Operator algebra in critical loop models and non-rational Conformal Field Theories

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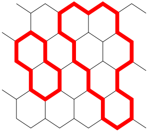
1. Introduction

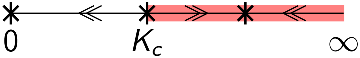
The O(n) loop model

[Nienhuis 82]

- Definition:
  - $C =$  $\longrightarrow \; W(C) = n^{\#\text{loops}(C)} \; K^{\#\text{edges}(C)}$
  - Partition function: $Z(n, K) = \sum_{C} W(C)$
- Correlations:

$$\langle \mathcal{O}_1(\vec{r}_1) \dots \mathcal{O}_p(\vec{r}_p) \rangle = \frac{1}{Z} \sum_{C} W(C) \times \mathcal{O}_1(\vec{r}_1) \dots \mathcal{O}_p(\vec{r}_p)$$

- Phase diagram for $-2 < n \leq 2$:



- Examples:
  - $n \longrightarrow 0$: single closed self-avoiding walk
  - $n = 1$: domain walls of the 2d Ising model

Scaling theory of the O(n) loop model

[Nienhuis 84, Dotsenko-Fateev 84, Nienhuis-Foda 89]

- Compactified boson coupled to scalar curvature:

$$A[\phi] = \int d^2 r \, \frac{\sqrt{g}}{4\pi} \left(\partial_\mu \phi \partial^\mu \phi + 2 i \alpha_0 \mathcal{R} \phi \right) , \quad \phi \equiv \phi + 2\pi R$$

- Parameters: $n = -2\cos\frac{\pi}{R^2}$ $\quad R > \frac{1}{\sqrt{2}}$ $\quad \alpha_0 = \frac{1}{2}(R - R^{-1})$

- Central charge: $c = 1 - 24\alpha_0^2$

- Vertex operators: $V_\alpha(z, \bar{z}) = \exp[i\alpha\phi(z, \bar{z})]$

  ↖ well-defined only if $\alpha \in \mathbb{Z}/R$

- "Magnetic" defect of charge $m \in \mathbb{Z}$ at z_0:

$$\phi(z_0 + e^{2i\pi}\epsilon,\ \bar{z}_0 + e^{-2i\pi}\bar{\epsilon}) = \phi(z_0 + \epsilon,\ \bar{z}_0 + \bar{\epsilon}) + 2\pi m R$$

Spectrum of the O(n) loop model

[Di Francesco-Saleur-Zuber 87]

- Kac notations: $\alpha_{rs} = \frac{(1-r)}{2R} - \frac{(1-s)R}{2}$, $h_{rs} = \alpha_{rs}^2 - 2\alpha_0\alpha_{rs}$
- The spectrum:
  - Zero-defect sector: $V_k = V_{\alpha_{2k+1,1}}$ $h = \bar{h} = h_{2k+1,1}$ $k \in \mathbb{N}$
  - m-defect sector: $\mu_{e,m}$ $\begin{cases} h = h_{e,+m} \\ \bar{h} = h_{e,-m} \end{cases}$ $e \in \mathbb{N}/m$
- Non-rational CFT:

$$Z_{\rm torus}(q) = \frac{1}{|\eta(q)|^2} \sum_{\{h,\bar{h}\}} N_{h,\bar{h}}\, q^h\, \bar{q}^{\bar{h}} \qquad q = e^{2i\pi\tau} \qquad N_{h,\bar{h}} \notin \mathbb{N}$$

- Non-local interaction:

$$Z_{\rm torus}(q) = \widetilde{\rm Tr}\left(q^{L_0 - c/24}\ \bar{q}^{\bar{L}_0 - c/24}\right) \neq {\rm Tr}\left(q^{L_0 - c/24}\ \bar{q}^{\bar{L}_0 - c/24}\right)$$

Operator algebra in the zero-defect sector

[Dotsenko-Fateev 84]

