Effective temperatures in quantum quenches

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Aim of this talk

in two sentences

Advocate the use of fluctuation-dissipation relations as:

Tests of Gibbs-Boltzmann equilibration.

A means to measure the GGE effective temperatures in integrable systems.

Plan

1. Some words about glassiness, classical and quantum.

   Fluctuation dissipation relations and effective temperatures.

2. Quantum quenches in isolated systems.

   Example: the Ising chain

   L. Foini, LFC & A. Gambassi 11-12

3. FDRs and Generalized-Gibbs-Ensemble effective temperatures.

   LFC, L. Foini, A. Gambassi & R. Konik 16

4. Work in progress.

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Disordered spin systems

Quantum fully-connected p-spin model

$$\boxed{\hat{H}_{\rm syst} = \sum_{i_1<\dots<i_p}^{N} J_{i_1\dots i_p} \hat{\sigma}^x_{i_1} \dots \hat{\sigma}^x_{i_p} + \Gamma \sum_{i=1}^{N} \hat{\sigma}^z_i}$$

$\hat{\sigma}^a_i$ with $a = 1, 2, 3$ the Pauli matrices, $[\hat{\sigma}^a_i, \hat{\sigma}^b_i] = 2\mathrm{i}\epsilon_{abc}\hat{\sigma}^c_i$.

Γ transverse field. It measures quantum fluctuations.

In the limit $\Gamma \to 0$ the classical limit should be recovered.

Sum over all p-uplets on a complete graph (extensions to random graphs)

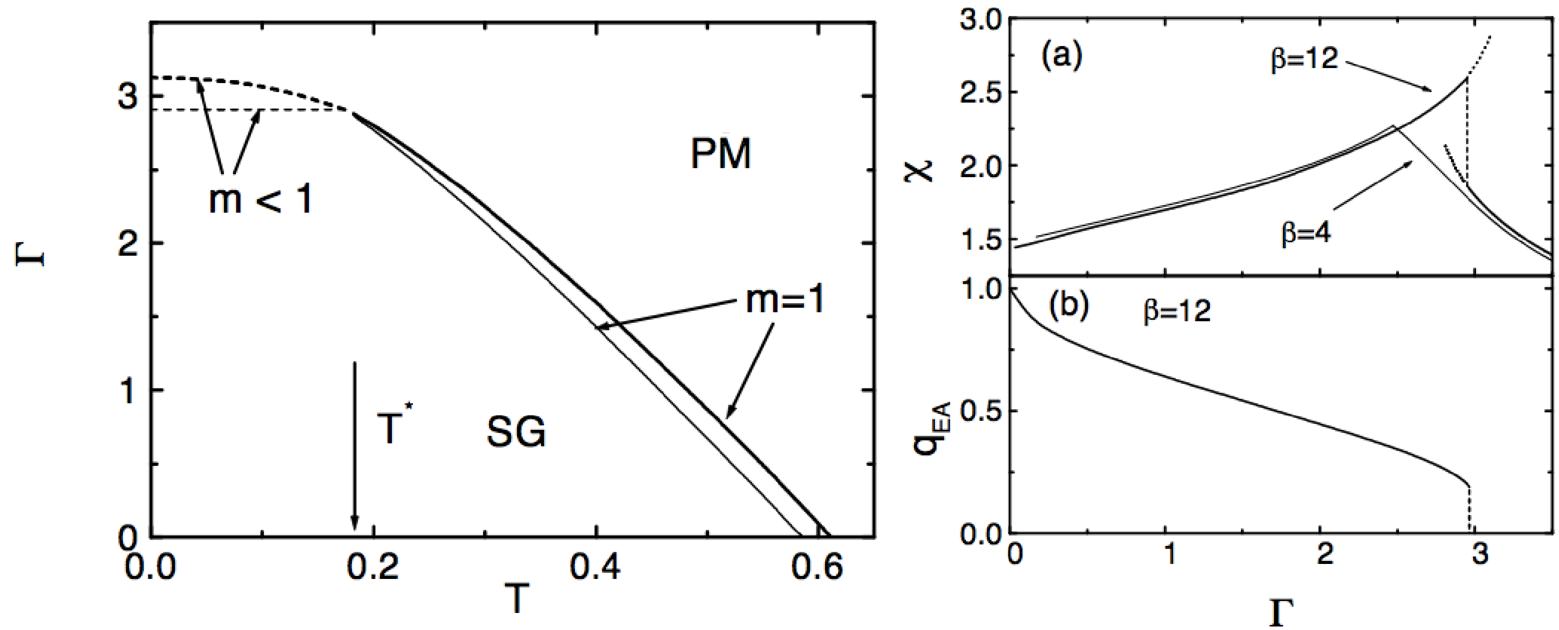
$$P(J_{i_1 i_2 \dots i_p}) = e^{-p!\, J^2_{i_1 i_2 \dots i_p}/(2N^{p-1}J^2)}$$

$p \geq 2$ Ising: quantum Sherrington-Kirkpatrick and p-spin models.

$p \geq 2$ continuous variables: quantisation achieved by adding a kinetic energy.

Phase transitions

Quantum fully-connected $p \geq 3$ spin model



Jump in the susceptibility across the dashed line: 1st order phase trans.

LFC, Grempel & da Silva Santos 00

In dilute disordered $p \geq 3$ models, review:

Bapst, Foini, Krzakala, Semerjian & Zamponi 12

Real-time dynamics

The system is coupled to a bath.

Simple model for the bath: independent harmonic oscillators.

Schwinger-Keldysh closed-time path-integral for the quantum case, similar formalism in the classical limit.

Gaussian integration over the bath oscillator variables $\Rightarrow$

Two-time long-range interactions.

Typical initial conditions $\hat{\rho}_{\rm bath} \otimes \hat{\rho}_{\rm syst}$

$\hat{\rho}_{\rm bath}$ bath in Boltzmann equilibrium

$\hat{\rho}_{\rm syst}$ system in a 'random' state:

no need of replica trick to average over disorder.

