One-dimensional non-interacting lattice fermions

1 Tight-binding Hamiltonian, low energy spectrum

1. We consider a system of non-interacting fermions in one space dimension, on a lattice of N sites. The lattice Hamiltonian, with parameters $\lambda > 0$ and μ, is given by

$$H = \sum_{j=1}^{N} \left[-\lambda (c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) - \mu c_j^\dagger c_j \right] , \tag{1}$$

where $c_j^\dagger$ is the fermion creation operator on site j. We impose antiperiodic boundary conditions:

$$c_{N+1} := -c_1 \,, \qquad c_{N+1}^\dagger := -c_1^\dagger \,. \tag{2}$$

▶ Give the physical interpretation of the parameters λ and μ.

2. Such a quadratic model is straightforward to solve. In terms of the Fourier modes

$$\widehat{c}_k^\dagger = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} e^{ikj} c_j^\dagger \tag{3}$$

the Hamiltonian is diagonal

$$H = \sum_k \varepsilon_k \widehat{c}_k^\dagger \widehat{c}_k \,. \tag{4}$$

▶ For a single-particle state, what are the possible values of the momentum k?

▶ Compute the associated dispersion relation ε_k.

3. The ground state of H is given by the Fermi sea, *i.e.* the many-particle state where all modes with negative energy are occupied.

▶ For what values of the parameters λ and μ is the model gapped ? Gapless ?

4. We will focus on the gapless case, and we will argue that the low energy/long distance behaviour is captured by a conformal field theory.

▶ Argue that the low energy/long distance physics is dominated by the momenta k close to the Fermi surface, and therefore one can linearise the dispersion relation :

$$\varepsilon_{\pm k_F + \delta k} \sim \pm v_F \, \delta k \,, \tag{5}$$

for small enough δk. Compute the Fermi velocity v_F.

For the rest of the problem we set $\mu = 0$ (thus $k_F = \pi/2$), and we assume that the system size is a multiple of four : $N = 4n$.

5. ▶ What is the ground state of the system ? Using the Euler-MacLaurin formula, show that the finite-size behaviour of the ground-state energy corresponds to the CFT expectation:

$$E_0(N) = Ne_0 - \frac{\pi v_F c}{6N} + O(N^{-2}) \,, \tag{6}$$

where e_0 is the ground-state energy density per lattice site and c is the central charge. Give e_0 in the form of a single integral.

▶ What is the value of the central charge c ? Can you explain the appearance of the Fermi velocity in the above formula ?

6. Consider the excited state $|\phi_p\rangle$ obtained from the ground state by shifting all momenta $k \to k + 2\pi p/N$, where $p \ll N$ is a finite integer. Compute the total momentum of this state. Show that its energy is of the form:

$$E_p(N) = E_0(N) + \frac{2\pi v_F p^2}{N} + O(1/N^2) \,. \tag{7}$$

From these results, determine the conformal dimensions $(h_{2\pi p}, \bar{h}_{2\pi p})$ associated to the state $|\phi_p\rangle$ in the scaling limit. Is this state degenerate under the Virasoro algebra ? Can you think of a way to change boundary conditions to allow real values of the parameter $p = \alpha/(2\pi)$?

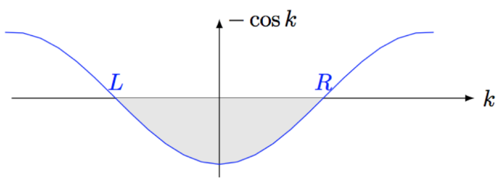
2 Effective low-energy Hamiltonian

We are concerned with the thermodynamic limit of this simple model of one-dimensional fermions. Before taking the thermodynamic limit, we introduce the lattice spacing a, so that sites are located at positions $x = aj$ with $j = 1, 2, \ldots, N$, and the total chain length is $L = Na$. The Fermi velocity now acquires the correct dimension $v_F \to av_F$. The thermodynamic limit of a correlation function of local operators $\langle O_1(j_1) \ldots O_m(j_m)\rangle$ is obtained by taking $a \to 0$, $N \to \infty$, keeping $x_i = aj_i$ and L constant. We label the momenta close to the Fermi surface as $k = \pm\pi/2 + aq$.