- Degenerate primary fields $\Phi_{2k+1,1} \equiv V_k$
- Decoupling null states, e.g.

$$\chi_{31} = (L_{-3} + xL_{-2}L_{-1} + yL_{-1}^3)\Phi_{31}$$
$$\langle \chi_{31} \ldots \rangle = 0 \;\Rightarrow\; \chi_{31} \equiv 0$$

- Fusion rules: $V_k \times V_\ell = V_{|k-\ell|} + V_{|k-\ell|+1} + \cdots + V_{k+\ell}$
- Differential equations and conformal blocks:

$$\mathcal{D}_{2k+1} \langle V_k(0) V_k(z) \Phi_h(1) \Phi_h(\infty) \rangle = 0$$

Solutions: $\{\mathcal{F}_0(z), \mathcal{F}_1(z) \ldots \mathcal{F}_{2k}(z)\}$ with $\mathcal{F}_\ell(z) \underset{z \to 0}{\sim} z^{-2h_k + h_\ell}$

Operator algebra in the zero-defect sector (2)

[Dotsenko-Fateev 84]

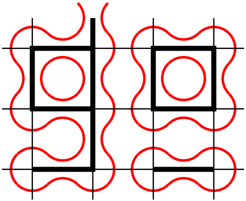
- [DF84]: Integral representation of the $\mathcal{F}_\ell$'s
- Physical correlation function $G(z,\bar{z}) = \sum_{\ell=0}^{2k} X_\ell \mathcal{F}_\ell(z)\overline{\mathcal{F}_\ell(z)}$
- Crossing symmetry $(z \to 1-z) \Rightarrow$ explicit OPE coeffs

$$V_j(z,\bar{z})V_k(0) \underset{z\to 0}{\sim} \sum_\ell C_{jk}^\ell V_\ell(0) + \dots$$

Connectivity of percolation clusters

[Delfino-Viti 2010]

- Critical dense phase of $O(n=1) \equiv$ percolation, with $c=0$



- Loops = boundaries of perco. clusters [Baxter-Kelland-Wu 76]
- Proba. of sitting on the same cluster: $P_c(\vec{r}_1, \vec{r}_2) \sim \frac{\text{const}}{|\vec{r}_1-\vec{r}_2|^{4h_{1/2,0}}}$
- Universal ratio:

$$\frac{P_c(\vec{r}_1, \vec{r}_2, \vec{r}_3)}{\sqrt{P_c(\vec{r}_1, \vec{r}_2)P_c(\vec{r}_1, \vec{r}_3)P_c(\vec{r}_2, \vec{r}_3)}} \sim C(\Phi_{1/2,0}, \Phi_{1/2,0}, \Phi_{1/2,0})$$

- **But** $\Phi_{1/2,0}$ does not correspond to a state in the spectrum. . .

Connectivity of percolation clusters (2)

[Delfino-Viti 2010]

- Take OPE constant from "time-like" Liouville:
  - Central charge $c = 1 - 6(b - b^{-1})^2$
  - $$C(V_{\alpha_1}, V_{\alpha_2}, V_{\alpha_3}) = \frac{\Upsilon(2b - b^{-1} + \alpha_{123}) \prod_{(ijk)} \Upsilon(b + \alpha_{ij}^k)}{\sqrt{\prod_i \Upsilon(b + 2\alpha_i)\Upsilon(2b - b^{-1} + 2\alpha_i)}}$$

  $$\alpha_{123} = \alpha_1 + \alpha_2 + \alpha_3 \,, \qquad \alpha_{ij}^k = \alpha_i + \alpha_j - \alpha_k$$

- Specialise to $c = 0$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_{1/2,0}$
- Perfect agreement with numerics !
  See also [Delfino-Picco-Santachiara-Viti 2012]
- More general relation O(n) model $\leftrightarrow$ timelike Liouville?