Real-time dynamics

The system equilibrates (à la Gibbs-Boltzmann) with the environment in the disordered (PM) phase.

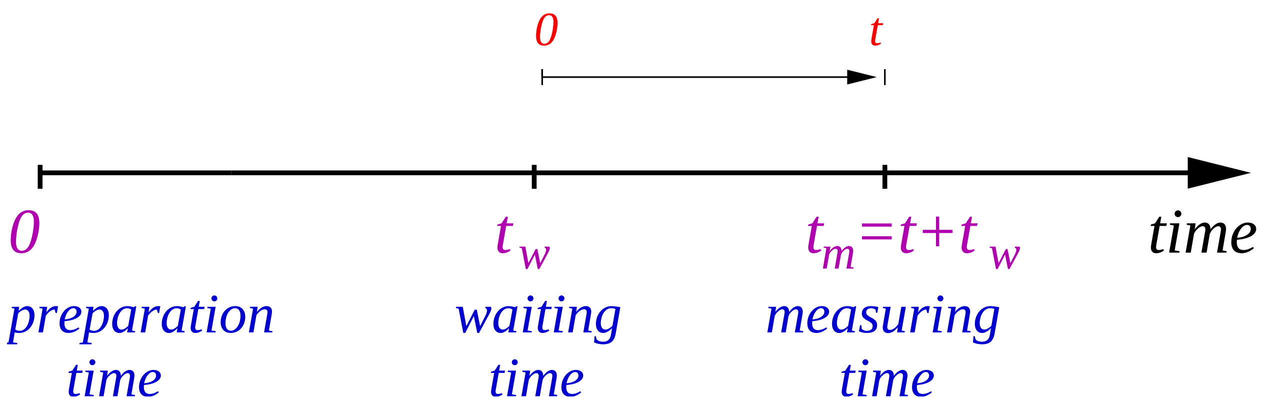
It does not equilibrate in the SG phase if times are not scaled with the system size and the thermodynamic limit is taken first, meaning

$$\boxed{\lim_{t\to\infty} \lim_{N\to\infty}}$$

(It should equilibrate with a convenient scaling of times $t(N)$)

Real-time dynamics

Two-time observables



Correlation

$$C(t+t_w, t_w) \equiv \langle [\hat{O}(t+t_w), \hat{O}(t_w)]_+ \rangle$$

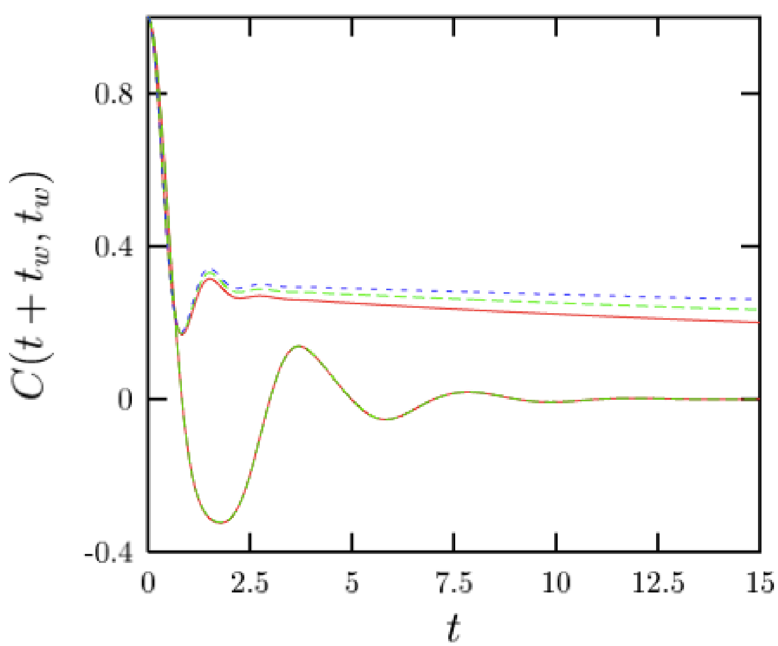
Linear response

$$R(t+t_w, t_w) \equiv \left.\frac{\delta \langle \hat{O}(t+t_w) \rangle}{\delta h(t_w)}\right|_{h=0} = \langle [\hat{O}(t+t_w), \hat{O}(t_w)]_- \rangle$$

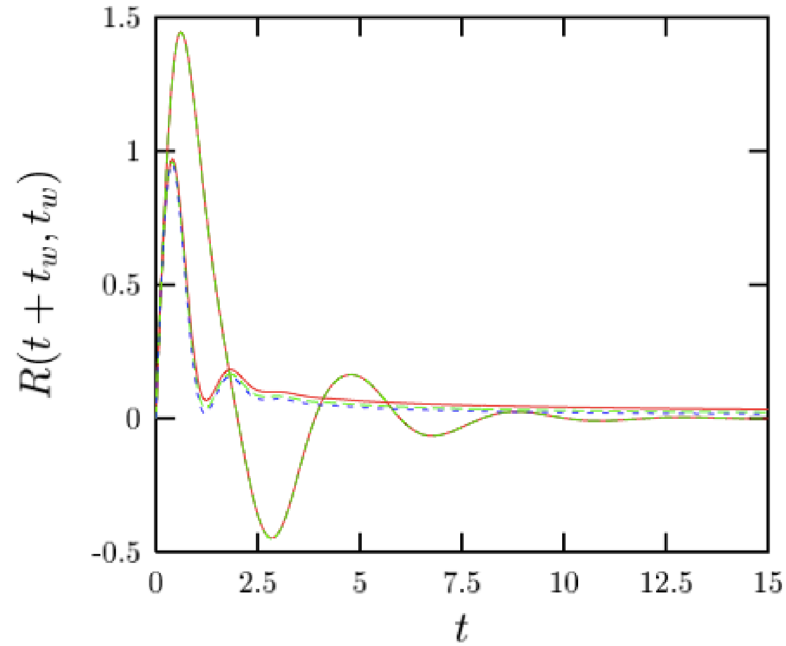
Real-time dynamics

In the ordered and disordered phases

Symmetric correlation



Linear response



Comparison between (PM) and (SG): stationary vs. aging

figs. from **LFC, Grempel, Lozano, Lozza & da Silva Santos 02**

FDT & FDR

In and out of equilibrium

The fluctuation-dissipation theorem is a model-independent equilibrium relation between the linear response and correlations of the corresponding spontaneous fluctuations in equilibrium.

