Master ICFP 2017-2018

Lectures Notes on String Theory

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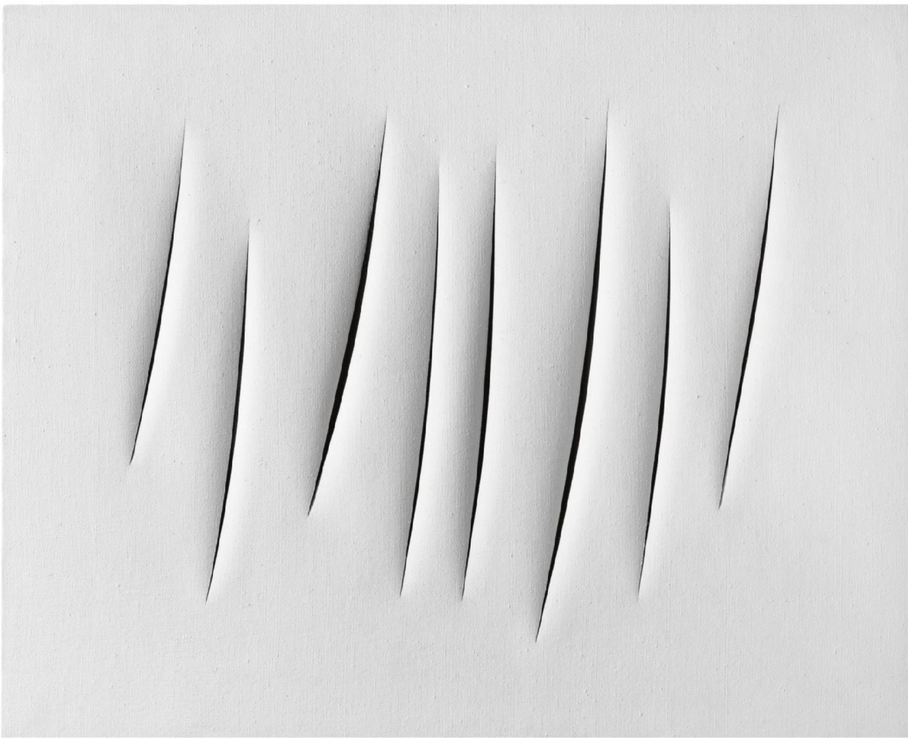

Lucio Fontana, « Concetto Spaziale »

Foreword

These lecture notes are based upon a series of courses given at the master program ICFP in 2018 by the author. Comments and suggestions are welcome. Some references that can complement these notes are

- *Superstring Theory* (Green, Schwarz, Witten) [1,2]: the classic textbook from the eighties, naturally outdated on certain aspects but still an unvaluable reference on many topics including the Green-Schwarz string and compactifications on special holonomy manifolds.
- *String Theory* (Polchinski) [3,4]: the standard textbook, with a very detailed derivation of the Polyakov path integral and strong emphasis on conformal field theory methods.
- *String Theory in a Nutshell* (Kiritsis) [5]: a concise presentation of string and superstring theory which moves quickly to rather advanced topics
- *String Theory and M-Theory: A Modern Introduction* (Becker, Becker, Schwarz) [6]: a good complement to the previous references, with a broad introduction to modern topics as AdS/CFT and flux compactifications.
- *A first course in String theory* (Zwiebach) [7]: an interesting and different approach, making little use of conformal field theory methods, in favor of a less formal approach.
- The lectures notes of David Tong (http://www.damtp.cam.ac.uk/user/tong/string.html) are rather enjoyable to read, with a good balance between mathematical rigor and physical intuition.
- The very lively online lectures of Shiraz Minwalla: http://theory.tifr.res.in/ minwalla/

Conventions

- The space-time metric is chosen to be of signature $(-,+,\ldots,+)$.
- We work in units $\hbar = \mathsf{c} = 1$

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Contents

Chapter 1

Introduction

In Novembrer 1994, Joe Polchinski published on the ArXiv repository a preliminary version of his celebrated textbook on String theory, based on lectures given at Les Houches, under the title *"What is string theory?"* [1]. If he were asked the same question today, the answer would probably be rather different as the field has evolved since in various directions, some of them completely unexpected at the time.

One may try to figure out what string theory is about by looking at the program of *Strings 2017*, the last of a series of annual international conferences about string theory that have taken place at least since 1989, all over the world. Among the talks less than half were about string theory proper (*i.e.* the theory you will read about in these notes) while the others pertained to a wide range of topics, such as field theory amplitudes, dualities in field theory, theoretical condensed matter or general relativity.

The actual answer to the question raised by Joe Polchinski, *"What is string theory?"*, may be answered at different levels:

- *litteral:* the quantum theory of one-dimensional relativistic objects that interact by joining and splitting.
- *historical:* before 1974, a candidate theory of strong interactions; after that date, a quantum theory of gravity.
- *practical:* a non-perturbative quantum unified theory of fundamental interactions whose degrees of freedom, in certain perturbative regime, are given by relativistic strings.
- *sociological:* a subset of theoretical physics topics studied by people that define themselves as doing research in string theory.

In these notes, we will provide the construction of a consistent first quantized theory of interacting quantum strings. We will show that such theory automatically includes a (perturbative) theory of quantum gravity, and can easily incorporate as well gauge interactions hence the building blocks of the Standard model of Particle physics.

Near the end we will venture a little bit into more advanced topics such as D-branes and gauge theories, strong coupling dynamics and non-perturbative dualities, and few words about AdS/CFT. We will conclude with a brief overview of current research in the area.

Along the way we will introduce some concepts and techniques that are as useful in other areas of theoretical physics as they are in string theory, for instance conformal field theories, BRST quantization of gauge theories or supersymmetry.

1.1 Gravity and quantum field theory

String theory has been investigated by a significant part of the high-energy theory community for more than forty years as it provides a compelling answer – and maybe *the* answer – to the following outstanding question: what is the quantum theory of gravity?

A successful theory of quantum gravity from the theoretical physics viewpoint should at least satisfy the following properties:

1. the theory should reproduce classical gravity, *i.e.* general relativity, in an appropriate semi-classical , weak-coupling regime;

2. the theory should be *predicative* at energies accessible to experiments, hence either UV-finite or renormalizable;

3. it should satisfy the basic requirements for any quantum theory, such as unitarity;

4. it should explain the origin of black hole entropy, possibly the only current prediction for quantum gravity;

5. last but not least, the physical predictions should be compatible with experiments, in particular with the Standard Model of particle physics, astrophysical and cosmological observations.

According to our current understanding, string theory passes successfully the first four tests. Whether string theory reproduces accurately at energies accessible to experiments the known physics of fundamental particles and interaction is still unclear, given that such physics occurs in a regime of the theory that is beyond our current analytical control (to draw an analogy, one cannot reproduce analytically the physics of condensed matter systems from the microscopic quantum mechanical description in terms of atoms). At least it is clear that the main ingredients are there: chiral fermions, non-Abelian gauge interactions and Higgs-like bosons.

The problem with quantizing general relativity

The classical theory of relativistic gravity in four space-time dimensions, or Einstein theory, follows in the absence of matter from the Einstein-Hilbert (EH) action in four space-time dimensions, that takes the form

$$\mathcal{S}_{\text{EH}} = \frac{1}{2\kappa^2} \int_M d^4x \sqrt{-g} \left(\mathcal{R}(g) - 2\Lambda \right) , \tag{1.1}$$

where $\mathcal{R}$ is the Ricci scalar associated with the space-time manifold M, endowed with a metric g, and Λ the cosmological constant that has no *a priori* reason to vanish. The coupling constant κ of the theory is related to the Newton's constant through $\kappa = \sqrt{8\pi G}$; by dimensional analysis it has dimension of length. Its inverse defines the Planck mass $M_{\text{PL}} \sim 10^{19}$ GeV.

Quantizing general relativity raises a number of deep conceptual issues, that can be raised even before attempting to make any explicit computation. Some of them are:

- Because of diffeormorphism invariance, there are no *local observables* in general relativity.
- A path-integral formulation of quantum gravity should include, by definition, a sum over space-time geometries. Which geometries should be considered? Should we specify boundary conditions?
- A Hamiltonian formulation of quantum gravity would require a foliation of space-time in terms of space-like hypersurfaces. Generically, such foliation does not exist.
- Classical dynamics of general relativity predicts the formation of event horizons, shielding regions of space-time from the exterior. This challenges the unitarity of the theory, through the *black hole information paradox.*

Quantum gravity with a positive cosmological constant – which seems to be relevant to describe the Universe – raises a number of additional conceptual issues that will be ignored in the rest of the lectures. We will mainly focus on theories with a vanishing cosmological constant, and discuss briefly the case of negative cosmological constant that played an important role in recent developments.

Perturbative QFT for gravity

One may try to ignore these conceptual problems and build a quantum field theory of gravity in the usual way, *i.e.* by defining propagators, vertices, Feynman rules, etc..., from the non-linear EH action, eqn. (1.1). With vanishing cosmological constant, one considers fluctuations of the metric around a reference Minkowski space-time metric:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,. \tag{1.2}$$

Linearizing the equations of motion that follows from (1.1), in the absence of sources, we arrive to:

$$\Box\bar{h}_{\mu\nu} - 2\partial^\rho\partial_{(\mu}\bar{h}_{\nu)\rho} + \eta^{\mu\nu}\partial^\rho\partial^\sigma\bar{h}_{\mu\nu} = 0\,, \tag{1.3}$$

where we have defined the traceless symmetric tensor $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}h^\rho{}_\rho\,\eta_{\mu\nu}$. This theory possesses a gauge invariance that comes from the diffeomorphism invariance of the full theory. The equations of motion are invariant under

$$h_{\mu\nu} \;\mapsto h_{\mu\nu} + \partial_\mu\zeta_\nu + \partial_\nu\zeta_\mu\,. \tag{1.4}$$

One can choose to work in a Lorentz gauge, defined by $\partial^\mu\bar{h}_{\mu\nu} = 0$, in which case the field equations (1.3) amounts to a wave equation for each component, $\Box\bar{h}_{\mu\nu} = 0$.

The solutions of these equations are naturally plane waves $\bar{h}_{\mu\nu}(x^\rho) = h^0_{\mu\nu}\exp(ik_\rho x^\rho)$, and the Lorentz gauge condition means that they are transverse. Finally, the residual gauge invariance that remains in the Lorentz gauge, corresponding to vector fields ζ_μ satisfying the wave equation $\Box\zeta_\mu = 0$, can be fixed by choosing the longitudinal gauge $\bar{h}_{0\mu} = 0$. As a result, the gravitational waves have two independent transverse polarizations. The corresponding quantum theory is a theory of free *gravitons* that are massless bosons of helicity two.

The interactions between gravitons are added by expanding the EH action around the background (1.2) in powers of $h_{\mu\nu}$. In pure gravity one obtains three-graviton and four-graviton vertices, that have a rather complicated form. For instance the four-graviton vertex looks roughly like:

$$G^{\mu_1\nu_1,\ldots,\mu_4\nu_4}(k_1,\ldots,k_4) = \kappa^2\Big(k_1\cdot k_2\,\eta^{\mu_1\nu_1}\cdots\eta^{\mu_4\nu_4} + k_1^{\mu_3}k_2^{\nu_3}\eta^{\mu_1\mu_2}\eta^{\nu_1\nu_2} + \cdots\Big) \tag{1.5}$$

Using these vertices one can define Feynman rules for the quantum field theory of gravitons and compute loop diagrams like the one below.

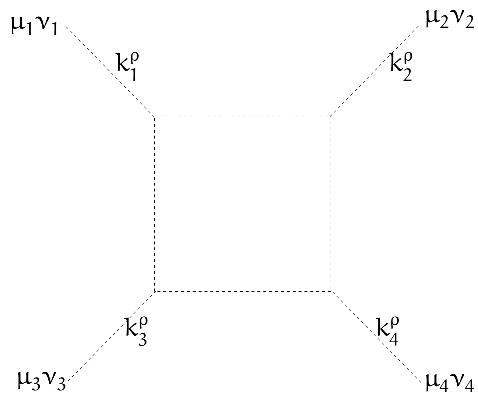

Figure 1.1: *Graviton scattering*

As in most quantum field theories, such loops integrals diverge when the internal momenta propagating in the loop become large, and should be regularized. By dimensional analysis, the regularized loop diagrams will be weighted by positive powers of $(\Lambda_{\text{uv}}/M_{\text{PL}})$, where Λ_{uv} is the ultraviolet cutoff.

In *renormalizable* QFTs as quantum chromodynamics, such high-energy – or ultraviolet – divergences can be absorbed into redefinitions of the couplings and fields of the theory, which leads to theories with predictive power. In contrast, this cannot be done for general relativity, for the simple reason that the coupling constant is dimensionfull (it has the dimension of length). Therefore, the divergences cannot be absorbed by redefining fields and couplings in the original two-derivative action; rather higher derivative terms should be included to do so. General relativity is thus a prominent example of *non-renormalizable* quantum field theory. Still it doesn't mean that such a theory is meaningless in the Wilsonian sense; it can describe the low-energy dynamics, well below the Planck scale M_{PL}, of an "ultraviolet" theory of quantum gravity that is not explicitly known. However, as in any non-renormalizable theory, this effective action has no predictive power, as higher loop divergences need to be absorbed in extra couplings that were not present in the original action,[1] but become less important as the energy becomes lower. As we shall see string theory solves the problem in a rather remarkable way, by removing *all* the ultraviolet divergences of the theory.

1.2 String theory: historical perspective

This is a very sketchy account about the history of string theory; interested reader may consult the book [2] for first-hand testimonies.

String theory as a theory of quantum gravity came almost by accident, after being proposed as a theory of strong interactions. The prehistory of string theory occurred during the sixties. At that time, general relativity was not, with few exceptions, a topic of interest for theoretical physicists, but was rather a playground for mathematicians.

Quantum field theory itself didn't have the central role that it has today in our understanding of fundamental interactions. While quantum electrodynamics was acknowledged as the appropriate description of electromagnetic interactions, most physicists thought that it was an inappropriate tool to solve the big problems of the time, the physics of strong and weak interactions. This was especially true for the strong interactions, as the experiments were finding a growing number of hadronic particles, with large masses and spins. These particles were mostly resonances *i.e.* particles with a finite lifetime. Defining a QFT including all of these resonances did not seem, rightfully, a sensible idea. Quarks did make their appearance in the theoretical physicists' lexicon, however they were thought as mathematical tools rather than as actual elementary particles – the fact that they cannot be observed individually supported this point of view.

A different approach, called the *S-matrix program*, was widely popular back then. The idea was to construct directly the S-matrix of the theory using some general physical principles (unitarity, analyticity,...), as well as some experimental input from the specific theory that was considered, without any reference to a "microscopic" Lagrangian.

One crucial experimental observation was that hadronic resonances could be classified in families along curves in the mass-angular momentum plane (M, J) called *Regge trajectories*:

$$J = \alpha(0) + \alpha' M^2 \tag{1.6}$$

The value of the parameter $\alpha(0)$, or *intercept*, was determining a given family of resonances, while the slope α' was universal – with one exception – and given experimentally by

$$\alpha' \simeq 1\text{GeV}^{-2} \tag{1.7}$$

in natural units.

Among important requirements imposed upon the S-matrix, was that all the hadrons along the Regge trajectories should appear on the same footing, and both as intermediate particles (resonances) in the s-channel or as virtual exchanged particles in the t-channel, see

[1] Strictly speaking, the one-loop divergence of pure GR can be absorbed by field redefinition. This not the case when matter is present, and from two-loops onwards for pure gravity.

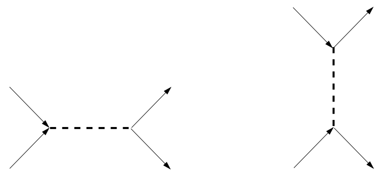

Figure 1.2: *Channel duality: s-channel (left) and t-channel (right)*

fig. 1.2; actually either point of view was expected to give a complete description of the scattering process. This *channel duality* property of the S-matrix, together with the other physical constraints, led Gabriele Veneziano to write down, in 1968, an essentially unique solution to the problem for the decay $\omega \to \pi^+ + \pi^0 + \pi^-$ [3]:

$$T = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))} + \Big((s,t) \to (s,u)\Big) + \Big((s,t) \to (t,u)\Big) \,, \tag{1.8}$$

where $\alpha(s) = \alpha_0 + \alpha' s$ describes a Regge trajectory. This amplitude has remarkable properties; it exhibits an infinite number of poles in the s- and t-channels, and its ultraviolet behavior is softer than of any quantum field theory.

This breakthrough was the starting point for lot of activity in the theoretical physics community, and remarkably lot of progress was done without having any microscopic Lagrangian to underlie this physics. For instance the generalization to N-particle S-matrices was obtained, the addition of $SU(N)$ quantum numbers, the analysis of the unitarity of the theory (by looking at the signs of the residues) and even loop amplitudes.

Soon however people discovered strange properties of what was known at the time as the *dual resonance model*. In order to avoid negative norm states, the intercept of the Regge trajectory had to be tuned in such a way that unexpected massless particles of spin 1,2,... appeared in the theory. Embarrassingly, it was also needed that the dimension of space-time was 26! Around the same time it was realized, finally, that the states of the theory were describing the quantized fluctuations of relativistic strings by Nambu, Nielsen and Susskind in 1970.

Another problem was the appearance of a tachyon, *i.e.* an imaginary mass particle, in the spectrum. This was solved soon after, following the work of Neveu, Schwarz [4] and Ramond [5], who introduced fermionic degrees of freedom on the string in 197 (bringing the space-time dimension to 10) by Gliozzi, Scherk[2] and Olive, who obtained the first consistent *superstring* theories in 1976 [6].

At the same time that these remarkable achievements were obtained, the non-Abelian quantum field theory of the strong interactions, or quantum chromodynamics, was recognized as the valid description of the hadronic world and, together with the electroweak theory, gave to quantum field theory the central role in theoretical high-energy physics that it has today.

[2]Joël Scherk (1946-1980) was a remarkable French theoretical physicist who made many key contributions to string theory and supergravity in the seventies, and died tragically when he was 33 years old only, leaving an indelible imprint in the field. The library at the LPTENS is dedicated to his memory.

It could had been the end of string theory, however, by a remarkable change of perspective, Scherk and Schwarz proposed in 1974 that string theory, instead of a theory of strong interactions, was providing a theory of quantum gravity [7]. From this point of view the annoying massless spin two particule of the dual resonance model was corresponding to the graviton, and they show that it has indeed the correct interactions.

The six extra dimensions of the superstring could be considered in this context as *compact* dimensions, given that the geometry was now dynamical, resurrecting the old idea of Kaluza [8], and Klein [9] from the twenties. The value of the Regge slope should be radically different from was it was considered before in the hadronic context, in order to account for the observed magnitude of four-dimensional gravity. One was considering

$$\alpha' \simeq 10^{-38}\text{Gev}^{-2}\,, \tag{1.9}$$

or equivalently strings of a size smaller by 19 orders of magnitude than the hadronic string, *i.e.* impossible to resolve directy by current or foreseeable experiments. Despite that string theory was able to fullfill an old dream – quantizing general relativity – research in string theory remained rather confidential before the next turning point of its history,

Between 1984 and 1986, several important discoveries occurred and changed the fate of the theory: the invention of heterotic string [10] (which made easy to incorporate non-Abelian gauge interactions in string theory), the *Green-Schwarz* anomaly cancellation mechanism [11] which strengthened the link between string theory and low-energy supergravity, thereby making the former more convincing, and finally the discovery of Calabi-Yau compactifications [12] and orbifold compactifications [13] which allowed to get at low energies models of particle physics with $\mathcal{N} = 1$ supersymmetry in four dimensions. After that string theory became more mainstream, as many theoretical physicists started to realize that it was a promising way of unifying all fundamental particles and interactions in a consistent quantum theory.

Thirty years and a second revolution after, we haven't yet achieved this goal fully but tremendous progress has been made, the hallmarks being the discoveries of D-branes [14], of strong/weak dualities [15–17], of holographic dualities [18] and of flux compactifications [19, 20] to name a few. We still have a long way to go, and it is certainly worth trying.

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Chapter 2

Bosonic strings: action and path integral

Bosonic string theory, which is the most basic form of string theory, describes the propagation of one-dimensional relativistic extended objects, the *fundamental strings*, and their interactions by joining and splitting.

Quantum field theories of point particles are obtained by starting with a classical action, and quantizing the fluctuations around a given classical solution of the equations of motion. Upon quantization one gets field operators acting on the Fock space of the theory by creating or annihilating particles at a given point in space. An analogous *string field theory* exists, but is still poorly understood. In such theory one should have operators creating a *loop* in space, which is certainly more difficult to describe mathematically.

Rather the practical way to handle string theory is to follow the propagation in time of a single string in a fixed reference space-time. As restrictive as it looks like, this *first-quantized* formalism does not prevent for studying the interactions between strings, computing loop amplitudes and make a large number of predictions. As we will see below this "first-order" formalism can be used for point particles as well, as an alternative to QFT Feynman diagrams that allows to perform perturbative computations; however it misses important aspects as solitons or instantons that can be handled semi-classically from a field theory, and is not suited for all types of computations.

2.1 Relativistic particle in the worldline formalism

We consider a relativistic particle of mass m and charge q in a given d-dimensional space-time $\mathcal{M}$ of metric G and background electromagnetic field. Its dynamics is governed by the action

$$\mathcal{S} = -m \int_{\mathfrak{l}} ds - q \int_{\mathfrak{l}} A \,, \tag{2.1}$$

where $\mathfrak{l}$ is the worldline of the particle, s the proper time and $A(x^\mu) = A(x^\mu)_\rho dx^\rho$ the gauge potential. Under a gauge transformation, $A \mapsto A + d\Lambda$, the worldline action (2.1) is invariant up to possible boundary terms.