Outline

1. Introduction ✓

2. The O(n) model and the timelike Liouville CFT

3. OPEs of non-scalar fields in the O(n) model

2. The O(n) model and the timelike Liouville CFT

The timelike Liouville theory

[Al. Zamolodchikov 2005]

- Action
  - $A[\phi] = \int d^2r \, \frac{\sqrt{g}}{4\pi} \left(\partial_\mu\phi\partial^\mu\phi + iQ\mathcal{R}\phi + \kappa e^{-ib\phi}\right)$
  - $\mathcal{R}$ = scalar curvature
  - background charge $Q = b^{-1} - b$ with $b \in \mathbb{R}$
  - $e^{-ib\phi} = V_{-b}$ = screening charge with $h = \bar{h} = 1$

- Central charge: $c = 1 - 6Q^2$

- Non-compact target space: $\phi \in \;]-\infty, +\infty[$

- Spectrum of primary ops: $\{V_\alpha = e^{i\alpha\phi}, \alpha \in \mathbb{R}\}$

- Consistent non-rational CFT with $c \leq 1$
  [Ribault-Santachiara 2015]

Conformal bootstrap in (timelike) Liouville

[Dorn-Otto 94, Zamolodchikov-Zamolodchikov 96]

- Assume CFT has degenerate primary fields Φ_{21} and Φ_{12}
- Consider correlation function of scalar ops (for any $\alpha_1, \alpha_2, \alpha_3$):

$$G(z,\bar{z}) = \langle V_{\alpha_1}(0)\Phi_{21}(z,\bar{z})V_{\alpha_2}(1)V_{\alpha_3}(\infty)\rangle$$

- Null-vector cond. $(L_{-2} + \eta L_{-1}^2)\Phi_{21} = 0$
  $\Rightarrow G$ satisfies hypergeometric ODE
- Decompositions $\begin{cases} G(z,\bar{z}) & = X_1 I_1(z)\overline{I_1(z)} + X_2 I_2(z)\overline{I_2(z)} \\ G(z,\bar{z}) & = Y_1 J_1(z)\overline{J_1(z)} + Y_2 J_2(z)\overline{J_2(z)} \end{cases}$

$$I_1(z) = {}_2\mathrm{F}_1(a_1, b_1; c_1|z)\,, \qquad I_2(z) = {}_2\mathrm{F}_1(a_2, b_2; c_2|z)$$
$$J_1(z) = {}_2\mathrm{F}_1(a_1, b_1; c_1|1-z)\,, \qquad J_2(z) = {}_2\mathrm{F}_1(a_2, b_2; c_2|1-z)$$

- $\Rightarrow \frac{X_1}{X_2} = \frac{\gamma_b(\alpha_1-\alpha_1')\gamma_b(\alpha_3-\alpha_1-\alpha_2'+\frac{b}{2})\gamma_b(\alpha_3-\alpha_1+\alpha_2+\frac{b}{2})}{\gamma_b(\alpha_1'-\alpha_1)\gamma_b(\alpha_3-\alpha_1'-\alpha_2'+\frac{b}{2})\gamma_b(\alpha_3-\alpha_1'+\alpha_2+\frac{b}{2})} \equiv f_b(\alpha_1,\alpha_2,\alpha_3)$

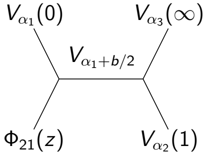
$$\gamma_b(x) = \Gamma(bx)/\Gamma(1-bx) \qquad \alpha_j' = Q - \alpha_j$$

Conformal bootstrap in (timelike) Liouville (2)

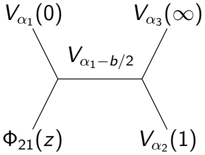
[Dorn-Otto 94, Zamolodchikov-Zamolodchikov 96]

- Interpretation of conformal blocks:

$I_1(z) =$



$I_2(z) =$



- $\Rightarrow X_1/X_2 = \frac{C(\alpha_1,\alpha_{12},\alpha_1+b/2)C(\alpha_1+b/2,\alpha_2,\alpha_3)}{C(\alpha_1,\alpha_{12},\alpha_1-b/2)C(\alpha_1-b/2,\alpha_2,\alpha_3)} = f_b(\alpha_1,\alpha_2,\alpha_3)$

- Same exercise with Φ_{12}:
$$\frac{C(\alpha_1,\alpha_{12},\alpha_1-b^{-1}/2)C(\alpha_1-b^{-1}/2,\alpha_2,\alpha_3)}{C(\alpha_1,\alpha_{12},\alpha_1+b^{-1}/2)C(\alpha_1+b^{-1}/2,\alpha_2,\alpha_3)} = f_{-1/b}(\alpha_1,\alpha_2,\alpha_3)$$

- Unique solution to functional equations:
$$C(\alpha_1,\alpha_2,\alpha_3) = \frac{\Upsilon(2b-b^{-1}+\alpha_{123})\prod_{(ijk)}\Upsilon(b+\alpha_{ij}^k)}{\sqrt{\prod_i \Upsilon(b+2\alpha_i)\Upsilon(2b-b^{-1}+2\alpha_i)}}$$

$\alpha_{123}=\alpha_1+\alpha_2+\alpha_3\,,\quad \alpha_{ij}^k=\alpha_i+\alpha_j-\alpha_k$ $\qquad \Upsilon =$ ad hoc special function

Generic vertex operators in the O(n) model

- Reminder:
  - Loop model $\equiv$ compact boson coupled to curvature
  - Allowed vertex ops in zero-defect sector : $V_k = V_{\alpha_{2k+1,1}}$
- Two-point function of generic vertex ops :

$$\langle V_{\alpha_1}(\vec{r}_1) V_{2\alpha_0-\alpha_1}(\vec{r}_2)\rangle_{\mathrm{Liouville}} = \langle V_{\alpha_1}(\vec{r}_1) V_{2\alpha_0-\alpha_1}(\vec{r}_2)\rangle_{\mathrm{comp.\ boson}} \sim \frac{Z_{n_1}(\vec{r}_1,\vec{r}_2)}{Z}$$

- Modified partition sum: $\quad Z_{n_1}(\vec{r}_1,\vec{r}_2) = \sum_C K^{|C|} n^{\ell(C)}\ n_1^{\ell_1(C)}$
- $\ell(C)=$ # loops which do not split $\vec{r}_1$ and $\vec{r}_2$
  $\ell_1(C)=$ # loops which split $\vec{r}_1$ and $\vec{r}_2$
- Loop weights: $n = -2\cos\frac{\pi}{R^2} \qquad n_1 = 2\cos\frac{2\pi(\alpha_1-\alpha_0)}{R}$

Three-point functions

- We propose the three-point function:

$$Z_{n_1,n_2,n_3}(\vec{r}_1,\vec{r}_2,\vec{r}_3) = \sum_C K^{|C|} n^{\ell(C)}\ n_1^{\ell_1(C)}\ n_2^{\ell_2(C)}\ n_3^{\ell_3(C)}$$

- $\ell(C)=$ # loops which do not split $\{\vec{r}_1,\vec{r}_2,\vec{r}_3\}$
  $\ell_1(C)=$ # loops which split $\vec{r}_1$ from $\{\vec{r}_2,\vec{r}_3\}$
  $\ell_2(C)=$ # loops which split $\vec{r}_2$ from $\{\vec{r}_1,\vec{r}_3\}$
  $\ell_3(C)=$ # loops which split $\vec{r}_3$ from $\{\vec{r}_1,\vec{r}_2\}$

- We argue that:

$$\boxed{\langle V_{\alpha_1}(\vec{r}_1)V_{\alpha_2}(\vec{r}_2)V_{\alpha_3}(\vec{r}_3)\rangle_{\text{Liouville}} \sim \frac{Z_{n_1,n_2,n_3}(\vec{r}_1,\vec{r}_2,\vec{r}_3)}{Z}}$$