The FDT applies to any pair of observables.

The FDT involves the temperature but no other characteristic of the system.

Whenever the FDT does not apply, the system is out of equilibrium.

One can still look at the relation between linear response and correlations out of equilibrium and see what happens : construct a fluctuation-dissipation relation (FDR).

FDT

Gibbs-Boltzmann density operator $\hat{\rho} = Z^{-1} e^{-\beta \hat{H}}$

One proves the KMS relations: $\tilde{C}_{BA}(-\omega) = e^{\beta\omega} \tilde{C}_{AB}(\omega)$

and then

$$\boxed{\mathrm{Im}\tilde{R}^{AB}(\omega) = [\hbar^{-1} \tanh(\beta\hbar\omega/2)]^{\pm 1}\, \tilde{C}_{\pm}^{AB}(\omega)}$$

Bosons

Fermions

In the classical limit: $\mathrm{Im}\tilde{R}^{AB}(\omega) = \beta\omega\, \tilde{C}^{AB}(\omega)$

FDR

Any evolution

Just measure

$$\boxed{\text{Im}\tilde{R}^{AB}(\omega)} \qquad \boxed{\tilde{C}_{\pm}^{AB}(\omega)}$$

take the ratio and extract $\tanh(\beta_{\text{eff}}^{AB}(\omega)\hbar\omega/2)$

$$\boxed{\text{In equilibrium all } \beta_{\text{eff}}^{AB}(\omega) \text{ are equal to the same constant } \beta}$$

Ideas exploited in the glassy context taking care of separation of time-scales

FDR

Any evolution

Just measure

$\boxed{\mathrm{Im}\tilde{R}^{AB}(\omega)}$ $\boxed{\tilde{C}_{\pm}^{AB}(\omega)}$

take the ratio and extract $\tanh(\beta_{\rm eff}^{AB}(\omega)\hbar\omega/2)$

In equilibrium all $\beta_{\rm eff}^{AB}(\omega)$ are equal to the same constant β

e.g. quantum p-spin model coupled to an equilibrium bath (β) :

$$\tanh(\beta_{\rm eff}(\omega)\hbar\omega/2) \simeq \begin{cases} \tanh(\beta\hbar\omega/2) & \omega t_w \gg 1 \quad \text{FDT} \\ \beta^*\hbar\omega/2 & \omega t_w \ll 1 \end{cases}$$

Effective temperatures

What happens in glasses ?

Glasses are out of equilibrium.

There is a separation of time-scales in their relaxation,

with a crossover at, roughly, ωt_w

The FDRs take a very special form :

$\omega t_w \ll 1$ quasi-stationary relation and FDT OK.

$\omega t_w \gg 1$ non-stationary relation and a single constant $T_{\rm eff}$.

