Discrete and continuous operators in critical models

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July 2020

Abstract

In this thesis, I summarise two main directions of my recent research in the field of 2d critical phenomena, both related to operators in critical models. The first axis is concerned with the systematic construction, through the underlying quantum group structure, of discretely holomorphic parafermions, a class of lattice objects which satisfy a discrete version of the Cauchy-Riemann equations. The second axis is the development of the analytic 2d conformal bootstrap in the context of non-rational Conformal Field Theories with Virasoro or W_N symmetry, which include non-scalar primary operators in their spectrum.

Acknowledgements

I dedicate this work to my family.

The work presented in this thesis was done at the LPTHE, in collaboration with Benoît Estienne, Jesper Jacobsen, Hubert Saleur, Hirohiko Shimada, Robert Weston, Michael Wheeler and Paul Zinn-Justin. It was very much stimutaled by enlighting discussions with colleagues in the Paris area : Jean-Bernard Zuber, Olivier Babelon, Fedor Smirnov, Vladimir Dotsenko, Kolya Kitanine, Véronique Terras, Raoul Santachiara, Jacopo Viti, Dima Chelkak.

I am particularly grateful to Jean-Bernard Zuber and Robert Weston for the careful reading of the manuscript.

Contents

Introduction

Since their early developments in the 80's, the rich algebraic structures of quantum groups [1, 2, 3, 4, 5, 6] and Conformal Field Theory (CFT) [7, 8, 9] have found important motivations and applications in the field of two-dimensional (2d) critical phenomena in Statistical Mechanics. The minimal models of conformal invariance have provided a classification scheme for 2d phase transitions with a finite number of "basic" local operators, whereas the notions of quantum groups and quantum inverse scattering [10] (i) have emerged as a formal framework to describe the exact solutions [11] of lattice models found before, especially the Bethe Ansatz [12, 13, 14, 15, 16, 17, 18], (ii) have allowed the systematic construction of new families of exactly solvable lattice models [19, 20, 21, 22, 23], and (iii) have provided powerful tools to study correlation functions on the lattice [24].

A particular class of non-local problems, associated to critical interfaces, has led to interesting developments. Indeed, correlation functions of non-local objects may be considered even in the context of statistical models with short-range interactions, such as the Ising model with nearest-neighbour interaction. Typical examples are the geometry of Ising domain walls, percolation clusters, polymers and self-avoiding walks. All these examples are encoded in the O(n) loop model [25, 26]. First, on the lattice, the study of these extended objects has motivated the introduction of *diagram algebras*, such as the Temperley-Lieb algebra [27, 28], which turn out to play an important role [2] in the quantum group underlying the lattice model. Also, they constitute a typical situation where the (imaginary-time) evolution operator becomes non-diagonalisable, which is the defining feature of a logarithmic CFT (see the review [29] and references therein). Finally, they are a well-suited setting where the relation between boundary CFT [30] and the theory of Schramm-Loewner evolution (SLE) [31] can be established.

In this thesis, I summarise two main directions of my own research in the field of 2d critical phenomena.

Lattice parafermions. In classical Statistical Mechanics, parafermions are operators whose correlation functions pick up a non-trivial phase (different from 1 and -1) when one winds around the other. They where identified in clock models with $\mathbb{Z}_N$ symmetry [32], and later, the corresponding critical points were described in terms of a parafermionic conformal algebra [33]. At the lattice level, it was noticed that, at the critical point, some of these operators satisfy local linear relations analogous to the Cauchy-Riemann equations: they are thus called discretely holomorphic operators (see [34]). This observation dates back to [35] for fermions associated to the Ising model, and it was rediscovered and extended, in particular, to Ising [36], loop [37] and clock models [38]. The main motivation for the latter studies was to provide the basic ingredient for rigorous proofs of

the conformal invariance of critical lattice models. The main steps of such proofs are as follows:

1. Exhibit a discretely holomorphic operator $\psi(z)$ coupled to a critical interface γ.

2. Prove that the lattice average value $\langle \psi(z) \dots \rangle$ converges to a holomorphic function $F(z)$ in the scaling limit, and solve the boundary-value problem for $F(z)$.

3. Construct from $F(z)$ a martingale associated to the random curve γ, and characterise γ as an SLE process (or one of its variants).

This program has been fully realised only for the percolation [39] and Ising [36] models, where specific additional features are used in the argument. In more general cases, *e.g.* clock models with $\mathbb{Z}_N$ symmetry or the O(n) loop model with $n \neq 1$, the main obstruction to complete this type of rigorous proof can be stated very simply. On a portion of the square lattice consisting of N faces, the lattice parafermions live on edges, and the discrete Cauchy-Riemann equations are associated to faces, so we get an underdetermined system of N equations for $2N$ unknowns – except for the Ising model and the percolation problem on the triangular lattice, where additional linear relations hold trivially. The practical consequence is that one does not control the convergence of $\langle \psi(z) \dots \rangle$ well enough to prove holomorphicity in the scaling limit. Hence, finding the "missing half" of discrete Cauchy-Riemann equations remains an important open problem, whose solution would open the way to the completion of the above program for a large variety of 2d critical models.

The Ising model is certainly the one for which the consequences of discrete holomorphicity have been exploited to give the most fruitful studies. This approach was used to prove rigorously conformal invariance of interfaces [36] and the covariance of spin and energy correlation functions [40]. Isoradial lattices were first considered, but much more general lattices can actually be treated, leading especially to the concept of S-embeddings. This is a currently active subject for the Probability Theory community working on Statistical Mechanics: see [41] and references therein.

In [42], following [36, 37, 38] we empirically found some new discretely holomorphic parafermions in a variety of integrable loop models. At that stage it became clear that discrete holomorphicity was related to integrability. In a subsequent series of papers [43, 44, 45], we proved that the discretely holomorphic parafermions could be obtained from the non-local conserved currents of the underlying quantum group symmetry, following a general construction introduced by [46] in a different context. This provides a systematic way of deriving the discretely holomorphic parafermions in an integrable model, as opposed to the empirical approach of earlier studies.

The operator algebra in non-rational CFTs. The most studied critical models (Ising, Potts, RSOS ...) correspond to *rational* CFTs in the scaling limit [7], *i.e.* models with a finite number of primary fields, organised in the Kac table. In contrast, examples of non-rational CFTs have been known for a long time. Their spectrum of conformal dimensions may be continuous, as for the Liouville CFT (see [47] and references therein), or discrete, as in the case of the CFT associated to the O(n) loop model [48].

The recent renewal of interest for non-rational models in the Statistical Mechanics community is related to the introduction of the imaginary (or *timelike*) Liouville CFT [49,

50, 51] and its study through the analytical conformal bootstrap [52, 53, 54, 55]: indeed, the range for the central charge of this model is $c \leq 1$, which coincides with the typical critical lattice models quoted above. Moreover, Delfino and Viti [56] made the important observation that the three-point amplitude for percolation cluster connectivity is correctly predicted by the OPE structure constant of imaginary Liouville CFT at central charge $c = 0$. This was, in my opinion, a surprising and stimulating result, since the CFT describing these percolation cluster connectivities is known to have a discrete spectrum [48], and its conformal bootstrap analysis is clearly not the same as in imaginary Liouville.

The discovery of Delfino and Viti motivated further studies of three- and four-point correlation functions of cluster connectivities in the Fortuin-Kasteleyn (FK) cluster model [57], and in the O(n) loop model [58]. Also, the conformal bootstrap approach for the quantum Liouville CFT was further developped: existence of a solution to crossing symmetry equations for complex values of the central charge [59], rigorous construction of vertex correlation functions for $c \leq 25$ [60].

Inspired by some of these ideas, in [61] we studied the operator algebra of the O(n) loop model for generic values of n: the corresponding CFT is non-rational, with a discrete, *non-diagonal* spectrum: some of the primary operators are non-scalar, *i.e.* they have distinct holomorphic and anti-holomorphic conformal dimensions $h \neq \bar{h}$. Our study shows that the OPEs are consistent with the presence of a single degenerate operator Φ_{21} at level two (and *not* Φ_{12}), as predicted in [48]. As an important consequence, after imposing crossing symmetry on four-point functions, only one shift equation is obtained for the OPE coefficients, instead of two equations as in the standard Liouville case. Nevertheless, we were able to derive analytically some OPE coefficients by adapting the bootstrap approach to this non-diagonal situation: for a certain class of non-scalar operators, the OPE coefficients are simply given by the geometric mean of the holomorphic and anti-holomorphic parts.

Another line of research that we followed, was the investigation of the conformal bootstrap for abstract CFTs (*i.e.* possibly not related to a critical lattice model) with non-diagonal spectra, and as many primary fields as possible. This was initiated in [62] for the case of the Virasoro algebra, where a classification of primary fields according to their fusion rules with Φ_{12} and Φ_{21} was proposed. Considering the case of the W_n algebra, which encodes an internal $\mathfrak{S}_n$ permutation symmetry group, we generalised these ideas [63], and showed that the primary fields are classified by the conjugacy classes of $\mathfrak{S}_n$.

Outline. This manuscript is organised as follows. In Chapter 1 are recalled, using examples, the main elements of formalism that are useful for the rest of the text, namely (i) quantum groups and the various types of associated lattice models, (ii) the Coulomb-Gas approach to derive the CFT description of critical models in the scaling limit, and (iii) the analytical conformal bootstrap to compute OPE coefficients from the crossing symmetry constraints. In Chapter 2, the construction of lattice parafermions as quantum-group conserved currents is explained, and the cases of vertex, loop, face and clock models are treated. In Chapter 3, the study of OPEs in two types of non-rational CFTs is exposed: the scaling theory of the O(n) loop model, and the non-diagonal imaginary Toda CFT.