- Loop weights: $n_j = 2\cos\frac{2\pi(\alpha_j-\alpha_0)}{b}$

Numerical checks

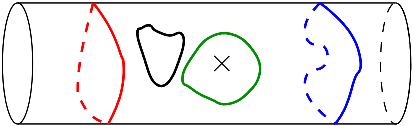
- Percolation (dense $n=1$) [Delfino-Viti 2010] :

$$Z_{n_{1,2,3}=0}(\vec{r}_1, \vec{r}_2, \vec{r}_3) \propto P_c(\vec{r}_1, \vec{r}_2, \vec{r}_3) \qquad \checkmark$$

- Generic values of n, n_1, n_2, n_3: transfer-matrix algorithm

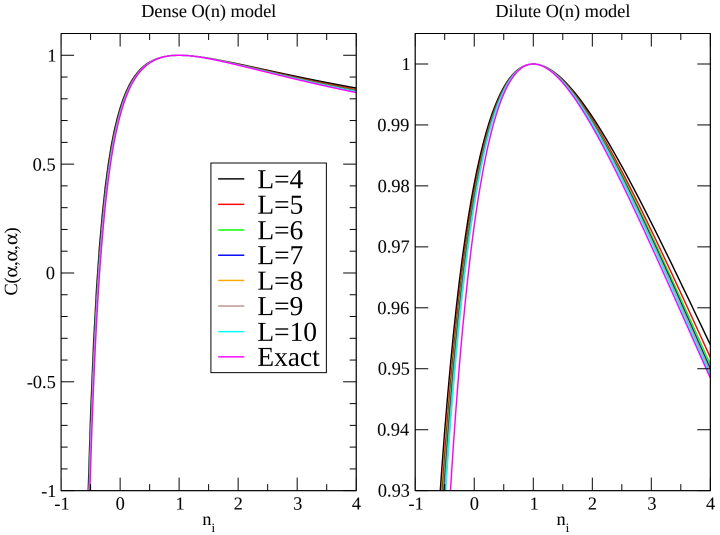
- Infinite cylinder geometry:

$$Z^{\mathrm{cyl}}_{n_1,n_2,n_3}(-\infty, 0, +\infty) = \langle \psi_{n_1} | \widehat{\psi}_{n_2} | \psi_{n_3} \rangle$$



  - $|\psi_{n_j}\rangle$: g.s. of transfer matrix with weight[non-triv. loops]$=n_j$
  - Scalar product $\langle \alpha | \widehat{\psi}_{n_2} | \beta \rangle$ incorporates weights n_1, n_2, n_3

Numerical results



3. OPEs of non-scalar fields in the O(n) model

Conformal half-bootstrap

- We want to determine OPE constants $C(\mu_{em}, \dots)$
- In timelike Liouville, min. models: Φ_{21} and Φ_{12} in spectrum
- In O(n) model $\equiv$ compact boson: only Φ_{21} allowed
- Conformal bootstrap with Φ_{21} and $h_i \neq \bar{h}_i$

Conformal half-bootstrap (2)

- $G(z,\bar{z}) = \langle \Phi_{h_1,\bar{h}_1}(0)\Phi_{21}(z,\bar{z})\Phi_{h_2,\bar{h}_2}(1)\Phi_{h_3,\bar{h}_3}(\infty)\rangle$

- Decomposition: $G(z,\bar{z}) = X_1 I_1(z)\bar{I}_1(\bar{z}) + X_2 I_2(z)\bar{I}_2(\bar{z})$

$$I_1(z) = {}_2\mathrm{F}_1(a_1, b_1; c_1|z) \qquad I_2(z) = {}_2\mathrm{F}_1(a_2, b_2; c_2|z)$$
$$\bar{I}_1(\bar{z}) = {}_2\mathrm{F}_1(\bar{a}_1, \bar{b}_1; \bar{c}_1|\bar{z}) \qquad \bar{I}_2(z) = {}_2\mathrm{F}_1(\bar{a}_2, \bar{b}_2; \bar{c}_2|\bar{z})$$

