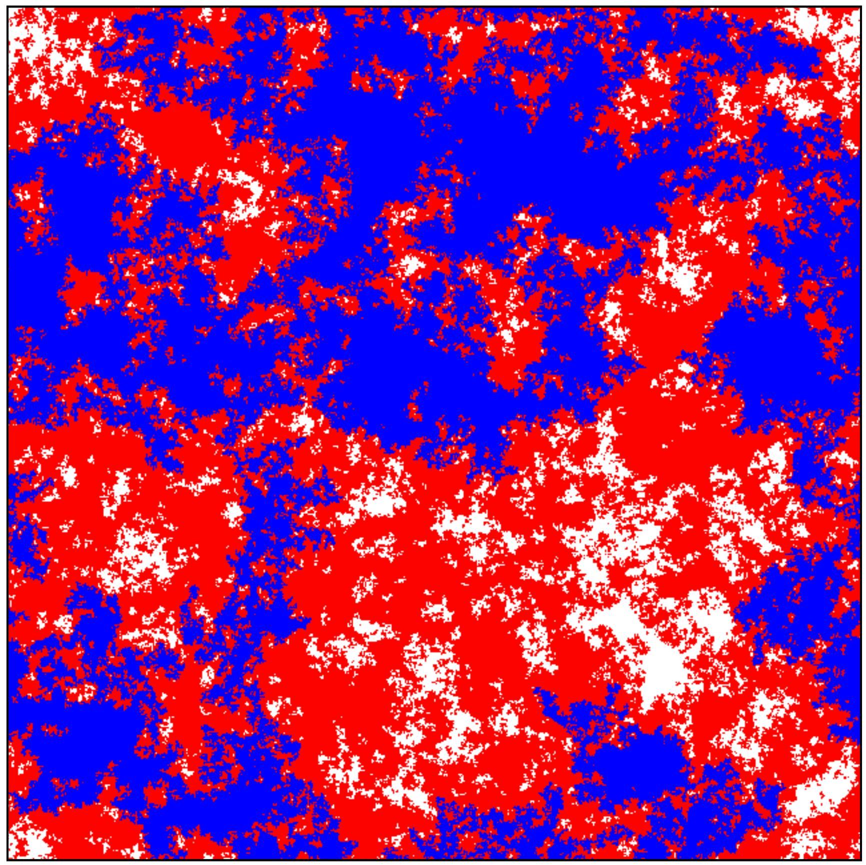
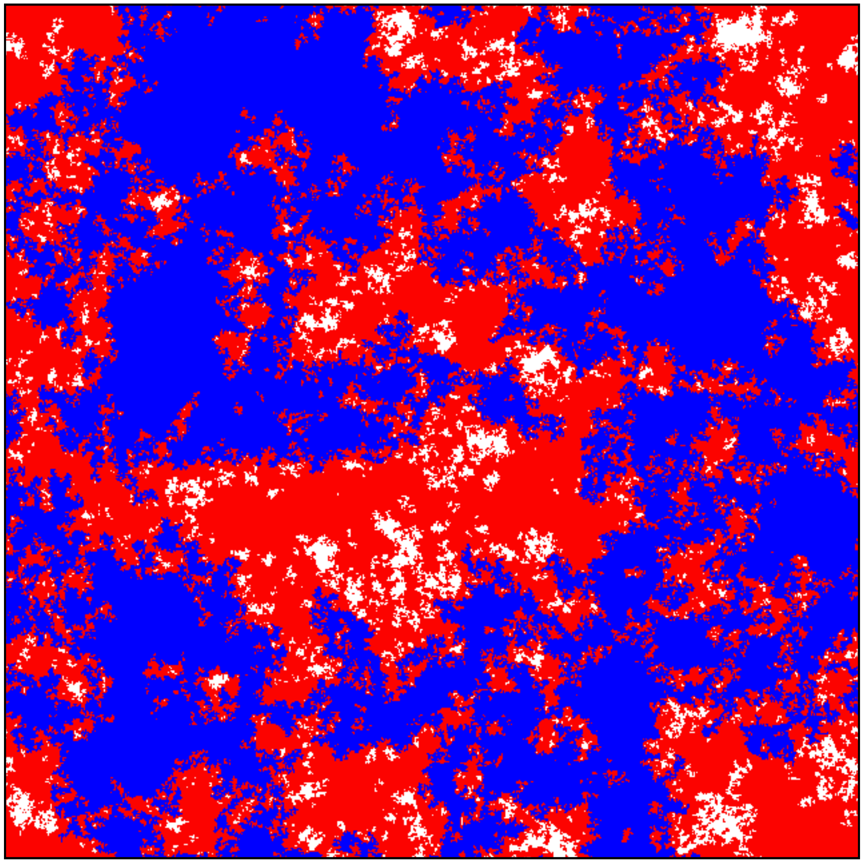