In order to capture the low energy/long distance physics, we drop the fast moving degrees of freedom and only keep the low-energy terms in the Fourier expansion of the fermion operator

$$c_j = \frac{1}{\sqrt{N}} \sum_k \widehat{c}_k e^{ikj} \to \sqrt{\frac{a}{L}} \left(e^{i\frac{\pi}{2}j} \sum_q \widehat{c}_{\frac{\pi}{2}+aq} e^{iqx} + e^{-i\frac{\pi}{2}j} \sum_q \widehat{c}_{-\frac{\pi}{2}+aq} e^{iqx} \right),$$

where $x = ja$ and the sum over q should be understood as a sum over all values of q in a "small enough window" (for $\delta k = aq$) around the Fermi surface $k = \pm\pi/2$. But as we send $a \to 0$ this "small window" contains more and more values of q, and in the thermodynamic limit this sum becomes an infinite one.



Through this procedure, we get a left moving fermion field around $k = -\frac{\pi}{2}$, and a right moving one around $k = \frac{\pi}{2}$. The reason they are called left (resp. right) moving will become clear in question 9.

$$\Psi_R(x) = \sqrt{\frac{1}{L}} \sum_q \underbrace{\widehat{c}_{\frac{\pi}{2}+aq}}_{\widehat{c}_R(q)} e^{iqx}, \qquad \Psi_L(x) = \sqrt{\frac{1}{L}} \sum_q \underbrace{\widehat{c}_{-\frac{\pi}{2}+aq}}_{\widehat{c}_L(q)} e^{iqx}$$

Thus

$$c_j = \sqrt{a}\left[e^{-i\pi j/2}\Psi_L(x) + e^{i\pi j/2}\Psi_R(x)\right], \qquad x = ja\,.$$

7. Check that $\Psi_\eta(x+L) = -\Psi_\eta(x)$, where η stands for L or R. In the thermodynamic limit $N \to \infty$, the sum over q becomes an infinite sum. Check that the two fermions Ψ_L and Ψ_R obey

$$\{\Psi_\eta^\dagger(x), \Psi_{\eta'}(x')\} = \delta_{\eta,\eta'}\delta(x-x')$$

that is, they become fully fledged fermionic operators in the continuum. Thus the lattice fermion operator c_j yields two fermion fields in the continuum ! This phenomenon, which goes under the name of *fermion doubling*, is due to the fact that there are two momentum regions in the low-energy limit.

8. Show that in the continuum limit, the non-interacting fermionic Hamiltonian becomes

$$H = iv_F \int_0^L dx\, \left(\Psi_L^\dagger(x)\partial_x\Psi_L(x) - \Psi_R^\dagger(x)\partial_x\Psi_R(x)\right), \tag{8}$$

$$= v_F \sum_q q\left(c_R^\dagger(q)c_R(q) - c_L^\dagger(q)c_L(q)\right). \tag{9}$$

9. In the Heisenberg picture (with Planck's constant $\hbar = 1$), show that

$$\Psi_R(x,t) = \Psi_R(x - v_F t), \qquad \Psi_L(x,t) = \Psi_L(x + v_F t)\,, \tag{10}$$

where t denotes time. In imaginary time $\tau = it$ this means that the operators $\Psi = \Psi_R$ and $\overline{\Psi} = \Psi_L$ are respectively holomorphic and anti-holomorphic in the complex variable $z = x + iv_F\tau$.

10. This effective Hamiltonian has two U(1) symmetries : both left and right fermion numbers are conserved. Is it surprising considering the initial lattice model ?

3 The complex fermion

We admit that the associated Euclidean action is given by the "complex fermion":

$$S = \frac{1}{4\pi}\int d^2r\, \left(\Psi^\dagger\partial_{\bar{z}}\Psi + \Psi\partial_{\bar{z}}\Psi^\dagger + \bar{\Psi}^\dagger\partial_z\bar{\Psi} + \bar{\Psi}\partial_z\bar{\Psi}^\dagger\right), \tag{11}$$

where

$$\Psi = \psi_1 + i\psi_2\,, \qquad \bar{\Psi} = \bar{\psi}_1 + i\bar{\psi}_2\,, \tag{12}$$

$$\Psi^\dagger = \psi_1 - i\psi_2\,, \qquad \bar{\Psi}^\dagger = \bar{\psi}_1 - i\bar{\psi}_2\,, \tag{13}$$

with independent Grassmann variables $\psi_1, \psi_2, \bar{\psi}_1, \bar{\psi}_2$.

11. At the classical level, what should be the scale dimension of the fields Ψ and $\overline{\Psi}$ so that one gets a scale invariant action ?

