Entanglement entropies in minimal models

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This talk is based on:

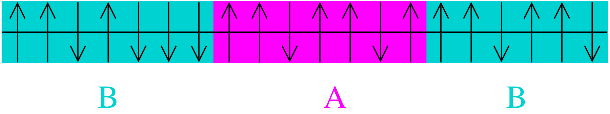
Thomas Dupic, Benoît Estienne and YI
Entanglement entropies in minimal models from null vetors
`arXiv:1709.09270`
accepted for publication in SciPost Physics

Outline

1. Introduction
2. The null-vector approach
3. Entanglement entropies in the Yang-Lee model
4. Further studies of the cyclic orbifold

1. Introduction

Entanglement entropies in quantum systems



- Density matrix of the whole system $A + B$
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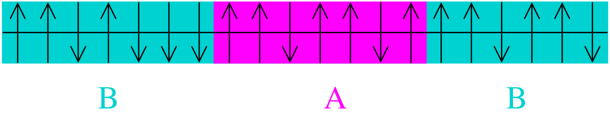

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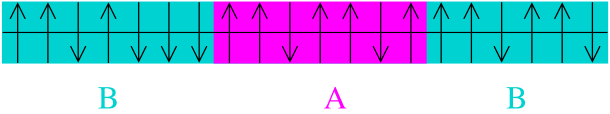

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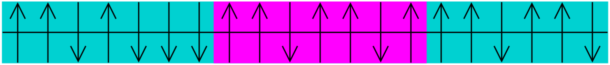

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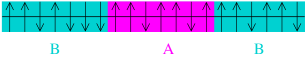
- $\neq$ Gapless systems in $d = 1+1$: **Conformal Field Theory** [Holzhey-Larsen-Wilczek '94, Calabrese-Cardy '04]
  $$S(A) \sim \frac{c}{3} \log \ell_A$$

The path-integral formalism for Rényi entropies

[Calabrese-Cardy '04]

- $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$:



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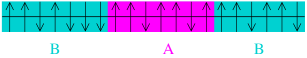

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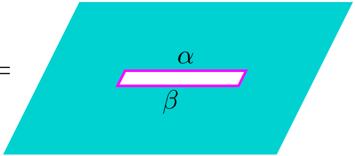

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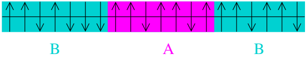

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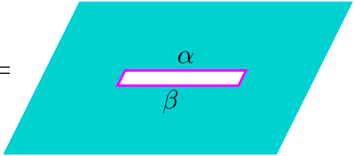
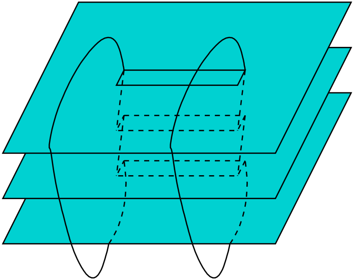


- Rényi entropy

$$\mathrm{Tr}[(\rho_A)^N] =$$



$$= Z(\mathcal{R}_N)$$

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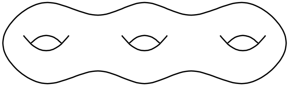
$\mathcal{R}_N \;\; \cong$ 

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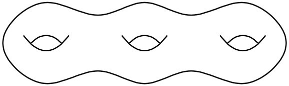
  $\mathcal{R}_N \ \tilde{=}$ 
- For $c = 1$: results available from [Zamolodchikov '87], [Dixon, Friedan, Martinec, Shenker '87], [Alvarez-Gaumé, Gost, Moore, Nelson, Vafa '87], [Dijkgraaf, Verlinde, Verlinde '88]