Chapter 1

General background

1.1 Lattice models

1.1.1 Vertex models and quantum affine algebras

The six-vertex model

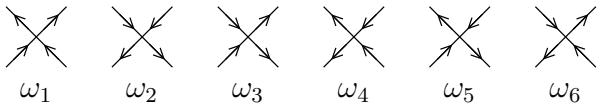


Figure 1.1: The configurations of the six-vertex model.

Vertex models are lattice models where the degrees of freedom live on the edges of the lattice, and the rules (and weights) for the allowed configurations are defined around the vertices. Let us describe the simplest example: the six-vertex (6V) model [15, 16, 17], shown in Fig. 1.1. The configurations of a vertex can be encoded in the R-matrix, oriented from bottom to top, in the basis $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$:

$$R = \begin{pmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_3 & \omega_5 & 0 \\ 0 & \omega_6 & \omega_4 & 0 \\ 0 & 0 & 0 & \omega_2 \end{pmatrix} . \tag{1.1}$$

This matrix is an endomorphism of $V_1 \otimes V_2$, where V_1 and V_2 are two-dimensional vector spaces generated by $|\uparrow\rangle$ and $|\downarrow\rangle$. A related object is the $\check{R}$-matrix, given by $\check{R} = \pi R$, where π is the permutation of V_1 and V_2. If properly parametrised, this $\check{R}$-matrix becomes an intertwiner of evaluation representations of the $U_q(\widehat{\mathfrak{sl}}_2)$ quantum affine algebra [1].

The $U_q(\widehat{\mathfrak{sl}}_2)$ quantum affine algebra

The $U_q(\widehat{\mathfrak{sl}}_2)$ is a Hopf algebra, generated by $\{e_0, e_1, f_0, f_1, t_0, t_1\}$ subject to the relations:

$$\begin{aligned}
&t_i t_i^{-1} = t_i^{-1} t_i = \mathbf{1}\,, \qquad t_0 t_1 = t_1 t_0\,,\\
&t_i e_j t_i^{-1} = q^{d_i A_{ij}} e_j\,, \qquad t_i f_j t_i^{-1} = q^{-d_i A_{ij}} f_j\,, \qquad [e_i, f_j] = \delta_{ij}\,\frac{t_i - t_i^{-1}}{q^{d_i} - q^{-d_i}}\,,\\
&\sum_{k=0}^{1-A_{ij}} (-1)^k \left[\begin{array}{c} 1-A_{ij} \\ k \end{array}\right]_{q^{d_i}} e_i^{1-A_{ij}-k}\, e_j\, e_i^k = 0\,,\\
&\sum_{k=0}^{1-A_{ij}} (-1)^k \left[\begin{array}{c} 1-A_{ij} \\ k \end{array}\right]_{q^{d_i}} f_i^{1-A_{ij}-k}\, f_j\, f_i^k = 0\,,
\end{aligned} \tag{1.2}$$

where the Cartan matrix A and the gradation d are

$$A = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}\,, \qquad d_0 = d_1 = 1\,, \tag{1.3}$$

and we have used the q-binomial coefficients:

$$\left[\begin{array}{c} m \\ n \end{array}\right]_q = \frac{(q^m - q^{-m}) \dots (q^{m-n+1} - q^{-m+n-1})}{(q^n - q^{-n}) \dots (q - q^{-1})}\,. \tag{1.4}$$

The coproduct in $U_q(\widehat{\mathfrak{sl}}_2)$ is given by:

$$\Delta(e_i) = e_i \otimes \mathbf{1} + t_i \otimes e_i\,, \qquad \Delta(f_i) = f_i \otimes t_i^{-1} + \mathbf{1} \otimes f_i\,, \qquad \Delta(t_i) = t_i \otimes t_i\,. \tag{1.5}$$

The evaluation representations π_z are a family of two-dimensional irreducible representations of $U_q(\widehat{\mathfrak{sl}}_2)$, labelled by a complex number z (the spectral parameter), with the matrices:

$$\begin{aligned}
&\pi_z(e_0) = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}\,, \qquad \pi_z(f_0) = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}\,, \qquad \pi_z(t_0) = \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix}\,,\\
&\pi_z(e_1) = \begin{pmatrix} 0 & z \\ 0 & 0 \end{pmatrix}\,, \qquad \pi_z(f_1) = \begin{pmatrix} 0 & 0 \\ z^{-1} & 0 \end{pmatrix}\,, \qquad \pi_z(t_1) = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}\,.
\end{aligned} \tag{1.6}$$

We denote by V_z the two-dimensional space for the representation π_z.

To describe the intertwining relation, we denote the tensor-product representations as:

$$\pi_{12} = (\pi_{z_1} \otimes \pi_{z_2}) \circ \Delta\,, \qquad \text{and} \qquad \pi_{21} = (\pi_{z_2} \otimes \pi_{z_1}) \circ \Delta\,. \tag{1.7}$$

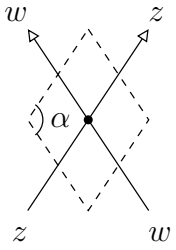
The R-matrix

The $\check{R}$-matrix given by

$$\check{R}(z) = \begin{pmatrix} qz - (qz)^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & z - z^{-1} & 0 \\ 0 & z - z^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & qz - (qz)^{-1} \end{pmatrix} \tag{1.8}$$

shall be depicted as

$$\check{R}(z/w) = \quad .$$



(1.9)

A discussion on the opening angle α can be found in the next paragraph. The $\check{R}$-matrix satisfies the intertwining relation

$$\forall x \in U_q(\widehat{\mathfrak{sl}}_2)\,, \qquad \check{R}(z_1/z_2)\,\pi_{12}(x) = \pi_{21}(x)\,\check{R}(z_1/z_2)\,. \tag{1.10}$$

More generally, for two representations V, V' of $U_q(\widehat{\mathfrak{sl}}_2)$, the matrix $\check{R}_{V,V'}$ is the intertwiner, with the defining property

$$\forall x \in U_q(\widehat{\mathfrak{sl}}_2)\,, \qquad \check{R}_{V,V'}\,[(\pi_V \otimes \pi_{V'})\Delta(x)] = [(\pi_{V'} \otimes \pi_V)\Delta(x)]\,\check{R}_{V,V'}\,. \tag{1.11}$$

Note that this intertwiner does not always exist (for instance, in the case of cyclic representations).

An important property of the $\check{R}$-matrix is the Yang-Baxter equation, which is actually a consistency relation for intertwiners of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$:

$$(\check{R}(z) \otimes \mathbf{1})(\mathbf{1} \otimes \check{R}(zw))(\check{R}(w) \otimes \mathbf{1}) = (\mathbf{1} \otimes \check{R}(w))(\check{R}(zw) \otimes \mathbf{1})(\mathbf{1} \otimes \check{R}(z))\,. \tag{1.12}$$

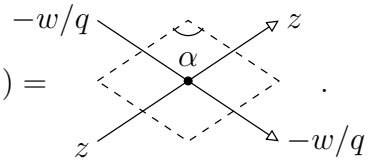
More general solutions of the Yang-Baxter equation can be constructed from the intertwiners of representations of algebras $U_q(\widehat{\mathfrak{g}})$, where $\mathfrak{g}$ is a semi-simple Lie algebra, or a Lie superalgebra.

Finally, let us mention the "crossing symmetry" under $z \mapsto -(qz)^{-1}$. The weights of the transformed matrix

$$\check{R}'(z) = \check{R}[-(qz)^{-1}] = \begin{pmatrix} z - z^{-1} & 0 & 0 & 0 \\ 0 & q - q^{-1} & qz - (qz)^{-1} & 0 \\ 0 & qz - (qz)^{-1} & q - q^{-1} & 0 \\ 0 & 0 & 0 & z - z^{-1} \end{pmatrix} \tag{1.13}$$

correspond to those of the original matrix $\check{R}(z)$, rotated by 90°. The corresponding picture is

$$\check{R}(-w/(qz)) = \quad .$$



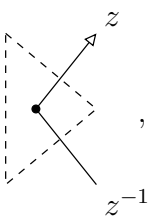
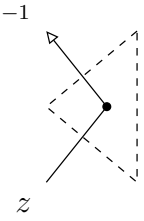
(1.14)

The K-matrices

Integrable boundary Boltzmann weights are encoded by K-matrices [64]. In this thesis, we shall only consider diagonal K-matrices of the form

$$K(z;b) = \begin{pmatrix} z + bz^{-1} & 0 \\ 0 & z^{-1} + bz \end{pmatrix}\,, \tag{1.15}$$

where z is the spectral parameter and b is an external boundary parameter. We shall depict it on the left and right boundary, respectively, as

$K_\ell(z;b) =$  , $K_r(z;b) =$  . (1.16)

This matrix is *not* an intertwiner between representations of $U_q(\widehat{\mathfrak{sl}}_2)$. Indeed, the general property defining a K-matrix is that it should intertwine between representations of a coideal subalgebra of $U_q(\widehat{\mathfrak{sl}}_2)$. A subalgebra B of a Hopf algebra A is a left (resp. right) coideal iff it satisfies the stability property $\Delta : B \to B \otimes A$ (resp. $\Delta : B \to A \otimes B$). In the case of $A = U_q(\widehat{\mathfrak{sl}}_2)$, the subalgebras $B_b^{(\ell)}, B_b^{(r)}$ generated by