The worldline of the particle in space-time $\mathcal{M}$ corresponds to an embedding map[1]

$$\mathbb{R} \hookrightarrow \mathcal{M} \tag{2.2}$$

$$\tau \mapsto x^\mu(\tau)\,, \tag{2.3}$$

where τ is an affine parameter and $\{x^\mu,\ \mu = 0, \ldots, d-1\}$ a set of coordinates on $\mathcal{M}$. The proper time differential can be expressed as $ds^2 = G_{\mu\nu}\dot{x}^\mu \dot{x}^\nu d\tau^2$, therefore the action (2.1) can be rewritten as

$$\mathcal{S} = -m \int d\tau \sqrt{-\dot{x}^\mu(\tau)\dot{x}^\nu(\tau) G_{\mu\nu}[x^\rho(\tau)]} - q \int_{\mathfrak{l}} A_\mu[x^\mu(\tau)] dx^\mu(\tau)\,, \tag{2.4}$$

This action is invariant under diffeomorphisms of the worldline, *i.e.* under any differentiable reparametrization

$$\tau \mapsto \tilde{\tau}(\tau) \tag{2.5}$$

[1]Strictly speaking, this is in general valid in an open set of $\mathcal{M}$ where the coordinates $\{x^\mu,\ \mu = 0, \ldots, d-1\}$ are well-defined.

The embedding is now given by definition by the set of differentiable functions $\{\tilde{x}^\mu(\tilde{\tau}) = x^\mu(\tau),\ \mu = 0, \dots, d-1\}$.

Let us consider the variation of the particle action (2.4) under the infinitesimal change of the path, namely

$$x^\mu \mapsto x^\mu + \delta x^\mu \tag{2.6a}$$

$$G_{\mu\nu} \mapsto G_{\mu\nu} + \partial_\sigma G_{\mu\nu} \delta x^\sigma . \tag{2.6b}$$

At first order, one gets

$$\delta S = m \int d\tau \left\{ \frac{G_{\mu\sigma}\dot{x}^\mu \delta\dot{x}^\sigma}{\sqrt{-\dot{x}^\mu\dot{x}^\nu G_{\mu\nu}}} + \frac{\dot{x}^\mu\dot{x}^\nu\partial_\sigma G_{\mu\nu}\delta x^\sigma}{\sqrt{-\dot{x}^\mu\dot{x}^\nu G_{\mu\nu}}} - \frac{q}{m}\left(A_\sigma\delta\dot{x}^\sigma + \partial_\sigma A_\mu \delta x^\rho \dot{x}^\mu\right)\right\} \tag{2.7}$$

After integration by parts of the first and third term, and trading the integral over the affine parmeter τ for the integral over the proper time s, one gets

$$\delta S = m \int ds \left[\frac{d^2x^\nu}{ds^2} + \Gamma^\nu_{\rho\sigma}\frac{dx^\rho}{ds}\frac{dx^\sigma}{ds} - \frac{q}{m}F^\nu{}_\mu \frac{dx^\mu}{ds}\right]\delta x_\nu \tag{2.8}$$

Not surprisingly, one obtains the relativistic equation of motion of a massless charged particle, *i.e.* the geodesic equation plus the coupling to the electromagnetic field strength $F = dA$.

In order to make more explicit the diffeomorphism invariance of the worldline action, one can introduce an independent *worldline metric* as $ds^2 = h_{\tau\tau}(\tau)d\tau^2$. In the one-dimensional analogue of the tetrad formalism of general relativity, one defines the *einbein* $e(\tau) = \sqrt{-h_{\tau\tau}}$. The action (2.4) can be then rewritten in a classically equivalent way as:

$$\begin{aligned}\mathcal{S}_e &= -\frac{1}{2}\int d\tau \sqrt{-h_{\tau\tau}}\left(h^{\tau\tau}\partial_\tau x^\mu \partial_\tau x^\nu G_{\mu\nu} + m^2\right) - q\int_l A_\mu dx^\mu \\ &= \frac{1}{2}\int d\tau \left(\frac{1}{e}G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e\right) - q\int_l A_\mu dx^\mu\end{aligned} \tag{2.9}$$

where one can see that $e(\tau)$ play the role of a Lagrange multiplier field $e(\tau)$ – *i.e.* a non-dynamical field that enforces a constraint in field space. Its equation of motion is simply $0 = -e^{-2}G_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2$, which, upon replacing e by the solution in the action (2.9), gives back the original action (2.4).

One can view this action as a one-dimensional theory of gravity coupled to a set of free scalar fields $x^\mu(t)$ (there is naturally no curvature term in one dimension). Notice that the coupling of the particle to the electromagnetic four-potential A – the last term in equation (2.9) – is independent of the worldline metric. In this sense this coupling is of *topological* nature. Under diffeorphisms $\tau \mapsto \tilde{\tau}(\tau)$ the einbein transforms according to

$$e(\tau)d\tau = \tilde{e}(\tilde{\tau})d\tilde{\tau} \tag{2.10}$$

which gives, for an infinitesimal transformation

$$\tau \mapsto \ \tilde{\tau} = \tau + \epsilon(\tau) \tag{2.11}$$

$$e(\tau) \mapsto \tilde{e}(\tilde{\tau}) = e(\tilde{\tau}) - \frac{d}{d\tilde{\tau}}\big(e(\tilde{\tau})\epsilon(\tilde{\tau})\big) . \tag{2.12}$$

As in four-dimensional gravity, this reparametrization invariance is a gauge symmetry, *i.e.* a redundancy in the description of the system that will eventually remove some degrees of freedom from the theory.[2]

Path integral quantization

The quadratic action (2.9) for the relativistic charged particle is a convenient starting point for quantizing the theory through the path integral formalism. It is convenient as well to analytically continue the space-time to Euclidean signature $x^0 \mapsto ix^0$, as well as the worldline time $\tau \mapsto i\tau$. We will consider a particle moving in flat space, *i.e.* with metric $G_{\mu\nu} = \delta_{\mu\nu}$, in the absence of electromagnetic field.

The one-particle vacuum energy in Euclidean space deduced from the wordline action is given by summing over closed paths of the particle, which is given schematically by the functional integral:

$$Z_1 = \int \frac{\mathcal{D}e}{\text{Vol}(\text{diff})} \int_{x(0)=x(1)} \mathcal{D}x \exp\left\{ -\frac{1}{2} \int_0^1 d\tau \left(\frac{1}{e}\dot{x}^2 + m^2 e \right) \right\} , \tag{2.13}$$

where one has to divide the functional integral over the einbein (or equivalently over the one-dimensional metrics) by the infinite volume of the group of diffeomorphisms of the worldline. This group contains the transformations of the vielbein given infinitesimally by (2.12). On top of this shifts of τ by a constant, $\tau \mapsto \tau + \tau_0$, are diffeomorphisms that are not fixed by the choice of a reference einbein. The volume of this factor of the gauge group is finite and given by T, the invariant length of the closed path of the particle, see eq. (2.14) below. We choose finally the parameter τ to be in the interval $[0, 1]$, and, the path being closed, the einbein is a periodic function: $e(\tau + 1) = e(\tau)$.

Gauge symmetry

To carry the functional integral over the "gauge field" $e(\tau)$ one starts by slicing the field space into gauge orbits, *i.e.* einbeins that are related to each other by a diffeomorphism. The ratio of the integral over the whole field space over the volume of the group of diffeomorphisms is then equivalent to a functional integral over a slice in field space that cuts once each orbit, see figure 2.1 up to the Jacobian of the change of coordinates in field space; this is the Faddeev-Popov method.[3]

To start with pick a gauge choice corresponding to a reference einbein $\hat{e}$. For convenience we can take the reference einbein to be $\hat{e} = 1$. This reference einbein $\hat{e}$ generates a gauge orbit, the family $\{\hat{e}_\alpha\}$ of all einbeins obtained from $\hat{e}$ by some diffeomorphism α: $\hat{e} \mapsto \hat{e}_\alpha$. Using eqn. (2.10), starting from an arbitrary einbein e one can reach in principle the reference einbein $\hat{e} = 1$ with a diffeomorphism α that satisfies $\frac{d}{d\tau}\alpha(\tau) = e(\tau)$ hence is seems that all

[2]This redundancy was already explicit in the original description of the theory, eq. (2.1), as one could have chosen the gauge $x^0(\tau) = \tau$ to start with.

[3]This method is rather overkill for dealing with a free particle but will be used again in the case of the string.

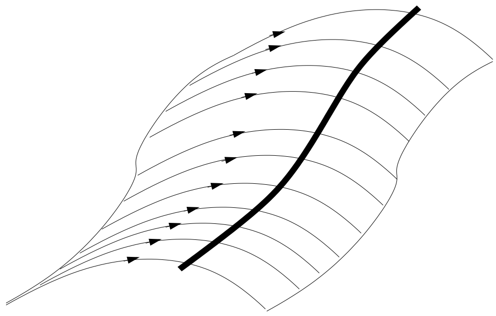

Figure 2.1: *Foliation of the space of gauge fields into gauge orbits. A slice through field space intersecting all orbits once is represented in bold.*

metrics on the worldline can be brought to the reference metric by a diffeomorphism. However the periodicity of the einbein is not preserved, as $\alpha(1) \neq 1$ for a generic diffeomorphism; this is a global obstruction for all metrics on the closed worldline being diffeomorphic-equivalent. The invariant length of the path is, as its name suggests, invariant under diffeomorphisms:

$$T = \int_0^1 e(\tau)d\tau = \int_0^{\tilde{\tau}(1)} \tilde{e}(\tilde{\tau})d\tilde{\tau}\,. \tag{2.14}$$

Hence the positive parameter T labels gauge-equivalent classes of metrics over closed worldlines; it is called a *modulus*. If one fixes the integration domain $[0, 1]$ to preserve the periodicity, the reference einbein should be defined accordingly. We choose then our reference einbein, in a class of metrics of invariant length T, as $\hat{e}(T) := T$, such that $\int_0^1 \hat{e}(T)(\tau)d\tau = T$.

In the path integral, one should perform the ordinary integral over all possible values of T, as we integrate over all possible geometries of the worldline. In other words, the functional integral measure $\mathcal{D}e$ splits into a gauge-invariant measure $\mathcal{D}\alpha$ over the gauge group and an integral over the modulus, $\int dT$.

As in ordinary gauge theories like quantum chromodynamics in four space-time dimensions, one introduces then the Faddeev-Popov determinant [1] (FP for short) through the relation:

$$\frac{1}{\Delta_{\text{FP}}(e)} := \int dT \int \mathcal{D}\alpha\, \delta(e - \hat{e}_\alpha(T))\delta(\alpha(0))\,, \tag{2.15}$$

The distribution $\delta(e - \hat{e}_\alpha(T))$ will eventually project the integral over the metrics onto an integral over the chosen gauge slice in field space, which is a one-parameter family of diffeomorphism-inequivalent reference einbeins $\hat{e}(T)$ depending of the moduli T of the closed path.

The Faddeev-Popov determinant should be thought of as the Jacobian of the change of coordinates from $\mathcal{D}e$, the integral over all one-dimensional einbeins, to $dT\,\mathcal{D}\alpha$, the integral over the gauge slice on the one hand, and over the directions orthogonal to it – in other words over the gauge orbits – on the other hand.

Finally, the translation symmetry of the closed path $\tau \mapsto \tau + \nu$ is gauge-fixed by the term $\delta(\alpha(0))$ setting arbitrarily the origin of the path at $\tau = 0$; the diffeomorphisms α that we integrate over are requested to keep this origin fixed.

One can plug this expression into the path integral (2.13) and integrate readily over the einbein e:

$$\begin{aligned} Z_1 &= \int \frac{\mathcal{D}e\, \mathrm{d}T\, \mathcal{D}\alpha\, \mathrm{d}L}{\mathrm{Vol}(\mathrm{diff})} \Delta_{\mathrm{FP}}(e)\delta(e-\hat{e}_\alpha(T))\delta(\alpha(0)) \int \mathcal{D}x \exp\left\{-\frac{1}{2}\int_0^1 \mathrm{d}\tau \left(\frac{1}{e}\dot{x}^2 + m^2 e\right)\right\} \\ &= \int \mathrm{d}T \int \frac{\mathcal{D}\alpha}{\mathrm{Vol}(\mathrm{diff})} \Delta_{\mathrm{FP}}(\hat{e}_\alpha(T))\delta(\alpha(0)) \int \mathcal{D}x \exp\left\{-\frac{1}{2}\int_0^1 \mathrm{d}\tau \left(\frac{1}{\hat{e}_\alpha(T)}\dot{x}^2 + m^2 \hat{e}_\alpha(T)\right)\right\} . \end{aligned} \tag{2.16}$$

The last expression can be simplified further by noticing that *(i)* the Faddeev-Popov is gauge-invariant (being defined as an average over the gauge group) and and *(ii)* that by trading the functional integral over x^μ by the functional integral over the transformed field x^μ_α under the diffeomorphism α, the integrand of the integral over the gauge group is actually gauge-independent, hence the integral $\int \mathcal{D}\alpha$ factors out and cancels the volume of the gauge group, except the factor T corresponding to the group of translations – as we have gauge-fixed this symmetry – giving finally:

$$Z_1 = \int_0^T \frac{\mathrm{d}T}{T}\, e^{-\frac{m^2 T}{2}} \Delta_{\mathrm{FP}}(T) \int \mathcal{D}x \exp\left\{-\frac{1}{2T}\int \mathrm{d}\tau\, \dot{x}^2\right\} . \tag{2.17}$$

The determinant $\Delta_{\mathrm{FP}}(T)$ can be expressed as a functional determinant as follows. Using the infinitesimal expression (2.12) for the gauge transformation of the einbein, together with an infinitesimal variation of the loop modulus $T \mapsto T + \chi$, one can write

$$\delta(\hat{e} - T_\alpha) = \delta\left(T\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} - \chi\right) = \int \mathcal{D}\beta\, e^{-2i\pi T \int_0^1 \mathrm{d}\tau\, \beta\left(\frac{\mathrm{d}\epsilon}{\mathrm{d}\tau} - \chi/T\right)} , \tag{2.18}$$

where one has introduced a path integral over a Lagrange multiplier field β to implement the desired constraint in field space. Similarly one can write

$$\delta(\epsilon(0)) = \int \mathrm{d}\lambda\, \exp 2i\pi\lambda\epsilon(0) . \tag{2.19}$$

To obtain the Faddeev-Popov determinant, rather than its inverse, as a functional integral, one can trade $(\beta, \epsilon, \chi, \lambda)$ for Grassmann variables (b, c, ψ, ρ), *i.e.* fermionic variables, and write, after some rescaling of the fields

$$\Delta_{\mathrm{FP}}(T) = \int \mathcal{D}b\mathcal{D}c\mathcal{D}\psi \mathrm{d}\rho\; e^{-T\int_0^1 \mathrm{d}\tau\, b\left(\frac{\mathrm{d}c}{\mathrm{d}\tau} - \psi/T\right) - \rho c(0)} . \tag{2.20}$$

One can perform immediately the integral over the Grassmann variables ψ and ρ, which gives finally

$$\Delta_{\mathrm{FP}}(T) = \int \mathcal{D}b\mathcal{D}c \left(\int_0^1 \mathrm{d}\tau\, b\right)\, c(0)\;\, e^{-T\int_0^1 \mathrm{d}\tau\, b\frac{\mathrm{d}c}{\mathrm{d}\tau}} . \tag{2.21}$$

In other words, one has inserted into the path integral over (b, c) the mean value of $b(\tau)$ over the worldline, $\mathbf{i.e.}$ the zero-mode of the field, as well as $c(0)$; both insertions are actually necessary to cancel the integration over the zero-modes of the fields in the path integral as we will see shortly.

Functional determinants

We need now to perform the functional integral over the coordinate fields x^μ. One considers then the path integral

$$\int \mathcal{D}x\, e^{-\frac{1}{2T}\int_0^1 d\tau \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau}}\,. \tag{2.22}$$

One expands then x^μ over a complete set of eigenfunctions of the positive-definite operator $-\frac{1}{T}\partial_\tau^2$ satisfying the right boundary conditions. It is convenient to separate the zero-modes, *i.e.* the classical solutions of the equations of motion, from the fluctuations:

$$x^\mu(\tau_E) = x_0^\mu + q^\mu(\tau)\,, \tag{2.23}$$

where $q(\tau)$ satisfies the Dirichlet boundary conditions $q(0) = q(1) = 0$, such that $x(0) = x(1) = x_0$. The norm in field space for the fluctuations is naturally

$$||q||^2 = \frac{1}{2}\int_0^1 \hat{e}_T(\tau) d\tau\, q^2(\tau) = \frac{T}{2}\int_0^1 d\tau\, q^2(\tau)\,. \tag{2.24}$$

One has then the expansion on an orthonormal basis on eigenfunctions of the differential operator $D = -\frac{1}{T}\partial_\tau^2$ with the right boundary conditions:

$$q^\mu(\tau_E) = \sum_{n=1}^\infty c_n^\mu \frac{2}{\sqrt{T}} \sin \pi n \tau\,. \tag{2.25}$$

and the measure of integration is

$$\mathcal{D}x = dx_0 \prod_{n=1}^\infty dc_n\,. \tag{2.26}$$

The ordinary integral over the zero-mode x_0 gives the (infinite) volume V of space-time, while from the integration over the non-zero modes one gets

$$\left(\int \prod_n dc_n e^{-\frac{\pi^2 n^2}{T^2} c_n}\right)^d = \left(\prod_n \frac{\pi n^2}{T^2}\right)^{-d/2} \tag{2.27}$$

This product of eigenvalues is divergent and needs to be regularized.

Functional determinants of the form $\det(D) = \prod_{n=1}^\infty \lambda_n$ can be evaluated using the *zeta-function regularization.* One assumes that $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots$ and defines the spectral zeta-function as

$$\zeta_D(z) = \sum_{n=1}^\infty \lambda_n^{-z}\,, \tag{2.28}$$

which converges provided $\Re(z)$ is large enough. It can be analytically continued to the whole z plane except possibly at a finite set of points. Next we notice that

$$\log\det D = \sum_{n=1}^\infty \log \lambda_n = -\zeta_D'(0)\,, \tag{2.29}$$

and define the regularized functional determinant as

$$\prod_n \lambda_n = e^{-\zeta_D'(0)} \,. \tag{2.30}$$

In the present case one has

$$\zeta_D = \sum_{n=1}^{\infty} \left(\frac{\pi n^2}{T^2}\right)^{-z} = \left(\frac{T^2}{\pi}\right)^{z} \zeta(2z) \,, \tag{2.31}$$

in terms of the Riemann zeta-function ζ. Since $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\ln 2\pi$, the path integral over $x^\mu(\tau)$ gives finally, dropping the infinite volume factor and after some T-independent rescaling

$$\int \mathcal{D}x \; e^{-\frac{1}{2T}\int_0^1 dt\, \dot{x}^2} = T^{-d/2} \,. \tag{2.32}$$

This result can be obtained – in a perhaps simpler way – by viewing the path integral over a closed loop in Euclidean time as a partition function. One has

$$Z_x = \int \mathcal{D}x \; e^{-\int_0^1 d\tau \frac{1}{2T}\dot{x}^2} = \mathrm{Tr}\left(e^{-\beta H}\right) \;, \quad \beta = 1 \,. \tag{2.33}$$

The Hamiltonian $H = \frac{T}{2}p^2$ is the same as a free (non-relativistic) particle of mass $1/T$ in D spatial dimensions, and the computation of the partition function gives simply

$$Z_x = \int \frac{d^D p}{(2\pi)^D} e^{-T\frac{p^2}{2}} = (2\pi T)^{-D/2} \,. \tag{2.34}$$

We now turn to the evaluation of the ghost path integral. We start with the expression of the FP determinant that we have obtained before:

$$\Delta_{FP}(T) = \int \mathcal{D}b\mathcal{D}c \left(\int_0^1 d\tau\, b\right) c(0) \;\; e^{-T\int_0^1 d\tau\, b\frac{dc}{d\tau}} \,. \tag{2.35}$$

Because b and c are ghosts, with are dealing with fermionic variables with *periodic* (rather than anti-periodic) boundary conditions, hence having zero-mode. We recall here the rules of integration over Grassmann variables:

$$\int d\theta = 0 \;, \quad \int d\theta\, \theta = 1 \;, \quad \int d\theta\, f(\theta) = f'(0) \,. \tag{2.36}$$

which implies that, to get a non-zero answer, the integrand should contain the right number of zero-modes to cancel the corresponding zero-mode integration measure. Fortunately, the path integral (2.21 contains the right number of insertions of ghosts zero modes. The integral over the zero-modes yields

$$\int db_0\, dc_0\, b_0 c_0 = 1 \,, \tag{2.37}$$

Computing the integral over the non-zero modes is a little bit subtle, however by looking at this problem from a statistical mechanics point of view one can get the result easily. The equations of motions for the b and c classical fields are

$$\dot{b} = \dot{c} = 0 \tag{2.38}$$

hence, with periodic boundary conditions, one has two zero-modes b_0 and c_0. In the quantum theory, since b can be seen as the canonical momentum conjugate to c, one has the anti-commutator

$$\{b_0, c_0\} = 1\,. \tag{2.39}$$

Since the Hamiltonian vanishes the Hilbert space contains two states of zero energy, $|\pm\rangle$, that satisfy

$$\begin{aligned} b_0|-\rangle &= 0\,, \quad b_0|+\rangle = |-\rangle \\ c_0|+\rangle &= 0\,, \quad c_0|-\rangle = |+\rangle\,. \end{aligned} \tag{2.40}$$

From these relations one learns that b_0c_0 projects onto the ground state $|-\rangle$. Then the path integral on a Euclidean circle of length T is interpreted as a thermal average of b_0c_0 at inverse temperature $\beta = 1$, and one has:

$$\int \mathcal{D}b\mathcal{D}c b_0 c_0\; e^{-\int_0^1 d\tau\, T\, b \frac{dc}{d\tau}} = \langle -|e^{-H}|-\rangle = 1\,. \tag{2.41}$$

We will make a more detailed derivation of a similar result for the string path integral.[4]

Worldline versus QFT

Collecting all pieces, the path integral computation of the vacuum amplitude gives, up to an overall normalization factor,

$$Z_1 = \int_0^\infty \frac{dT}{T^{1+d/2}} e^{-\frac{m^2 T}{2}} \tag{2.42}$$

This integral is clearly divergent for $T \to 0$, *i.e.* when the particle loop shrinks to zero-size. The QFT picture will give us a more familiar understanding of this divergence.