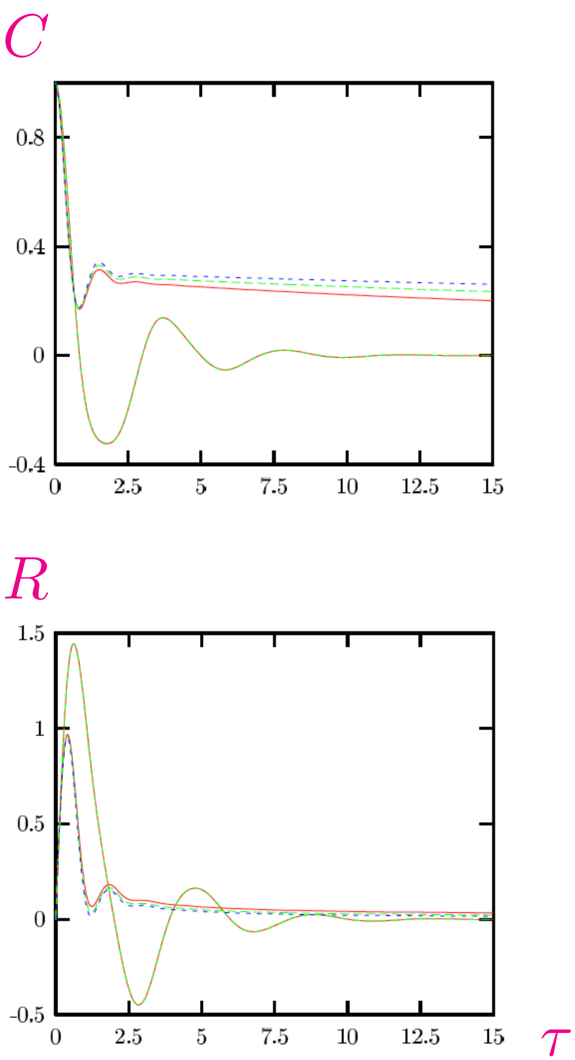
$T_{\rm eff}$ depends upon

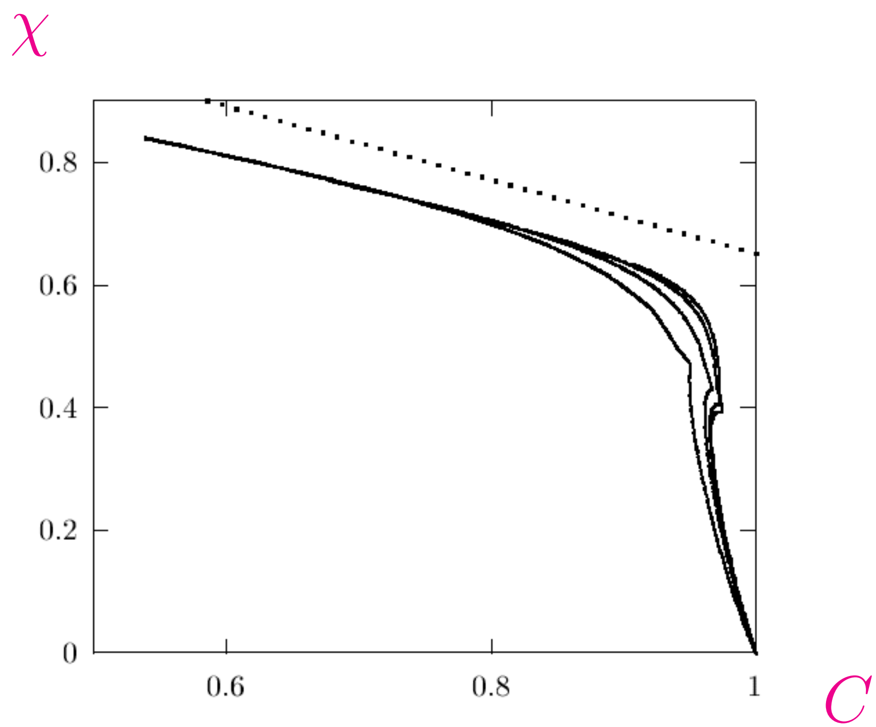
the initial condition before the quench (disordered vs. ordered) ;

weakly on other parameters of the systems.

Dissipative quantum glasses

Quantum p-spin coupled to a bath of harmonic oscillators





Out of equilibrium decoherence

LFC & Lozano 98

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Isolated quantum systems

Quantum quenches

- Take an isolated quantum system with Hamiltonian $\hat{H}_0$
- Initialize it in, say, $|\psi_0\rangle$ the ground-state of $\hat{H}_0$ (or any $\hat{\rho}(t_0)$)
- Unitary time-evolution with $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ with a Hamiltonian $\hat{H}$.

$\boxed{\text{Does the system reach some steady state?}}$

$\boxed{\text{Are at least some observables described by thermal ones?}}$

$\boxed{\text{When, how, which?}}$

Quantum quenches

Questions

Does the system reach a thermal equilibrium density matrix ?

Do their dynamics satisfy the equilibrium rules ?

different cases of interest : non-integrable *vs.* integrable systems ; role of initial states ; non critical *vs.* critical quenches, *etc.*

- Definition of T_e from $\langle \psi_0 | \hat{H} | \psi_0 \rangle = \langle \hat{H} \rangle_{T_e} = Z^{-1}_{\beta_e}$ Tr $\hat{H} e^{-\beta_e \hat{H}}$

Just one number, it can always be done

- Comparison of dynamic and thermal correlation functions, *e. g.*

$$C(r,t) \equiv \langle \psi_0 | \hat{\phi}(\vec{x},t) \hat{\phi}(\vec{y},t) | \psi_0 \rangle \text{ vs. } C(r) \equiv \langle \hat{\phi}(\vec{x}) \hat{\phi}(\vec{y}) \rangle_{T_e} .$$

Calabrese & Cardy ; Rigol et al ; Cazalilla & Iucci ; Silva et al, etc.

But the functional form of correlation functions can be misleading !

Quantum quenches

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Calabrese & Cardy ; Rigol et al ; Cazalilla & Iucci ; Silva et al, etc.

Proposal : put qFDT to the test to check whether $T_{\text{eff}} = T_e$ exists

FDRs

Quantum SU(2) Ising chain (integrable)

The initial Hamiltonian $$\boxed{\hat{H}_{\Gamma_0} = -\sum_i \hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + \Gamma_0 \sum_i \hat{\sigma}_i^z}$$

The initial state $|\psi_0\rangle$ ground state of $\hat{H}_{\Gamma_0}$

Instantaneous quench in the transverse field $\Gamma_0 \to \Gamma$

Evolution with $\hat{H}_{\Gamma}$.

Iglói & Rieger 00

Reviews : **Karevski 06 ; Polkovnikov et al. 10 ; Dziarmaga 10**

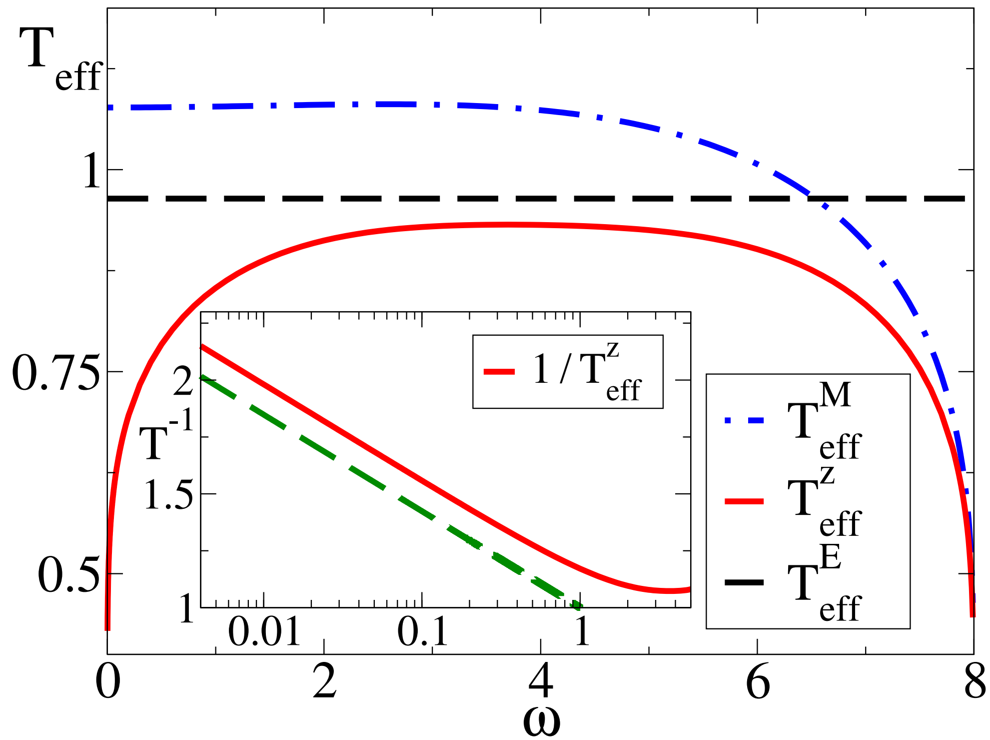
Specially interesting case $\Gamma = \Gamma_c$ the critical point. **Rossini et al. 09**