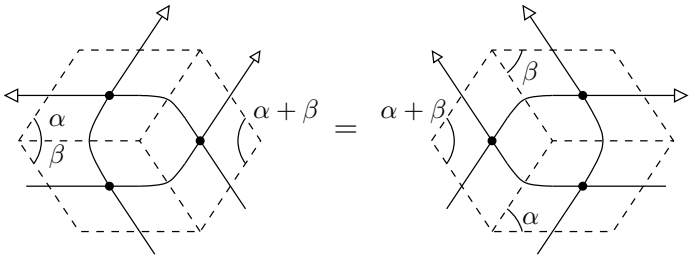
$$\begin{aligned} &\{\ t_0\,, \quad t_1\,, \quad \mu_\ell = e_1 + bqt_0f_0\,, \quad \bar{\mu}_\ell = qt_1f_1 + be_0\ \}\,, \\ &\{\ t_0\,, \quad t_1\,, \quad \mu_r = be_1 + q^{-1}t_1f_0\,, \quad \bar{\mu}_r = bq^{-1}t_0f_1 + e_0\ \}\,, \end{aligned} \tag{1.17}$$

are, respectively, left and right coideals. The intertwining properties then read:

$$\begin{aligned} &\forall x \in B_b^{(\ell)}\,, \qquad K(z;b)\,\pi_{z^{-1}}(x) = \pi_z(x)\,K(z;b)\,, \\ &\forall x \in B_b^{(r)}\,, \qquad K(z;b)\,\pi_z(x) = \pi_{z^{-1}}(x)\,K(z;b)\,. \end{aligned} \tag{1.18}$$

Rhombic embedding

The relation between the spectral parameters and opening angle α defining the rhombic embedding (1.9) is fixed by demanding that the YBE equation (1.12) be represented by adjacent rhombi, and by the crossing symmetry described above. The YBE equation can be depicted as:

 , (1.19)

which imposes a relation of the form $z/w = \exp(u\alpha)$, where u is some constant. Crossing symmetry under $z \mapsto -(qz)^{-1}$ in turn yields the correct value of u. As a results, one gets for the opening angle corresponding to $\check{R}(z/w)$ (1.9):

$$z/w = e^{i\lambda\alpha/\pi}\,, \tag{1.20}$$

where we have set $q = -e^{-i\lambda}$ with $0 < \lambda < \pi$.

Relation with $U_q(\mathfrak{sl}_2)$ and the Temperley-Lieb algebra

When q is generic (i.e. q is not a root of unity), the $\check{R}$-matrix (1.8) can also be analysed in terms of the underlying $U_q(\mathfrak{sl}_2)$ quantum group symmetry as follows. Let us recall first

the defining relations of $U_q(\mathfrak{sl}_2)$, with generators $\{e, f, t\}$:

$$\begin{aligned} t.t^{-1} &= t^{-1}.t = \mathbf{1}\,, & t.e.t^{-1} &= q^2 e\,, \\ t.f.t^{-1} &= q^{-2} f\,, & [e,f] &= \frac{t-t^{-1}}{q-q^{-1}}\,, \end{aligned} \tag{1.21}$$

and the coproduct:

$$\Delta(e) = e\otimes\mathbf{1} + t\otimes e\,, \quad \Delta(f) = f\otimes t^{-1} + \mathbf{1}\otimes f\,, \quad \Delta(t^{\pm 1}) = t^{\pm 1}\otimes t^{\pm 1}\,. \tag{1.22}$$

For generic q, the irreducible representations and the fusion rules of $U_q(\mathfrak{sl}_2)$ are the same as those of $\mathfrak{sl}_2$, but they are not unitary. Denoting by V_ℓ the $(2\ell+1)$-dimensional representation of $U_q(\mathfrak{sl}_2)$, we have:

$$V_k \otimes V_\ell \cong V_{k+\ell} \oplus V_{k+\ell-1} \oplus \cdots \oplus V_{|k-\ell|}\,. \tag{1.23}$$

In particular, the two-dimensional representation $V_{1/2}$ is given by the matrices:

$$\pi_{1/2}(e) = \sigma^+\,, \quad \pi_{1/2}(f) = \sigma^-\,, \quad \pi_{1/2}(t) = q^{\sigma^z}\,. \tag{1.24}$$

The product of two representations $V_{1/2}$ decomposes as

$$V_{1/2}\otimes V_{1/2} \cong V_0 \oplus V_1\,, \tag{1.25}$$

and we denote the corresponding projectors as $|0\rangle\langle 0|$ and P_1. The "singlet" state $|0\rangle$ and its dual are:

$$|0\rangle = |\uparrow\downarrow\rangle - q|\downarrow\uparrow\rangle\,, \qquad \langle 0| = \frac{1}{q+q^{-1}}(q^{-1}\langle\uparrow\downarrow| - \langle\downarrow\uparrow|)\,, \tag{1.26}$$

and the projector onto the V_1 part reads, in the basis $(|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle)$:

$$P_1 = \frac{1}{q+q^{-1}}\begin{pmatrix} q+q^{-1} & 0 & 0 & 0 \\ 0 & q & 1 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 0 & q+q^{-1} \end{pmatrix}. \tag{1.27}$$

After a change of basis $\mathcal{U}_{ij} = U_{z_i}\otimes U_{z_j}$, where $U_w = w^{\sigma^z/2}$ on each rapidity line, the $\check{R}$-matrix takes the form:

$$\mathcal{U}_{21}^{-1}\check{R}(z)\mathcal{U}_{12} = \begin{pmatrix} qz-(qz)^{-1} & 0 & 0 & 0 \\ 0 & (q-q^{-1})z & z-z^{-1} & 0 \\ 0 & z-z^{-1} & (q-q^{-1})z^{-1} & 0 \\ 0 & 0 & 0 & qz-(qz)^{-1} \end{pmatrix}, \quad z = z_1/z_2\,. \tag{1.28}$$

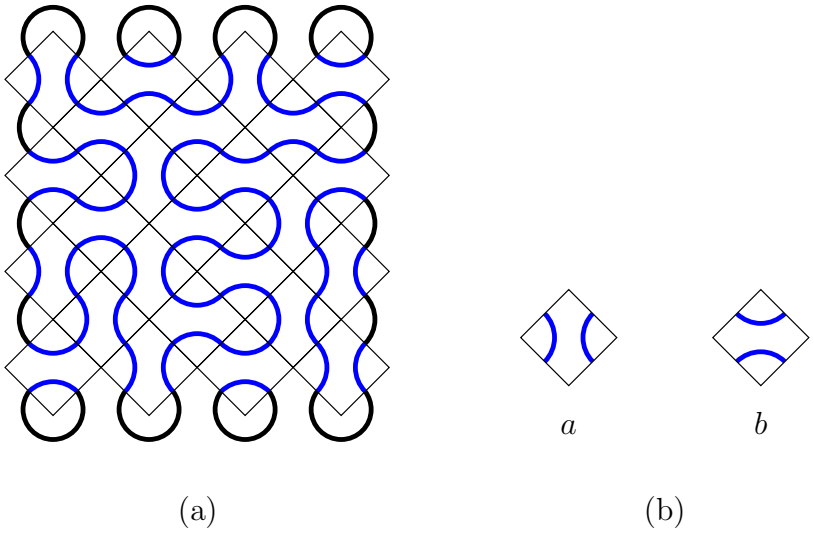
In this form, it admits the decomposition:

$$\mathcal{U}_{21}^{-1}\check{R}(z)\mathcal{U}_{12} = (qz-(qz)^{-1})\,\mathbf{1} - (z-z^{-1})(q+q^{-1})|0\rangle\langle 0|\,. \tag{1.29}$$

An interesting fact is that, on a chain of L "spins" in representation $V_{1/2}^{\otimes L}$, the operators e_j acting as $e_j = -(q+q^{-1})|0\rangle\langle 0|$ on spins j and $j+1$ obey the Temperley-Lieb (TL) algebra [27]:

$$\begin{aligned} &e_j^2 = -(q+q^{-1})e_j\,, \\ &e_j e_{j\pm 1} e_j = e_j\,, \\ &e_j e_k = e_k e_j \qquad \text{if } |j-k|>1. \end{aligned} \tag{1.30}$$

As we shall see below, the TL algebra is an example of a diagram algebra, naturally associated to a loop model.



(a) (b)

Figure 1.2: (a) An example configuration of the Temperley-Lieb loop model, with reflecting boundary conditions. (b) The two types of elementary plaquettes of the Temperley-Lieb loop model.

1.1.2 Loop models

Loop models [26] are lattice models of closed polygon configurations satisfying certain local rules, and with Boltzmann weights associated to the number of polygons. The simplest example is the Temperley-Lieb (TL) loop model, and it is related to the six-vertex model. The TL loop model is defined on the square lattice, with two possible elementary plaquettes: see an example configuration in Fig. 1.2. The Boltzmann weight of a lattice configuration is given by:

$$n^{\#\text{closed loops}}\, a^{N_a}\, b^{N_b}\,, \tag{1.31}$$

where n is the loop fugacity, a and b are the local Boltzmann weights, and N_a (resp. N_b) denotes the number of plaquettes of type a (resp. b). It will be convenient to introduce the parameterisation of the loop model weights:

$$n = 2\cos\lambda\,, \qquad q = -e^{-i\lambda}\,, \qquad a = qz - (qz)^{-1}\,, \qquad b = z - z^{-1}\,. \tag{1.32}$$

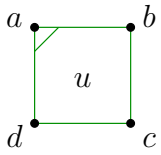
The direct relation [11] between the TL loop model and the 6V model works on a rhombic embedding of the square lattice on the plane: see (1.20). We start from the TL loop model with the above weights. Each loop may be oriented independently anti-clockwise or clockwise, and we give it the corresponding factor $e^{i\lambda}$ or $e^{-i\lambda}$, respectively, so that the total loop weight is indeed $n = 2\cos\lambda$. These turning factors can be distributed locally : any left (resp. right) loop turn with angle $2\pi\delta$ is assigned the factor $e^{i\lambda\delta}$ (resp. $e^{-i\lambda\delta}$). Each plaquette then carries a configuration of the 6V model, together with a pairing of the midedges. When summing over these pairings, and keeping track of the turning factors, one obtains exactly the 6V model with weights (1.8). Note that we have taken the convention that the loop lines enter/leave orthogonally to the sides of the rhombus.