    Inside a correlation function, the ψ_j's are holomorphic, and the $\bar{\psi}_j$'s are antiholomorphic, and so we write them as $\psi_j(z)$ and $\bar{\psi}_j(\bar{z})$, and similarly for $\Psi, \Psi^\dagger, \bar{\Psi}, \bar{\Psi}^\dagger$. From the above quadratic action, one can show (by standard integration over Grassmann variables) that the two-point functions are

$$\langle \psi_1(z)\psi_1(w)\rangle = \langle \psi_2(z)\psi_2(w)\rangle = \frac{1}{z-w}\,, \qquad \langle \psi_1(z)\psi_2(w)\rangle = 0\,, \tag{14}$$

    and similarly for $\bar{\psi}_1, \bar{\psi}_2$. Compute $\langle \Psi(z)\Psi(w)\rangle$, $\left\langle \Psi^\dagger(z)\Psi^\dagger(w)\right\rangle$ and $\left\langle \Psi(z)^\dagger\Psi(w)\right\rangle$. What are the left and right conformal dimensions of Ψ and $\Psi^\dagger$?

12. The corresponding stress-energy tensor is

$$T(z) = -\frac{1}{4}\left[\,:\!\Psi^\dagger(z)\partial_z\Psi(z)\!: \,+\, :\!\Psi(z)\partial_z\Psi^\dagger(z)\!:\right]\,. \tag{15}$$

    Using Wick's theorem, compute explicitly (i) the OPEs $T(z).\Psi(w)$ and $T(z).\Psi^\dagger(w)$, and (ii) the OPE $T(z).T(w)$. Show that Ψ and $\Psi^\dagger$ are primary, and that the results are consistent with the value of the central charge $c = 1$.

13. Show that this action has a U(1) symmetry. What is the physical meaning of this symmetry ? Check that the associated current is

$$J(z) = :\!\Psi^\dagger(z)\Psi(z)\!:\,, \qquad \overline{J}(\bar{z}) = :\!\overline{\Psi}^\dagger(\bar{z})\overline{\Psi}(\bar{z})\!:\,. \tag{16}$$

    What is its conformal dimension? Is it surprising ?

4 Charge fluctuation

14. We consider the periodic system defined in the first question, with system size $L = Na$, where a is the lattice step. We are interested in the "full-counting statistics", *i.e.* the quantum statistics of the number of fermions in a given interval. If $m \le m'$ are two points on midedges of the lattice ($m, m' \in \{1/2, 3/2, \ldots, N-1/2\}$), we introduce

$$n_f(m, m') = \sum_{j=m+1/2}^{m'-1/2} (c_j^\dagger c_j - 1/2)\,. \tag{17}$$

    We consider the scaling regime, where both N and $|m'-m|$ tend to infinity, $a \to 0$, with both physical lengths $L = Na$ and $\ell = (m'-m)a$ staying fixed and finite. We admit that, for any $\alpha \in [-\pi, \pi]$, the quantity $\exp[i\alpha\, n_f(m, m')]$ is given by a product of scaling operators $V_\alpha(m)V_{-\alpha}(m')$, where $V_{\pm\alpha}(m)$ scales to a primary operator $v_{\pm\alpha}$, with conformal dimensions $h_\alpha = \bar{h}_\alpha = (\alpha/\pi)^2/8$. Relate the probability distribution of $n_f(m, m')$ in the ground-state, to the CFT two-point correlation function $\langle v_\alpha v_{-\alpha}\rangle$ on an infinite cylinder. Compute this function explicitly, and deduce that this probability distribution tends to a Gaussian. What is the variance ?

15. Let $|\Phi\rangle$ be an excited state of the periodic system. We assume $|\Phi\rangle$ scales to a scalar primary state $|\phi_h\rangle$ in the CFT limit. Express the expectation value $\langle\Phi|V_\alpha(m)V_{-\alpha}(m')|\Phi\rangle$ as a CFT correlation function on an infinite cylinder, and relate it to the four-point correlation function on the complex plane:

$$\langle \phi_h(\infty) v_\alpha(1) v_{-\alpha}(z, \bar{z}) \phi_h(0)\rangle\,, \tag{18}$$

and express the variable z in terms of the physical lengths ℓ and L. If ϕ_h is degenerate under the Virasoro algebra, express this correlation function in terms of conformal blocks.

16. Using a similar argument, compute explicitly the expectation value $\langle T|V_\alpha(m)V_{-\alpha}(m')|T\rangle$, where $|T\rangle$ is the excited state corresponding to $L_{-2}|0\rangle$ in the scaling limit.