Claims of thermal equilibration due to gapless spectrum.

Quantum quench

e.g., $T_{\rm eff}$ from $\hat\sigma^z$ and $\hat M = L^{-1}\sum_i \hat\sigma_i^z$ (finite L chain)

$$\hbar\,\mathrm{Im}R^{z,M}(\omega) = \tanh\left(\frac{\beta_{\rm eff}^{z,M}(\omega)\omega\hbar}{2}\right) C_+^{z,M}(\omega)$$



$\beta_{\rm eff}^z(\omega) \neq \beta_{\rm eff}^M(\omega) \neq$ ct: no thermal equilibrium

Ising chain

Non-interacting integrable model

After Jordan-Wigner and Bogoliubov transformations:

$$\boxed{\hat{H}_\Gamma = -\sum_k \epsilon_k(\Gamma)\, \hat{\eta}_k^\dagger \hat{\eta}_k}$$

with fermionic creation and annihilation operators $\hat{\eta}_k^\dagger$ and $\hat{\eta}_k$,

and charges leading to conserved quantities

$$\boxed{\hat{n}_k = \hat{\eta}_k^\dagger \hat{\eta}_k}$$

independently of the initial state.

Ising chain

Non-interacting integrable model

The Gibbs-Boltzmann measure should be generalized to

The Generalized-Gibbs Ensemble (GGE)

$$\boxed{\hat{\rho}_{\mathrm{GGE}} = Z_{GGE}^{-1}\, e^{-\sum_k \beta_k^{GGE}\, \epsilon_k(\Gamma)\, \hat{n}_k}}$$

with effective inverse temperatures β_k^{GGE} fixed by

$$\boxed{\langle \psi_0 | \hat{n}_k | \psi_0 \rangle = \langle \hat{n}_k \rangle_{GGE}}$$

Applied to the quenched Ising chain this condition yields $\beta_k^{GGE}(\Gamma_0, \Gamma)$ for $k = \pm\pi(2n+1)/L$ and $n = 0, \dots, L/2 - 1$

Ising chain

Non-interacting integrable model

Take, for example, $\hat{M} = L^{-1} \sum_{i=1}^{L} \hat{\sigma}_i^z$

recall that $\hat{\sigma}_i^z = 1 - 2\hat{\eta}_i^\dagger \hat{\eta}_i$

In the FDR the frequency ω selects each mode k such that $\omega = 2\epsilon_k$ and

$$\boxed{\beta_{\rm eff}^{M}(\omega) = \beta_k^{GGE} \qquad\qquad \omega = 2\epsilon_k}$$

One can 'read' the GGE effective temperatures from the FDR

Same mechanism in other non-interacting integrable systems.

In interacting integrable cases ?

Summary

Fluctuation-dissipation relations

- Use of fluctuation-dissipation relations in the dynamics of closed quantum systems to **check for Gibbs-Boltzmann equilibrium**.

  Foini, Gambassi & LFC 11-12

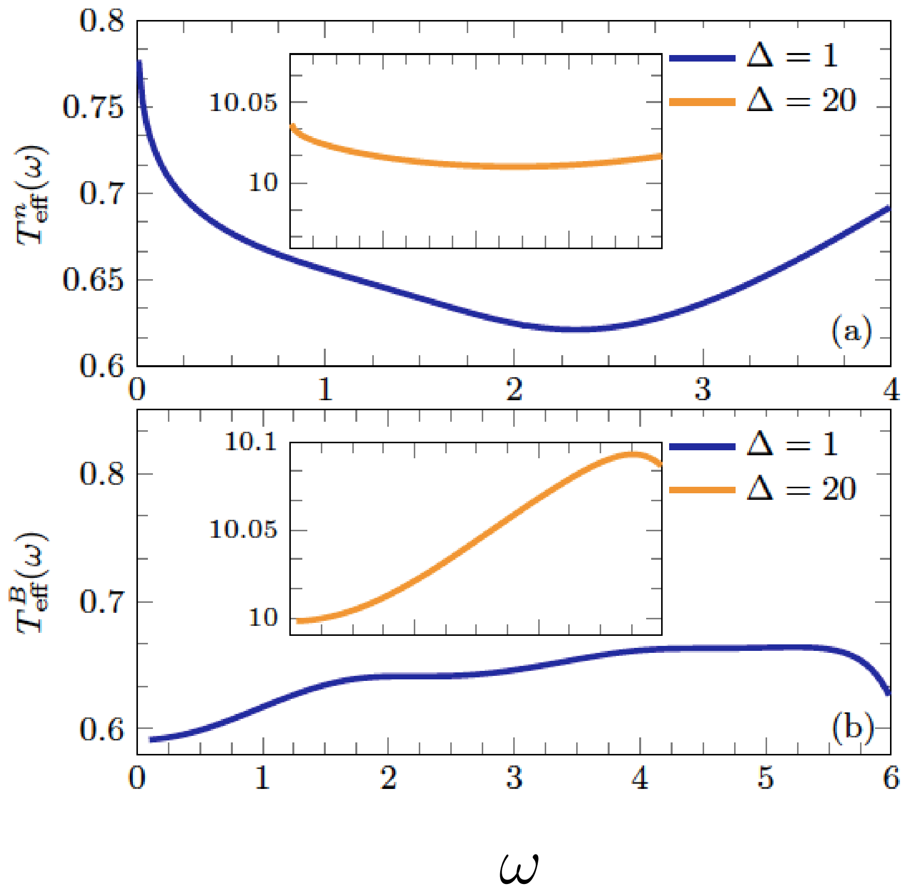
- Use of fluctuation-dissipation relations to **measure the GGE effective temperatures**