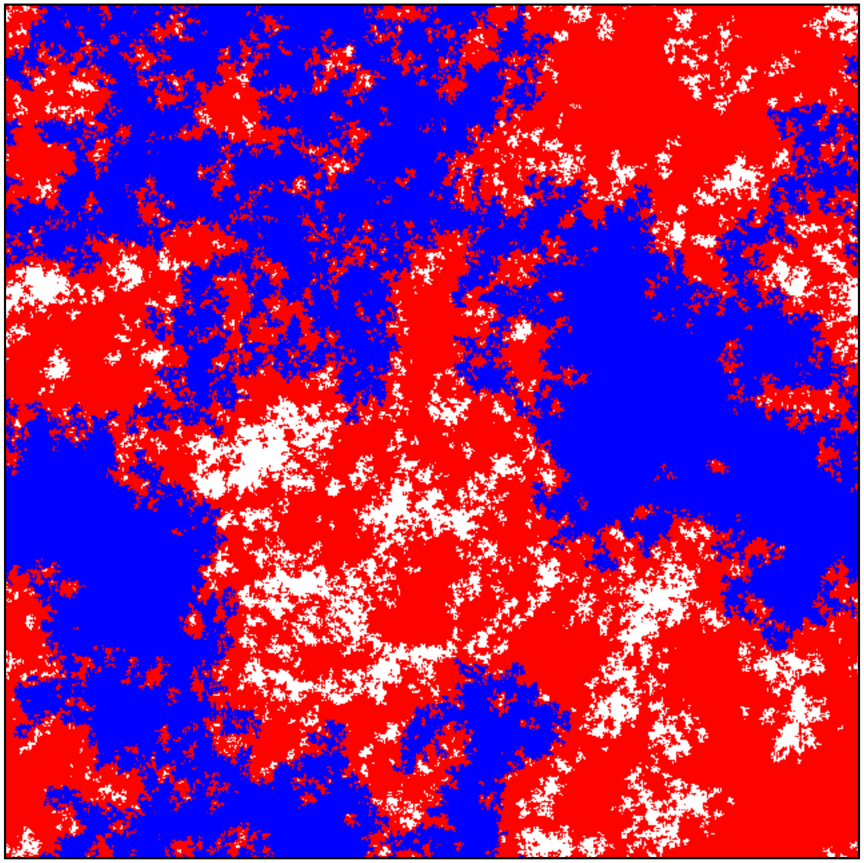