Let us then consider the one-particle contribution to the vacuum energy of a free massive Klein-Gordon QFT. One has

$$Z_{\text{KG}} = \log \int \mathcal{D}\phi e^{-\frac{1}{2}\int d^4x\, \phi(-\nabla^2+m^2)\phi} = \log\left(\det(-\nabla^2 + m^2)\right)^{-1/2} = -\frac{1}{2}\int \frac{d^Dp}{(2\pi)^D}\log(p^2+m) \tag{2.43}$$

We can now move to Schwinger parametrization by using the simple identity

$$\frac{1}{x^a} = \frac{1}{\Gamma(a)}\int_0^\infty \frac{dT}{T^{1-a}} e^{-Tx}\,, \quad a > 0\,. \tag{2.44}$$

[4] To be precise, the Euclidean fermionic path integral with *periodic* boundary conditions rather than anti-periodic is not exactly the partition function but rather $\mathrm{Tr}\left[(-1)^F \exp(-\beta H)\right]$, where $F = b_0c_0$ counts the number of fermionic excitations.

which allows to get the general result

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2+m^2)^a} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{dT}{T^{1-a}} \frac{d^D p}{(2\pi)^D} e^{-T(p^2+m^2)} = \frac{1}{\Gamma(a)} \int_0^\infty \frac{dT}{T^{1-a}} e^{-Tm^2} (4\pi T)^{-D/2} . \tag{2.45}$$

At first order in the expansion in the parameter a one gets formally

$$Z_{\text{KG}} = -\frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \log(p^2+m^2) = \frac{1}{2} \lim_{a\to 0} \int_0^\infty \frac{dT}{T^{1-a}} e^{-Tm^2} (4\pi T)^{-D/2} . \tag{2.46}$$

In other words,

$$Z_{\text{KG}} = \frac{1}{2} \int \frac{d^D p}{(2\pi)^D} \int_0^\infty \frac{dT}{T} e^{-\frac{T}{2}(p^2+m^2)} = \frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-\frac{m^2}{2}T} (2\pi T)^{-D/2} , \tag{2.47}$$

which is the same, up to the numerical normalization factor that we did not computed precisely, the same as the worldline computation (2.42).

As was noticed before, the expressions (2.42,2.47) present a divergence for $T \to 0$; its origin is clear from eqn. (2.46). In the momentum-space expression (2.43), it is understood as the usual UV divergence of the loop integral for $\|p\| \to +\infty$. One can compute directly the integral in (2.45) and get:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{(p^2+m^2)^a} = (4\pi)^{-D/2} \frac{\Gamma(a-D/2)}{\Gamma(a)} m^{D-2a} . \tag{2.48}$$

In the $a \to 0$ limit, matching the $\mathcal{O}(a)$ terms on both sides yields to

$$\int \frac{d^D p}{(2\pi)^D} \log(p^2+m^2) = (4\pi)^{-D/2} \frac{2}{D} \Gamma(1-D/2) m^D . \tag{2.49}$$

Hence the ultraviolet divergence can be removed by dimensional regularization, as usual.

By analoguous computations, one can get the Klein-Gordon propagator by considering open worldlines between two points x and x' in Euclidean space-time. It is also interesting to consider the worldline path integral in the presence of an electromagnetic potential, starting from the more general action (2.9); in this way one can get for instance the photon N-point function in scalar QED, in particular the vacuum polarization ($N = 2$). Choosing a constant electric field, one finds also the famous Schwinger formula for production of charged particles/antiparticle pairs in an electric field [2].

We have learned with from this little exercise two important lessons that will be important in the forthcoming study of relativistic strings:

1. quantum mechanics (or equivalently 0+1 dimensional QFT) on the worldline of a massive relativistic charged particle provides an equivalent formulation of massive scalar QFT in an external electromagnetic field;

2. The UV loop divergences of QFT are mapped in the worldline formalism to closed wordlines shrinking to zero size.

There exists several extensions of this worldline formalism, in order to include spinors, non-Abelian gauge interactions, etc... As this is not the main topic of the lectures we will not comment further but refer the interested reader to [3].

2.2 Relativistic strings

We start our journey in string theory by considering the exact analogue of the relativistic point particle in the worldline formalism for relativistic objects with an extension in a space-like direction – the fundamental strings of string theory. These objects have a mass per unit length, or *tension* $\mathcal{T}$, which is by tradition parametrized as

$$\mathcal{T} = \frac{1}{2\pi\alpha'}\,, \tag{2.50}$$

where the parameter α', the *Regge slope*, has dimensions of length squared. The interested reader may consult section 1.2 to have some idea about where this terminology comes from.

In order to have a finite energy, the strings should have a finite length. It leaves two possibilities, topologically speaking: a loop or an interval, which are called *closed strings* and *open strings* respectively, see figure 2.2. In both cases the position along the string is parametrized by σ.

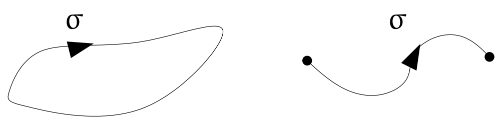

Figure 2.2: *Closed strings (left) and open strings (right).*

Open strings are a little bit more subtle to handle, as one has to specify what are the boundary conditions at the end of the strings. We will therefore deal exclusively in this chapter with *closed strings*, postponing the construction of open string sectors to chapter 7. Another rationale for this choice is that the closed string sector contains quantum gravity, the main string theory achievement.

Classical closed strings

A propagating relativistic closed string swaps in spacetime a two-dimensional surface $\mathfrak{s}$, or *worldsheet*, in analogy with the worldline of point particles. It has the topology of a cylinder, parametrized by the σ for the space-like direction and τ for the time-like direction, see fig. 2.3. For closed strings, the coordinate σ is periodic. We choose the convention

$$\sigma \sim \sigma + 2\pi\,. \tag{2.51}$$

The worldsheet of a relativistic closed string in space-time $\mathcal{M}$ of metric g corresponds to an embedding map

$$S^1 \times \mathbb{R} \;\hookrightarrow\; \mathcal{M} \tag{2.52}$$

$$(\sigma,\tau) \;\mapsto\; x^\mu(\sigma,\tau)\,, \quad x^\mu(\sigma+2\pi,\tau) = x^\mu(\sigma,\tau)\,, \tag{2.53}$$

where the set of functions $\{x^\mu(\sigma,\tau), \mu = 0,\ldots,D-1\}$ should be periodic in σ. For convenience we will use the notation $(\sigma^0,\sigma^1) = (\tau,\sigma)$. The codomain of the map, *i.e.* the space-time $\mathcal{M}$ where the string leaves, is usually called the *target space* of the string.

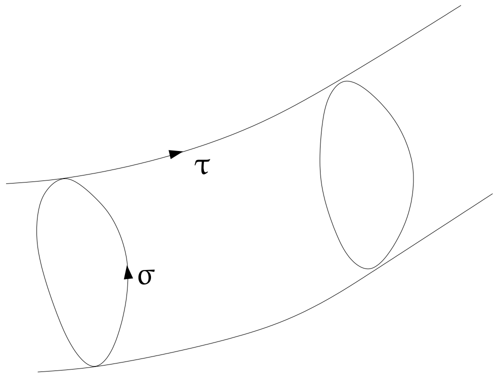

Figure 2.3: *Closed string worldsheet.*

The space-time metric $G_{\mu\nu}(x^\rho)$ induces a metric h on the world-sheet since along the surface parametrized by the affine parameters σ and τ we have:

$$\begin{aligned} ds^2 = G_{\mu\nu}dx^\mu dx^\nu &= G_{\mu\nu}\frac{\partial x^\mu(\sigma^k)}{\partial\sigma^i}\frac{\partial x^\nu(\sigma^k)}{\partial\sigma^j}d\sigma^i\,d\sigma^j \\ &=: h_{ij}d\sigma^i\,d\sigma^j \end{aligned} \tag{2.54}$$

Therefore the surface element on an integral over the world-sheet is $\sqrt{-\det h}$ as the tangent space of the surface can be split into a time-like and a space-like directions over every point.

In complete analogy with the relativistic particle case, see eq. (2.4), one postulates a string action of the form

$$S_{\text{NG}} = -\frac{1}{2\pi\alpha'}\int d\sigma^0\int_0^{2\pi}d\sigma^1\sqrt{-\det\,\frac{\partial x^\mu(\sigma^k)}{\partial\sigma^i}\frac{\partial x^\nu(\sigma^k)}{\partial\sigma^j}}\,. \tag{2.55}$$

This action is known as the *Nambu-Goto* action [4, 5]. It is invariant under diffeomorphisms of the worldsheet as it should:

$$\sigma^i \mapsto \tilde{\sigma}^i(\sigma_j) \tag{2.56a}$$

$$d\sigma^0 d\sigma^1 \mapsto (\det\partial_i\tilde{\sigma}^j)^{-1}d\tilde{\sigma}^0 d\tilde{\sigma}^1 \tag{2.56b}$$

$$\frac{\partial x^\mu}{\partial\sigma^i}\frac{\partial x^\nu}{\partial\sigma^j} \mapsto \frac{\partial x^\mu}{\partial\tilde{\sigma}^l}\frac{\partial x^\nu}{\partial\tilde{\sigma}^m}\partial_i\tilde{\sigma}^l\partial_j\tilde{\sigma}^m\,. \tag{2.56c}$$

In Minkowski space-time ($G_{\mu\nu} = \eta_{\mu\nu}$) the action is also invariant under space-time Poincaré transformations $x^\mu \mapsto \lambda^\mu{}_\nu x^\nu + a^\mu$.

In the point particle case, there was a natural coupling to the electromagnetic potential, $i.e.$ to the one form $A(x^\mu) = A(x^\mu)_\rho dx^\rho$. There exists an analogous coupling allowed for the string, but this time to a two-form potential[5] $B(x^\mu) = \frac{1}{2}B_{\rho\sigma}(x^\mu)dx^\mu \wedge dx^\nu$, which is called the

[5] For basic facts about differential forms, see appendix 11.1.

Kalb-Ramond potential [6]:[6]

$$S_{\text{KR}} = -\frac{1}{2\pi\alpha'}\int_{\mathfrak{s}} B = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} B_{\rho\sigma}[x^{\mu}(\tau)]\frac{\partial x^{\mu}}{\partial\sigma^{i}}\frac{\partial x^{\nu}}{\partial\sigma^{j}}d\sigma^{i}\wedge d\sigma^{j}$$
$$= -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d\sigma^{0}d\sigma^{1}\,\epsilon^{ij}B_{\rho\sigma}\frac{\partial x^{\mu}}{\partial\sigma^{i}}\frac{\partial x^{\nu}}{\partial\sigma^{j}}\,. \tag{2.57}$$

As its one-dimensional cousin, the Kalb-Ramond coupling is independent of the parametrization of the worldsheet as it does not depend explicitly on the two-dimensional induced metric h. Furthermore, one can naturally associate to the coupling (2.57) a gauge invariance

$$B_{\mu\nu}\,dx^{\mu}\wedge dx^{\nu} \;\mapsto B_{\mu\nu}\,dx^{\mu}\wedge dx^{\nu} + d(\Lambda_{\nu}dx^{\nu}) = B_{\mu\nu}\,dx^{\mu}dx^{\nu} + \partial_{\mu}\Lambda_{\nu}\,dx^{\mu}\wedge dx^{\nu}\,, \tag{2.58}$$

where the parameter of the gauge transformation is a one-form $\Lambda = \Lambda_{\mu}dx^{\mu}$. This transformation leaves invariant (2.57) up to boundary terms using Stokes' theorem:

$$\int_{\mathfrak{s}} B \mapsto \int_{\mathfrak{s}} B + \int_{\mathfrak{s}} d\Lambda = \int_{\mathfrak{s}} B + \int_{\partial\mathfrak{s}} \Lambda\,. \tag{2.59}$$

Polyakov action

The non-linear Nambu-Goto action (2.55) is not a convenient starting point for quantizing the theory. In analogy with the relativistic particle case, we will introduce a dynamical worldsheet metric γ and consider instead the action known as the *Polyakov action* [7]:

$$S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^{2}\sigma\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu}\partial_{i}x^{\mu}(\sigma^{k})\partial_{j}x^{\nu}(\sigma^{k})\,, \tag{2.60}$$

where the non-dynamical field $\gamma_{ij}(\sigma^{k})$ is determined by its equation of motion. The Polyakov action can be understood as the minimal coupling of a two-dimensional metric to a set of scalar fields, hence is automatically invariant under diffeomorphisms of the worldsheet $\sigma^{i} \mapsto \tilde{\sigma}^{i}(\sigma^{k})$.

Let us now prove that the equation of motion of the dynamical metric γ in the Polyakov action (2.60) gives back the Nambu-Goto action (2.55). Under an infinitesimal variation $\gamma \mapsto \gamma + \delta\gamma$, one finds that at first order

$$\delta S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^{2}\sigma\sqrt{-\det\gamma}\left[\frac{\delta(\sqrt{-\det\gamma})}{\sqrt{-\det\gamma}}\gamma^{kl}h_{kl} + \delta\gamma^{ij}G_{\mu\nu}\partial_{i}x^{\mu}\partial_{j}x^{\nu}\right]\,. \tag{2.61}$$

Given that $\gamma^{ij}\gamma_{jk} = \delta^{i}{}_{k}$ one finds at first order that

$$\gamma^{ij}\delta\gamma^{jk} + \gamma_{jk}\delta\gamma_{ij} = 0 \implies \delta\gamma^{jk} = -\gamma^{ij}\gamma^{kl}\delta\gamma_{il} \tag{2.62}$$

[6] We use the two-dimensional epsilon symbol with non-zero components $\epsilon^{01} = -\epsilon^{10} = 1$. Note that $(-\det\gamma)^{-1/2}\epsilon^{ij}$ is a two-index contravariant antisymmetric tensor.

and

$$\frac{\delta(\sqrt{-\det\gamma})}{\sqrt{-\det\gamma}} = \frac{1}{2}\delta\ln(-\det\gamma) = \frac{1}{2}\gamma^{ij}\delta\gamma_{ij}\,. \tag{2.63}$$

We obtain then

$$\delta S_{\text{P}} = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\sqrt{-\det\gamma}\left[\frac{1}{2}\gamma^{ij}\gamma^{kl}h_{kl} - \gamma^{ik}\gamma^{jl}h_{kl}\right]\delta\gamma_{ij}\,. \tag{2.64}$$

The vanishing of the first order variation leads therefore to

$$\frac{1}{2}\gamma^{ij}\gamma^{kl}h_{kl} = \gamma^{ik}\gamma^{jl}h_{kl} \quad\Longrightarrow\quad h_{ij} = \frac{1}{2}\gamma^{kl}h_{kl}\gamma_{ij}\,. \tag{2.65}$$

The determinant of this relation gives

$$\det h = \left(\frac{1}{2}\gamma^{kl}h_{kl}\right)^2\det\gamma\,, \tag{2.66}$$

from which we deduce that

$$\begin{aligned} S_{\text{P}} &= -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det\gamma}\,\gamma^{ij}h_{ij} = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-\det h}\left(\frac{1}{2}\gamma^{kl}h_{kl}\right)^{-1}\gamma^{kl}h_{kl} \\ &= -\frac{1}{2\pi\alpha'}\int d^2\sigma\sqrt{-\det h} = S_{\text{NG}} \end{aligned} \tag{2.67}$$

Hence, at least classically, the Nambu-Goto and the Polyakov actions give equivalent dynamics for the relativistic strings.

The Polyakov action (2.60) is certainly not the most general action on the string worldsheet that one can write. First, the coupling to the Kalb-Ramond field, eq. (2.60), is independent of the worldsheet metric hence takes the same form in the Nambu-Goto and Polyakov formalisms. Second, an acute reader may have wondered why, in the Polyakov action, we did not include a "cosmological constant" term

$$-\frac{1}{4\pi}\int d^2\sigma\sqrt{-\gamma}\,\Lambda\,, \tag{2.68}$$

that would be analogous to the mass term $\int d\tau\, e(\tau)\, m^2$ in the worldline action (2.9) for the relativistic particle. Such a term would imply that

$$\frac{1}{2}\gamma^{ij}\gamma^{kl}h_{kl} + \frac{1}{2}\alpha'\Lambda\gamma^{ij} = \gamma^{ik}\gamma^{jl}h_{kl}\,. \tag{2.69}$$

Contracting this equation with γ_{ij} gives

$$\gamma^{kl}h_{kl} + \alpha'\Lambda = \gamma^{kl}h_{kl} \tag{2.70}$$

which has no solutions unless $\Lambda = 0$. This is a peculiarity of string actions, which is not shared with actions of particles or higher-dimensional extended objects.[7] We will understand shortly its significance.

There exists a last possible coupling of the relativistic string that has no analogue in the particle case. From the two-dimensional worldsheet metric γ one can construct the Ricci scalar $\mathcal{R}[\gamma]$, and write down a last term in the action of the Einstein-Hilbert type (after all we are considering a dynamical worldsheet metric):

$$\chi(\mathfrak{s}) = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\,\mathcal{R}[\gamma]\,. \tag{2.71}$$

In short, one has traded the problem of quantizing gravity in four dimensions to the problem of quantizing gravity in two dimensions! Einstein gravity is two dimensions is much simpler, as first it has no propagating degrees of freedom because of diffeorphism invariance (standard counting gives -1 degrees of freedom). The Einstein-Hilbert action is actually a topological invariant of the two-dimensional manifold $\mathfrak{s}$, known as the *Gauss-Bonnet term.* In Euclidean space it is equal to the *Euler characteristic* $\chi(\mathfrak{s})$ of the two-dimensional worldsheet. If the worldsheet is an oriented surface without boundaries, it is given by

$$\chi(\mathfrak{s}) = 2 - 2g \tag{2.72}$$

where h is the number of handles, or "holes", of the surface. For the sphere $g = 0$, the torus $g = 1$, etc... We will come back latter to the significance of these topologies.

There exists a generalization of the Einstein-Hilbert term that involves a coupling to a scalar field in space time $\Phi(x^\mu)$ and that is not topological:

$$S_{\rm D} = \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\Phi[x^\mu(\sigma^i)]\mathcal{R}[\gamma]\,. \tag{2.73}$$

The field $\Phi(x^\mu)$, which plays an important role in string theory, is called the *dilaton.*

To summarize this discussion, the general fundamental string action is given by the sum of (2.60), (2.60) and (2.73), hence a (1+1)-dimensional quantum field theory on the worlsheet given by:

$$\boxed{\mathcal{S} = -\frac{1}{4\pi\alpha'}\int_{\mathfrak{s}} d^2\sigma\left(\sqrt{-\det\gamma}\,\gamma^{ij}G_{\mu\nu} + \epsilon^{ij}B_{\mu\nu}\right)\partial_i x^\mu\partial_j x^\nu - \frac{1}{4\pi}\int d^2\sigma\sqrt{-\det\gamma}\,\Phi[x^\mu(\sigma^i)]\mathcal{R}[\gamma]} \tag{2.74}$$

This action describes the propagation of a single relativistic string in a background specified by a metric G, a Kalb-Ramond field B and a dilaton Φ. The later two have no obvious interpretation at this stage; note that in four dimensions the Kalb-Ramond field is actually equivalent to a real pseudo-scalar field as its field strength $H = dB$ is Hodge-dual to the differential of a scalar field: $\star H = da$.[8]

[7] Indeed for a p-dimensional extended object, $p+1$ being the dimension of its worldvolume, everything is the same except the contraction with γ_{ij} which gives $\frac{p+1}{2}(\gamma^{kl}h_{kl} + \alpha'\Lambda) = \gamma^{kl}h_{kl}$.

[8] Without using the differential form notation, one has $H_{\mu\nu\rho} = \partial_{[\mu}B_{\nu\rho]}$ and $\partial_\mu a = G_{\mu\alpha}\epsilon^{\alpha\nu\rho\sigma}H_{\nu\rho\sigma}$.

2.3 Symmetries

We now turn to the path integral quantization of the bosonic string. To start, one has to pay attention to the symmetries of the theory, in particular to the gauge symmetries that need to be carefully taken care of, as in the example of the point particle that we have dealt with in section 2.1. As there we will consider the path integral with an imaginary time coordinate, *i.e.* we will consider an Euclidean worldsheet of coordinates $(\sigma^1, \sigma^2) = (\sigma^1, -i\sigma^0)$ endowed with an Euclidean metric γ. However we will keep the signature of space-time to be $(-, +, \cdots, +)$.[9]

To simplify the discussion, consider the action of a string action with vanishing Kalb-Ramond field and constant dilaton field:

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int_{\mathfrak{s}} d^2\sigma \sqrt{\det\gamma}\, \gamma^{ij} G_{\mu\nu} \partial_i x^\mu \partial_j x^\nu + \frac{\Phi_0}{4\pi} \int_{\mathfrak{s}} d^2\sigma \sqrt{\det\gamma}\, \mathcal{R}[\gamma] \,. \tag{2.75}$$

The symmetries of the theory splits into worldsheet and target space symmetries. We will start by looking at the latter. One has first *target-space symmetries* of the action (2.75) corresponding to symmetries of space-time.

If the space-time is Minskowki space-time ($g_{\mu\nu} = \eta_{\mu\nu}$) the action is invariant under Poincaré transformations:

$$x^\mu \mapsto \Lambda^\mu{}_\nu x^\mu + a^\mu \ , \quad \Lambda \in SO(1, d-1) \,. \tag{2.76}$$

These are *global symmetries* of the two-dimensional field theory.

The Polyakov action (2.60) is actually invariant under diffeomorphisms in space-time *i.e.* infinitesimal coordinate transformations $\delta x^\mu = r^\mu(x^\rho)$ accompanied with a change of the metric $\delta G_{\mu\nu} = -2\nabla_{(\mu} r_{\nu)}$. In the present context it expresses the invariance of the theory under field redefinitions.

2.3.1 Worldsheet symmetries

The string action (2.75), or its more general version (2.74), is by design invariant under coordinate transformations, or diffeomorphims of the worldsheet,

$$\sigma^i \mapsto \tilde{\sigma}^i(\sigma^k) \,, \tag{2.77a}$$

$$\gamma_{ij} \mapsto \tilde{\gamma}_{ij} = \frac{\partial\sigma^k}{\partial\tilde{\sigma}^i} \frac{\partial\sigma^l}{\partial\tilde{\sigma}^j} \gamma_{k\ell} \,, \tag{2.77b}$$

being a theory of two-dimensional gravity minimally coupled to scalar fields $x^\mu(\sigma)$.