1.1.3 Face models

Unrestricted Solid-On-Solid models

Face models [also known as Interaction-Round-a-Face (IRF) or Solid-On-Solid (SOS) models] are lattice models with spins (or heights) living on the sites of a square lattice, and Boltzmann weights are defined around the faces [65, 66, 67, 68, 69, 22]. The allowed height configurations are given through an adjacency graph $\mathcal{G}$: two heights sitting on neighbouring sites of the square lattice must take two adjacent values on $\mathcal{G}$. The face weights

$$W\left(\begin{array}{cc} a & b \\ d & c \end{array}\middle| u\right) = \qquad (1.33)$$

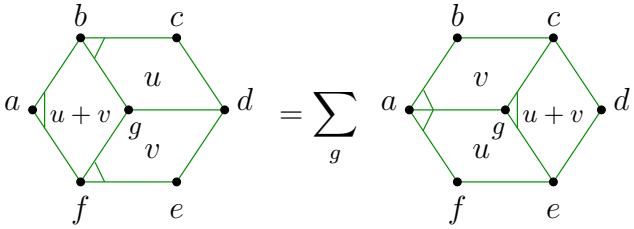


are then chosen in such a way that they satisfy the Yang-Baxter equation:

$$\begin{aligned} &\sum_g W\left(\begin{array}{cc} f & g \\ e & d \end{array}\middle| v\right) W\left(\begin{array}{cc} a & b \\ f & g \end{array}\middle| u+v\right) W\left(\begin{array}{cc} b & c \\ g & d \end{array}\middle| u\right) \\ &\quad = \sum_g W\left(\begin{array}{cc} a & g \\ f & e \end{array}\middle| u\right) W\left(\begin{array}{cc} g & c \\ e & d \end{array}\middle| u+v\right) W\left(\begin{array}{cc} a & b \\ g & c \end{array}\middle| v\right) . \end{aligned} \qquad (1.34)$$

where the sums run over the allowed heights, i.e. the vertices of $\mathcal{G}$. The Yang-Baxter equation may be depicted as:

$$\sum_g \qquad = \sum_g \qquad . \qquad (1.35)$$



The simplest choice for the adjacency graph is $\mathcal{G} = a_0 + \mathbb{Z}$, where a_0 is an arbitrary complex number. This gives rise to the unrestricted SOS models. A solution of (1.34) with trigonometric functions is then given by [22]

$$\begin{aligned} W\left(\begin{array}{cc} a & a\pm 1 \\ a\pm 1 & a\pm 2 \end{array}\middle| u\right) &= \sin(u+\eta)\,, \\ W\left(\begin{array}{cc} a & a\pm 1 \\ a\mp 1 & a \end{array}\middle| u\right) &= \frac{\sin u\, \sin[(a\pm 1)\eta]}{\sin(a\eta)}\,, \\ W\left(\begin{array}{cc} a & a\pm 1 \\ a\pm 1 & a \end{array}\middle| u\right) &= \frac{\sin\eta\, \sin(a\eta \mp u)}{\sin(a\eta)}\,. \end{aligned} \qquad (1.36)$$

These weights are related to the 6V ones through the *vertex-IRF correspondence*. The central objects for this correspondence are the Baxter's intertwiners [70] and their duals (see also [71]):

$$\phi(a, a\pm 1|u) = \begin{bmatrix} \exp\frac{i}{2}(-u \pm a\eta) \\ \exp\frac{i}{2}(+u \mp a\eta) \end{bmatrix}\,, \qquad (1.37)$$

$$\phi^*(a, a\pm 1|u) = \frac{\pm 1}{2\sin a\eta}\begin{bmatrix} \exp\frac{i}{2}(+iu \pm ia\eta) & -\exp\frac{i}{2}(-iu \mp ia\eta) \end{bmatrix}\,. \qquad (1.38)$$

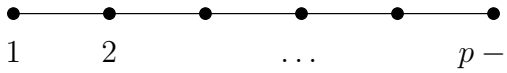
If we set $q = e^{i\eta}$ and $z = e^{iu}$, the six-vertex R-matrix related to (1.8) and the Baxter intertwiners obey the following relations

$$R(u-v)\,[\phi(a,b|u)\otimes\phi(b,c|v)] = \sum_d [\phi(d,c|u)\otimes\phi(a,d|v)]\;\; W\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, u-v\right)\,, \quad (1.39)$$

$$[\phi^*(d,c|u)\otimes\phi^*(a,d|v)]R(u-v) = \sum_b W\left(\begin{matrix} a & b \\ d & c \end{matrix}\,\middle|\, u-v\right)[\phi^*(a,b|u)\otimes\phi^*(b,c|v)]\,. \quad (1.40)$$

Restricted Solid-On-Solid models

In the case $\eta = \pi p'/p$, where p, p' are coprime integers, if we set $a_0 = 0$ the weights (1.36) vanish if $1 \leq a, c, d \leq p-1$ and $b = 0$ or $b = p$. Hence, if the boundary heights are fixed in the interval $\{1, \ldots, p-1\}$, all the heights on the lattice also have this property: this is the restriction mechanism. Then the weights (1.36) also provide a solution to the YBE for the adjacency graph $\mathcal{G} = A_{p-1}$:

$A_{p-1} =$  .

The corresponding face model is known as the Andrews-Baxter-Forrester (ABF) model [65], or A_{p-1} Restricted Solid-On-Solid (RSOS) model. However, the intertwining relations (1.39–1.40) do not hold for this RSOS model.

Cyclic Solid-On-Solid models

Still in the case $\eta = \pi p'/p$, if we keep a_0 nonzero, we can notice that the weights (1.36) are periodic under $a \mapsto a+p$. Thus, an SOS model may be defined on a circle $a \in a_0 + \mathbb{Z}/p\mathbb{Z}$, with the Boltzmann weights given by the same expression as (1.36). This defines the Cyclic Solid-On-Solid model (CSOS) [72].

Relation to the TL algebra

This paragraph summarises the Pasquier construction [66, 67]. Face models can also be used to construct representations of the TL algebra. Consider a generic, finite adjacency graph $\mathcal{G}$, and denote by $\mathcal{A}$ its adjacency matrix:

$$\mathcal{A}_{ab} = \begin{cases} 1 & \text{if } a \text{ and } b \text{ are adjacent on } \mathcal{G}, \\ 0 & \text{otherwise.} \end{cases} \quad (1.41)$$

Since $\mathcal{A}$ is a real, symmetric matrix, it admits an orthogonal basis of eigenvectors $\{S^{(p')}\}$, and we denote by $\beta_{p'}$ the corresponding eigenvalues:

$$\sum_b \mathcal{A}_{ab}\, S_b^{(p')} = \beta_{p'}\, S_a^{(p')}\,. \quad (1.42)$$

If $\mathcal{G}$ is the Dynkin diagram of an algebra in the ADE family, then the eigenvalues take the form:

$$\beta_{p'} = -2\cos\frac{\pi p'}{p}\,, \quad (1.43)$$

where p is associated to the rank of the algebra, and p' can only take integer values. If we choose an eigenvector $S^{(p')}$ and define the face "operator" E as

$$E\begin{pmatrix} a & b \\ d & c \end{pmatrix} = \delta_{ac} \frac{\sqrt{S_b^{(p')} S_d^{(p')}}}{S_a^{(p')}}, \tag{1.44}$$

then it is easy to show that the E's acting at various locations on the lattice generate a TL algebra with loop fugacity $\beta_{p'}$:

$$\begin{aligned} \sum_e E\begin{pmatrix} a & b \\ e & c \end{pmatrix} E\begin{pmatrix} a & e \\ d & c \end{pmatrix} &= \beta_{p'} E\begin{pmatrix} a & b \\ d & c \end{pmatrix}, \\ \sum_e E\begin{pmatrix} a & b \\ e & c \end{pmatrix} E\begin{pmatrix} e & c \\ c' & f \end{pmatrix} E\begin{pmatrix} a & e \\ d & c \end{pmatrix} & \\ = \sum_e E\begin{pmatrix} a & b \\ e & c \end{pmatrix} E\begin{pmatrix} f & c \\ c' & e \end{pmatrix} E\begin{pmatrix} a & e \\ d & c \end{pmatrix} &= \delta_{cc'} E\begin{pmatrix} a & b \\ d & c \end{pmatrix}. \end{aligned} \tag{1.45}$$

We introduce the parameterisation $\eta = \pi p'/p$. Then, the face weight is defined as

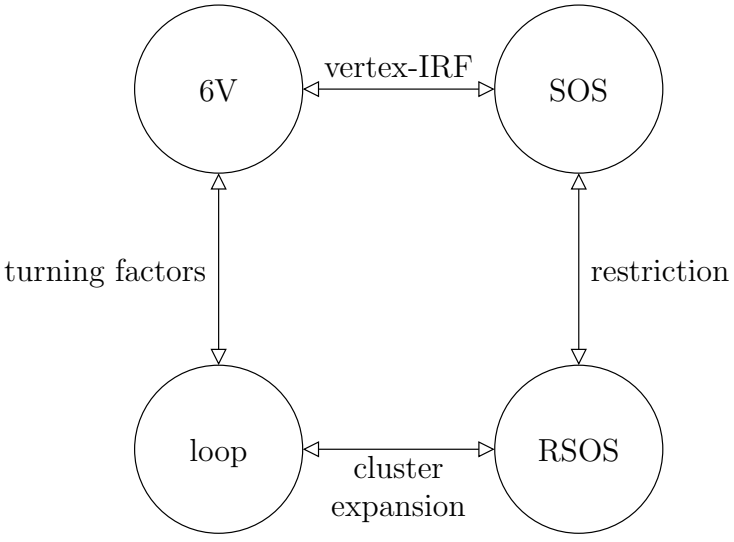
$$W\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) = \sin(\eta + u)\, \delta_{bd} + \sin u\, E\begin{pmatrix} a & b \\ d & c \end{pmatrix}, \tag{1.46}$$

and it satisfies the YBE (1.34). Thus, any choice of a finite graph $\mathcal{G}$ and an eigenvector of its adjacency matrix $\mathcal{A}$ defines an RSOS model with an underlying TL algebra. In other words, the graphical cluster expansion of the RSOS model following (1.46) gives rise to the TL loop model.

