The conformal bootstrap for the Ising model

1 Kramers-Wannier duality and disorder operators

Consider the Ising model on a domain of the square lattice $\mathcal{L}$, where each spin $\sigma_i = \pm 1$, and with Boltzmann weights:

$$W[\sigma] = \prod_{\langle ij \rangle} e^{J\sigma_i\sigma_j}\,, \tag{1}$$

where the product is over all nearest neighbour pairs $\langle ij \rangle$, and J is the coupling constant. We will denote by V the number of vertices (or sites) of the domain, and by E the number of edges.

Using the identity

$$e^{J\sigma_i\sigma_j} = \cosh J + \sinh J\,\sigma_i\sigma_j = \sum_{k=0,1} x_k\,(\sigma_i\sigma_j)^k\,, \tag{2}$$

where $x_0 = \cosh J$ and $x_1 = \sinh J$, we can rewrite the partition as

$$Z(\mathcal{L}, J) = \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle} \sum_{k=0,1} x_k (\sigma_i\sigma_j)^k = \sum_{\{\sigma_i\}} \sum_{\{k_{ij}\}} \prod_{\langle ij \rangle} x_{k_e} (\sigma_i\sigma_j)^{k_{ij}}\,. \tag{3}$$

► Perform the sum over the original spin variables $\{\sigma_i\}$, and show that the allowed configurations for the k_{ij}'s are those which satisfy the condition

$$\forall i\,, \qquad \sum_{\langle ij \rangle} k_{ij} \equiv 0 \mod 2\,, \tag{4}$$

where the sum runs over the nearest neighbours j of the site i. Show that, as a consequence, the partition function can be written as

$$Z(\mathcal{L}, J) = 2^{V-1} A^E \sum_{\{q_\alpha\}} \prod_{\langle \alpha\beta \rangle} e^{\widehat{J} q_\alpha q_\beta}\,, \tag{5}$$

where the q_α's are Ising spins living at the middle of the faces of $\mathcal{L}$. Find the value of the constants $A(J)$ and $\widehat{J}$. This transformation is called *Kramers-Wannier duality.* Argue that the critical point is such that $J = \widehat{J}$, and find the critical value of J.

Consider the *disorder operator* μ_α which inserts the extremity of a path γ at the middle of the face α, so that the Boltzmann weights are modified by $J \mapsto -J$ for all edges $\langle ij \rangle$ crossing the path γ. For instance, the two-point function of disorder operators reads

$$\langle \mu_u \mu_v \rangle = \frac{1}{Z(\mathcal{L}, J)} \sum_{\{\sigma_i\}} \prod_{\langle ij \rangle \notin \gamma^\perp} e^{J\sigma_i\sigma_j} \prod_{\langle ij \rangle \in \gamma^\perp} e^{-J\sigma_i\sigma_j}\,, \tag{6}$$

where $\gamma^\perp$ denotes the set of edges crossing the path γ.

▶ Show that $\langle \mu_u \mu_v \rangle$ is independent of the choice of the path γ. Perform the Kramers-Wannier duality on this two-point function. Generalise to n-point functions of disorder operators.

▶ In the scaling limit, for a system defined on the full plane, compute the two-point functions $\langle \sigma(r_1)\sigma(r_2) \rangle$ and $\langle \mu(r_1)\mu(r_2) \rangle$, using your knowledge on the Ising CFT. Consider a mixed correlation function

$$\langle \sigma(r_1) \dots \sigma(r_n) \cdot \mu(r'_1) \dots \mu(r'_m) \rangle \ . \tag{7}$$

What is the behaviour of this function as r_i winds around r'_j ? Consider the OPE of σ and μ: argue that it should contain a holomorphic operator ψ and an anti-holomorphic operator $\bar{\psi}$, and give the conformal dimensions of these operators.

2 Correlation functions

We want to analyse the correlation function of spin operators

$$F(z, \bar{z}) = \langle \sigma(\infty)\sigma(1)\sigma(z, \bar{z})\sigma(0) \rangle \ . \tag{8}$$

Since σ is degenerate at level 2, F satisfies a differential equation deriving from the null-state equation. We admit that the precise form of this differential equation is

$$\left[\frac{4}{3}\partial_z^2 + \frac{1-2z}{z(1-z)}\partial_z - \frac{1}{16z^2(1-z)^2} \right] F(z, \bar{z}) = 0 \tag{9}$$

We want to find a basis of holomorphic solutions of the form

$$F(z) = z^\lambda \sum_{n=0}^{\infty} a_n z^n \ . \tag{10}$$

▶ Express the differential equation in terms of the operator $\theta = z\partial_z$, find the two possible values for λ, and derive the recursion relation for the coefficients a_n.

The basis of holomorphic solutions constructed above corresponds to the conformal blocks of $\langle \sigma\sigma\sigma\sigma \rangle$ in the channel $z \to 0$. We give the explicit form of these conformal blocks (NB: they do not obey the standard normalisation)

$$I_1(z) = \frac{\sqrt{1 + \sqrt{1-z}}}{z^{1/8}\,(1-z)^{1/8}} \ , \qquad I_2(z) = \frac{\sqrt{1 - \sqrt{1-z}}}{z^{1/8}\,(1-z)^{1/8}} \ . \tag{11}$$

▶ Find the internal primary operators in these conformal blocks. Arguing on the monodromies of $I_1(z), I_2(z)$ around $z = 0$ and $z = 1$, construct the physical correlation function $F(z, \bar{z})$. Derive the OPE coefficient $C^\epsilon_{\sigma\sigma}$.

▶ Perform a similar study for the mixed correlation function

$$\langle \sigma(\infty)\sigma(1)\mu(z, \bar{z})\mu(0) \rangle \ .$$

Find the conformal blocks in the channel $z \to 1$, and identify their internal dimensions. Compute the OPE coefficients $C^\psi_{\sigma\mu}$ and $C^{\bar{\psi}}_{\sigma\mu}$.