The Polyakov action has an extra local symmetry, which corresponds to *Weyl transformations* of the world-sheet metric, *i.e.* local scale transformations:

$$\gamma_{ij} \ \mapsto \ e^{2\omega(\sigma^i)} \gamma_{ij} \ , \tag{2.78}$$

[9] As we will see the price to pay will be the appearance of negative norm states. The latter can be fortunately removed by using the gauge symmetry of the theory.

where $\omega(\sigma^i)$ is an arbitrary differentiable function on the two-dimensional manifold $\mathfrak{s}$. This property comes from the scaling of the determinant of the metric in d dimensions

$$\det\gamma \overset{\text{Weyl}}{\mapsto} e^{2d\omega}\det\gamma\,, \tag{2.79}$$

which, specialized to two dimensions, implies that the action (2.60) invariant under Weyl transformations. The Kalb-Ramond coupling (2.57) is also by definition invariant being independent of the metric.

The Ricci scalar transforms simply under a Weyl rescaling of the metric. We leave as an exercise to show that, in d dimensions,

$$\gamma \mapsto e^{2\omega}\gamma \tag{2.80a}$$

$$R[\gamma] \mapsto e^{-2\omega}\left(R[\gamma]-(d-1)\nabla^2\omega-2(d-2)(d-1)\partial_a\omega\partial^a\omega\right)\,. \tag{2.80b}$$

In two dimensions, this expression implies that $\sqrt{\det\gamma}\,R[\gamma]$ transforms as a total derivative, since $\sqrt{\det\gamma}\,\nabla^2\omega=\partial_i(\sqrt{\det\gamma}\,\nabla^i\omega)$. One concludes that, at the classical level, the dilaton action (2.73) is not invariant under Weyl transformations, unless Φ is a constant, in which case it was expected since the two-dimensional Einstein-Hilbert term (2.71) is topological. We will see later on that, in the quantum theory, the story is somewhat altered by the presence of anomalies.

A careful reader would have noticed that the Weyl symmetry is not present in the original Nambu-Goto action (2.55). One can trace back its origin to the equation of motion for the worldsheet metric, eq. (2.66), which is invariant under Weyl transformations. This feature of the Polyakov action is not problematic. The Weyl symmetry is a gauge symmetry, hence does not really correspond to a symmetry but rather to a redundancy of our description of the theory. In the path integral quantization of the theory, this gauge symmetry will need to be taken care of properly, as the diffeomorphism invariance.

Finally, we notice that the cosmological constant term (2.68) that we considered to include in the action is not Weyl invariant, which explains why this term is forbidden in the first place by the gauge symmetries of the problem.

2.3.2 Gauge choice

The Euclidean path integral of the fundamental string is defined by a functional integral over the fields $\{x^\mu\}$ as well as over the two-dimensional metrics γ – *i.e.* over Euclidian worldsheet geometries – moded out by the volume of the gauge group of the theory, made of two-dimensional diffeomorphisms and Weyl transformations.

As we have done for the point particle, we will properly define this path integral by gauge-fixing and introducing the corresponding Faddeev-Popov determinant. To start with, one associates to each two-dimensional metric γ its *gauge orbit*, the set of its images $\{\gamma^\Xi\}$ under gauge transformations $\Xi=(\Sigma,\Omega)$ composed of diffeomorphisms and Weyl rescalings:

$$\Xi\ :\left\{\begin{array}{ll}\sigma^i & \mapsto\ \Sigma^i(\sigma^k)\\ \gamma_{ij} & \mapsto \exp(2\Omega)\ \times\ \frac{\partial\sigma^k}{\partial\Sigma^i}\frac{\partial\sigma^l}{\partial\Sigma^j}\gamma_{k\ell}\end{array}\right. \tag{2.81}$$

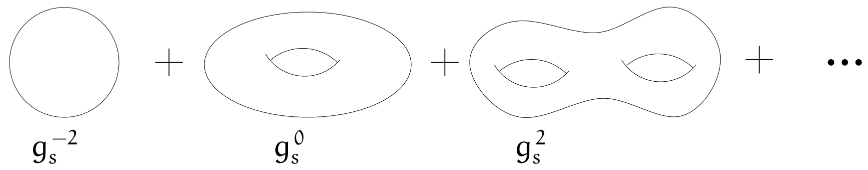

Figure 2.4: *Perturbative expansion of string theory.*

To define properly the gauge-fixing condition, one has then to understand how to classify all metrics over two-dimensional surfaces into equivalence classes under gauge transformations.

The coarser classification of compact two-dimensional surfaces is according to their *topology*. If we restrict ourselves to oriented surfaces without boundaries, their topology is completely specified by the number of handles in the surface, which is called its *genus* g. Explicitly, surfaces with $g = 0$ have the topology of a sphere, surfaces of genus $g = 1$ have the topology of a torus, etc... As we have no reasons to restrict ourselves to a particular type of surfaces, the Euclidean path integral contains a sum over *topologies* of the worldsheet. For surfaces with fixed topology, the value of the second term of the action (2.75) is fixed:

$$\frac{\Phi_0}{4\pi}\int_{\mathfrak{s}} \mathrm{d}^2\sigma\sqrt{\det\gamma}\,\mathcal{R}[\gamma] = \Phi_0\chi(\mathfrak{s}) = \Phi_0(2-2g)\,. \tag{2.82}$$

We have learned something very interesting. Let us define

$$g_s = \exp\Phi_0\,. \tag{2.83}$$

The path integral, in the sector of genus g surfaces, is weighted by the factor g_s^{2-2g}. In other words, the sum over topologies is nothing that the *perturbative expansion*, or loop expansion, of the theory! The parameter g_s is the string coupling constant. This is summarized on figure 2.4.

Having set the topology of the surface by its genus g, one has to find simple representatives in each gauge orbit under the action of diffeomorphisms and Weyl transformations. Locally, as the two-dimensional metric has three independent components, one can use the reparametrization invariance (*i.e.* the two functions $\Sigma^{1,2}(\sigma^k)$) to bring the metric in a conformally flat form:

$$\gamma_{ij}(\sigma_i) \mapsto \exp(2\Omega(\Sigma^i))\delta_{ij}\,. \tag{2.84}$$

This is called the *conformal gauge*. The conformal factor $\exp(2\Omega(\Sigma^i))$ can be naturally offset by a Weyl transformation, leaving a flat Euclidian metric. It will turn out to be convenient to use complex coordinates $w = \sigma^1 + i\sigma^2$ and $\bar{w} = \sigma^1 - i\sigma^2$, and the reference metric

$$\mathrm{d}s_{\mathfrak{s}}^2 = 2\,\widehat{\gamma}_{w\bar{w}}\,\mathrm{d}w\,\mathrm{d}\bar{w} = \mathrm{d}w\,\mathrm{d}\bar{w}\,. \tag{2.85}$$

In these complex coordinates the integration measure over the worldsheet is

$$\int \mathrm{d}w\mathrm{d}\bar{w} = 2\int \mathrm{d}^2\sigma \tag{2.86}$$

and the holomorphic and anti-holomorphic derivatives:

$$\partial = \partial_w = \tfrac{1}{2}(\partial_1 - i\partial_2)\ , \quad \bar{\partial} = \partial_{\bar{w}} = \tfrac{1}{2}(\partial_1 + i\partial_2)\,. \tag{2.87}$$

A generic infinitesimal gauge transformation (i.e. an infinistesimal diffeomorphism together with an infinitesimal Weyl transformation) around the flat metric is

$$\delta\gamma_{ij} = 2\delta\omega\delta_{ij} - \delta_{jk}\partial_i\delta\sigma^k - \delta_{ik}\partial_j\delta\sigma^k\,, \tag{2.88}$$

which gives in complex coordinates

$$\delta\gamma_{w\bar{w}} = \delta\omega - \tfrac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w})\,, \tag{2.89a}$$

$$\delta\gamma_{ww} = -\partial\delta\bar{w}\,, \tag{2.89b}$$

$$\delta\gamma_{\bar{w}\bar{w}} = -\bar{\partial}\delta w\,, \tag{2.89c}$$

where $\delta\omega$, δw and $\delta\bar{w}$ are arbitrary differentiable functions of w and $\bar{w}$.

Finally, the Polyakov action (2.60) in the flat gauge – or in the conformal gauge as it is invariant under Weyl rescalings – takes the form

$$\mathcal{S} = \frac{1}{2\pi\alpha'}\int dw d\bar{w}\ G_{\mu\nu}\partial x^\mu \bar{\partial} x^\nu\,. \tag{2.90}$$

2.3.3 Residual symmetries and moduli

In the study of the point particle path integral, we have seen two global aspects of the worldline geometry that played an important role: the existence of parameters, or *moduli*, that couldn't be gauged away by the choice of reference metric and the existence of gauge transformations that did not change the one-dimensional metric. In the string theory case these two aspects are still relevant, and needs a little bit more effort to be understood.

The *moduli* are, by definition, given by changes of the metric that are orthogonal to gauge transformations, *i.e.* that cannot be compensated for by a combination of a diffeomorphism and a Weyl transformation. In other words we consider a change of the metric $\delta\gamma_{ij}$ such that

$$\int d^2\sigma\left(2\delta\omega\delta_{ij} - \partial_i\delta_{jk}\delta\sigma^k - \partial_j\delta_{ik}\delta\sigma^k\right)\delta\gamma^{ij} = 0\,. \tag{2.91}$$

This should hold true for *any* $\delta\omega$ and $\delta\sigma^k$, hence it leads to a pair of independent relations:

$$\mathrm{Tr}\,(\delta\gamma) = 0 \implies \delta\gamma_{\bar{w}w} = 0 \tag{2.92a}$$

$$\partial^i\delta\gamma_{ij} = 0 \implies \begin{cases} \bar{\partial}\delta\gamma_{ww} = 0 \\ \partial\delta\gamma_{\bar{w}\bar{w}} = 0 \end{cases} \tag{2.92b}$$

Solutions of these equations are called *holomorphic quadratic differentials*. The number of independent solutions will give the number of *moduli* of the surface n_μ. In the mathematical literature, the space spanned by these moduli is called the *Teichmüller space*.

A second mismatch between the space of metric and the space of gauge transformations corresponds to combinations of diffeomorphisms and Weyl transformations that leaves the metric invariant. From equations (2.89b,2.89c) we learn that they correspond to diffeomorphisms satisfying

$$\partial\delta\bar{w} = \bar{\partial}\delta w = 0\,, \tag{2.93}$$

while the compensating Weyl transformation $\delta\omega$ is unambiguously determined by eq. (2.89a).

The solutions of these equations are *holomorphic vector fields*, *i.e.* vectors fields that are locally in any open set of the form $f(w)\partial_w$ where f is a holomorphic function. The key point here is that we need to find vectors fields that satisfy *globally* this condition on the whole surface, which is a rather strong constraint. These solutions form the *conformal Killing group* of the surface; its dimension will be called n_k.

The number of moduli and of conformal Killing vectors of a surface are not independent but related to each other by the *Riemann-Roch theorem*. The number n_μ of moduli and the number n_k of gauge transformations leaving invariant the metric is related to the Euler characteristic of the surface, which specifies its topology, through the relation

$$n_\mu - n_k = -3\chi(\mathfrak{s}) = 6(g-1)\,. \tag{2.94}$$

The sphere and the two-torus

We will now move away from this rather abstract discussion and derive in detail the moduli and conformal Killing vectors for the two most useful examples, the sphere and the torus.

Genus zero surfaces have the topology of a two-dimensional sphere. The Riemann-Roch theorem (2.94) indicates that $n_\mu - n_k = -6$. A conformally flat metric on the two-dimensional unit sphere is given, in complex coordinates, by

$$\mathrm{d}s^2 = \frac{\mathrm{d}w\mathrm{d}\bar{w}}{1+w\bar{w}}\,. \tag{2.95}$$

The coordinates $(w,\bar{w})$ are defined in a patch that excludes the "south pole" of the sphere at $w \to \infty$. A patch including the south pole is covered by the coordinates $(z,\bar{z}) = (1/w, 1/\bar{w})$. The compactification of the complex plane $\bar{\mathbb{C}} = \mathbb{C} \cap \{\infty\}$ is topologically equivalent to the two-sphere; in a way the patch containing the south pole has ben shrunk to the point at infinity.

The sphere has no moduli (in particular the radius can be absorbed by a constant Weyl transformation) and six conformal Killing vectors. Three of them are easy to identify, the generators of the Lie algebra $\mathfrak{so}(3)$. To find all of them, one needs to study the holomorphic vector fields on this manifold. Let us assume that the holomorphic vector field δw admits a holomorphic power series expansion around the north pole $w = 0$:

$$\delta w = c_0 + c_1 w + c_2 w^2 + c_3 w^3 + \cdots \tag{2.96}$$

This holomorphic vector field should be defined everywhere, in particular in the patch around the south pole. Under the coordinate transformation $w \mapsto z = 1/w$, one finds that

$$\delta z = \frac{\partial z}{\partial w}\delta w = -z^2(c_0 + c_1/z + c_2/z^2 + c_3/z^3 + \cdots)\,, \tag{2.97}$$

hence one gets a globally well-defined holomorphic vector field, in particular at the south pole $z = 0$, provided that $c_n = 0$ for $n \geqslant 3$. The three complex parameters $\{c_0, c_1, c_2\}$ parametrize the conformal Killing group of the sphere around the identity. This group is isomorphic to the Möbius group, *i.e.* the group of fractional linear transformations

$$z \mapsto \frac{az+b}{cz+d}\,, \quad a,b,c,d \in \mathbb{C}\,, \quad ad-bc \neq 0\,. \tag{2.98}$$

Given that the map is invariant under rescalings of the parameters, one can set $ad - bc = 1$ and these transformations define a group isomorphic to $\mathsf{PSL}(2, \mathbb{C})$, the group of complex 2×2 matrices M of determinant one identified under its center $M \mapsto -M$.

Genus one surfaces are topologically equivalent to a two-torus. The Euler characteristic of the two-torus vanishes, hence a two-torus can be endowed with a flat metric $ds^2 = dwd\bar{w}$. The torus has an obvious discrete $\mathbb{Z}_2$ symmetry $w \mapsto -w$ as well as two conformal Killing vectors corresponding to translations along the two one-cycles of the torus. They are described simply by the constant holomorphic vector field $\delta w = c_0$. According to the Riemann-Roch theorem, one expects that the torus has two real moduli.

The torus can be described conveniently as the complex plane quotiented by the discrete identifications

$$w \sim w + 2\pi n u_1 + 2\pi m u_2\,, \quad n, m \in \mathbb{Z}\,, \tag{2.99}$$

where u_1 and u_2 are complex parameters. By a rescaling of w, accompanied by a constant Weyl transformation, one can get rid of the former hence we consider the quotient

$$w \sim w + 2\pi n + 2\pi m\tau\,, \quad n, m \in \mathbb{Z}\,, \tag{2.100}$$

where we have adopted the standard notation $\tau \in \mathbb{C}$ for the torus modulus. As for the circular worldline in the point particle case, see the discussion above eqn. (2.14), an alternative way to think about the torus is to consider the metric

$$ds^2 = |d\sigma^1 + \tau d\sigma^2|^2\,, \tag{2.101}$$

with the standard identifications $\sigma^i \sim \sigma^i + 2\pi$. There exists some discrete ambiguity in the identification of the parameter τ. First, the metric (2.101) is invariant under complex conjugation of the parameter τ, hence we can restrict the discussion to $\tau_2 = \Im(\tau) > 0$ (the case $\Im(\tau) = 0$ being degenerate) *i.e.* to the upper half plane $\mathbb{H}$. For a square torus, $\Re(\tau) = 0$, while in general the real part of τ represents the way the circle parametrized by σ^1 is "twisted" before identifying the two enpoints of the cylinder.

The two-torus, being defined as a quotient of the complex plane, is nothing but a two-dimensional lattice, see fig. 2.5. It is obvious that the same lattice is described by replacing τ by $\tau + 1$. According to the metric (2.101) it amounts to redefine $\sigma^1 \to \sigma_1 + \sigma_2$, which is compatible with the periodicities of the coordinates. It it slightly less obvious to realize that another equivalent parametrization of the torus is given by $\tau \mapsto -1/\tau$, if one allows a Weyl rescaling of the metric. From the metric (2.101), one sees that it amounts to replace $\sigma^1 \to \sigma^2$ and $\sigma^2 \to -\sigma^1$. In other words it exchanges the role of Euclidean worldsheet time and of

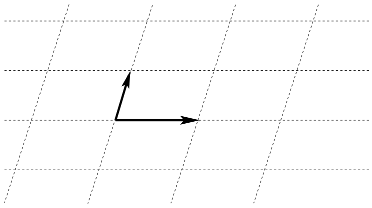

Figure 2.5: *Two-torus as a quotient of the complex plane.*

the space-like coordinate along the string. These two transformations generate the *modular group* $\mathsf{PSL}(2,\mathbb{Z})$, which acts on the modular parameter as

$$\tau \mapsto \frac{a\tau+b}{c\tau+d}\,, \qquad a,b,c,d \in \mathbb{Z} \tag{2.102}$$

This group is indeed the group $\mathsf{SL}(2,\mathbb{Z})$ of 2×2 integer matrices M of determinant one

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,, \quad ad-bc=1\,, \quad a,b,c,d \in \mathbb{Z} \tag{2.103}$$

quotiented by its center, *i.e.* with the identification $M \sim -M$, as replacing (a,b,c,d) by $(-a,-b,-c,-d)$ does not change the action (2.102). To avoid an over-counting in the path integral, we will choose to select a representative of the modular parameter τ into each orbit of the modular group. One can show that every point is the upper half plane $\mathbb{H}>0$ has a unique antecedent under the modular group in the *fundamental domain* $\mathfrak{F}$, defined by the conditions

$$\mathfrak{F} = \{\tau \in \mathbb{H}\,,\ |\mathfrak{R}(\tau)| \leqslant \tfrac{1}{2}\,,\ |\tau| \geqslant 1\}\,, \tag{2.104}$$

where the boundaries for $\mathfrak{R}(\tau)>0$ and $\mathfrak{R}(\tau)<0$ are identified, see fig. (2.6). To anticipate a little bit, this technical point will have drastic consequences. Remember that the torus diagram represents the one-loop contribution in the perturbative expansion in string theory, much as a circle represented the one-loop contribution in the point particle case, see section 2.1. We have found there that the UV divergences in QFT were related to circle of size L going to zero size. In the string theory case, it would correspond to the limit $\mathfrak{I}(\tau)\to 0$, which is completely excluded from the path integral if we choose to integrate over $\mathfrak{F}$, the fundamental domain! This remarkable feature of string theory, which persists to higher order, indicates that the theory is UV-finite.

2.3.4 Conformal symmetry

Our discussion of conformal Killing vectors – *i.e.* of metric-preserving gauge symmetries – had been *global* as we only focused on holomorphic vector fields (2.93) defined everywhere on the manifold. This was important as we wanted to knew exactly the mismatch between integrating over worldsheet metrics and over diffeomorphisms × Weyl gauge symmetries. There are however important properties of the quantum field theory defined on the worldsheet

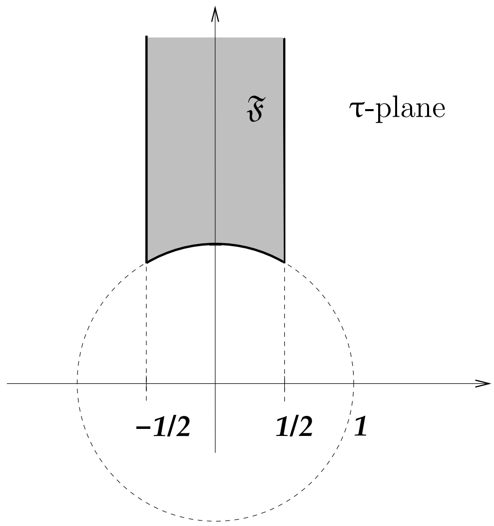

Figure 2.6: *Fundamental domain* $\mathfrak{F}$ *of the modular group.*

of the string by the gauge-fixed Polyakov action (2.90) that depend only on the symmetries of the problem in an open set of the complex plane $\mathbb{C}$.

If we allow the transformations (2.93) to be defined only in some patch of the worldsheet, they generate a much bigger group of symmetry. If we consider finite transformations rather than infinitesimal ones, what these equations tell us is that for any holomorphic function f, the change of coordinates

$$w \mapsto f(w) \tag{2.105a}$$

$$\bar{w} \mapsto \bar{f}(\bar{w}) \tag{2.105b}$$

$$dw\, d\bar{w} \mapsto \left|\partial_w f(w)\right|^{-2} dw\, d\bar{w} \tag{2.105c}$$

can be compensated by a Weyl transformation of parameter $\Omega = \log|\partial f|$ such as to leave invariant the flat two-dimensional metric.

This infinite-dimensional symmetry will play a crucial role in the following. The holomorphic coordinate transformations (2.105a,2.105b) preserve the metric up to a local scale transformation as can be seen from eqn. (2.105c), and the set of such transformations corresponds to the *conformal group*, which is indeed of infinite dimension precisely for a space of dimension two.

We will obtain Ward identities associated with these symmetries for the two-dimensional quantum field theory defined on the worldsheet, as such identities do not care about the well-definiteness of the symmetry at the global level. What we will obtain is an infinite number of constraints on this QFT, which is every theoretical physicist dream!

2.4 Polyakov path integral

We have gathered the ingredients to define properly the path integral associated with the Polyakov action (2.90), which is a path integral over two-dimensional scalar fields x^μ and over two-dimensional metrics γ on Euclidean worldsheets.

We have learned first in section 2.3 that the path integral splits into a sum over topologies, *i.e.* over surfaces of different genera g. In each sector we integrate over metrics γ_g of surfaces with a given genus. Accordingly we are considering the formal vacuum amplitude

$$Z_1 = \sum_{g=0}^{\infty} g_s^{2g-2} \int \frac{\mathcal{D}\gamma_g}{\text{Vol}(\text{diff} \times \text{Weyl})} \int \mathcal{D}x \, \exp\Big(- S_{\text{P}}[\gamma_g, x]\Big)\,, \tag{2.106}$$

that receives contributions from single string worldsheets.

As in the point particle case, we have formally divided the integral over the metrics γ_g by the volume of the gauge group Vol(diff $\times$ Weyl) in order to take care of the gauge redundancies of the formulation of the theory. As there we will define properly this integral by the Faddeev-Popov method.