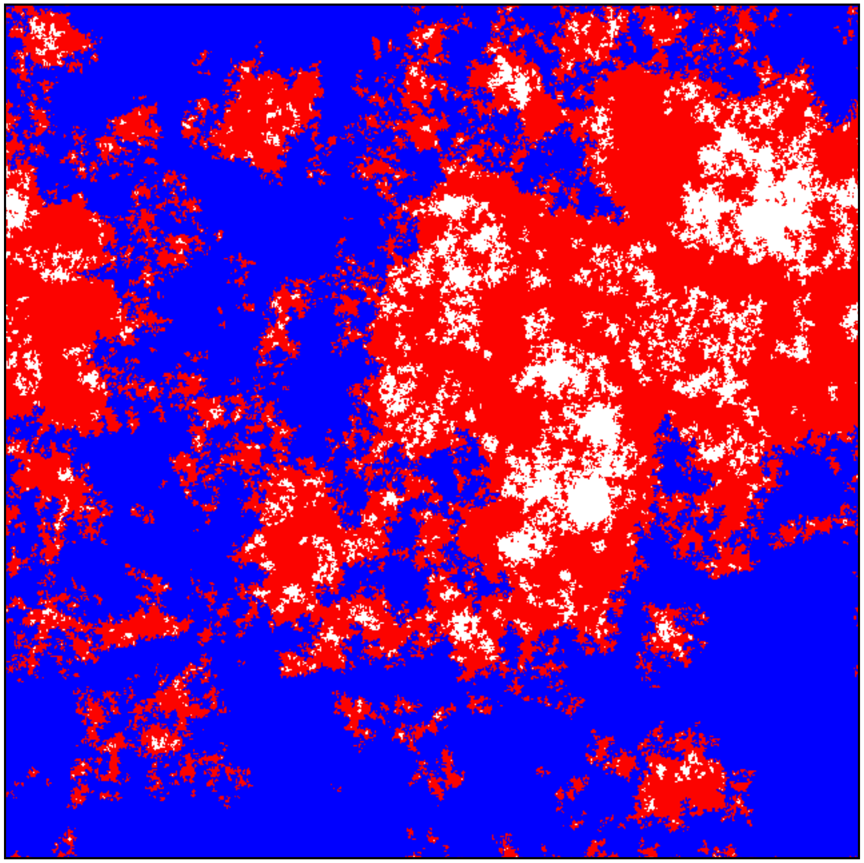
Square lattice 2dVM



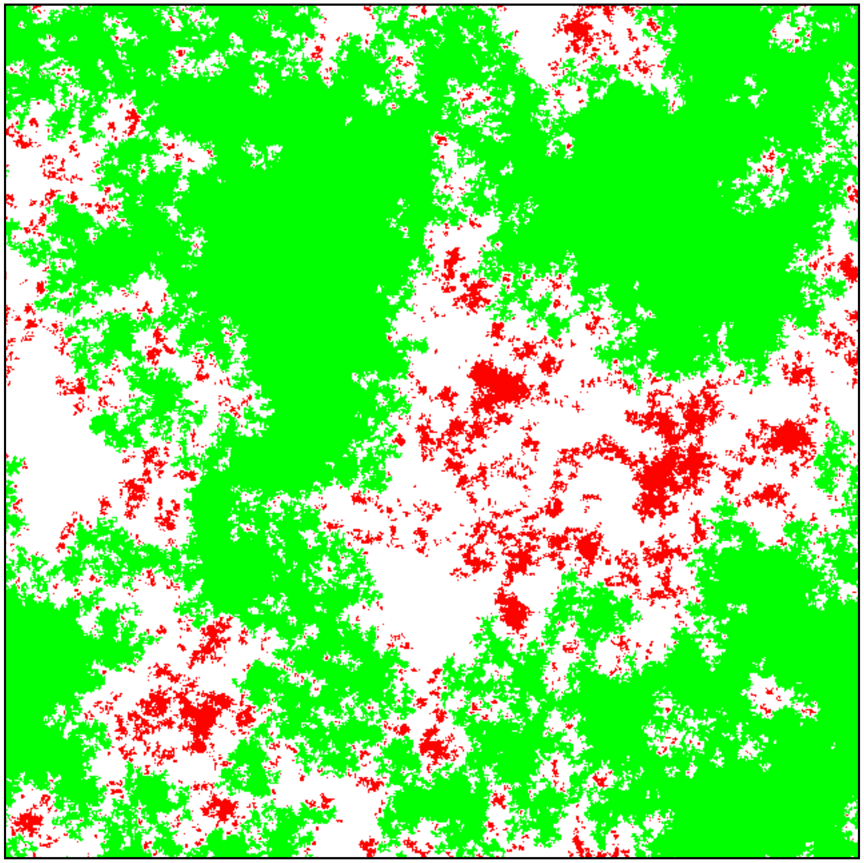
Square lattice 2dVM



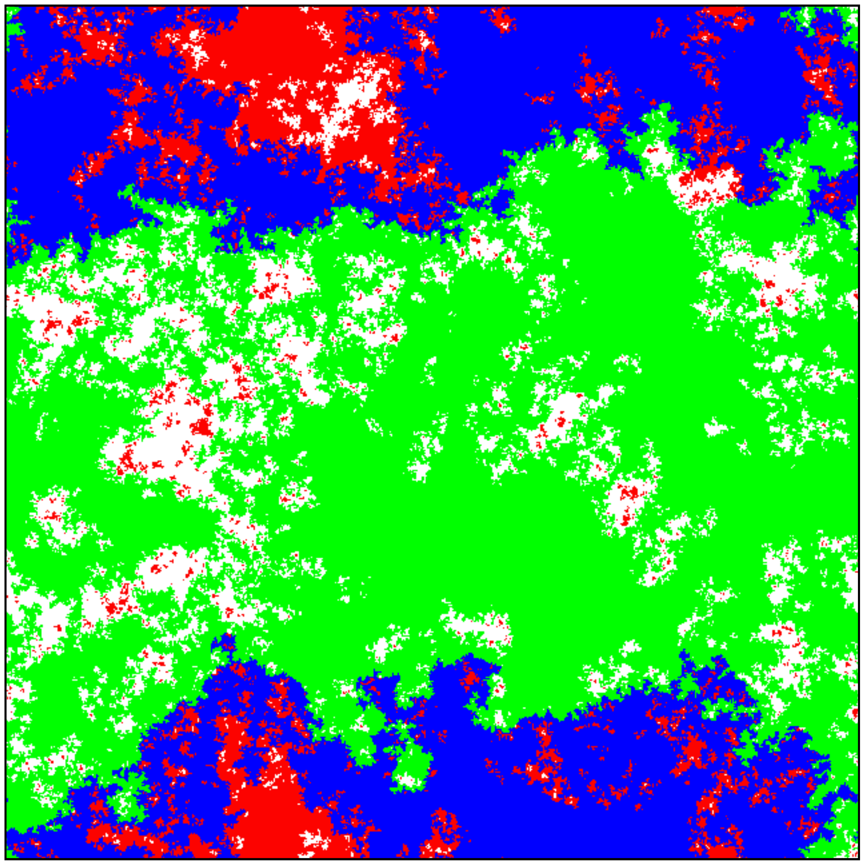
Square lattice 2dVM



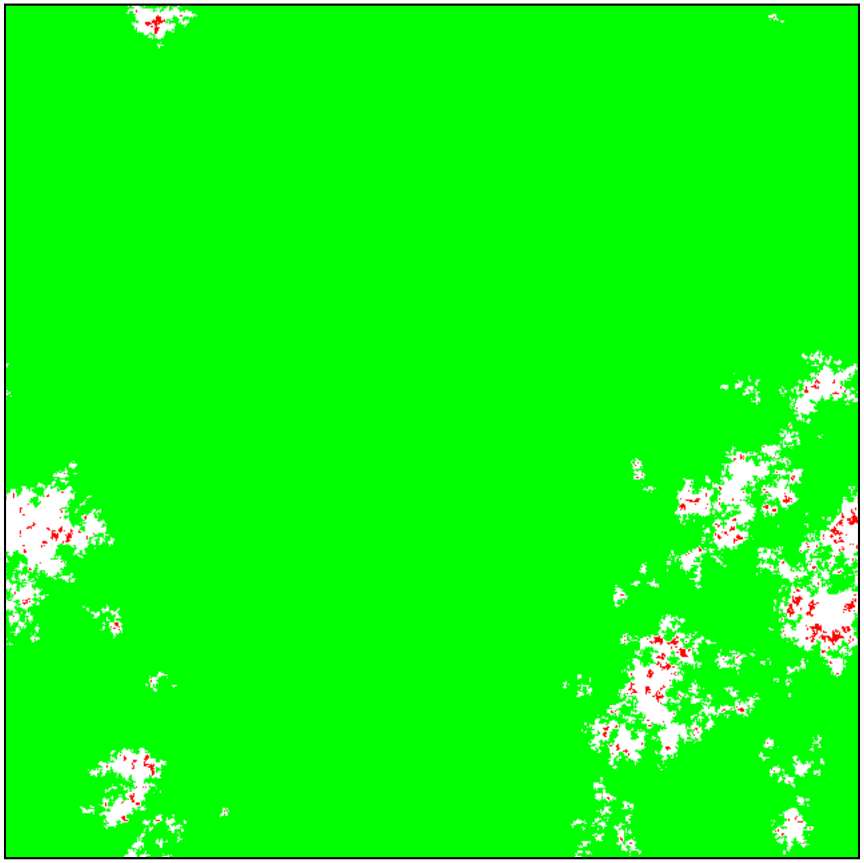