1.1.4 Summary of equivalences

The 6V, loop and face models all provide representations of the TL algebra: this is apparent in expressions (1.29) and (1.46). Note that these equivalences are strictly valid only on planar domains, but may become more complicated or even invalid on curved or higher-genus surfaces.

The direct equivalences between these three models can be summarised as follows:



1.1.5 Clock models and the chiral Potts model

Clock models and the Kramers-Wannier duality

Generically, a $\mathbb{Z}_N$ clock model [32] is a lattice model where the spins $n_i \in \mathbb{Z}_N$ live on the sites of the lattice, and the Boltzmann weights are given by the interaction functions (or edge weights) W_{ij} on the edges of the lattice:

$$\prod_{\langle ij \rangle} W_{ij}(n_i - n_j)\,,$$

and the edge weight W_{ij} is N-periodic. In the simplest case, W_{ij} is taken as an even function, and the model is defined on any unoriented lattice. The local symmetry of Boltzmann weights is then enhanced to the dihedral group $\mathbb{D}_N$. However, the case when the W_{ij}'s are *not* even functions is also interesting, since it describes phase transitions with intrisically chiral degrees of freedom.

We introduce the notation: $\omega = \exp(2i\pi/N)$, and the discrete Fourier transform:

$$\widehat{W}(k) = \sum_{n=0}^{N-1} \omega^{-kn}\, W(n)\,, \qquad W(n) = \frac{1}{N} \sum_{k=0}^{N-1} \omega^{kn}\, \widehat{W}(k)\,. \tag{1.47}$$

Consider for simplicity the case of the square lattice $\mathcal{L}$. One can show that the following partition functions are equal:

$$\sum_{\{n_i\}} \prod_{\langle ij \rangle} W_{ij}(n_i - n_j) = \sum_{\{k_i\}} \prod_{\langle ij \rangle} \widehat{W}_{ij}(k_i - k_j)\,, \tag{1.48}$$

where the spins n_i (resp. k_i) live on the sites (resp. dual sites) of $\mathcal{L}$.

The $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ algebra and its cyclic representations

The underlying symmetry of the chiral Potts model [73, 74, 75] is the $U_q(\widehat{\mathfrak{sl}}_2)$ quantum affine algebra [76]. More precisely, it is a slight generalisation [23], called the $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$, with the deformation parameter

$$q = -\exp(i\pi/N)\,.$$

It is a Hopf algebra generated by $\{e_0, e_1, f_0, f_1, t_0, t_1, z_0, z_1\}$ with multiplication rules (1.2) from the standard $U_q(\widehat{\mathfrak{sl}}_2)$, and where z_0, z_1 are two central elements, and the coproduct is

$$\begin{aligned} \Delta(e_i) &= e_i \otimes \mathbf{1} + z_i t_i \otimes e_i\,, \\ \Delta(f_i) &= f_i \otimes t_i^{-1} + z_i^{-1} \otimes f_i\,, \\ \Delta(t_i) &= t_i \otimes t_i\,, \\ \Delta(z_i) &= z_i \otimes z_i\,. \end{aligned} \tag{1.49}$$

Note that the evaluation representation (1.6) associated to the 6V model can be promoted to a representation of $\widetilde{U}_q(\widehat{\mathfrak{sl}}_2)$ by taking $\pi_z(z_i) = c_i \mathbf{1}$, where c_i is any constant.

The representations relevant to the chiral Potts model are N-dimensional cyclic representations [23] denoted $V_{aa'}$ and parametrised by a pair of points $(a, a') \in \mathcal{C}_k \times \mathcal{C}_k$. Here $\mathcal{C}_k$ is the algebraic curve given by $a = (x, y, \mu) \in \mathbb{C}^3$ such that

$$x^N + y^N = k(1 + x^N y^N)\,, \quad \mu^N = \frac{k'}{1 - kx^N} = \frac{1 - ky^N}{k'}\,, \tag{1.50}$$

where $k^2 + k'^2 = 1$. If $a = (x, y, \mu)$ and $a' = (x', y', \mu')$ are two elements of $\mathcal{C}_k$, then the representation on $V_{aa'}$ is given by

$$\begin{aligned}
\pi_{aa'}(e_0) &= \frac{q}{(q^2-1)^2} X^{-1} \left(\frac{y}{\mu\mu'} Z^{-1} - x'\mathbf{1} \right), & \pi_{aa'}(t_0) &= \frac{1}{c_0 \mu\mu'} Z^{-1}, \\
\pi_{aa'}(f_0) &= \left(\frac{c_0 \mu\mu'}{x'} Z - \frac{q^2}{c_0 y} \mathbf{1} \right) X, & \pi_{aa'}(z_0) &= c_0 \mathbf{1}, \\
\pi_{aa'}(e_1) &= \frac{q}{(q^2-1)^2} (x\mu\mu' Z - y'\mathbf{1}) X, & \pi_{aa'}(t_1) &= c_0 \mu\mu' Z, \\
\pi_{aa'}(f_1) &= \frac{c_0}{xx'\mu\mu'} X^{-1} (yZ^{-1} - x'\mu\mu'\mathbf{1}), & \pi_{aa'}(z_1) &= c_0^{-1} \mathbf{1},
\end{aligned} \tag{1.51}$$

where $c_0 = q\sqrt{\frac{xx'}{yy'}}$. Here, the objects X and Z are $N \times N$ matrices, such that

$$ZX = \omega XZ, \qquad X^N = Z^N = \mathbf{1}, \qquad \omega = \exp\left(\frac{2i\pi}{N}\right).$$

We shall fix X and Z as

$$X = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \omega & 0 & \cdots & 0 & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & \omega^{N-1} \end{pmatrix}, \qquad Z = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}. \tag{1.52}$$

The intertwiner between two representations $V_{aa'}$ and $V_{bb'}$ has the factorised form

$$\check{R}(aa', bb') = S_{a'b}(T_{a'b'} \otimes T_{ab}) S_{ab'}, \tag{1.53}$$

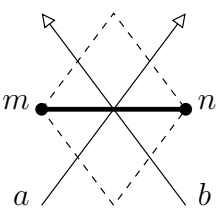
where

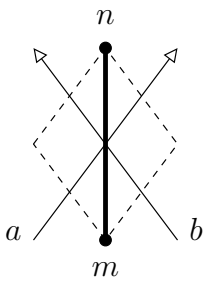
$$S_{ab'} : \begin{cases} V_{aa'} \otimes V_{bb'} \to V_{b'a'} \otimes V_{ba} \\ |m\rangle \otimes |n\rangle \mapsto W_{ab}(n-m)\, |m\rangle \otimes |n\rangle \end{cases}, \quad T_{ab} : \begin{cases} V_{ba} \to V_{ab} \\ |m\rangle \mapsto \sum_{n=0}^{N-1} \overline{W}_{ab}(m-n) |n\rangle \end{cases} \tag{1.54}$$

and

$$W_{ab}(n) = \left(\frac{\mu_a}{\mu_b}\right)^n \times \prod_{\ell=1}^{n} \frac{y_b - x_a\omega^\ell}{y_a - x_b\omega^\ell}, \quad \overline{W}_{ab}(n) = (\mu_a\mu_b)^n \times \prod_{\ell=1}^{n} \frac{x_a\omega - x_b\omega^\ell}{y_b - y_a\omega^\ell}. \tag{1.55}$$

Graphically, we represent them as:

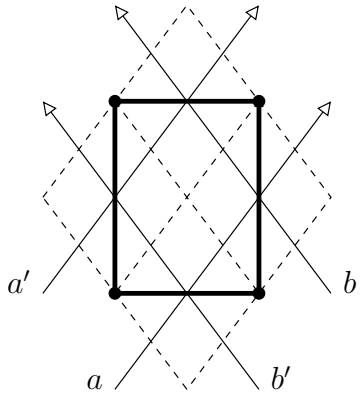
$W_{ab}(m-n) =$ 

, $\overline{W}_{ab}(m-n) =$ 

. (1.56)

The R-matrix (1.53) is depicted as:

$$\check{R}(aa',bb') = \quad .$$



(1.57)

Parameterisation of the spectral curve

A convenient parameterisation of a triplet $a = (x, y, \mu)$ on the curve $\mathcal{C}_k$ (1.50) is given by:

$$x = e^{i(u+\phi)/N}\,, \qquad y = e^{i(u-\phi+\pi)/N}\,, \qquad \mu = e^{i(\bar{\phi}-\phi)/N}\,, \tag{1.58}$$

where the variables $\phi, \bar{\phi}, u$ are now related by

$$\sin\phi = -k\sin u\,, \qquad \sin\bar{\phi} = -\frac{ik}{k'}\cos u\,, \qquad \cos\phi = k'\ \cos\bar{\phi}\,. \tag{1.59}$$

The isotropic, critical case

In the case $k = 0$, we have $\phi = \bar{\phi} = 0$, and the curve $\mathcal{C}_{k=0} = \mathbb{C}$. If we write $x_a = e^{iu/N}$ and $x_b = e^{iv/N}$, then the Boltzmann weights take the form:

$$W(n) = \prod_{\ell=1}^{n} \frac{\sin\frac{(2\ell-1)\pi+u-v}{2N}}{\sin\frac{(2\ell-1)\pi-u+v}{2N}}\,, \qquad \overline{W}(n) = \prod_{\ell=1}^{n} \frac{\sin\frac{(2\ell-2)\pi-u+v}{2N}}{\sin\frac{2\ell-\pi+u-v}{2N}}\,. \tag{1.60}$$

These are the weights of the Fateev-Zamolodchikov $\mathbb{Z}_N$ model [77]. They are self-dual under Kramers-Wannier transformation, and reflection symmetric $[W(-n) = W(n)]$. The continuum limit of this model is described by the $\mathbb{Z}_N$-parafermionic CFT [33], with central charge $c = 2(N-1)/(N+2)$. Together with the stress-energy tensor $T(z)$, this CFT possesses $(N-1)$ additional holomorphic currents $\psi_1, \ldots \psi_{N-1}$, with fusion rules reflecting the $\mathbb{Z}_N$ symmetry:

$$\psi_p \times \psi_{p'} \to \begin{cases} \psi_{p+p'} & \text{if } p+p' \neq 0 \mod N, \\ \mathbf{1} & \text{otherwise,} \end{cases} \tag{1.61}$$

and the conformal dimension of ψ_p is $p(N-p)/N$.