- Crossing symmetry $\Rightarrow$ explicit expression for X_1/X_2

- Results:

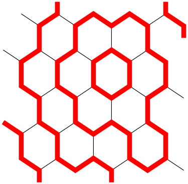
$$C(\mu_{em}, \mu_{+e,-m}, V_k) = C_{\mathcal{L}}(h_{em}, h_{e,-m}, h_{2k+1,1})$$
$$C(\mu_{em}, \mu_{-e,-m}, V_k) = \sqrt{C_{\mathcal{L}}(h_{em}, h_{e,m}, h_{2k+1,1})C_{\mathcal{L}}(h_{e,-m}, h_{e,-m}, h_{2k+1,1})}$$
$$\frac{C(\mu_{e+1,m}, \Phi_1, \Phi_2)}{C(\mu_{e-1,m}, \Phi_1, \Phi_2)} = \sqrt{\frac{C_{\mathcal{L}}(h_{e+1,m}, h_1, h_2)C_{\mathcal{L}}(h_{e+1,-m}, \bar{h}_1, \bar{h}_2)}{C_{\mathcal{L}}(h_{e-1,m}, h_1, h_2)C_{\mathcal{L}}(h_{e-1,-m}, \bar{h}_1, \bar{h}_2)}}$$

- Numerical checks with transfer-matrix algorithm

Extension to the W_3-algebra

[Dupic-Estienne-YI] work in progress . . .

- Fully-Packed Loop model (FPL) on honeycomb lattice [Reshetikhin 91, Kondev-Nienhuis-de Gier 96]



- FPL $\leftrightarrow \mathrm{U}_q(\widehat{\mathfrak{sl}}_3)$ vertex model [Reshetikhin 91]

- Extended conformal algebra [Fateev-Zamolodchikov 87]:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}$$

$$[L_n, W_m] = (2n-m)W_{n+m}$$

$$[W_n, W_m] = \#\delta_{n+m,0} + \#\Lambda_{n+m} + \#L_{n+m}$$

$$\Lambda_n = \sum_{k\in\mathbb{Z}} :L_k L_{n-k}: + \#L_n$$

- Stat. Mech. models: 3-Potts, dimers . . .

Extension to the W_3-algebra (2)

[Dupic-Estienne-YI] work in progress ...

- Free-field realisation of W_3: two-component free boson coupled to curvature [Fateev-Zamolodchikov 87]

- Central charge $c = 2 - 24\alpha_0^2$

- "Liouville" counterpart: $\mathfrak{sl}_3$ Toda CFT
  - Action:
    $A_{\mathrm{Toda}}[\phi] = \int d^2 r \frac{\sqrt{g}}{4\pi} \left[\langle \partial_\mu \phi, \partial^\mu \phi \rangle + \langle Q, \phi \rangle \mathcal{R} + \kappa (e^{b\langle e_1, \phi \rangle} + e^{b\langle e_2, \phi \rangle}) \right]$
  - e_1, e_2: simple roots of $\mathfrak{sl}_3$
  - Primary fields: $e^{\langle \alpha, \phi \rangle}$

- OPE coefficients from bootstrap [Fateev-Litvinov 2007]

- Our results ...
  - Discrete spectrum of FPL loop model, Z_{torus}
  - Half-bootstrap $\to$ (ratios of) OPE coefficients with $h \neq \bar{h}$
  - Define timelike $\mathfrak{sl}_3$ Toda CFT
    $\to \langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} \rangle_{t-\mathrm{Toda}} = Z_{\mathrm{FPL}}(n_1, n_2, n_3)$

References

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- Y.I., J. L. Jacobsen and H. Saleur
  Three-point functions in $c \leq 1$ *Liouville theory and conformal loop ensembles*
  `arXiv:1509.03538`

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  in preparation

Thank you for your attention!