Inequivalent gauge orbits of metrics γ_g under diff$\times$Weyl gauge transformations are labeled by a finite set of n_μ parameters m_ℓ, the *moduli*. Along each of these orbits one can choose a reference metric $\widehat{\gamma}_g(m_\ell)$, whose image under a transformation Ξ will be denoted $\widehat{\gamma}_g^{\Xi}(m_\ell)$.

We want to trade the integral over the metrics γ_g by the product of an integral over the moduli and an integral over the gauge group, but the latter contains elements that do not change the reference metric, the *conformal Killing vectors*. In the point particle case this extra symmetry was taken care of by fixing the image of the origin under the diffeomorphisms of the worldline. In the present case we have to discuss separately the different topologies, as the outcome will be different. As before we will focus on the two most important cases, the sphere and the torus.

2.4.1 Path integral on the sphere

Surfaces of genus zero have positive curvature, see eqn. (2.71), and are all diff.$\times$ Weyl equivalent to the round unit two-sphere, as the latter has no moduli, see section 2.3.

The conformal Killing vectors on the two-sphere form a group isomorphic to $\mathsf{PSL}(2,\mathbb{C})$, the Möbius group, which is a non-compact Lie group of complex dimension three. While it is easy to gauge-fix part of the rotation subgroup $\mathsf{SO}(3) \subset \mathsf{PSL}(2,\mathbb{C})$ by leaving fixed the origin of the coordinates $w = \bar{w} = 0$ under the diffeomorphisms that we integrate over – exactly as we did for the point particle path integral – there is no way to gauge-fix the whole Möbius symmetry for the vacuum amplitude on the sphere.

The path integral (2.106), which contains the inverse of the volume of the Diff. $\times$ Weyl gauge group, will therefore retain a factor of $1/\text{Vol}\big(\mathsf{PSL}(2,\mathbb{C})\big)$ which vanishes, as the Möbius group is a non-compact group. We learn that the vacuum amplitude on the sphere, *i.e.* at tree level, vanishes, as it should be because otherwise it would mean that we don't expand around a vacuum of the theory.[10]

Of course, string theory, as QFT, is not just about computing the vacuum amplitude. Physical observables correspond generically to time-ordered correlation functions of gauge-invariant observables of the theory. Because of diffeomorphism invariance, the observables take the form

$$\mathcal{O}_k = \int d^2 w \sqrt{\det \gamma}\, \mathcal{V}_k[x^\mu(w,\bar{w})]\,, \tag{2.107}$$

where $\mathcal{V}_k$ is some functional of the fields $x^\mu(w,\bar{w})$ that transforms as a scalar under world-sheet diffeomorphisms; other constraints should be imposed on these functionals, and will be discussed later. Then we have to consider path integrals of the form

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=0} \sim g_s^{-2} \int \mathcal{D}\gamma_0 \mathcal{D}x \prod_{k=1}^{n} \int d^2 w_k \sqrt{\det \gamma}\, \mathcal{V}_k e^{-S_P[x,\gamma_0]}\,. \tag{2.108}$$

In this context the gauge-fixing problem that we had for the vacuum amplitude is easy to solve. If the number n of operators is larger or equal to three, the Möbius symmetry is completely fixed by setting the positions $(w_k, \bar{w}_k)$ of three operators to arbitrary fixed values $(\widehat{w}_k, \widehat{\bar{w}}_k)$ instead of integrating over them.

To formulate the problem a bit differently, the integral $\int \mathcal{D}\gamma_0 \prod_{k=1}^3 d^2 w_k$ over the metrics and the position of three operators covers the whole gauge group, hence can be traded for an integral $\int \mathcal{D}\Xi$ which cancels out completely the volume of the gauge group in the path integral (2.106).

We define then the Faddeev-Popov determinant of string theory on the sphere in terms of the path integral over the gauge group

$$\frac{1}{\Delta_{\text{FP}}(\gamma_0)} := \int \mathcal{D}\Xi\ \delta(\gamma_0 - \widehat{\gamma}_0^{\Xi}) \prod_{k=1}^{3} \delta\left(w_k - \widehat{w}_k^{\Xi}\right) \tag{2.109}$$

where $\widehat{w}_k$ are arbitrary positions and $\widehat{w}_k^{\Xi}$ their images under diff. $\times$ Weyl gauge transformations. We obtain for the full path integral at tree-level

$$\begin{aligned}
\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=0} &= g_s^{-2} \int \frac{\mathcal{D}\gamma_0\, \mathcal{D}\Xi}{\text{Vol}(\text{diff} \times \text{Weyl})} \Delta_{\text{FP}}(\gamma_0) \prod_{k=1}^{n} \int d^2 w_k \sqrt{\det \gamma_0} \int \mathcal{D}x \prod_{k=1}^{n} \mathcal{V}_k[x(w_k,\bar{w}_k)] \\
&\qquad \times \exp\Big(- S_P[\gamma_0, x]\Big)\, \delta(\gamma_0 - \widehat{\gamma}_0^{\Xi}) \prod_{k=1}^{3} \delta\left(w_k - \widehat{w}_k\right) \\
&= g_s^{-2} \int \mathcal{D}x\ \exp\Big(- S_P[\widehat{\gamma}_0, x]\Big)\, \Delta_{\text{FP}}(\widehat{\gamma}_0) \prod_{k=1}^{3} \sqrt{\det \widehat{\gamma}_0} \mathcal{V}_k[x(\widehat{w}_k, \widehat{\bar{w}}_k)] \prod_{K=3}^{n} \mathcal{O}_K\,.
\end{aligned} \tag{2.110}$$

By diffeomorphisms and Weyl transformations, one can bring the metric on the sphere to a flat metric, the price to pay being that the south pole is mapped to $|w| \to \infty$. This is not

[10] As we will see below, the same conclusion can be obtained by noting that the zero-modes of the ghost path integral measure will not be "saturated" by the appropriate ghost insertions.

a problem as long as we consider the compactification of the complex plane, $\bar{\mathbb{C}} := \mathbb{C} \cap \infty$, which is topologically a two-sphere.

In order to evaluate the Faddeev-Popov determinant, we first recall that the argument of the distribution $\delta(\gamma_0 - \widehat{\gamma}_0^{\bar{\xi}})$ around the reference metric is given by eqns. (2.89). We have then

$$\frac{1}{\Delta_{\text{FP}}(\gamma_0)} = \int \mathcal{D}\delta\omega \mathcal{D}\delta w \mathcal{D}\delta\bar{w} \; \delta\left[\delta\omega - \tfrac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w})\right] \delta\left[\partial\delta\bar{w}\right] \delta\left[\bar{\partial}\delta w\right] \prod_{k=1}^{3} \delta\left(\delta w(\widehat{w}_k, \widehat{\bar{w}}_k)\right) \, \delta\left(\delta\bar{w}(\widehat{w}_k, \widehat{\bar{w}}_k)\right) \, . \quad (2.111)$$

All these Dirac distributions are exponentiated by means of a corresponding Lagrange multiplier:

$$\frac{1}{\Delta_{\text{FP}}(\widehat{\gamma}_0)} = \int \mathcal{D}\delta\omega \mathcal{D}\delta w \mathcal{D}\delta\bar{w} \mathcal{D}\eta \; \mathcal{D}\beta \; \mathcal{D}\bar{\beta} \; \prod_{k=1}^{3} \mathrm{d}\phi_k \mathrm{d}\bar{\phi}_k \; e^{2i\pi \int d^2z \, \eta \left(\delta\omega - \frac{1}{2}(\partial\delta w + \bar{\partial}\delta\bar{w})\right)} e^{2i\pi \int d^2z \, \beta \bar{\partial}\delta w} e^{2i\pi \int d^2z \, \bar{\beta}\partial\delta\bar{w}} e^{2i\pi\phi_k \delta w(\widehat{w}_k, \widehat{\bar{w}}_k)} e^{2i\pi\bar{\phi}_k \delta\bar{w}(\widehat{w}_k, \widehat{\bar{w}}_k)} \quad (2.112)$$

We eventually need to insert the FP determinant rather than its inverse as for the particle, therefore we substitute for the variables $(\delta\omega, \delta w, \delta\bar{w}, \eta, \beta, \bar{\beta}, \phi_k, \bar{\phi}_k)$, the Grassmann variables $(\kappa, c, \widetilde{c}, \zeta, b, \widetilde{b}, \psi_k, \bar{\psi}_k)$. We can compute immediately the integrals over κ, ζ, ψ_k and $\bar{\psi}_k$ which gives, after a rescaling of the fields, the relatively simple expression

$$\Delta_{\text{FP}}(\widehat{\gamma}_0) = \int \mathcal{D}b \mathcal{D}\widetilde{b} \mathcal{D}c \mathcal{D}\widetilde{c} \; e^{-\int \frac{d^2w}{2\pi} (b\bar{\partial}c + \widetilde{b}\partial\widetilde{c})} \prod_{k=1}^{3} c(\widehat{w}_k)\widetilde{c}(\widehat{\bar{w}}_k) \quad (2.113)$$

To summarize the sphere path integral of string theory is given in its full glory by the expression ($n > 3$):

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=0} = g_s^{-2} \int \mathcal{D}x \, \mathcal{D}b \mathcal{D}\widetilde{b} \mathcal{D}c \mathcal{D}\widetilde{c} \; e^{-\int \frac{d^2w}{2\pi\alpha'} \, g_{\mu\nu}\partial x^\mu \bar{\partial} x^\nu} e^{-\int \frac{d^2w}{2\pi} (b\bar{\partial}c + \widetilde{b}\partial\widetilde{c})} \prod_{k=1}^{3} c(\widehat{w}_k)\widetilde{c}(\widehat{\bar{w}}_k) \prod_{k=1}^{3} \mathcal{V}_k[x(\widehat{w}_k, \widehat{\bar{w}}_k)] \prod_{K=3}^{n} \mathcal{O}_K \, , \quad (2.114)$$

where $\{\widehat{w}_k\}$ are arbitrary positions, that we take usually to be $\{0, 1, +\infty\}$.

2.4.2 Path integral on the torus

The second important case is the two-torus, that corresponds to the one-loop amplitude in string theory, and present some differences with the previous case.

Gauge-inequivalent surfaces of genus one correspond to two-tori characterized by one complex parameter τ in the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}, \Im(z) > 0\}$. Because of the ambiguity in assigning a modular parameter τ to a given two-torus – or equivalently to a given two-dimensional lattice – one restricts further τ to be in the fundamental domain $\mathfrak{F}$ of the modular group, see eqn. (2.104). A two-dimensional metric on the torus of complex modular parameter

$$\tau = \tau_1 + i\tau_2 \,, \quad |\tau_1| \leqslant \tfrac{1}{2} \,, \quad \tau_1^2 + \tau_2^2 \geqslant 1 \tag{2.115}$$

is then given by the flat metric (2.101) with $\sigma^i \sim \sigma^i + 2\pi$.

The two-torus has two real conformal Killing vectors, that are easy to understand, as they correspond to translations along σ^1 and along σ^2. These translations form a group isomorphic to $U(1)^2$, whose volume is $4\pi^2\tau_2$, the area of the torus of metric (2.101). It means that the integral $\int d^2\tau\, \mathcal{D}\Xi$ over the moduli and over the gauge group covers more that the integral $\int \mathcal{D}\gamma_1$ over the metrics on genus one surfaces.

Fixing this extra symmetry is possible, even for the vacuum amplitude in the following way (this is the same method that we used for the point particle vacuum amplitude). A general diffeomorphism is given by two functions $\Sigma^i(\sigma^k)$ that will not generically map the origin of the coordinates $\sigma^1 = \sigma^2 = 0$ to the origin, as $\Sigma^i(0,0) \neq 0$ in general. Because of translation invariance is possible to restrict the path integral to diffeomorphisms preserving the origin, *i.e.* such that $\Sigma^i(0,0) = 0$ or, say differently, for every diffeomorphism Σ^i one can translate back the image of the origin to $\Sigma^i(0,0) = 0$, which will select an element of the translation group . In this way, the spurious gauge freedom will be removed. In addition to these continuous symmetries, the torus metric is naturally invariant under the $\mathbb{Z}_2$ symmetry $\sigma^i \mapsto -\sigma^i$, which doubles the volume of the group of metric-preserving gauge transformations.

These considerations lead to the following expression for the Faddeev-Popov determinant on the two-torus, as an integral over the gauge transformations $\Xi = (\Sigma, \Omega)$:

$$\frac{1}{\Delta_{\text{FP}}(\gamma_1)} := \int_{\mathfrak{F}} d\tau \int \mathcal{D}\Xi\; \delta\Big(\gamma_1 - \widehat{\gamma}_1^{\Xi}(\tau)\Big)\delta\big(\Sigma^1(0)\big)\delta\big(\Sigma^2(0)\big)\,, \tag{2.116}$$

with the reference metric $\widehat{\gamma}_1(\tau)$ obtained from eqn. (2.101)

$$\widehat{\gamma}_1(\tau) = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} . \tag{2.117}$$

Expanding the Dirac distribution near the reference metric gives

$$\begin{aligned} \delta\Big(\gamma_1 - \widehat{\gamma}_1^{\Xi}(\tau)\Big) &= \\ &\delta\Big(2\delta\omega - 2\partial_1(\delta\sigma^1 + \tau_1\delta\sigma^2)\Big)\delta\Big(2|\tau|^2\delta\omega - 2\partial_1(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) + 2(\tau_1\delta\tau_1 + \tau_2\delta\tau_2)\Big) \\ &\quad \delta\Big(2\delta\omega\tau_1 - \partial_1(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) - \partial_2(\delta\sigma^1 + \tau_1\delta\sigma^2) + 2\delta\tau_1\Big) \end{aligned} \tag{2.118}$$

As before we introduce Lagrange multipliers fields. Compared to the sphere case as the reference metric is not diagonal it will be technically slightly more cumbersome. We introduce a symmetric two-index tensor of Lagrange multipliers of components β^{ij} as well as a one-form of components η_i. It gives

$$\frac{1}{\Delta_{\rm FP}(\widehat{\gamma}_1(\tau))} = \int d\delta\tau\, \mathcal{D}\delta\omega\, \mathcal{D}\delta\sigma^i\, \mathcal{D}\beta^{ij} d\eta^i \, \exp\left(2i\pi\eta_i\delta\sigma^i(0)\right)$$

$$\exp 2i\pi\tau_2 \int d^2\sigma \Big\{ \beta^{11}\big(\delta\omega - \partial_1(\delta\sigma^1 + \tau_1\delta\sigma^2)\big) + \beta^{22}\big(|\tau|^2\delta\omega - \partial_1(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) + 2(\tau_1\delta\tau_1 + \tau_2\delta\tau_2)\big)$$

$$+ 2\beta^{12}\big(2\delta\omega\tau_1 - \partial_1(|\tau|^2\delta\sigma^2 + \tau_1\delta\sigma^1) - \partial_2(\delta\sigma^1 + \tau_1\delta\sigma^2) + 2\delta\tau_1\big)\Big\} \quad (2.119)$$

Integrating over $\delta\omega$ imposes that the tensor β is traceless, *i.e.* that $\beta^{ij}\widehat{\gamma}_1(\tau)_{ij} = 0$. The remaining path integral in $\mathcal{D}b$ will be therefore on traceless tensors only. In order to get the FP determinant rather that its inverse, we replace $(\beta^{ij}, \delta\sigma^i, \eta^i, \delta\tau_\ell)$ by Grassmann variables $(b^{ij}, c^i, \psi^i, \kappa_\ell)$ and get after rescaling of the fields

$$\Delta_{\rm FP}(\widehat{\gamma}_1(\tau)) = \int d\kappa\, \mathcal{D}c^i\, \mathcal{D}b^{ij} d\psi_i \, \exp\left(\psi_i c^i(0)\right) \exp\Big\{ -\frac{1}{2\pi}\int d^2\sigma \sqrt{\det \widehat{\gamma}_1(\tau)}\; b^{ij}\partial_i c_j \Big\}$$

$$\times \exp\Big\{ \frac{1}{2\pi}\int d^2\sigma \sqrt{\det \widehat{\gamma}_1(\tau)}\kappa_\ell b^{ij}\partial_{\tau_\ell}\widehat{\gamma}_1(\tau)_{ij}\Big\}, \quad (2.120)$$

which can be further simplified by integrating over κ_ℓ and over ψ^i.

Inserting this result in the Polyakov path integral (2.106) one gets finally the vacuum amplitude as[11]

$$Z_1 = \int_{\mathfrak{F}} \frac{d\tau d\bar{\tau}}{16\pi^2\tau_2} \int \mathcal{D}x^\mu\, \mathcal{D}b^{ij}\, \mathcal{D}c_i \, \exp\Big\{ -\frac{\tau_2}{4\pi}\int d^2\sigma \left(b^{ij}\partial_i c_j + \tfrac{1}{\alpha'} g_{\mu\nu}\widehat{\gamma}_1^{ij}\, \partial_i x^\mu \partial_j x^\nu\right)\Big\}$$

$$c^1(0)c^2(0) \times \frac{\tau_2}{4\pi}\int d^2\sigma\, b^{ij}\partial_\tau\widehat{\gamma}_1(\tau)_{ij} \times \frac{\tau_2}{4\pi}\int d^2\sigma\, b^{ij}\partial_{\bar{\tau}}\widehat{\gamma}_1(\tau)_{ij} \quad (2.121)$$

The dependence in $1/\tau_2$ in the measure of integration over the modulus τ comes from the volume of the group of translations along the torus, since have we have explained we don't integrate over this part of the gauge group. As we will see this factor ensures that the result is invariant under the modular group $\mathsf{PSL}(2,\mathbb{Z})$.

It is finally convenient to come back to complex coordinates $(w, \bar{w}) = (\sigma^1 + \tau\sigma^2, \sigma^1 + \bar{\tau}\sigma^2)$. We write then

$$b_{ww} = b\ , \quad b_{\bar{w}\bar{w}} = \widetilde{b}\ , \quad c^w = c\ , \quad c^{\bar{w}} = \widetilde{c} \quad (2.122)$$

The insertion in $b(w,\bar{w})$ and $\widetilde{b}(w,\bar{w})$ takes in this basis the form

$$\frac{1}{2\pi\tau_2}\int d^2w\, b(w,\bar{w})\ \frac{1}{2\pi\tau_2}\int d^2w\, \widetilde{b}(w,\bar{w})\ , \quad (2.123)$$

[11]The exact normalization of the result could be obtained by checking carefully how the insertions in the path integral are normalized, however since the original path integral is ill-defined, and, since we did several field rescalings while manipulating formally the path integral, the honest way to get the factor right is to ask that the end result is properly normalized if interpreted as a partition function. Concretely if one chooses a fully compact space-time one can request that the vacuum appears with degeneracy one.

As we will see later, one can replace $b(w, \bar{w})$ by its value at any given point, for instance $w = \bar{w} = 0$, as the non-zero modes of the field don't contribute to the path integral (2.121). Therefore we can replace $\int d^2w\, b \to 4\pi^2\tau_2 b(0)$, and the same for the $\widetilde{b}$ insertion.

We get the final result for the vacuum amplitude of string theory at one-loop as

$$\boxed{Z_1 = \int_{\mathfrak{F}} \frac{d^2\tau}{4\tau_2} \int \mathcal{D}x^\mu\, \mathcal{D}b\mathcal{D}\widetilde{b}\, \mathcal{D}c\mathcal{D}\widetilde{c}\;\; b(0)\widetilde{b}(0)\, c(0)\widetilde{c}(0)\, e^{-\int \frac{d^2w}{4\pi}\left(b\bar{\partial}c + \widetilde{b}\partial\widetilde{c} + \frac{1}{\alpha'}g_{\mu\nu}\partial x^\mu \bar{\partial} x^\nu\right)}} \qquad (2.124)$$

The dependence of the integrand in the modulus τ is hidden in the periodicity of the complex variable $w \sim w + 1 \sim w + \tau$.

If one wants to compute a different observable as an n-point function $\langle \mathcal{O}_1 \cdots \mathcal{O}_n \rangle_{g=1}$, it is enough to insert the operators $\mathcal{O}_k$ in the path integral above, as no gauge-fixing of the position of some operators is needed in the present case.

This result is very close to the sphere path integral (2.114), the differences reflecting the number of moduli and conformal Killing vectors in each case. One can generalize of course this discussion to surfaces of higher genera, but a rigorous presentation would be rather technical.

In this chapter we have assumed that the gauge symmetries of the classical theories, diffeomorphisms and Weyl transformations, were also valid in the quantum theory. As we shall see, the latter may be violated by an *anomaly* that put the theory in danger of being inconsistent. Before proceeding to this computation, we will introduce in the next chapter the powerful methods of two-dimensional conformal field theory.

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Chapter 3

Conformal field theory

Two-dimensional conformally-invariant quantum field theories were introduced by Belavin, Polyakov, and Zamolodchikov in 1984 [1]. These theories play also an essential role in statistical physics, for the description of critical phenomena in two dimensions. The book [2] is probably the most comprehensive book on the subject, and reference [3] is geared towards string applications.

After gauge-fixing of the diffeomorphisms and Weyl transformations to the flat worldsheet metric $ds^2 = dw d\bar{w}$, the Euclidean Polyakov action

$$\mathcal{S} = \frac{1}{4\pi\alpha'} \int_{\mathfrak{s}} d^2w \, G_{\mu\nu}[X^\rho] \, \partial X^\mu \bar{\partial} X^\nu \tag{3.1}$$

has an infinite-dimensional residual symmetry that consists in the coordinates transformation

$$w \mapsto f(w) \, , \quad \bar{w} \mapsto \bar{f}(\bar{w}) \, , \tag{3.2}$$

where f is a holomorphic function, that leaves the metric invariant up to a conformal factor:

$$dw d\bar{w} \mapsto \left(\frac{\partial f(w)}{\partial w} \right) \left(\frac{\partial \bar{f}(\bar{w})}{\partial \bar{w}} \right) dw d\bar{w} \tag{3.3}$$

As we have seen previously, the topology of the worldsheet restricts severely the transformations of this type that are allowed globally, *i.e.* the conformal Killing vectors of the surface. On the sphere we have found that they corresponded to the Möbius group $PSL(2, \mathbb{C})$, while on the two-torus the only holomorphic functions periodic around both of the one-cycles are constants. As a result, the transformations (3.2) are not properly speaking local symmetries of string theory itself. In addition we will see shortly that the local Weyl symmetry might not hold in the quantum theory due to an anomaly.