  LFC, Foini, Gambassi & Konik soon

- Also useful to **distinguish** (or not) **glassiness from MBL** ?

Another example

$1d$ hard-core bosons in a super-lattice potential



(local) density operator

$$\hat{A} = \hat{B} = \hat{n}_i \hat{n}_i$$

(non-local) boson operator

$$\hat{A} = \hat{B} = \hat{b}_i^\dagger \hat{b}_i$$

Bortolin & Iucci 15

Similar ideas in models of photon/polariton condensates,

Chiocchetta, Gambassi, Carusotto 15

Asymptotic limit

of the dynamics isolated many-body systems

— Stationary measure reached ?

— In one or several time-regimes ?

— Which one(s) ?

— Thermal *à la* Gibbs-Boltzmann or other ?

All these questions can be posed, and are difficult to answer, in both classical and quantum systems.

In the following : equilibrium $\equiv$ Gibbs-Boltzmann equilibrium.

Dynamics in equilibrium

Conditions on quantum systems

Equilibrium is a matter of statics,

instantaneous probability density $\hat{\rho}(t_0)$

but also of dynamics,

evolution operators $\hat{U}(t_0 \to t)$

$\hat{\rho} \mapsto e^{-\beta \hat{H}}/Z$ and $\hat{U} \mapsto e^{-i\hat{H}(t-t_0)/\hbar}$ to ensure that the system reaches Gibbs-Boltzmann equilibrium at a given time t_0.

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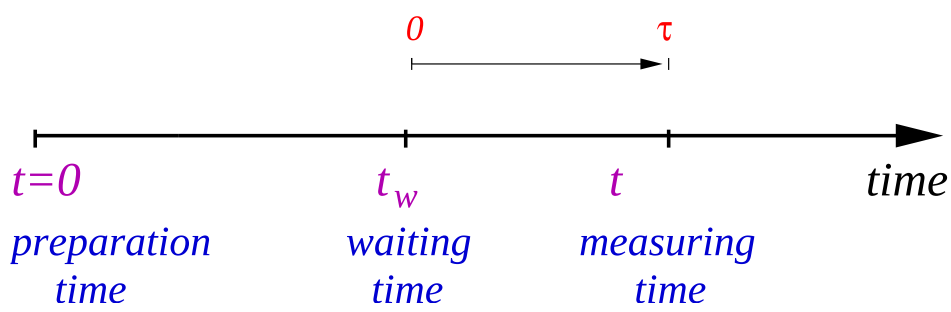
Two-time observables

Correlations



The two-time correlation between two observables $\hat{A}(t)$ and $\hat{B}(t_w)$ is

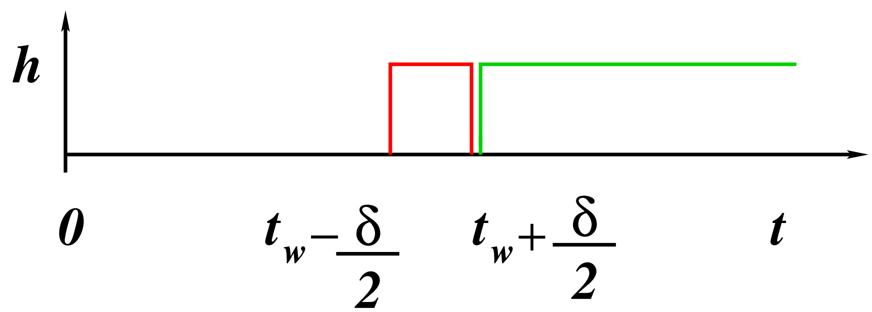
$$C_{AB}(t, t_w) \equiv \langle\, \hat{A}(t)\hat{B}(t_w)\,\rangle$$

expectation value in a quantum system, $\langle \ldots \rangle = \mathrm{Tr} \ldots \hat{\rho}/\mathrm{Tr}\hat{\rho}$

or the average over realizations of the dynamics (initial conditions, random numbers in a MC simulation, thermal noise, etc.) in a classical system.

Two-time observables

Linear response



The perturbation couples linearly to the observable $\hat{B}$ at time t_w

$$\hat{H} \;\to\; \hat{H} - h(t_w)\hat{B}$$

The linear instantaneous response of another observable $\hat{A}(t)$ is

$$R_{AB}(t,t_w) \equiv \left.\frac{\delta\langle \hat{A}(t)\rangle_h}{\delta h(t_w)}\right|_{h=0}$$

Similarly in a classical system

Linear response

In an asymptotic steady case

The dynamics are stationary

$$C_{AB} \to C_{AB}(t - t_w) \text{ and } R_{AB} \to R_{AB}(t - t_w)$$

Fourier transforms

$$\tilde{C}_{AB}(\omega) \text{ and } \tilde{R}_{AB}(\omega)$$

Kubo formula, just linear response, to obtain

$$\boxed{-\pi^{-1}\mathrm{Im}\tilde{R}_{AB}(\omega) = \tilde{C}_{AB}(\omega) \mp \tilde{C}_{BA}(-\omega)}$$