1.2 The Coulomb-Gas approach

1.2.1 The compactified boson and the 6V model

Compact boson action

We take the 6V model with weights (1.8):

$$\omega_1 = \omega_2 = \sin(\lambda - u)\,, \qquad \omega_3 = \omega_4 = \sin u\,, \qquad \omega_5 = \omega_6 = \sin\lambda\,, \tag{1.62}$$

and embed it on the rhombic lattice as in (1.20). To any vertex configuration, we associate a height configuration $\{\varphi_j\}$ living on face centers, such that two neighbouring heights vary by $\pm\pi b$ according to the arrow between them. In the scaling limit, the discrete height φ renormalises [78] to a free compactified boson $\phi(r)$ governed by the action[1] :

$$\begin{aligned} A_0[\phi] &= \frac{1}{2\pi}\int d^2r\,\sqrt{h}\,h^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi\,, \\ \phi &\equiv \phi + 2\pi b\,, \end{aligned} \tag{1.63}$$

where $h^{\mu\nu}$ is the metric. This model has central charge $c = 1$. The value of b is not fixed by the present argument. By comparing, *e.g.* the energy exponent with exact lattice computation of the singular part of the free energy [11], one obtains the relation:

$$b^2 = \frac{\pi - \lambda}{\pi}\,. \tag{1.64}$$

Operator content

We denote by V_α the vertex operators, and by Δ_α their conformal dimensions:

$$V_\alpha = :\exp(2i\alpha\phi):\qquad \Delta_\alpha = \alpha^2\,. \tag{1.65}$$

Due to the compactification condition, the vertex charges (or spin-waves) which are allowed in the spectrum of the transfer matrix are of the form $V_{e/2b}$ with $e \in \mathbb{Z}$. A vortex of charge $m \in \mathbb{Z}$ placed at position r is a winding configuration where the field undergoes a shift $\phi \to \phi + 2\pi mb$ around r, and the corresponding operator has dimension $\widehat{\Delta}_m = m^2b^2/4$.

A generic primary operator, denoted W_{em}, is given by the combination of a spin-wave of charge e and a vortex operator of charge m, and has conformal dimensions:

$$\Delta_{e,m} = \frac{1}{4}\left(\frac{e}{b} - mb\right)^2\,,\qquad \bar{\Delta}_{e,m} = \frac{1}{4}\left(\frac{e}{b} + mb\right)^2\,. \tag{1.66}$$

The full operator content of the 6V model in the continuum limit is

$$\{W_{em}\,,\quad (e,m) \in \mathbb{Z}^2\}\,. \tag{1.67}$$

This operator content is similar to that of the XY model at the Kosterlitz-Thouless transition point: see [78].

Fractional vertex charges

The fractional vertex charges $\alpha = \kappa/2b$ with $\kappa \notin \mathbb{Z}$ do not appear in the spectrum. However, in the 6V model, they are generated non-locally by inserting a twist line made of a chain of $e^{i\pi\kappa\sigma^z}$ operators. Consider for instance the two-point function of these operators:

$$\langle V_{\kappa/2b}(0)V_{-\kappa/2b}(r)\rangle_0\,.$$

Due to the commutation relation:

$$(e^{i\pi\kappa\sigma^z}\otimes e^{i\pi\kappa\sigma^z})\,R(u) = R(u)\,(e^{i\pi\kappa\sigma^z}\otimes e^{i\pi\kappa\sigma^z})\,, \tag{1.68}$$

[1]The elementary defect is produced by flipping a single arrow, and hence it has an amplitude $\delta\varphi = 2\pi b$.

the correlation function is independent of the choice of path joining 0 and r. More generally, any correlation function of the form:

$$\langle V_{\kappa_1/2b}(r_1)V_{\kappa_2/2b}(r_2)\dots V_{\kappa_N/2b}(r_N)\rangle_0 \tag{1.69}$$

with the neutrality condition $\kappa_1+\kappa_2+\dots\kappa_N=0$ can be represented by a configuration of twist lines in the 6V model. Note that the lattice operator $e^{i\kappa\varphi/2b}$ inserting the endpoint of a twist line only depends on $\pi\kappa$ modulo π. Hence, in the scaling limit, it generates all the vertex operators with charge $\alpha\in(\kappa+\mathbb{Z})/2b$.

1.2.2 The Coulomb-Gas formalism for the loop model

Coulomb-Gas action

On a planar domain, the TL loop model is equivalent to the 6V model through the turning factor trick (see Sec. 1.1.2). Hence, the compactified boson action (1.63) captures correctly the loop fluctuations *on a flat surface*. However, on a generic surface, any loop enclosing a non-trivial curvature $\mathcal{R}$ is associated to a turning factor $e^{\pm i\lambda(1-\mathcal{R})}$ rather than the wanted factor $e^{\pm i\lambda}$. This discrepancy can be corrected by inserting a factor

$$\exp[i\lambda\mathcal{R}(r)\,\phi(r)]$$

into the Boltzmann weight. Moreover, a generic $2\pi b$-periodic potential on the lattice generates vertices of the form $V_{\alpha\in\mathbb{Z}/2b}$ in the action. In the scaling limit, one obtains the action [8, 79]

$$\begin{aligned} A_L[\phi] &= \frac{1}{2\pi}\int d^2r\,\sqrt{h}\,\left(h^{\mu\nu}\,\partial_\mu\phi\,\partial_\nu\phi+\frac{iQ}{2}\,\mathcal{R}\,\phi+\mu_b\,e^{2i\phi/b}\right), \\ \phi &\equiv \phi+2\pi b\,, \end{aligned} \tag{1.70}$$

where μ_b is a non-universal coupling constant. The term $e^{2i\phi/b}$ is the called a screening charge, and should be marginal (the other possible terms $V_{\alpha\in\mathbb{Z}/2b}$ are either relevant, in which case their critical coupling constant is zero, or irrelevant – in both cases, they are not included in the action). The model has central charge

$$c=1-24Q^2\,. \tag{1.71}$$

The vertex operators and their conformal dimensions read:

$$V_\alpha=:\!\exp(2i\alpha\phi)\!: \qquad h_\alpha=\alpha(\alpha-2Q)\,. \tag{1.72}$$

The screening charge $V_{1/b}$ is marginal under the condition:

$$2Q=b^{-1}-b\,, \tag{1.73}$$

and b is given by (1.64).

Operator content of the loop model

Because of the compactification condition in (1.70), the spectrum of primary operators is discrete, like in the 6V model. Let us use the Kac notations for vertex charges and conformal dimensions:

$$\alpha_{rs}=\frac{(1-r)}{2b}-\frac{(1-s)b}{2}\,, \qquad h_{rs}=\frac{(r/b-sb)^2-(1/b-b)^2}{4}\,. \tag{1.74}$$

The spectrum of primary operators of the TL loop model is given by [48]:

- Vertex operators $V_{\alpha_{r,1}}$ where $r \in \mathbb{Z}$, with dimensions $(h_{r,1}, h_{r,1})$.
- Mixed operators, W_{em} with dimensions $(h_{em}, h_{e,-m})$, where the vortex charge is $m \in \{1, 2, 3, \dots\}$ and the spin-wave (or vertex charge) is $e \in \mathbb{Z}/m$.

Let us discuss the case of vertex operators V_α which do not belong to the above spectrum. Let us denote the lattice vertex operators as

$$v_\kappa = \exp(i\kappa\varphi/b)\,.$$

Through the turning factor argument, one sees that $v_\kappa(r)$ changes the turning factors of any loop surrounding r to $\exp[\pm i(\lambda - \pi\kappa)]$, instead of $\exp(\pm i\lambda)$. Hence, this class of loops has a modified fugacity $2\cos(\lambda - \pi\kappa)$. Since this effect is unchanged (up to a sign) under $\kappa \to \kappa + 1$, in the scaling limit v_κ generates all the vertex operators V_α with charges

$$\alpha = \frac{\kappa + \ell}{2b}\,, \qquad \ell \in \mathbb{Z}\,. \tag{1.75}$$

Note that the lattice operators v_κ and $v_{2\lambda/\pi-\kappa}$ have the same effect on the Boltzmann weights, consistently with the invariance of h_α under $\alpha \to 2Q - \alpha$.