At this stage we will be interested in a slightly different problem, the properties of two-dimensional quantum field theories defined on the complex plane $(w, \bar{w})$ with *global* symmetries including the two-dimensional conformal transformations (3.2). In this context two-dimensional gravity is no longer dynamical, and we work in a theory with a fixed two-dimensional flat metric $ds^2 = dz d\bar{z}$. In the string theory applications we will consider the compactification of the complex plane by adding the point at infinity, $\bar{\mathbb{C}} = \mathbb{C} \cap \{\infty\}$, which is related to the two-dimensional sphere (2.95) by a Weyl transformation $\omega = \frac{1}{2} \log(1 + w\bar{w})$.

In general, quantum field theory with conformal invariance are called *conformal field theories*. In two dimensions this symmetry is often powerful enough to solve exactly the theory, without using any perturbative expansion.

3.1 The conformal group in diverse dimensions

The conformal group in two-dimensions is of infinite dimension, unlike the conformal group in higher dimensions. To understand this we will study the conformal group in arbitrary Euclidean space-time of dimension D.

Conformal transformations are defined as coordinate transformations that preserve the metric up to a conformal factor, *i.e.*

$$x^i \mapsto \tilde{x}^i(x^j)\ , \quad \delta_{ij}dx^i dx^j \mapsto \delta_{ij}\frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^\ell}d\tilde{x}^j d\tilde{x}^\ell = \exp\left(2\Omega(\tilde{x}^k)\right)\delta_{ij}d\tilde{x}^j d\tilde{x}^\ell\,. \tag{3.4}$$

Here the metric is fixed, hence such transformation is really changing the geometry. The defining propery of conformal transformations is to preserve angles between vectors of tangent space over any given point.

Let us consider a generic differentiable change of coordinates $x^i = x^i + \delta x^i(x^j)$. At linear order in δx, this change generates a conformal transformation of the D-dimensional metric γ provided that

$$\exists \alpha \in \mathcal{C}^0(\mathbb{R}^D)\ , \quad \delta(\gamma_{k\ell}) = -(\partial_\ell \delta x_k + \partial_k \delta x_\ell) = \alpha(\sigma^i)\delta_{k\ell}\,. \tag{3.5}$$

Taking the trace of tells us that $-2\partial_i\delta x^i = D\alpha$, hence one has the equation

$$(\partial_\ell \delta x_k + \partial_k \delta x_\ell) - \tfrac{2}{D}(\partial_i \delta x^i)\delta_{k\ell} = 0\,. \tag{3.6}$$

Acting on this equation with ∂^ℓ finally gives

$$\Box \delta x_k + \left(1 - \tfrac{2}{D}\right)\partial_k(\partial_i \delta x^i) = 0\,. \tag{3.7}$$

Let us assume a power series expansion:

$$\delta x^i = a^i + m^i{}_j x^j + b^i x^2 + x^i(c_j x^j) + \mathcal{O}(x^3) \tag{3.8}$$

and look for constraints on all the coefficients. While a is unconstrainted, eqn. (3.6) tells us that

$$m_{(ij)} - \tfrac{1}{D}\delta_{ij}\mathrm{Tr}\,(m) = 0\,, \tag{3.9}$$

hence m splits into a trace part and an antisymmetric part. Equation (3.7) becomes at this order

$$2Db^i + 2c^i + \left(1 - \tfrac{2}{D}\right)(2b^i + (D+1)c^i) = 0 \tag{3.10}$$

which is solved for $c^i = -2b^i$.

One can check that there are no solutions to the problem for higher order terms in (3.8) if $D > 2$. The space of solutions of these equations is then finite and corresponds to:

- translations: $\delta x^i = a^i$ constant
- rotations: $\delta x^i = r^i{}_j x^j$ with r in the vector representation of $\mathfrak{so}(D)$, i.e. the antisymmetric part of m in eqn. (3.8)
- dilatations: $\delta x^i = \lambda x^i$ with λ a non-vanishing constant, i.e. the trace part of m in eqn. (3.8)
- special conformal transformations $\delta x^i = b^i x^2 - 2x^i(b_j x^j)$.

These are actually a set of generators of the Lie algebra $\mathfrak{so}(D+1,1)$, *i.e.* of the Lie Algebra of the Lorentz group in $D+2$ dimensions.

In two dimensions, the conformal transformations are given, in complex coordinates, by $w \mapsto f(w)$, $\bar{w} \mapsto \bar{f}(\bar{w})$ with f holomorphic, as we have already noticed. Naturally this infinite-dimensional group contains as a subgroup the transformations existing in general dimensions. Explicitly:

- translations: $\delta w = a$, $\delta \bar{w} = \bar{a}$
- rotations: $\delta w = i\theta w$, $\delta \bar{w} = -i\theta \bar{w}$ with $\theta \in \mathbb{R}$.
- dilatations: $\delta w = \lambda w$, $\delta \bar{w} = \lambda \bar{w}$ with $\lambda \in \mathbb{R}$.
- special conformal transformations $\delta w = -\bar{b}\, w^2$, $\delta \bar{w} = -b\, \bar{w}^2$, $b \in \mathbb{C}$.

Following the general discussion this generates a Lie group isomorphic to $SO(3,1)$, i.e. the Lorentz group in three dimensions, whose component connected to the identity is isomorphic to the Möbius group $PSL(2,\mathbb{C})$ that appeared already in section 2.3. It was shown there that the Möbius group was the subgroup of two-dimensional conformal transformations that are globally defined on the two-sphere, or equivalently on the compactified complex plane $\bar{\mathbb{C}}$.

3.2 Radial quantization

The Euclidean two-dimensional conformal field theory associated with the worldsheet of a propagating string is naturally associated with a surface with the topology of a cylinder, *i.e.* parametrized by two coordinates (σ_1, σ_2) with $\sigma_1 \sim \sigma_1 + 2\pi$, or equivalently the complex coordinate $w = \sigma^1 + i\sigma^2$ with $w \sim w + 2\pi$. The coordinate σ^2, which runs from $-\infty$ to $+\infty$, is the Euclidean time, obtain through Wick rotation: $\sigma^2 = -i\tau$.

Even outside the string theory context, the natural starting point for canonical quantization of conformal field theories is on the cylinder, in order to avoid infrared problems in the case of an infinite space direction.

States of the quantum field theory are defined on a space-like slice, *i.e.* on a slice of constant σ_2 after Wick rotation to Euclidean space. In particular, an initial state $|\text{in}\rangle$ of the theory is defined on a slice with $\sigma_2 \to -\infty$.

Invariance of the theory under conformal transformation allows to give a different representation of the (Euclidean) time evolution of the QFT and of states of the theory. Let us consider the conformal mapping from the cylinder to the complex plane:

$$w \mapsto z = e^{-iw} \, , \quad \bar{w} \mapsto \bar{z} = e^{i\bar{w}} \, . \tag{3.11}$$

In this description, $\sigma^2 \to -\infty$ is mapped to the origin $z = 0$ and slices of constant σ^2 are mapped to circles around the origin, see fig. 3.1.

Time evolution correspond to dilatations or, said differently, the Hamiltonian operator corresponds to the dilatation operator. Likewise, time-ordered correlation functions of the QFT become *radial-ordered* correlation functions.

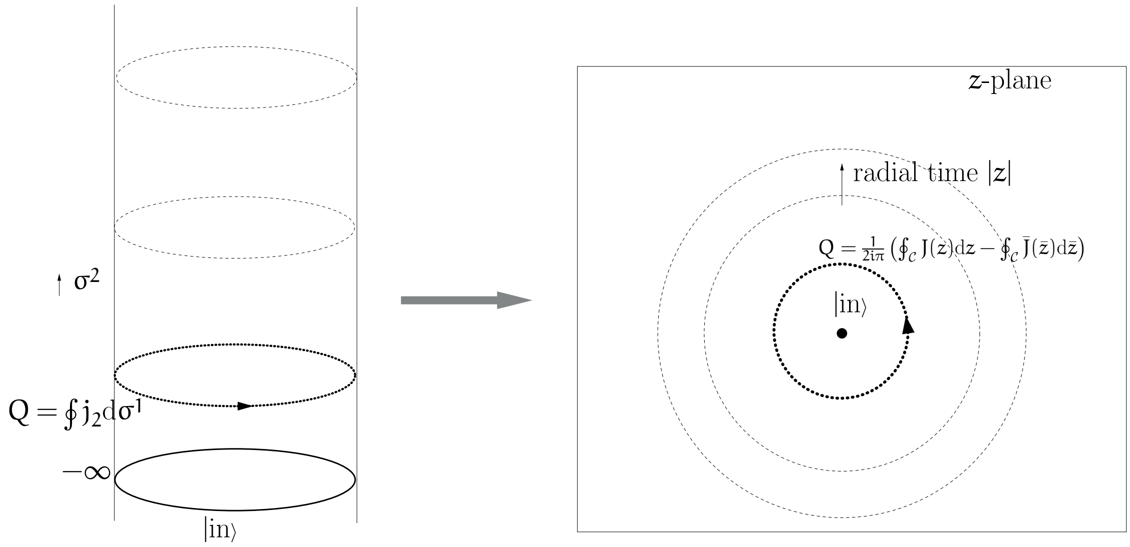

Figure 3.1: *Conformal mapping of the cylinder to the complex plane.*

3.2.1 State - operator correspondence

In quantum mechanics or quantum field theories, states and operators are very different kind of objects.

On the one hand, states characterize the system at a defined time and are implemented in the path integral formalism as boundary conditions for the functional integral. For instance, in quantum mechanics,

$$\langle q_1, t | q_0, 0\rangle = \int_{q(0)=q_0}^{q(t)=q_1} \mathcal{D}q \exp \frac{i}{\hbar}\int_0^t dt' \mathcal{L}[q(t')] \,. \tag{3.12}$$

If we prepare the system in a given state characterized by the wave-function $\Psi_0(q, 0)$ at $t = 0$, the wave-function at time t is given by

$$\Psi_1(q_1, t) = \int dq_0 \int_{q(0)=q_0}^{q(t)=q_1} \mathcal{D}q\, \Psi_0(q_0, 0) \exp \frac{i}{\hbar}\int_0^t dt' \mathcal{L}[q(t')] \,. \tag{3.13}$$

Likewise in quantum field theory a state is a functional of the fields defined on a space-like slice of space-time, in the present case on the circle parametrized by σ^1.

On the other hand, local operators in quantum field theories are defined as arbitrary local expressions constructed from the elementary fields of the theory and their derivatives, *i.e.* local functionals $\Psi[\hat{\phi}^i(x^\mu), \partial_\nu\hat{\phi}^i(x^\mu), \cdots]$.

The conformal mapping between the cylinder and the plane has some surprising consequence. Let us consider an initial state $|\text{in}\rangle$ of the CFT on the cylinder, defined on the circle parametrized by σ^1 in the infinite past $\sigma^2 \to -\infty$. Under the conformal mapping (3.11) it is mapped to the origin of the plane $z = \bar{z} = 0$. This means that the initial state is mapped to a local object at the origin, in other words a *local operator*, inserted at the origin. This is called the *state-operator correspondence*:

$$\widehat{A}|0\rangle \ \leftrightarrow \mathcal{O}_A(z=0,\bar{z}=0)\,, \tag{3.14}$$

which says that a state obtained by acting on the vacuum by some operator $\widehat{A}$ is equivalent to a local operator $\mathcal{O}_A(z,\bar{z})$ inserted at the origin of the complex plane. In this perspective the vacuum $|0\rangle$ corresponds to the identity operator on the right-hand side.

3.2.2 Conserved charges

Let us consider a field theory with a conserved current, $\partial_\mu j^\mu = 0$, and a certain space-like foliation of the ambient space-time. One can defined a conserved charge

$$Q = \int j_\mu d\Sigma^\mu \tag{3.15}$$

where $d\Sigma^\mu$ is the surface element over a space-like slice. For a conserved current on the cylindrical worldsheet of the string, it gives

$$Q = \oint j_2 \frac{d\sigma^1}{2\pi}\,. \tag{3.16}$$

Let us map the theory to the plane. A two-dimensional current has components $J_z(z,\bar{z})$ and $J_{\bar{z}}(z,\bar{z})$ in complex coordinates, and current conservation means that $\bar{\partial}J_z = 0$ and $\partial\bar{J}_z = 0$. Hence $J := J_z(z)$ is a holomorphic function and $\bar{J} := \bar{J}_{\bar{z}}(z)$ a anti-holomorphic one.

The integral around the cylinder $\int d\sigma^1 j_r$ becomes an integral over the polar angle, $\int d\theta\, J_r$. In complex coordinates, one obtains

$$Q = \frac{1}{2i\pi}\left(\oint_{\mathcal{C}} J(z)dz - \oint_{\mathcal{C}} \bar{J}(\bar{z})d\bar{z}\right)\,, \tag{3.17}$$

where $\mathcal{C}$ denotes a contour encircling the local operator corresponding to the state for which we compute the charge. In many cases one finds conserved holomorphic and anti-holomorphic charges, associated respectively to holomorphic and anti-holomorphic transformations.

We will consider below the main application of this formalism to conformal field theories, by looking at conservation laws associated with conformal transformation themselves.

3.3 Conformal invariance and Ward identities

The dynamics of quantum field theories with conformal invariance is severely constrained by this symmetry, especially in two dimensions where this symmetry is infinite dimensional.

3.3.1 Stress-energy tensor

In the classical theory, the stress-energy tensor obeys strong constraints for conformal invariance. Under a change of the background metric, the action of a field theory transforms as

$$\delta S = -\frac{1}{2\pi}\int d^D x\, \delta\gamma_{ij} \frac{\delta}{\delta\gamma_{ij}} \sqrt{-\det\gamma}\,\mathcal{L} = -\frac{1}{2\pi}\int d^D x\, \sqrt{-\det\gamma}\, T^{ij}\delta\gamma_{ij} \tag{3.18}$$

Consider a general diffeo.$\times$ Weyl transformation of the background metric, $\delta\gamma_{ij} = 2\gamma_{ij}\delta\omega - (\nabla_i\delta\sigma_j + \nabla_j\delta\sigma_i)$. Invariance under diffeomorphisms implies that the stress-energy tensor is conserved:

$$\nabla^i T_{ij} = 0\,, \tag{3.19}$$

while invariance under Weyl symmetries implies that the stress-energy tensor is traceless:

$$T_{ij}\gamma^{ij} = 0\,. \tag{3.20}$$

In two-dimensions, these expressions take a particularly simple form in complex coordinates with a flat background metric:

$$\bar\partial T_{z\star} = \partial T_{\bar z\star} = 0 \tag{3.21}$$

for the former, and

$$T_{z\bar z} = 0 \tag{3.22}$$

for the later. This means that the stress-energy tensor has only two non-vanishing components,

$$T := T_{zz}\ , \quad \widetilde{T} := T_{\bar z\bar z}\,, \tag{3.23}$$

and that they are respectively holomorphic and anti-holomorphic functions:

$$\bar\partial T_{zz} = 0 \implies T = T(z)\,, \tag{3.24a}$$

$$\partial T_{\bar z\bar z} = 0 \implies \widetilde{T} = \widetilde{T}(\bar z)\,. \tag{3.24b}$$

We will finally give the form of the Noether currents associated with conformal transformations. Let us consider the infinitesimal transformation

$$z \mapsto z + \rho(z,\bar z)\varepsilon(z)\ , \quad \bar z \mapsto \bar z + \bar\rho(z,\bar z)\bar\varepsilon(\bar z)\,, \tag{3.25}$$

which reduces to a conformal transformation for constant ρ and $\bar\rho$. According to eqn. (3.18) the variation of the action of the theory is then

$$\begin{aligned}\delta S &= -\frac{1}{2\pi}\int d^2 z\,\left\{T(z)\bar\partial\big(\rho(z,\bar z)\varepsilon(z)\big) + \widetilde{T}(\bar z)\partial\big(\bar\rho(z,\bar z)\bar\varepsilon(\bar z)\big)\right\}\\ &= -\frac{1}{2\pi}\int d^2 z\,\big(T(z)\varepsilon(z)\bar\partial\rho(z,\bar z) + \widetilde{T}(\bar z)\bar\varepsilon(\bar z)\partial\bar\rho(z,\bar z)\big)\end{aligned} \tag{3.26}$$

A very powerful point of view in two-dimensional Euclidean QFT is to consider w and $\bar{w}$ are independent variables. This means that we consider the analytic continuation of the Euclidean space $\mathbb{R}^2$ to $\mathbb{C}^2$, and the move from coordinates (σ^1, σ^2) to $(w, \bar{w})$ just as a change of basis in $\mathbb{C}^2$. Naturally at the end of the day one should enforce the reality condition $(\bar{z})^\star = z$.

From this point of view one can consider purely holomorphic conformal transformations, *i.e.* with $\bar{\varepsilon} = 0$. The associated Noether current is then

$$J^z = T(z)\varepsilon(z) \ , \quad J^{\bar{z}} = 0 \tag{3.27}$$

The non-zero component of the current, $J := J^z$ is then holomorphic: $\bar{\partial} J = 0$.

In the same way a purely anti-holomorphic conformal transformation, , *i.e.* with $\bar{\varepsilon} = 0$, gives the conserved current

$$\bar{J}^z = 0 \ , \quad \bar{J}^{\bar{z}} = \widetilde{T}(\bar{z})\bar{\varepsilon}(\bar{z}) \, , \tag{3.28}$$

whose non-zero component $\bar{J} := \bar{J}^{\bar{z}}$ is anti-holomorphic: $\partial \bar{J} = 0$.

As in other quantum field theories, these classical equations lead to functional equations in the quantum theory, known as *Ward identities* that we shall study now.

3.3.2 Ward identities

Let us consider a generic time-ordered N-point correlation function of *local operators* in a two-dimensional conformal field theory. By local operator we mean any operator that can be written as a local expression in the fundamental fields ϕ^i and their derivatives. The corresponding path integral is written as

$$\langle \mathcal{O}_1(z_1, \bar{z}_1) \cdots \mathcal{O}_N(z_N, \bar{z}_N) \rangle = \int \mathcal{D}\phi^i e^{-S[\phi^i]} \mathcal{O}_1[\phi^i](z_1, \bar{z}_1) \cdots \mathcal{O}_N[\phi^i](z_N, \bar{z}_N) \tag{3.29}$$

We consider a holomorphic conformal transformation with support in a small disc $\mathfrak{d}$ in the complex plane, *i.e.* a transformation (3.25) with ρ non-vanishing only in the neighborhood of some point $(z_0, \bar{z}_0)$ and $\bar{\rho} = 0$.

There are two cases to consider: either there are no local operators in this neighborhood, or there is (at least) one inserted at a point inside the disk, see fig. 3.2.

Let us start with the former case. By construction the local operators are not affected by the conformal transformation, hence the change in the path integral comes from the measure $\mathcal{D}\phi^i$ and from the action $S[\phi^i]$ only. We will make the assumption that the measure is invariant so the only variation comes from the action.[1] The path integral is then modified at first order as

$$\int \mathcal{D}\phi^i e^{-S[\phi^i]} \mathcal{O}_1 \cdots \mathcal{O}_N \ \mapsto \ \int \mathcal{D}\phi^i e^{-S[\phi^i]} \left(1 - \frac{1}{2\pi} \int d^2 z \, T(z) \varepsilon \bar{\partial} \rho \right) \mathcal{O}_1 \cdots \mathcal{O}_N \, . \tag{3.30}$$

[1] This assumption is very strong and turns out actually to be wrong in many cases; this is the path integral view of QFT anomalies. We will come back to this important issue later on.

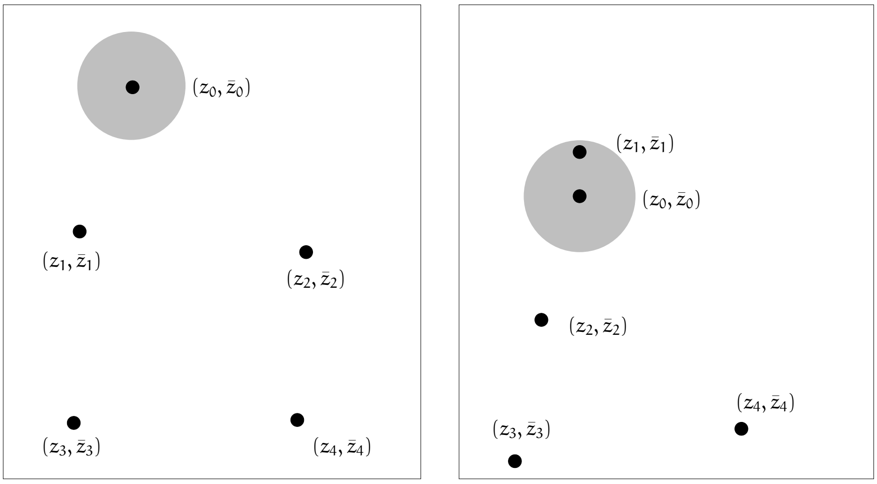

Figure 3.2: *Conformal transformation with support away from local operators (left panel) and including one local operator (right panel).*

The path integral should actually be completely independent of the transformation which should just be thought as some change of variables in the integral. Therefore one gets the constraint, after integration by parts

$$\int \mathcal{D}\phi^i e^{-S[\phi^i]} \int d^2z\, \rho(z,\bar{z})\, \bar{\partial}\Big(T(z)\varepsilon(z)\Big)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N) \tag{3.31}$$

Since this should hold for any choice of ρ, we are led to the condition that

$$\Big\langle \bar{\partial}\Big(T(z)\varepsilon(z)\Big)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\Big\rangle = 0\,. \tag{3.32}$$

This is the quantum version of Noether theorem; naturally there is an analogous formula for anti-holomorphic conformal transformations.