Overview of Entanglement Entropies in CFT

- Scaling argument $S \propto \frac{c}{3} \log \ell$ [Holzhey, Larsen, Wilczek '94]:
- Path-integral approach, EE by conformal mapping for $A = [u, v]$ [Calabrese,Cardy '04]
- Compute EE at $g > 0$ for $c = 1$ and/or Ising CFTs [Calabrese, Cardy, Tonni, Tagliacozzo, Alba, Datta, David, Misguich, Pasquier, Stéphan, Furukawa, Shiraishi, Essler, Campostrini, Nienhuis '06–'12]
- EE for excited states [Sierra, Alcaraz, Berganza, Palmai '12–'16]
- EE for integrable QFTs [Castro-Alvaredo, Doyon, Cardy, Blondeau-Fournier '07–'15]:
- EE, entanglement spectrum, fidelity using CTM [Franchini, Its, Korepin, Takhtajan, Evangelisti, Weston '11–'12]
- Entanglement after a quench [Cardy '11]
- EE for non-unitary CFTs [Castro-Alvaredo, Doyon, Ravanini, Bianchini, Levi, Couvreur, Jacobsen, Saleur '14–'17]
- Entanglement spectra in FQHE [Li, Haldane, Read, Rezayi, Dubail, Eisler, Peschel, Cardy, Tonni '08–'17]:
- ...

2. The null-vector approach

The $\mathbb{Z}_N$ cyclic orbifold CFT

[Crnković-Sotkov-Stanishkov '89, Klemm-Schmidt '90, Borisov-Halpern-Schweigert '98]

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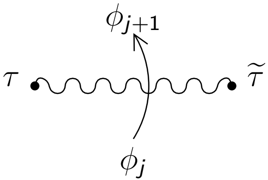

$$\tau(0).(\phi_1, \dots, \phi_{N-1}, \phi_N)(e^{2i\pi} z) = \tau(0).(\phi_2, \dots, \phi_N, \phi_1)(z)$$

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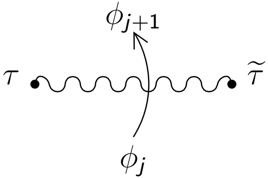

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- Examples:
  - $\langle \tau(u_1)\widetilde{\tau}(v_1) \dots \tau(u_p)\widetilde{\tau}(v_p) \rangle$
  - $\langle \Phi(\infty)\tau(u)\widetilde{\tau}(v)\Phi(0) \rangle \,, \qquad \Phi := \phi_{12} \otimes \cdots \otimes \phi_{12}$

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Generating algebra [Crnković-Sotkov-Stanishkov '89, Klemm-Schmidt '90, Borisov-Halpern-Schweigert '98]

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- Orbifold Virasoro algebra:

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- Basic example: $L_{-1}\mathbf{1} = 0 \quad \Rightarrow \quad \widehat{L}^{(-1)}_{-1/N}\tau = 0$

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From null vectors to differential equations

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- Strategy: obtain linear relation between

$$\langle(\widehat{L}^{(r_1)}_{m_1}\mathcal{O}_1)\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle\,,\langle\mathcal{O}_1(\widehat{L}^{(r_2)}_{m_2}\mathcal{O}_2)\mathcal{O}_3\mathcal{O}_4\rangle\,,$$
$$\langle\mathcal{O}_1\mathcal{O}_2(\widehat{L}^{(r_3)}_{m_3}\mathcal{O}_3)\mathcal{O}_4\rangle\,,\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3(\widehat{L}^{(r_4)}_{m_4}\mathcal{O}_4)\rangle$$

using **orbifold Ward identities** = Closed-contour relations:

$$\oint_C dz\,(z-1)^{m_2+1}\,(z-x)^{m_3+1}\,z^{m_4+1}$$
$$\times\,\langle \mathcal{O}_1(\infty)\mathcal{O}_2(1)\mathcal{O}_3(x,\bar{x})\,\widehat{T}^{(r)}(z)\mathcal{O}_4(0)\rangle = 0$$

where $\mathcal{O}_j \in [k_j] \Rightarrow m_j \in \mathbb{Z} + rk_j/N$

Summary of the null-vector approach

1. Consider eigenstate $|\psi\rangle$ of $H_{\mathcal{M}(p,q)}$
   - Write Rényi EE $\quad S_N(A) = \frac{1}{1-N} \log G(x, \bar{x})$
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6. Solve consistency for $\{X_i\} \leftrightarrow \{Y_j\}$ $\quad$ ("conformal bootstrap")

3. Entanglement entropies in the Yang-Lee model

The Yang-Lee edge singularity

[Yang-Lee '52]

- Classical Ising model in magnetic field :