1.2.3 RSOS models and minimal models

The Virasoro minimal model $\mathcal{M}(p, p')$, with $p' < p$ coprime integers, is realised on the lattice by the A_{p-1} RSOS model with weights (1.36) and parameter $\eta = \pi p'/p$. This is nicely seen from the Pasquier construction [66, 67, 68] described in the end of Sec. 1.1.3.

First, for the partition function, the graphical expansion of this RSOS model is a TL model with loop weight $n = -2\cos\pi p'/p$ (see Sec. 1.1.2), whose scaling limit is a CFT with central charge (1.71):

$$c = 1 - \frac{6(p - p')^2}{pp'}\,, \tag{1.76}$$

where we have used $b = \sqrt{p'/p}$. Let us now look at the operator content of the RSOS model. A basis of local operators acting on a single site of the RSOS model is given by:

$$v_a^{(k)} = \frac{S_a^{(k)}}{S_a^{(p')}}\,. \tag{1.77}$$

The key identity satisfied by these operators is:

$$\sum_e E\begin{pmatrix} a & b \\ e & c \end{pmatrix} v_e^{(k)} E\begin{pmatrix} a & e \\ d & c \end{pmatrix} = \beta_k\, v_a^{(k)}\, E\begin{pmatrix} a & b \\ d & c \end{pmatrix}\,. \tag{1.78}$$

This means that, in the loop expansion, any loop enclosing an operator $v^{(k)}$ gets a modified fugacity $\beta_k = -2\cos(\pi k/p)$. Thus, in terms of the Coulomb-Gas action (1.70), $v^{(k)}$ acts as a vertex operator $V_{\kappa/2b}$ with $\kappa = 1 + k/p$: see (1.75). Since p and p' are coprime, from the Bézout theorem, one may write the integer k as $k = (s-1)p' - rp$, with r and s integers. The associated vertex charge is α_{rs}: this shows that the local operators $v_a^{(k)}$ scale to the degenerate fields Φ_{rs} of the minimal model $\mathcal{M}(p, p')$.

1.2.4 The Coulomb Gas on Riemann surfaces

The Riemann sphere and the cylinder

For a compact Riemann surface Σ of genus g, the Gauss-Bonnet theorem relates the total curvature to the Euler characteristics $\chi = 2(1-g)$:

$$\frac{1}{8\pi}\int_\Sigma d^2r\,\sqrt{h}\,\mathcal{R} = 1-g\,. \tag{1.79}$$

Hence, a vertex correlation function $\langle V_{\alpha_1}(r_1)V_{\alpha_2}(r_2)\dots V_{\alpha_N}(r_N)\rangle$ in the imaginary Liouville model must satisfy the neutrality condition:

$$\sum_{j=1}^{N}\alpha_j + 2(g-1)Q = 0\,, \tag{1.80}$$

where $2(g-1)Q$ is the "background charge" corresponding to the total vertex charge generated by the term $iQ\mathcal{R}\phi/2$ in the CG action (1.70).

In the case $g=0$, the surface Σ is conformally equivalent to a sphere, and the background charge is $-2Q$. The partition function is obtained by inserting the operator $V_{+2Q}\equiv\mathbf{1}$ with dimension $h=0$ at any point, e.g. at the origin:

$$Z_\Sigma = \langle V_{+2Q}(0)\rangle \qquad \text{(genus zero)}\,. \tag{1.81}$$

More specifically, if we start from the Riemann sphere with a metric such that $\sqrt{h}\mathcal{R}(z) = 8\pi\delta(z-z_\infty)$, where z_∞ is the point located at infinity, and we apply the change of variable $z\mapsto w = \frac{L}{2\pi}\log z$, we obtain the infinite cylinder, with curvature $\lim_{M\to\infty}[4\pi\delta(w-M)+4\pi\delta(w+M)]$, corresponding to vertex charges $-Q$ at $\pm M$. The operator $V_{+2Q}(z = e^{-2\pi M/L}\to 0)$ is mapped to $w=-M$, and hence the partition function on the cylinder reads:

$$Z_{\rm cyl} \sim \langle V_Q(-M)V_{-Q}(M)\rangle_0\,, \tag{1.82}$$

where $\pm M$ are the endpoints of the cylinder, and $\langle\dots\rangle_0$ denotes the averaging with respect to the free boson action (1.63). On the lattice, this corresponds to the six-vertex model with twisted boundary conditions $\kappa = \lambda/\pi$.

The torus

In the case $g=1$, the surface Σ is conformally equivalent to a torus, and the background charge is zero. If we consider the torus of periods 1 and τ, with the flat metric, the curvature and screening terms in (1.70) do not contribute, and the partition function with periodic boundary conditions simply reads:

$$Z_0(b) = \int_{\phi\ \text{periodic}}[D\phi]\,\exp(-A_0) = \frac{b}{\sqrt{\operatorname{Im}\tau}\,\eta(\mathfrak{q})\,\eta(\bar{\mathfrak{q}})}\,.$$

To take the compactification condition $\phi\equiv\phi+2\pi b$ into account, one needs to sum over the possible windings of ϕ around the two periods of the torus:

$$Z_C(b) = \sum_{m,m'\in\mathbb{Z}^2} Z_{m,m'}(b)\,, \qquad Z_{m,m'}(b) := \int_{\substack{\phi(z+1)=\phi+2\pi mb\\ \phi(z+\tau)=\phi+2\pi m'b}}[D\phi]\,\exp(-A_0)\,.$$

A Poisson summation gives:

$$Z_C(b) = \frac{1}{\eta(\mathfrak{q})\,\eta(\bar{\mathfrak{q}})} \sum_{(e,m)\in\mathbb{Z}^2} \mathfrak{q}^{\Delta_{em}}\,\bar{\mathfrak{q}}^{\bar{\Delta}_{em}}\,, \tag{1.83}$$

where Δ_{em} and $\bar{\Delta}_{em}$ have been defined in (1.66). This is the toroidal partition function of the imaginary Liouville model with compactified ϕ.

Consider the loop model on the torus, with loop fugacity $n = -q - q^{-1} = 2\cos\lambda$ for contractible loops, and $\widetilde{n} = 2\cos\pi\kappa$ for non-contractible loops. To find the loop partition function, it turns out that one needs to insert a "topological factor" $\cos[\pi\kappa(m \wedge m')]$, where

$$x \wedge y = \begin{cases} \gcd(|x|,|y|) & \text{if } x \neq 0 \text{ and } y \neq 0, \\ x & \text{if } y = 0, \\ y & \text{if } x = 0. \end{cases}$$

The loop partition function reads [48]:

$$Z_{\text{loop}}(\lambda,\kappa) = \sum_{m,m'\in\mathbb{Z}^2} Z_{m,m'}(b)\,\cos[\pi\kappa(m\wedge m')]\,, \qquad b = \sqrt{\frac{\pi-\lambda}{\pi}}\,. \tag{1.84}$$

The Poisson summation is now much more involved. It reproduces the spectrum of the loop model exposed in Sec. 1.2.2, and takes the form:

$$Z_{\text{loop}}(\lambda,\kappa) = \frac{1}{\eta(\mathfrak{q})\,\eta(\bar{\mathfrak{q}})} \left[\sum_{r\in\mathbb{Z}} (\mathfrak{q}\bar{\mathfrak{q}})^{h_{r1}} + \sum_{m\in\mathbb{Z}\setminus\{0\}} \sum_{e\in\mathbb{Z}/m} \mathcal{N}_{em}\,\mathfrak{q}^{h_{em}}\,\bar{\mathfrak{q}}^{h_{e,-m}} \right], \tag{1.85}$$

where the conformal dimensions h_{em} are given in (1.74). The coefficients $\mathcal{N}_{em}$ have a complicated expression in terms of $\beta = 2\cos\pi\kappa$, and in general $\mathcal{N}_{em}$ *is not an integer*. The coefficient $\mathcal{N}_{em}$ should thus be interpreted as the "quantum multiplicity" of the module with dimensions $(h_{em}, h_{e,-m})$ in the Hilbert space of the loop model. In fact, the structure of these modules, even for generic λ, is quite involved, due to the fact that the evolution operator $L_0 + \bar{L}_0$ is not diagonalisable (see [80]).

1.3 Structure constants of the operator algebra

1.3.1 The Operator Product Expansion (OPE) in CFT

In a CFT, the Operator Product Expansion of the primary operators $\{\Phi_a\}$ takes the form, as $z \to 0$:

$$\Phi_a(0)\Phi_b(z,\bar{z}) = \sum_c C_{ab}^c\, z^{-h_a-h_b+h_c}\,\bar{z}^{-\bar{h}_a-\bar{h}_b+\bar{h}_c}\,\Phi_c(0,0) + \text{descendants}. \tag{1.86}$$

The structure constants C_{ab}^c, together with the set of conformal dimensions $(h_a, \bar{h}_a)$, constitute the basic data from which all the correlation functions may be reconstructed. The C_{ab}^c's are related simply to the three-point functions. Let us denote by Φ_a^* the charge-conjugate of Φ_a:

$$\langle \Phi_a^*(0)\Phi_a(z,\bar{z})\rangle = z^{-2h_a}\,\bar{z}^{-2\bar{h}_a}\,.$$

Then we have:

$$C_{ab}^{c^*} = \langle \Phi_a(0)\Phi_b(1)\Phi_c(\infty)\rangle := C(\Phi_a, \Phi_b, \Phi_c)\,, \tag{1.87}$$

and the above quantity is symmetric under any permutation of $\{a, b, c\}$. The standard way to derive the structure constants is to consider a four-point function of primary operators including one degenerate operator, and exploit the constraints imposed by the invariance under a monodromy transformation. This procedure is called *conformal bootstrap*.