We move now to the latter case. Let assume that the operator $\mathcal{O}(z_1,\bar{z}_1)$ is inserted at a point where $\rho \neq 0$, *i.e.* where the conformal transformation has support. Then on general grounds the operator transforms

$$\mathcal{O}(z_1,\bar{z}_1) \;\mapsto \mathcal{O}(z_1,\bar{z}_1) + \delta\mathcal{O}(z_1,\bar{z}_1)\,. \tag{3.33}$$

We insert this transformation in the path integral and, using similar arguments as before, we get at first order the relation

$$-\frac{1}{2\pi}\int d^2z\, \rho(z,\bar{z})\, \Big\langle \bar{\partial}\Big(T(z)\varepsilon(z)\Big)\mathcal{O}_1(z_1,\bar{z}_1)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\Big\rangle$$
$$= \Big\langle \delta\mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2(z_2,\bar{z}_2)\cdots\mathcal{O}_N(z_N,\bar{z}_N)\Big\rangle\,. \tag{3.34}$$

The function ρ being arbitrary, we can take it to be the indicator function of the disk $\mathfrak{d}$ around $(z_0,\bar{z}_0)$. The left-hand side can then be simplified using Stokes' theorem:

$$\int_{\mathfrak{d}} \left(\partial_z J^z + \partial_{\bar{z}} J^{\bar{z}}\right) = -i\oint_{\partial\mathfrak{d}} \left(J_z dz - J_{\bar{z}} d\bar{z}\right). \tag{3.35}$$

In the present case $J^{\bar{z}} = 0$ as we found before so eqn. (3.34) gives

$$\frac{i}{2\pi}\oint_{\partial\mathfrak{d}} dz \left\langle T(z)\varepsilon(z)\mathcal{O}_1(z_1,\bar{z}_1)\mathcal{O}_2\cdots\mathcal{O}_N\right\rangle = \left\langle \delta\mathcal{O}_1(z_1,\bar{z}_1)\,\mathcal{O}_2\cdots\mathcal{O}_N\right\rangle\,, \quad (z_1,\bar{z}_1)\in\mathfrak{d}\,. \tag{3.36}$$

This teaches us a very important result: the change of an operator $\delta\mathcal{O}$ under a conformal transformation is given by the residue of its product with the stress-energy tensor, namely

$$\delta\mathcal{O}_1(z_1,\bar{z}_1) = \frac{i}{2\pi}\oint_{\partial\mathfrak{d}} dz\, T(z)\varepsilon(z)\mathcal{O}_1(z_1,\bar{z}_1) = -\mathrm{Res}_{z\to z_1}\left(T(z)\varepsilon(z)\,\mathcal{O}_1(z_1,\bar{z}_1)\right) \tag{3.37}$$

The holomorphic function ε being arbitrary, this means that the operator product between $T(z)$ and $\mathcal{O}(z_1,\bar{z}_1)$ has a first order pole when they approach to each other:

$$T(z)\,\mathcal{O}_1(z_1,\bar{z}_1) = \cdots + \frac{\mathrm{Res}\,\left(T(z)\mathcal{O}_1(z_1,\bar{z}_1)\right)}{z-z_1} + \cdots\,. \tag{3.38}$$

There is naturally a similar story for the anti-holomorphic conformal transformations, which gives in particular:

$$\widetilde{T}(\bar{z})\,\mathcal{O}_1(z_1,\bar{z}_1) = \cdots + \frac{\mathrm{Res}\,\left(\widetilde{T}(\bar{z})\mathcal{O}_1(z_1,\bar{z}_1)\right)}{\bar{z}-\bar{z}_1} + \cdots\,. \tag{3.39}$$

These singularities when operators approach each other are a generic feature of quantum field theories. They have been studied more systematically under the name of *operator product expansions*. The operator product expansion exists in all quantum field theories, and provides the behavior of the theory when two operators $\mathcal{O}_1$ and $\mathcal{O}_2$ come close to each other. The basic idea is that, when the separation between them become infinitesimal, one can expand the product into a sum of local operators. While this is usually an asymptotic expansion only, one can show that in the case of conformal field theories the series converges, with the radius of convergence given by the distance to the nearest operator $\mathcal{O}_{n\neq 1,2}$.

3.4 Primary operators

The variation of a generic local operator $\mathcal{O}(z,\bar{z})$ under a holomorphic conformal transformation $\varepsilon(z)$ is contained in eqn (3.37). We can get some insight by looking at specific simple transformations.

Let us look first at a holomorphic translation defined as $z\mapsto z+a$, $\bar{z}\mapsto\bar{z}$. It acts on any local operator $\mathcal{O}$ as

$$\mathcal{O}(z,\bar{z}) \mapsto \mathcal{O}(z-a,\bar{z}) = \mathcal{O}(z,\bar{z}) - a\partial_z\mathcal{O}(z,\bar{z}) + \cdots \tag{3.40}$$

By identifying both sides of (3.37) one learns that

$$\partial_z\mathcal{O}(z,\bar{z}) = \mathrm{Res}_{z'\to z}\left(T(z')\mathcal{O}(z,\bar{z})\right) \tag{3.41}$$

or in other words, after doing the same for anti-holomorphic translations:

$$\mathsf{T}(z')\,\mathcal{O}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{\partial\mathcal{O}(z,\bar{z})}{z'-z} + \cdots \tag{3.42a}$$

$$\widetilde{\mathsf{T}}(\bar{z}')\,\mathcal{O}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{\bar{\partial}\mathcal{O}(z,\bar{z})}{\bar{z}'-\bar{z}} + \cdots \tag{3.42b}$$

We look now at infinitesimal scaling transformations, which are by definition the (non-holomorphic) transformations $z \mapsto (1+\delta\lambda)z$, $\bar{z} \mapsto (1+\delta\lambda)z$, with real $\delta\lambda$. We consider operators that are eigenstates of the dilatation operator; the corresponding eigenvalue is called the *scaling dimension* Δ of the operator:[2]

$$\mathcal{O} \mapsto \lambda^{-\Delta}\mathcal{O} \tag{3.43}$$

The transformation of an operator $\mathcal{O}_\Delta$ of scaling dimension Δ under an infinitesimal scale transformation is given by:

$$\delta\mathcal{O}_\Delta = -\delta\lambda\big(\Delta\mathcal{O}_\Delta(w,\bar{w}) + w\partial\mathcal{O}_\Delta + \bar{w}\bar{\partial}\mathcal{O}_\Delta\big)\,. \tag{3.44}$$

We consider finally infinitesimal rotations in the complex plane, given by $z \mapsto (1+i\delta\theta)z$, $\bar{z} \mapsto (1-i\delta\theta)\bar{z}$ with real $\delta\theta$. Consider again an operator which is an eigenstate of the rotation operator; its eigenvalue is called the *spin* s of the operator. The transformation law is given by

$$\delta\mathcal{O}_s = i\delta\theta\big(s\mathcal{O}_s(z,\bar{z}) + z\partial\mathcal{O}_\Delta + \bar{z}\bar{\partial}\mathcal{O}_\Delta\big)\,. \tag{3.45}$$

In order to use the holomorphic/antiholomorphic splitting of conformal transformations that we have used throughout, it is convenient to combine scalings and rotations into the complex transformations

$$\delta z = (\delta\lambda + i\delta\theta)z =: \delta\alpha\, z\,, \quad \delta\bar{z} = (\delta\lambda - i\delta\theta)\bar{z} =: \delta\bar{\alpha}\, z\,. \tag{3.46}$$

Consider common eigenstates of the dilatation and rotation operators. The eigenvalues are called the *conformal weights* $(h,\bar{h})$ of the state, and are related to the scaling dimension and spin through

$$\Delta = h + \bar{h}\,, \quad s = h - \bar{h}\,. \tag{3.47}$$

We obtain then from the transformations (3.44,3.46) the infinititesimal transformations:

$$\delta_\alpha\mathcal{O}_{h,\bar{h}} = -\delta\alpha\big(h\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + z\partial\mathcal{O}_{h,\bar{h}}\big)\,, \tag{3.48a}$$

$$\delta_{\bar{\alpha}}\mathcal{O}_{h,\bar{h}} = -\delta\bar{\alpha}\big(\bar{h}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \bar{z}\bar{\partial}\mathcal{O}_{h,\bar{h}}\big)\,. \tag{3.48b}$$

Under a finite scaling transformation and rotation, the operator of conformal weights $(h,\bar{h})$ transforms as

$$\mathcal{O}_{h,\bar{h}} \mapsto \lambda^{-\Delta}e^{-is\theta}\mathcal{O}_{h,\bar{h}} = \big(\lambda e^{i\theta}\big)^{-h}\big(\lambda e^{-i\theta}\big)^{-\bar{h}}\,\mathcal{O}_{h,\bar{h}}\,. \tag{3.49}$$

[2]At the classical level Δ is the same as the dimension in units of length coming from dimensional analysis. For instance, a free massless scalar field in dimension D has an action $\frac{1}{4\pi}\int d^D x\,(\partial\phi)^2$, hence ϕ has dimension $(\text{length})^{2-D}$, in other words scaling dimension $\Delta = D-2$.

The residue formula (3.37) will provide then the relation between the (anti)holomorphic scale transformation and the operator product with the stress-energy tensor. One has first

$$\begin{aligned} h\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + z\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) &= \mathrm{Res}_{z'\to z}\ (z'T(z')\,\mathcal{O}(z,\bar{z})) \\ &= \mathrm{Res}_{z'\to z}\ ((z'-z)T(z')\,\mathcal{O}(z,\bar{z})) + z\mathrm{Res}_{z'\to z}\ (T(z)\,\mathcal{O}(z,\bar{z})) \end{aligned} \tag{3.50}$$

which gives, using also eqn. (3.41)

$$h\mathcal{O}_{h,\bar{h}}(z,\bar{z}) = \mathrm{Res}_{z'\to z}\ ((z'-z)T(z')\,\mathcal{O}(z,\bar{z})) \tag{3.51}$$

In summary, we have learned that the *operator product expansion* of one of the components of the stress-energy tensor with an operator of conformal weight $(h,\bar{h})$ contains the terms

$$T(z')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{h}{(z'-z)^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(z'-z)}\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \cdots \tag{3.52a}$$

$$\widetilde{T}(\bar{z}')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{\bar{h}}{(\bar{z}'-\bar{z})^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \cdots \tag{3.52b}$$

The operator product with a generic operator of weights $(h,\bar{h})$ contains in principle more singular terms in the Laurent series expansion in powers of $(z-w)$.

Conformal primary operators

A distinguished class of local operators in a two-dimensional conformal field theory are operators for which the terms written in eqns. (3.52) exhaust all the possible singular terms in the expansion, namely

$$T(z')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \frac{h}{(z'-z)^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(z'-z)}\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \text{regular} \tag{3.53a}$$

$$\widetilde{T}(\bar{z}')\mathcal{O}_{h,\bar{h}}(z,\bar{z}) \overset{z'\to z}{\simeq} \frac{\bar{h}}{(\bar{z}'-\bar{z})^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \text{regular} \tag{3.53b}$$

These are called *primary operators*, or primaries for short. Given that the regular terms in the expansions do not contain essential information, we will often remove them from the expressions of the operator product expansions.

Primary operators are interesting because they have particularly simple transformation laws under a generic holomorphic conformal transformation $\delta z = \varepsilon(z)$. Using once again the residue formula (3.37) and the OPE (3.53a) for primary operators one finds that

$$\delta_\varepsilon\mathcal{O} = -\mathrm{Res}_{z'\to z}\left\{\varepsilon(z)\left(\frac{h}{(z'-z)^2}\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \frac{1}{(z'-z)}\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) + \text{regular}\right)\right\}. \tag{3.54}$$

The infinitesimal conformal transformation $\varepsilon(z)$ is by definition holomorphic in the neighborhood of z and can be Taylor-expanded there:

$$\varepsilon(z') = \varepsilon(z) + (z'-z)\partial\varepsilon(z) + \mathcal{O}\left((z'-z)^2\right) \tag{3.55}$$

Therefore transformation of a primary operator under an infinitesimal generic conformal transformation is given by

$$\delta_\varepsilon \mathcal{O}(z,\bar{z}) = -\varepsilon(z)\partial\mathcal{O}_{h,\bar{h}}(z,\bar{z}) - h\big(\partial\varepsilon(z)\big)\, h\mathcal{O}_{h,\bar{h}}(z,\bar{z})\,. \tag{3.56}$$

This transformation law can be generalized to a finite – rather than infinitesimal – transformation, namely $w \mapsto \tilde{w} = f(w)$, as well as to its anti-holomorphic counterpart $\bar{z} \mapsto \tilde{\bar{z}} = \bar{f}(\bar{z})$. One finds then:

$$\mathcal{O}(z,\bar{z}) \mapsto \tilde{\mathcal{O}}\big(\tilde{z},\tilde{\bar{z}}\big) = \left(\frac{\partial f}{\partial z}\right)^{-h}\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{-\bar{h}} \mathcal{O}(z,\bar{z})\,. \tag{3.57}$$

Correlation functions of primary operators

Conformal invariance severely constraints the form of correlation functions between primary operators. Let us start with the simplest case, the two-point function:

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle\,, \tag{3.58}$$

where ϕ_1 (resp. ϕ_2) is a conformal primary of weights $(h_1,\bar{h}_1)$ (resp. $(h_2,\bar{h}_2)$). Under a conformal transformation $z \mapsto y(z)$, $\bar{z} \mapsto \bar{y}(\bar{z})$ one should have

$$\begin{aligned}&\langle \phi_1(y_1,\bar{y}_1)\phi_2(y_2,\bar{y}_2)\rangle \\ &\quad = \left(\frac{\partial y}{\partial z}(y_1)\right)^{-h_1}\left(\frac{\partial \bar{y}}{\partial \bar{z}}(\bar{y}_1)\right)^{-\bar{h}_1}\left(\frac{\partial y}{\partial z}(y_2)\right)^{-h_2}\left(\frac{\partial \bar{y}}{\partial \bar{z}}(\bar{y}_2)\right)^{-\bar{h}_2}\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle\end{aligned} \tag{3.59}$$

Invariance under holomorphic and anti-holomorphic translations implies first that

$$\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = f(z_1-z_2,\bar{z}_1-\bar{z}_2)\,. \tag{3.60}$$

Next we investigate invariance under holomorphic and anti-holomorphic scale transformations. It leads to the functional identities

$$f\big(\lambda\times(z_1-z_2),\bar{z}_1-\bar{z}_2\big) = \lambda^{-h_1-h_2} f\big(z_1-z_2,\bar{z}_1-\bar{z}_2\big)\,, \tag{3.61a}$$
$$f\big(z_1-z_2,\bar{\lambda}\times(\bar{z}_1-\bar{z}_2)\big) = \bar{\lambda}^{-\bar{h}_1-\bar{h}_2} f\big(z_1-z_2,\bar{z}_1-\bar{z}_2\big)\,. \tag{3.61b}$$

hence f is homogeneous of degree $-(h_1+h_2)$ (resp. of degree $-(\bar{h}_1+\bar{h}_2)$) in z_1-z_2 (resp. in $\bar{z}_1-\bar{z}_2$). In other words,

$$f(z_1-z_2,\bar{z}_1-\bar{z}_2) = \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}}\,, \tag{3.62}$$

where C_{12} is some constant. We now impose invariance under the transformation $z \mapsto -1/z$, which is simpler to handle than the special conformal transformation $z \mapsto z/(1-\bar{b}\,z)$. One gets

$$(-1/z_1+1/z_2)^{-h_1-h_2} = \big(1/z_1^2\big)^{-h_1}\big(1/z_2^2\big)^{-h_2} \quad\Longrightarrow\quad (z_1z_2)^{h_1+h_2} = z_1^{2h_1}z_2^{2h_2}\,, \tag{3.63}$$

with a similar equation for the anti-holomorphic transformations. Therefore the two-point function can be non-zero only if $h_1 = h_2$ and $\bar{h}_1 = \bar{h}_2$:

$$\boxed{\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\rangle = \frac{C_{12}}{(z_1-z_2)^{h_1+h_2}(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2}}\delta_{h_1,h_2}\delta_{\bar{h}_1,\bar{h}_2}}\,. \tag{3.64}$$

Up to a constant, the two-point function is then completely fixed by invariance under the global conformal group $\mathsf{PSL}(2,\mathbb{C})$.[3]

The same can be said about the three-point function of primary operators. Invariance under $\mathsf{PSL}(2,\mathbb{C})$ reduces the three-point function computation to a single unknown coefficient:

$$\begin{aligned}\langle \phi_1(z_1,\bar{z}_1)\phi_2(z_2,\bar{z}_2)\phi_3(z_3,\bar{z}_3)\rangle = C_{123}&\frac{1}{(z_1-z_2)^{h_1+h_2-h_3}(z_2-z_3)^{h_2+h_3-h_1}(z_1-z_3)^{h_3+h_1-h_2}}\\ \times&\frac{1}{(\bar{z}_1-\bar{z}_2)^{\bar{h}_1+\bar{h}_2-\bar{h}_3}(\bar{z}_2-\bar{z}_3)^{\bar{h}_2+\bar{h}_3-\bar{h}_1}(\bar{z}_1-\bar{z}_3)^{\bar{h}_3+\bar{h}_1-\bar{h}_2}}\,.\end{aligned} \tag{3.65}$$

3.5 The Virasoro Algebra

The operator product expansions (3.52) are valid for all local operators that are eigenstates of the dilatation and rotation operators. By dimensional analysis, the stress-energy tensor T_{ij} given by the definition (3.18) has scaling dimension $\Delta = 2$ in two dimensions. Under a rotation $z \mapsto z' = e^{i\theta}z$ its non-vanishing components transform as $T_{zz} \mapsto T_{z'z'} = e^{-2i\theta}T_{zz}$ and $T_{\bar{z}\bar{z}} \mapsto T_{\bar{z}'\bar{z}'} = e^{2i\theta}T_{\bar{z}\bar{z}}$.

This analysis shows that $T = T_{zz}$ is an operator of conformal weights $(h,\bar{h}) = (2,0)$ while $\widetilde{T} = T_{\bar{z}\bar{z}}$ is an operator of conformal weights $(h,\bar{h}) = (0,2)$, see eq. (3.47). However nothing indicates that these are primary operators. Let us consider then the OPEs

$$T(z')T(z) \overset{z'\to z}{\simeq} \cdots + \frac{2}{(z'-z)^2}T(z) + \frac{1}{(z'-z)}\partial T(z) + \text{reg.} \tag{3.66a}$$

$$\widetilde{T}(\bar{z}')\widetilde{T}(\bar{z}) \overset{z'\to z}{\simeq} \cdots + \frac{2}{(\bar{z}'-\bar{z})^2}\widetilde{T}(\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\widetilde{T}(\bar{z}) + \text{reg.} \tag{3.66b}$$

3.5.1 The central charge

The missing information in the OPEs (3.66) are the possible terms more singular than $1/(z-w)^2$ in the expansion. The only universal operators that should appear in any conformal field theory are the components of the stress energy tensor and the identity operator which has naturally conformal dimensions $(h,\bar{h}) = (0,0)$. On general grounds, one should allow terms proportional to the identity on the right-hand side of eqns. (3.18). By dimensional analysis,

[3] Actually all is needed is that the operators transform as (3.57) under the action of $\mathsf{PSL}(2,\mathbb{C})$, which is a weaker statement than asking that this equation holds true for any holomorphic function. Such operators are called *quasi-primary* operators.

T being of dimension (2,0), this term should come with the power $(z'-z)^{-4}$ in the operator product expansion:

$$\mathsf{T}(z')\mathsf{T}(z) \overset{z'\to z}{\simeq} \frac{c}{2(z'-z)^4} + \frac{2}{(z'-z)^2}\mathsf{T}(z) + \frac{1}{(z'-z)}\partial\mathsf{T}(z) + \text{reg.} \tag{3.67a}$$

$$\widetilde{\mathsf{T}}(\bar{z}')\widetilde{\mathsf{T}}(\bar{z}) \overset{z'\to z}{\simeq} \frac{\bar{c}}{2(\bar{z}'-\bar{z})^4} + \frac{2}{(\bar{z}'-\bar{z})^2}\widetilde{\mathsf{T}}(z,\bar{z}) + \frac{1}{(\bar{z}'-\bar{z})}\bar{\partial}\widetilde{\mathsf{T}}(\bar{z}) + \text{reg.} \tag{3.67b}$$

These OPEs depend on two $\mathbb{C}$-numbers c and $\bar{c}$ that are called the *central charges* of the conformal field theory, and encode the failure of the components of the stress-energy tensor to be conformal primaries. They play a central role in the study of conformal field theories, characterizing in particular the number of degrees of freedom.

From the operator product expansion (3.67) one deduces the transformation of T under an arbitary infinitesimal holomorphic conformal transformation:

$$\begin{aligned}\delta_\varepsilon \mathsf{T}(z) &= -\text{Res}_{z'\to z}\left\{\varepsilon(z')\left(\frac{1}{(z'-z)^4} + \frac{h}{(z'-z)^2}\mathsf{T}(z) + \frac{1}{(z'-z)}\partial\mathsf{T}(z) + \text{reg.}\right)\right\}\\ &= -\frac{c}{12}\varepsilon'''(z) - 2\varepsilon'(z)\mathsf{T}(z) - \varepsilon(z)\partial\mathsf{T}(z)\,, \end{aligned} \tag{3.68}$$

and similarly

$$\delta_{\bar{\varepsilon}}\widetilde{\mathsf{T}}(\bar{z}) = -\frac{\bar{c}}{12}\bar{\varepsilon}'''(\bar{z}) - 2\bar{\varepsilon}'(z)\mathsf{T}(z) - \varepsilon(z)\bar{\partial}\widetilde{\mathsf{T}}(\bar{z})\,, \tag{3.69}$$

Notice that for infinitesimal dilatations, rotations and special conformal transformations the terms proportional to the central charges vanish.

For finite transformation $z \mapsto \tilde{z} = f(z)$ and $\bar{z} \mapsto \tilde{\bar{z}} = \bar{f}(\bar{z})$ the components of the stress-tensor transforms as

$$\mathsf{T}(z) \mapsto \tilde{\mathsf{T}}(\tilde{z}) = \left(\frac{\partial f(z)}{\partial z}\right)^{-2}\mathsf{T}(z) + \frac{c}{12}\{f(z), z\}\,, \tag{3.70a}$$

$$\widetilde{\mathsf{T}}(\bar{z}) \mapsto \tilde{\widetilde{\mathsf{T}}}(\tilde{\bar{z}}) = \left(\frac{\partial \bar{f}(\bar{z})}{\partial \bar{z}}\right)^{-2}\widetilde{\mathsf{T}}(\bar{z}) + \frac{\bar{c}}{12}\{\bar{f}(\bar{z}), \bar{z}\}\,, \tag{3.70b}$$

where one defines the *Schwarzian derivative*

$$\{f(z), z\} = \frac{2f'''(z)f'(z) - 3(f''(z))^2}{2(f'(z))^2}\,, \tag{3.71}$$

which is compatible with the composition of successive conformal transformations.