Bosons

Fermions

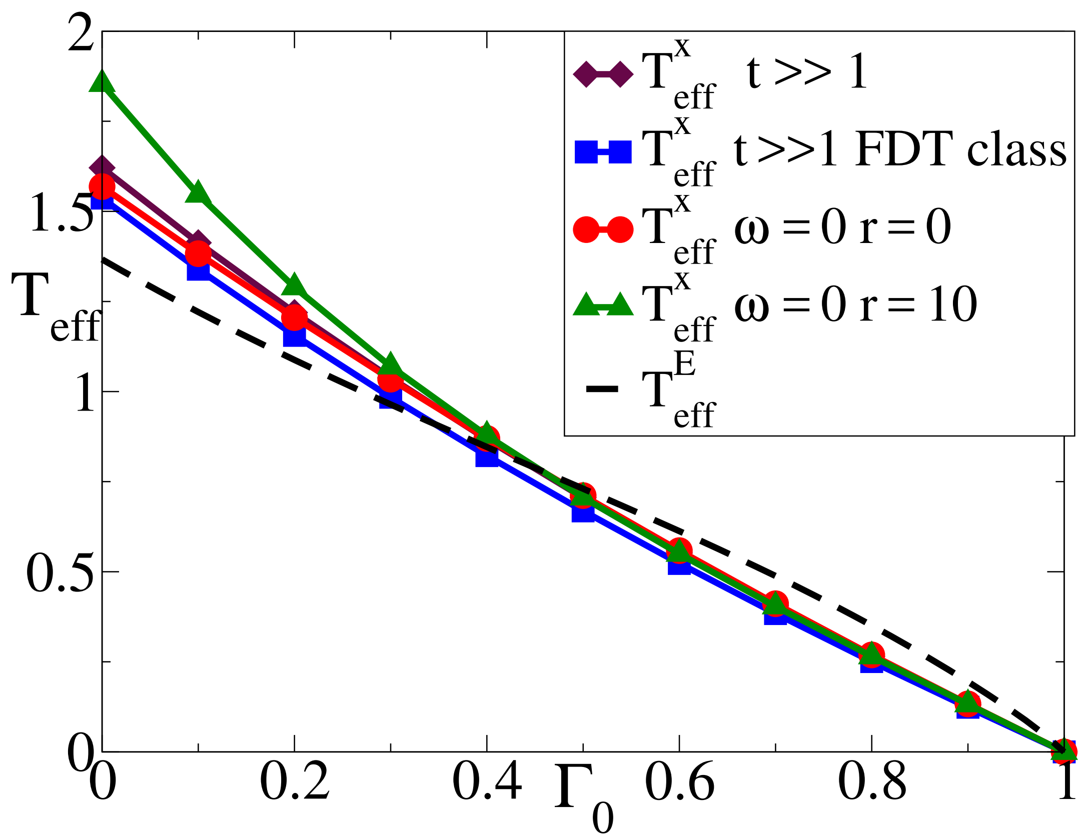
No need to use $\hat{\rho} = Z^{-1}e^{-\beta\hat{H}}$ to prove this relation.

Usual notation : $-\pi^{-1}\mathrm{Im}\chi_{AB}(\omega) = S_{AB}(\omega) \mp S_{BA}(-\omega) = [\hat{A}, \hat{B}]_{\mp}$

Quantum quench

No $T_{\rm eff}$ from FDT

A quantum quench $\Gamma_0 \to \Gamma_c = 1$ of the **isolated Ising chain**



Foini, LFC & Gambassi 11

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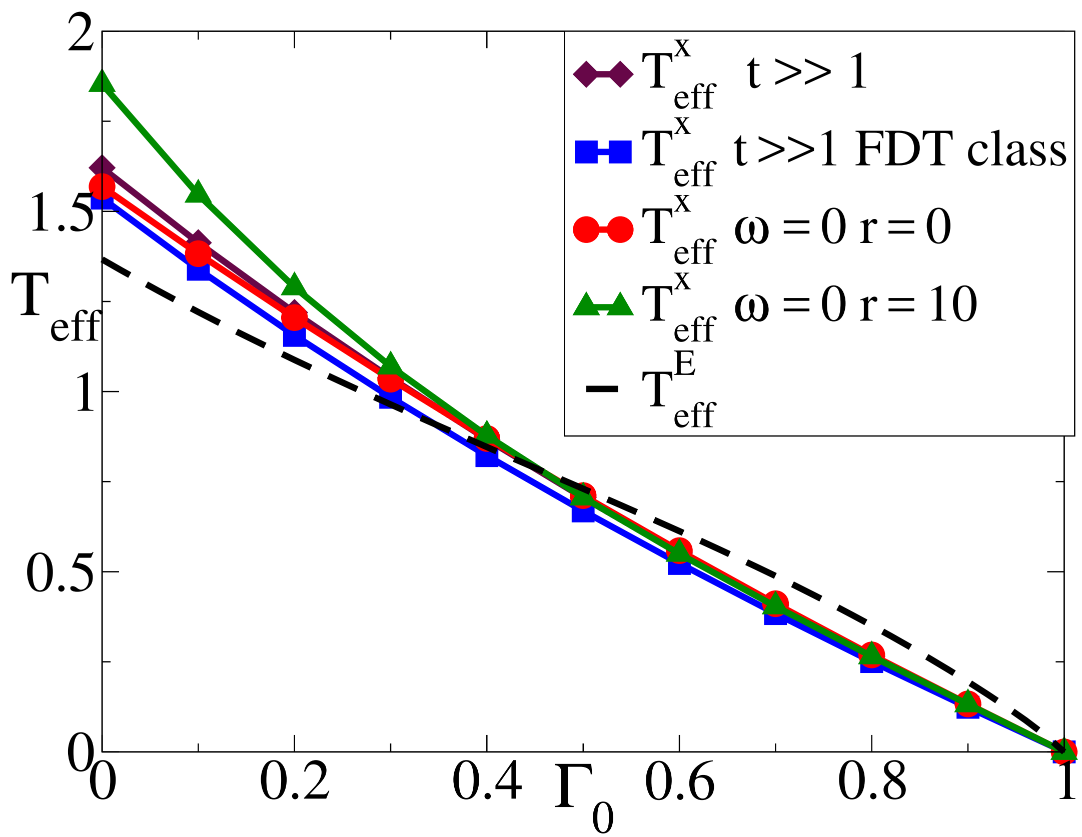