Square lattice 2dVM



Square lattice 2dVM



Square lattice 2dVM

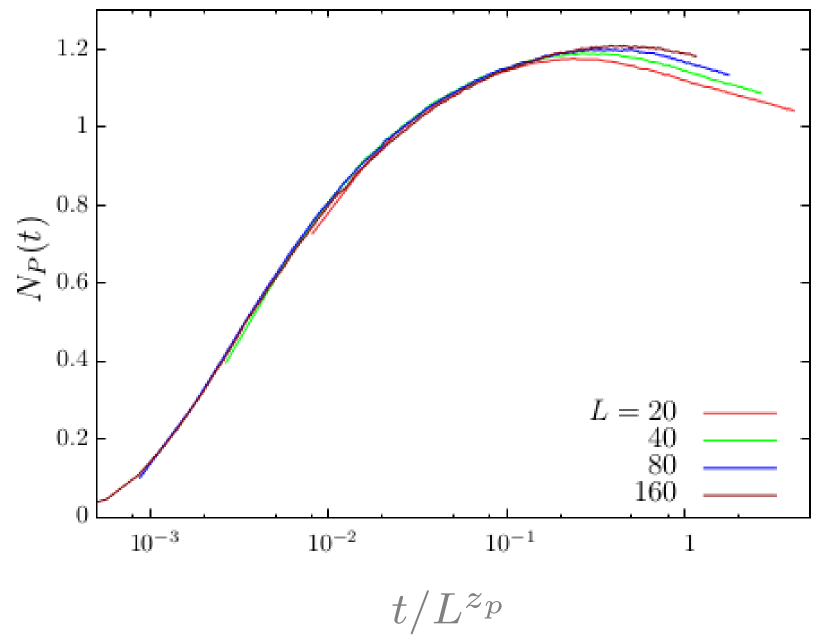


Square lattice 2dVM

Average number of wrapping clusters

The analyses of $\mathcal{N}(A,t)$ and $Q_{t_w(L)}(t)$ is hard.

Instead, we look at the dynamic finite-size scaling of the average $\#$ of wrapping clusters



$$N_p(t,L) = f(t/L^{z_p}) \quad \text{with} \quad z_p \simeq 1.667$$

Similar scaling in the Ising model, with the corresponding z_p.

Tartaglia, LFC & Picco 15