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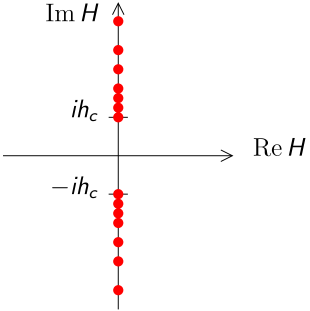

The Yang-Lee model as a minimal model of CFT

[Cardy '85]

- Scaling limit: fixed $T > T_c$ and $H = ih \to ih_c$ [Fisher '78]
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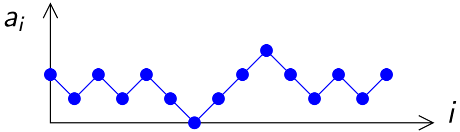
- Exactly solved RSOS model in same universality class [Andrews-Baxter-Forrester '84]

The RSOS quantum chain



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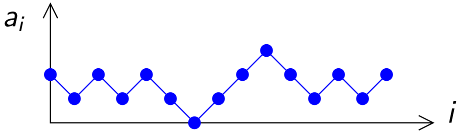

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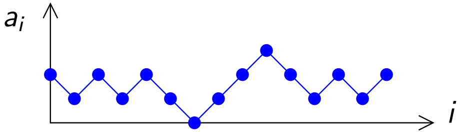

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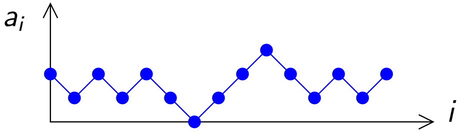

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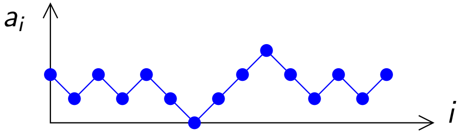

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- Hamiltonian: $\quad H = -\sum_{i=1}^{L} e_i \qquad$ with PBC
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$$e_i | \dots a_{i-1}, a_i, a_{i+1} \dots \rangle = \delta_{a_{i-1}, a_{i+1}} \sum_{a_i'} \frac{\sin \lambda a_i'}{\sin \lambda a_i} | \dots a_{i-1}, a_i', a_{i+1} \dots \rangle$$

- $\lambda = \frac{\pi(p-q)}{p}$ with $p > q$ coprime $\rightarrow$ scaling limit$=\mathcal{M}(p,q)$
- The YL case : $p = 5, \quad \lambda = \frac{3\pi}{5}$

EEs of a non-unitary model

see also [Bianchini,Castro-Alvaredo,Doyon,Levi,Ravanini '14]

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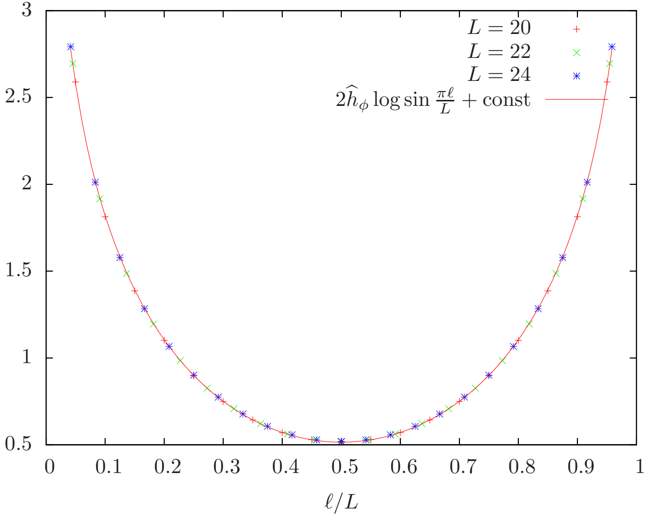
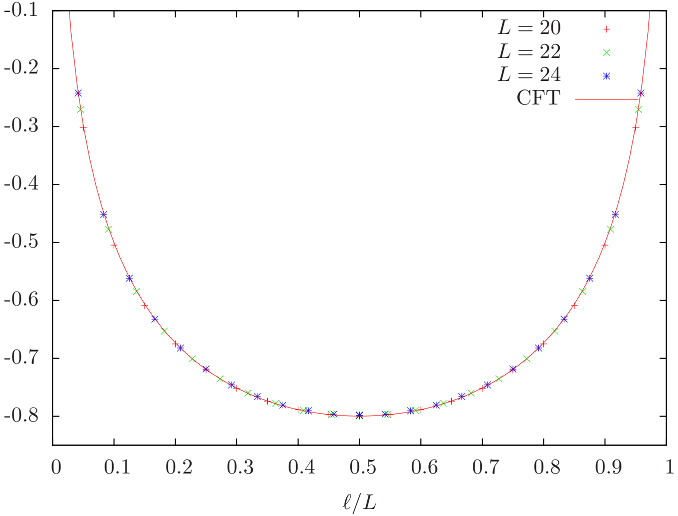
- EE in ground state $|\phi\rangle$:

$$\langle \Phi(\infty)\tau_\phi(1)\widetilde{\tau}_\phi(x,\bar{x})\Phi(0)\rangle = G(x,\bar{x}) \qquad \text{with } \Phi = \phi^{\otimes N}$$

$N = 3$ Rényi entropy in state $|\mathbf{1}\rangle$



$N = 3$ Rényi entropy in the ground state $|\phi\rangle$



4. Further studies of the cyclic orbifold

Coulomb Gas approach

- Dictionary between minimal CFTs and “imaginary Liouville” action [Dotsenko-Fateev ’84]:

$$A(\phi) = \int d^2x \left[(\nabla\phi)^2 + 2iQ\mathcal{R}\phi + \# e^{ib\phi} + \# e^{-i\phi/b} \right]$$

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  Vertex operators $V_\alpha = e^{i\alpha\phi}$, $\qquad h_\alpha = h_{2Q-\alpha} = \alpha^2 - 2Q\alpha$

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- Correlation functions: $\langle \Phi_1 \dots \Phi_n \rangle \leftrightarrow \langle V_{\alpha_1} \dots V_{\alpha_n} Q_+^k Q_-^\ell \rangle$
  Neutrality: $\sum_j \alpha_j + 2(g-1)Q + kb - \ell/b = 0$

Some working examples with the CG

- One-interval, generic N entropy in the state ϕ_{21}
  - Recall $\alpha_{21} = -b/2$
  - $\langle \Phi(0)\tau_h(z,\bar{z})\widetilde{\tau}_h(1)\Phi(\infty)\rangle$ with $\Phi = \phi_{21}^{\otimes N}$
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  - For $N = 2$: we obtain full action of monodromy group on conformal blocks

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- Two-interval $N = 2$ entropy in the vacuum state
  - Minimal model $\mathcal{M}(p,q)$: CG parameter $b = \sqrt{q/p}$
  - Four-point function $\langle \tau(u_1,\bar{u}_1)\widetilde{\tau}(v_1,\bar{v}_1)\tau(u_2,\bar{u}_2)\widetilde{\tau}(v_2,\bar{v}_2)\rangle$
  - Associate vertex charges $(0, 2Q)$ to $(\tau, \widetilde{\tau})$
  - Neutrality: $4Q + (p-2)b - (q-2)/b = 0$
  - $(p-1)(q-1)$ choices of contours?

Modular invariance for $\mathbb{Z}_N$ orbifolds

- In any rational CFT:
  - Torus partition function : $Z(\tau,\bar{\tau}) = \sum_j |\chi_j(\tau)|^2$
  - Modular S-matrix: $\chi_j(-1/\tau) = \sum_k S_{jk}\chi_k(\tau)$
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- Apply to $\mathbb{Z}_N$ orbifold of $\mathcal{M}(p,q)$ (for prime N):
  - Describe set of characters
  - Explicit S-matrix: (non-trivial even for $N=2$)
  - Obtain fusion rules for twisted and untwisted operators

Conclusions and perspectives

- Current results
  - "Standard" computations of EE: only $g = 0$ **or** $c \in \{\frac{1}{2}, 1\}$
  - New approach based on $\mathbb{Z}_N$-orbifold of Virasoro algebra
  - Works for minimal models (twist has two null vectors)
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- Future work
  - Find systematic derivation of differential equations ?
  - Use mod. invariance to get fusion rules in $\mathbb{Z}_N$-orbifold ?
  - Construct Coulomb-Gas formalism for conformal blocks in $\mathbb{Z}_N$-orbifold ? [joint with O. Blondeau-Fournier (Laval)]

Thank you!