3.5.2 The Virasoro Algebra

The information contained in the operator product expansions (3.67) can be recast in a different way that will turn out to be very useful, as it will allow to study the properties of the CFT using the tools of representation theory.

To start, the components of the stress tensor T and $\widetilde{\mathsf{T}}$ are respectively holomorphic and anti-holomorphic functions on the complex plane, hence admit a Laurent series expansion:

$$\mathsf{T} = \sum_{n\in\mathbb{Z}} \frac{\mathsf{L}_n}{z^{n+2}}\,, \quad \widetilde{\mathsf{T}} = \sum_{n\in\mathbb{Z}} \frac{\widetilde{\mathsf{L}}_n}{\bar{z}^{n+2}}\,. \tag{3.72}$$

The coefficients of the expansion can be found by a contour integral

$$\mathsf{L}_n = \oint_{\mathcal{C}} \frac{dz}{2i\pi}\, z^{n+1}\, \mathsf{T}(z) \tag{3.73a}$$

$$\widetilde{\mathsf{L}}_n = -\oint_{\mathcal{C}} \frac{d\bar{z}}{2i\pi}\, \bar{z}^{n+1}\, \widetilde{\mathsf{T}}(\bar{z}) \tag{3.73b}$$

where $\mathcal{C}$ is a contour encircling the origin counter-clockwise. The mode L_n corresponds to the conserved charge associated with the conformal transformation $\delta z = z^{n+1}$ following the discussion in subsection 3.2.2. In particular, $\mathsf{L}_0 + \widetilde{\mathsf{L}}_0$ is the dilatation generator, $\mathsf{L}_0 - \widetilde{\mathsf{L}}_0$ the rotation generator, while L_{-1} and $\widetilde{\mathsf{L}}_{-1}$ generate holomorphic and anti-holomorphic translations respectively.

In the quantum conformal field theory on the plane, we will consider the commutator of two Laurent modes of the stress-energy tensor. In other words, for any state $|\phi\rangle$ in the theory we consider the quantity

$$[\mathsf{L}_m, \mathsf{L}_n]|\phi\rangle = \mathsf{L}_m\mathsf{L}_n|\phi\rangle - \mathsf{L}_n\mathsf{L}_m|\phi\rangle\,. \tag{3.74}$$

The first term corresponds to applying first L_n to $|\phi\rangle$ and then L_m, while the second one corresponds to applying first L_m to $|\phi\rangle$ and then L_n.

Using the state-operator correspondence, the state $|\phi\rangle$ is mapped to an local operator $\mathcal{O}(z,\bar{z})$ that we can put at the origin. We have learned also in section 3.2 that the charge of a state w.r.t. a holomorphic current is computed by a contour integral around the corresponding local operator, see eqn. (3.17).

Let us consider the circular contours $\mathcal{C}$ of radius R and $\mathcal{C}'$ of radius $\mathsf{R}' > \mathsf{R}$, both around the origin. We have

$$\mathsf{L}_m\mathsf{L}_n\mathcal{O}(0,0) = \oint_{\mathcal{C}'} \frac{dz_1}{2i\pi} \oint_{\mathcal{C}} \frac{dz_2}{2i\pi}\, z_1^{m+1}\, z_2^{n+1}\, \mathsf{T}(z_1)\, \mathsf{T}(z_2)\mathcal{O}(0,0) \tag{3.75a}$$

$$\mathsf{L}_n\mathsf{L}_m\mathcal{O}(0,0) = \oint_{\mathcal{C}'} \frac{dz_2}{2i\pi} \oint_{\mathcal{C}} \frac{dz_1}{2i\pi}\, z_1^{m+1}\, z_2^{n+1}\, \mathsf{T}(z_1)\, \mathsf{T}(z_2)\mathcal{O}(0,0) \tag{3.75b}$$

In the following we will remove the operator $\mathcal{O}$ which plays no role in this computation; it is understood that the operators are applied to any local operator in the CFT inserted at the origin.

The only difference between the two equations (3.75) is that in (3.75a) the contour of integration over z_1 is around the contour of integration over z_2, while in (3.75b) it is just the opposite. To compute the commutator, the trick is to consider first the integration over z_1 for fixed z_2, see fig. 3.3. Going from expression (3.75a) to expression (3.75b) amounts to pass

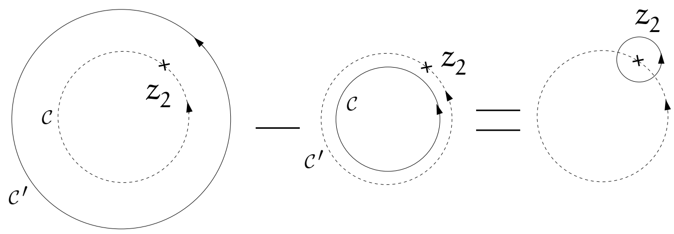

Figure 3.3: *Commutator of Virasoro generators.*

the contour of integration over z_1 (solid line) through the locus of the contour of integration over z_2 (dashed line).

As the operator product $T(z_1)T(z_2)$ has pole, one picks a residue when the contour of integration over z_1 crosses the position z_2 where T is inserted. One has finally to integrate this residue over z_2:

$$[L_m, L_n] = \oint \frac{dz_2}{2i\pi} z_2^{n+1} \operatorname{Res}_{z_1 \to z_2} \left(z_1^{m+1} T(z_1) T(z_2) \right) . \tag{3.76}$$

We now compute the residue using the operator product expansion (3.67a):

$$\begin{aligned} &\operatorname{Res}_{z_1\to z_2} \Big\{ \Big(z_2^{m+1} + (m+1)(z_1 - z_2) z_2^{m} + \tfrac{m(m+1)}{2}(z_1-z_2)^2 z_2^{m-1} + \tfrac{m(m^2-1)}{6} z_2^{m-2}(z_1-z_2)^3 + \cdots \Big) \\ &\qquad \times \Big(\tfrac{c}{2(z_1-z_2)^4} + \tfrac{2}{(z_1-z_2)^2} T(z_2) + \tfrac{1}{(z_1-z_2)} \partial T(z_2) + \text{reg.} \Big) \Big\} \\ &= \tfrac{m(m^2-1)c}{12} z_2^{m-2} + 2(m+1) z_2^{m} T(z_2) + z_2^{m+1} \partial T(z_2) . \end{aligned} \tag{3.77}$$

So we end up with the following integral

$$\begin{aligned} [L_m, L_n] &= \oint \frac{dz_2}{2i\pi} z_2^{n+1} \left(\tfrac{m(m^2-1)c}{12} z_2^{m-2} + 2(m+1) z_2^{m} T(z_2) + z_2^{m+1} \partial T(z_2) \right) \\ &= \oint \frac{dz_2}{2i\pi} \left(\tfrac{m(m^2-1)c}{12} z_2^{n+m-1} + 2(m+1) z_2^{m+n+1} T(z_2) - (n+m+2) z_2^{n+m+1} T(z_2) \right) \\ &= \oint \frac{dz_2}{2i\pi} \left(\tfrac{m(m^2-1)c}{12} z_2^{n+m-1} + (m-n) z_2^{m+n+1} T(z_2) \right) , \end{aligned} \tag{3.78}$$

where we have done an integration by parts of the last term in the second step. Now we simply have to express the right-hand side of the final expression in terms of the Laurent coefficients using equation (3.73a) and get the well-known *Virasoro algebra*:

$$\boxed{[L_m, L_n] = (m-n) L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n,0}} \tag{3.79}$$

If we did a similar computation, using the equation (3.73b) at the last step, we would find in the same way

$$[\widetilde{L}_m, \widetilde{L}_n] = (m-n) \widetilde{L}_{m+n} + \frac{\bar{c}}{12} m(m^2-1) \delta_{m+n,0} . \tag{3.80}$$

The Virasoro algebra (3.79) is very similar two an ordinary Lie algebra, except that it has an infinite number of generators, $\{L_n,\ n \in \mathbb{Z}\}$. Another characteristic feature is the presence of the constant term $\frac{\bar{c}}{12}m(m^2-1)\delta_{m+n,0}$, which commutes with all the generators; such a term is called a *central extension* of the algebra.

Finally a finite sub-algebra of the Virasoro algebra is obtained from the generators $\{L_{-1}, L_0, L_1\}$:

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}\ , \quad [L_{-1}, L_1] = 2L_0\,, \tag{3.81}$$

which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$; this is nothing but the Lie algebra of the *Möbius group*, the group of globally defined conformal transformations on the sphere $\bar{\mathbb{C}}$, *i.e.* of projective transformations $z \mapsto \frac{az+b}{cz+d}$, $a, b, c, d \in \mathbb{C}$, already discussed in subsection 2.3.3.

3.5.3 Conformally invariant vacuum and Casimir energy

When a conformal field theory theory is formulated on the cylinder, it is natural to expand the components of the stress energy tensor in terms of its Fourier modes:

$$T_{ww} = -\sum_{n\in\mathbb{Z}} T_n e^{inw} \tag{3.82a}$$

$$T_{\bar{w}\bar{w}} = -\sum_{n\in\mathbb{Z}} \widetilde{T}_n e^{-in\bar{w}} \tag{3.82b}$$

with

$$T_n = -\int \frac{d\sigma^1}{2\pi} e^{-in\sigma_1} T_{ww}(\sigma_1, 0)\,, \tag{3.83a}$$

$$\widetilde{T}_n = -\int \frac{d\sigma^1}{2\pi} e^{in\sigma_1} T_{\bar{w}\bar{w}}(\sigma_1, 0)\,. \tag{3.83b}$$

The conformal mapping (3.11) from the cylinder to the complex plane gives, using equations (3.70) and (3.71):

$$T_{ww}(w) = -z^2 T_{zz}(z) + \frac{c}{24} \tag{3.84a}$$

$$T_{\bar{w}\bar{w}}(\bar{w}) = -\bar{z}^2 T_{\bar{z}\bar{z}}(\bar{z}) + \frac{\bar{c}}{24} \tag{3.84b}$$

Hence the expansion (3.72) on the plane in terms of Laurent coefficients and the expansion (3.82) on the cylinder in terms of Fourier modes are related through

$$T_n = L_n - \frac{c}{24}\delta_{n,0}\ , \quad \widetilde{T}_n = \widetilde{L}_n - \frac{\bar{c}}{24}\delta_{n,0}\,. \tag{3.85}$$

In particular, the Hamiltonian of the conformal field theory on the cylinder is by definition the conserved charge associated with time translations:

$$H = \int \frac{d\sigma^1}{2\pi} T_{\sigma^2\sigma^2} = -\int \frac{d\sigma^1}{2\pi}\left(T_{ww} + T_{\bar{w}\bar{w}}\right) = L_0 + \widetilde{L}_0 - \frac{c+\bar{c}}{24}\,. \tag{3.86}$$

Conformally invariant vacuum

In radial quantization, the natural vacuum state of the conformal field theory on the complex plane, $|0\rangle$, is defined by inserting the identity operator at the origin $z = 0$.

According to the definition (3.72) of the Virasoro generators, in order for the components of the stress-tensor to be non-singular at the origin, one should have

$$L_n|0\rangle = 0 \ , \quad \widetilde{L}_n|0\rangle = 0 \ , \quad \forall n \geqslant -2 \, . \tag{3.87}$$

In particular, the vacuum is invariant under the action of $\{L_{-1}, L_0, L_1\}$ and $\{\widetilde{L}_{-1}, \widetilde{L}_0, \widetilde{L}_1\}$, *i.e.* under global conformal transformations. For this reason, it is called the $SL(2, \mathbb{C})$-*invariant vacuum.*

If we consider a two-dimensional CFT in the $PSL(2, \mathbb{C})$-invariant vacuum on the plane, equation (3.86) indicates that the corresponding energy of the ground state on the cylinder is given by

$$E_{cyl} = -\frac{c + \bar{c}}{24} \, , \tag{3.88}$$

which is interpreted as the Casimir energy of the QFT on the cylinder; notice that no regularization of high-energy divergences was needed to obtain this result. This remark provides another interpretation of the central charges of a conformal field theory.

3.5.4 Conformal primaries revisited

Primary operators were defined by the operator product expansion (3.53) with the stress-energy tensor. Let us consider some primary operator $\mathcal{O}_{h,\bar{h}}$. Plugging in the decompositions (3.73) in terms of Laurent modes, one finds that the corresponding state $|\mathcal{O}\rangle$ is annihilated by all positive modes of the Virasoro generators:

$$\forall n > 0 \ , \quad L_n|h, \bar{h}\rangle = 0 \ , \quad \widetilde{L}_n|h, \bar{h}\rangle = 0 \, . \tag{3.89}$$

Hence primary operators play the same role w.r.t. the Virasoro algebra as the highest weight vectors in representation theory of Lie algebras. For this reason they are also referred to as *highest weight states* of the Virasoro algebra.

The other states in the Virasoro representation associated with the highest weight state $|h, \bar{h}\rangle$ are obtained by applying repeatedly the the negative Virasoro modes $\{L_{-n}, n > 0\}$ (there is a similar story for anti-holomorphic generator). This generates what is known as a *Verma module*, that may or may not be an irreducible representation of the Virasoro algebra depending on the values of h and the central charge c.

The conformal dimension of these *descendant states*, obtained from the highest weight state using lowering operators L_{-n}, is easily computed:

$$L_0 L_{-n}|h, \bar{h}\rangle = n L_{-n}|h, \bar{h}\rangle + L_{-n} L_0|h, \bar{h}\rangle = (h + n)|h, \bar{h}\rangle, \quad n > 0 \, . \tag{3.90}$$

I would certainly agree with the reader that would tell me that the primary states should be called *lowest* dimension states rather than *highest* weight states!

3.5.5 Unitarity constraints

One important constraint on sensible quantum field theories is *unitarity*, the conservation of probabilities over time. The time evolution on the cylinder is given by the Hamiltonian density (with Minkowskian signature on the worldsheet):

$$\begin{aligned}\mathcal{H} = -(T_{ww} + T_{\bar{w}\bar{w}}) &= \sum_{n\in\mathbb{Z}} \left(T_n e^{in(\sigma^1+\tau)} + \widetilde{T}_n e^{-in(\sigma^1-\tau)} \right) \\ &= \sum_{n\in\mathbb{Z}} \left(L_n e^{in(\sigma^1+\tau)} + \widetilde{L}_n e^{-in(\sigma^1-\tau)} \right) - \frac{c+\bar{c}}{24} \,. \end{aligned} \tag{3.91}$$

Hence unitarity gives the constraints

$$(L_n)^\dagger = L_{-n} \quad , \qquad \left(\widetilde{L}_n\right)^\dagger = \widetilde{L}_{-n} \,. \tag{3.92}$$

These unitarity relations have two immediate consequences:

- consider a primary state $|h, \bar{h}\rangle$ of conformal dimensions $(h, \bar{h})$ in a unitary CFT.

  $$||L_{-1}|h,\bar{h}\rangle||^2 = \langle h, \bar{h}|[L_1, L_{-1}]|h,\bar{h}\rangle = 2\langle h,\bar{h}|L_0|h,\bar{h}\rangle \,, \tag{3.93}$$

  hence in a unitary QFT the conformal dimensions $(h, \bar{h})$ of primary operators – and, consequently, of all descendant states – should be non-negative. The unique state with $h = \bar{h} = 0$ in a unitary CFT is the $SL(2, \mathbb{C})$-invariant vacuum since it satisfies $L_{-1}|0,0\rangle = \widetilde{L}_{-1}|0,0\rangle = 0$ which means in operator language $\partial\mathcal{O}_{0,0} = \bar{\partial}\mathcal{O}(0,0) = 0$.

- $\forall n > 1$, one has

  $$||L_n|h,\bar{h}\rangle||^2 = \langle h,\bar{h}|[L_{-n}, L_n]|h,\bar{h}\rangle = \frac{c}{12}n(n^2-1) \,, \tag{3.94}$$

  and similarly for $\widetilde{L}_n|h,\bar{h}\rangle$, hence a unitary CFT has positive central charges: $c > 0$ and $\bar{c} > 0$. The only unitary CFT with $c = \bar{c} = 0$ is trivial and contains just the identity operator.

3.6 The Weyl anomaly

The failure of the components $(T, \widetilde{T})$ of the stress-energy tensor to be primary operators w.r.t. generic two-dimensional conformal transformations is a just a characteristic feature of conformal field theories. In the string theory context however, it leads to a potential disaster.

String theory was defined as a two-dimensional theory of gravity coupled to a set of scalar matter fields $\{x^\mu(\sigma^i)\}$. The theory has an enormous redundancy, since its gauge group corresponds to two-dimensional diffeomorphisms and Weyl transformations. After a suitable gauge-fixing procedure, we have obtained a path integral defined over a gauge slice, *i.e.* with a reference metric depending only on a handful of parameters. All this beautiful construction falls apart if the classical gauge symmetry of the theory is not satisfied at the quantum level.

The potential clash between gauge symmetry and quantum effects is associated with Weyl transformations. Heuristically it is not difficult to understand why. Computations in the two-dimensional quantum field theory we are dealing with have divergences, as in any other QFT, that should be regularized. If one uses dimensional regularization, the invariance under Weyl rescalings is lost, since it depends crucially on being in dimension two, see eqn. (2.79). Using alternatively a Pauli-Villars regularization, one would introduce a scale in the theory breaking explicitly scale invariance. As any sensible regulator breaks the Weyl symmetry, one may wonder if at the end of the computation, which is not covariant w.r.t. the Weyl gauge transformations, the invariance would be miraculously restored. It turns out not to be the case and the Weyl symmetry has potentially an *anomaly* of the gauge symmetry signaling the inconsistency of the theory.

To characterize this anomaly, one considers that the background geometry is fixed to some reference metric and consider whether the classical conservation laws associated with the symmetries of the theory are truly independent of the choice of reference metric. Classically, Weyl-invariance implies that the stress-energy tensor is traceless, see eqn. (3.20). Because of general covariance of the theory, a quantum violation of this condition is very restricted. In the quantum field theory context, one would like to see whether the operator ${T^i}_i$ inserted in an arbitrary correlation function gives zero. The more general parametrization is:

$$\langle {T^i}_i \cdots \rangle = a R[\hat{\gamma}] \langle \cdots \rangle \,, \tag{3.95}$$

where $R[\hat{\gamma}]$ is the Ricci scalar computed for the reference worldsheet metric $\hat{\gamma}$, given that the right-hand side of the equation should be a local expression, and of scaling dimension two; the parameter a is dimensionless.

Since the diffeomorphism symmetry is not anomalous, one can work in the conformal gauge, $ds^2 = \exp(2\omega) dw d\bar{w}$, in which case, the anomaly equation becomes, using eqn. (2.80b),

$$\langle T_{w\bar{w}} \cdots \rangle = -a \nabla^2 \omega \,. \tag{3.96}$$

The conservation of the stress-energy tensor gives $\nabla^{\bar{w}} T_{w\bar{w}} + \nabla^{w} T_{ww} = 0$. Hence taking the derivative of (3.96) with $\nabla^{\bar{w}}$ one reaches the equation:

$$\langle \nabla^w T_{ww} \cdots \rangle = 2a \nabla^{\bar{w}} \nabla^2 \omega \langle \cdots \rangle \,. \tag{3.97}$$

Let us now consider an infinitesimal Weyl rescaling around the flat metric, *i.e.* $\omega = 1 + \delta\omega(w, \bar{w})$, and compare the variations of the left- and right-hand sides of eqn. (3.97).

In the previous section we have considered the transformation of the components of the stress-tensor under an infinitesimal conformal transformation, *i.e.* the combination of a coordinate transformation $w \mapsto w + \varepsilon(w)$, $w \mapsto w + \varepsilon(w)$ and of a Weyl transformation $\delta\omega = \frac{1}{2}(\partial\varepsilon + \bar{\partial}\bar{\varepsilon})$:

$$-\delta T(w) = \underbrace{\frac{c}{12}\varepsilon'''(w)}_{\text{Weyl}} + \underbrace{2\varepsilon'(w)T(w) + \varepsilon(w)\partial T(w)}_{\text{diffeomorphism}} \,, \tag{3.98}$$

where we have indicated that the last two terms correspond to the coordinate transformation (*i.e.* the transformation of the tensor $T^{ij}(x^k) \mapsto \tilde{T}^{ij}(\tilde{x}^k) = (\partial\tilde{x}^i/\partial x^m)(\partial\tilde{x}^j/\partial x^n)T^{mn}(x^k)$), hence the first term corresponds to the effect of the Weyl transformation:

$$\delta_{\delta\omega} T = -\frac{c}{6}\partial_w^2 \delta\omega\,. \tag{3.99}$$

Plugging into eqn. (3.97) one finds, using $\nabla^z = 2\partial_{\bar{w}}$ and $\nabla^2 = 4\partial_w\partial_{\bar{w}}$,

$$-\frac{c}{3}\partial_{\bar{w}}\partial_w^2\delta\omega = 4a\partial_w^2\partial_{\bar{w}}\delta\omega \quad\Longrightarrow\quad a = -\frac{c}{12}\,. \tag{3.100}$$

The conclusion of this computation is that

$$\langle T^i{}_i \cdots \rangle = -\frac{c}{12}R[\hat{\gamma}]\langle \cdots \rangle\,. \tag{3.101}$$

This result is not problematic if one considers a conformal field theory outside of the string theory context, however in the present context it indicates that the theory is inconsistent *unless* $c = 0$. As we shall see this constraint will have far-reaching consequences.

A careful reader may have noticed that we could do exactly the same computation using $\nabla^w T_{w\bar{w}} + \nabla^{\bar{w}} T_{\bar{w}\bar{w}} = 0$, in which case we will reach the same equation as (3.101) with c replaced by $\bar{c}$. These two equations are not consistent with each other unless $c = \bar{c}$. It turns out that a conformal field theory with $c \neq \bar{c}$ cannot be coupled consistently to two-dimensional gravity; on top of the *Weyl anomaly* discussed here it has also a *gravitational anomaly.*

References

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