Foini, LFC & Gambassi 11

Another example

$1d$ hard-core bosons in a super-lattice potential

Fermionic representation :

$$\boxed{\hat{H}_0(\Delta) = -\sum_i \hat{f}_i^\dagger \hat{f}_{i+1} + \text{h.c.} + \Delta \sum_i (-1)^i \, f_i^\dagger f_i}$$

Quench from the ground state of $\hat{H}_0(\Delta)$ to $\hat{H} = \hat{H}_0(\Delta = 0)$.

Although $\hat{\rho} \mapsto \hat{\rho}_{\rm GGE} \approx \hat{\rho}_{\rm GB}$ for $\Delta \gg |\omega_k| = \mathcal{O}(1)$

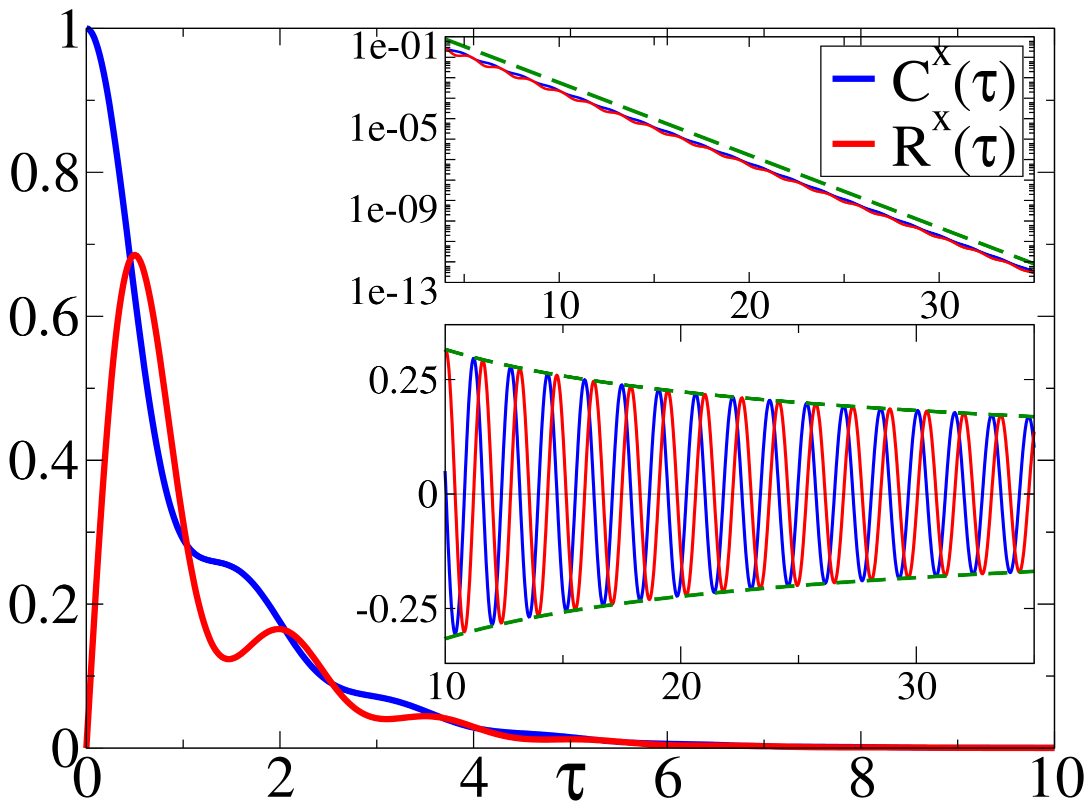
Chung, Iucci & Cazalilla 12

the FDT is not satisfied in this same limit, and different FDRs yield different $T_{\rm eff}$s.

Bortolin & Iucci 15

Quantum quench

$T_{\rm eff}$ from the longitudinal spin FDR



Insets

$e^{-\tau/\tau_{\rm c}}$

$\tau^{-2}\sin(4\tau+\phi)$

$$C^x(\tau) \simeq A_{\rm c} e^{-\tau/\tau_{\rm c}}[1 - a_{\rm c}\tau^{-2}\sin(4\tau+\phi_{\rm c})]$$

$$R^x(\tau) \simeq A_R e^{-\tau/\tau_{\rm c}}[1 - a_R\tau^{-2}\sin(4\tau+\phi_R)]$$

Foini, LFC & Gambassi 11

Quantum quench

T_{eff} from FDT ?

For sufficiently long-times such that one drops the power-law correction

$$\boxed{-\beta^x_{\text{eff}} \simeq \frac{R^x(\tau)}{d_\tau C^x_+(\tau)} \simeq -\frac{\tau_{\text{c}} A_R}{A_{\text{c}}}}$$

A constant consistent with a classical limit but

$$\boxed{T^x_{\text{eff}}(\Gamma_0) \neq T_e(\Gamma_0)}$$

Morever, a complete study in the full time and frequency domains confirms that $T^x_{\text{eff}}(\Gamma_0, \omega) \neq T^z_{\text{eff}}(\Gamma_0, \omega) \neq T_e(\Gamma_0)$ (though the values are close).

Fluctuation-dissipation relations as a probe to test thermal equilibration

No equilibration for generic Γ_0 in the quantum Ising chain