1.3.2 The conformal bootstrap in 2d CFT

Since the 1990's, an important activity was devoted to the full determination of structure constants of the Liouville CFT: see [52, 53, 54, 55]. In particular, it was shown in [55] that a careful analysis of four-point correlation functions through the conformal bootstrap yielded enough constraints to determine completely these constants. In this section, we review the approach of [55], but we adapt it to (rational and non-rational) CFTs with $c \leq 1$. In the rational case, this coincides with the early work of [8], whereas in the non-rational case, we recover the results on the imaginary Liouville CFT [49, 50, 51].

The Coulomb-Gas parameterisation of a CFT

Consider a generic CFT, where the spectrum-generating algebra is simply the Virasoro algebra, so that a primary field is specified by its conformal dimensions $h, \bar{h}$. Scalar primary fields will be denoted as Φ_h. When studying the correlation functions, it will be convenient to use the CG parameterisation [8] of the central charge and conformal dimensions:

$$Q := \frac{1}{2}(b^{-1} - b)\,, \qquad c = 1 - 24Q^2\,, \qquad h_\alpha = \alpha(\alpha - 2Q)\,. \tag{1.88}$$

The charges α associated to degenerate dimensions are given by the Kac formula:

$$\alpha_{rs} = \frac{(1-r)b^{-1}}{2} - \frac{(1-s)b}{2}\,, \tag{1.89}$$

and we denote by $\Phi_{rs}(z, \bar{z})$ the corresponding scalar primary field.

Four-point function satisfying a differential equation

We *assume* that the CFT under consideration contains the degenerate field Φ_{12}. We consider the correlation function of Φ_{12} with three scalar primary operators:

$$G(z, \bar{z}) = \langle \Phi_1(\infty)\Phi_2(1)\Phi_3(z, \bar{z})\Phi_{12}(0)\rangle\,, \tag{1.90}$$

and we denote by h_1, h_2, h_3 the conformal dimensions and $\alpha_1, \alpha_2, \alpha_3$ the parameters associated to Φ_1, Φ_2, Φ_3 respectively. The primary field Φ_{12} is degenerate at level 2:

$$(L_{-2} - b^{-2}L_{-1}^2)\Phi_{12} = 0\,, \tag{1.91}$$

and, for any four-point function $\langle \Phi_1(\infty)\Phi_2(1)\Phi_3(z, \bar{z})\Phi_4(0)\rangle$ where Φ_2 and Φ_3 are primary operators, the Virasoro modes act as differential operators:

$$\begin{aligned}
&\langle \Phi_1(\infty)\Phi_2(1)\Phi_3(z, \bar{z})(L_n\Phi_4)(0)\rangle - \langle (L_{-n}\Phi_1)(\infty)\Phi_2(1)\Phi_3(z, \bar{z})\Phi_4(0)\rangle \\
&\quad = \Big\{(1 - z^n)[z\partial_z + (n+1)h_3] \\
&\qquad + (h_4 - h_1) - n(h_2 + h_3)\Big\}\langle \Phi_1(\infty)\Phi_2(1)\Phi_3(z, \bar{z})\Phi_4(0)\rangle\,.
\end{aligned} \tag{1.92}$$

Thus, we get the second-order differential equation for (1.90):

$$z^2(z-1)^2\partial_z^2 G + z(z-1)[(2A_1+1)z - b^2]\partial_z G + (A_2 z^2 - b^2 h_3)G = 0\,, \tag{1.93}$$

where

$$A_1 = h_2 + h_3 - h_1 + \frac{b^2}{4}\,, \qquad A_2 = A_1^2 - b^2 h_2 - \frac{(b^2-1)^2}{4}\,,$$

and a similar equation with (z, ∂_z) replaced by $(\bar{z}, \partial_{\bar{z}})$.

As a guide to simplify the calculation, we can use the expected fusion rule:

$$\Phi_{12} \times \Phi_{h_\alpha} \to \Phi_{h_{\alpha - b/2}} + \Phi_{h_{\alpha+b/2}}\,. \tag{1.94}$$

Moreover, we have the simple identity $h_{\alpha+\beta} = h_\alpha + h_\beta + 2\alpha\beta$. Hence, in the limit $z \to 0$, we expect one of the solutions to behave as $z^{-h_{12}-h_3+h_{\alpha_3+b/2}} = z^{b\alpha_3}$. Similarly, the behaviour $(1-z)^{-h_2-h_3+h_{\alpha_1+b/2}} = (1-z)^{h_{12}-h_2-h_3+h_1+b\alpha_1}$ is expected. We thus introduce:

$$G(z,\bar{z}) = |z|^{2b\alpha_3}\,|1-z|^{2(h_{12}-h_2-h_3+h_1+b\alpha_1)}\,g(z,\bar{z})\,. \tag{1.95}$$

The function $g(z,\bar{z})$ satisfies the hypergeometric differential equation:

$$z(1-z)\partial_z^2 g + [C - (A+B+1)z]\partial_z g - AB\,g = 0\,, \tag{1.96}$$

with parameters

$$\begin{aligned} A &= 2h_{12} + b(\alpha_1+\alpha_2+\alpha_3)\,,\\ B &= 2h_{12} + b[\alpha_1 + (2Q-\alpha_2) + \alpha_3]\,,\\ C &= b^2 + 2b\alpha_3\,. \end{aligned} \tag{1.97}$$

Conformal blocks

The space of holomorphic solutions to (1.96) has basis:

$$\begin{aligned} I_1(z) &= {}_2\mathrm{F}_1(A,B;C|z)\,,\\ I_2(z) &= z^{1-C}\ {}_2\mathrm{F}_1(1+A-C,1+B-C,2-C|z)\,, \end{aligned} \tag{1.98}$$

where ${}_2\mathrm{F}_1(a,b;c|z)$ is the hypergeometric series

$${}_2\mathrm{F}_1(a,b;c|z) = \sum_{n=0}^{\infty} \frac{(a)_n\,(b_n)}{n!\,(c)_n}\,z^n\,, \qquad (x)_n = x(x+1)\ldots(x+n-1)\,. \tag{1.99}$$

After a change of variables $z \to 1-z$, one gets a different basis of solutions to (1.96)

$$\begin{aligned} J_1(z) &= {}_2\mathrm{F}_1(A,B;A+B-C+1|1-z)\,,\\ J_2(z) &= z^{C-A-B}\ {}_2\mathrm{F}_1(C-B,C-A,C-A-B+1|1-z)\,. \end{aligned} \tag{1.100}$$

The change of basis between $\{I_1, I_2\}$ and $\{J_1, J_2\}$ is well known:

$$I_i(z) = \sum_{j=1}^{2} p_{ij}\,J_j(z)\,, \tag{1.101}$$

where the p_{ij}'s are the entries of the fusion matrix:

$$P = \begin{bmatrix} \frac{\Gamma(C)\Gamma(D)}{\Gamma(C-A)\Gamma(C-B)} & \frac{\Gamma(C)\Gamma(-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(2-C)\Gamma(D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(2-C)\Gamma(-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} \end{bmatrix}, \tag{1.102}$$

with inverse:

$$P^{-1} = \begin{bmatrix} \frac{\Gamma(1-C)\Gamma(1-D)}{\Gamma(1-C+A)\Gamma(1-C+B)} & \frac{\Gamma(C-1)\Gamma(1-D)}{\Gamma(A)\Gamma(B)} \\ \frac{\Gamma(1-C)\Gamma(1+D)}{\Gamma(1-A)\Gamma(1-B)} & \frac{\Gamma(C-1)\Gamma(1+D)}{\Gamma(C-A)\Gamma(C-B)} \end{bmatrix}, \tag{1.103}$$

and where $D := C - A - B = 1 - b^2 - 2b\alpha_1$.

Crossing symmetry

The physical solution $g(z, \bar{z})$ must be a bilinear combination of the form:

$$g(z, \bar{z}) = \sum_{i,j} X_{ij}\, \overline{I_i(z)}\, I_j(z)\,. \tag{1.104}$$

This expression corresponds the conformal-block decomposition of $G(z, \bar{z})$ in the channel $z \to 0$. Similarly, in the channel $z \to 1$, we have

$$g(z, \bar{z}) = \sum_{k,\ell} Y_{k\ell}\, \overline{J_k(z)}\, J_\ell(z)\,. \tag{1.105}$$

From these two decompositions, and using (1.101), one gets the matrix relation $Y = P^\dagger X P$. Imposing a diagonal form for X and Y, and using the fact that the entries of P are real, one gets the linear relation for $X_1 := X_{11}$ and $X_2 := X_{22}$

$$p_{11}p_{12}\, X_1 + p_{21}p_{22}\, X_2 = 0\,. \tag{1.106}$$

Similarly, one gets a linear relation between $Y_1 := Y_{11}$ and $Y_2 := Y_{22}$. Finally, one obtains the ratios:

$$\begin{aligned} x_b(\alpha_1, \alpha_2, \alpha_3) &:= \frac{X_1}{X_2} = \frac{\gamma(2-C)}{\gamma(C)} \times \gamma(A)\gamma(B)\gamma(C-A)\gamma(C-B)\,, \\ y_b(\alpha_1, \alpha_2, \alpha_3) &:= \frac{Y_1}{Y_2} = \frac{\gamma(1+D)}{\gamma(1-D)} \times \frac{\gamma(A)\gamma(B)}{\gamma(C-A)\gamma(C-B)}\,. \end{aligned} \tag{1.107}$$

where $\gamma(x) = \Gamma(x)/\Gamma(1-x)$. After proper substitution of (1.97) into (1.107), one finds:

$$x_b(\alpha_1, \alpha_2, \alpha_3 + b/2) = \frac{\gamma(2 - b^2 - 2b\alpha_3)}{\gamma(b^2 + 2b\alpha_3)} \times \frac{\gamma(2b^2 - 1 + b\alpha_{123})\gamma(b^2 + b\alpha_{13}^2)\gamma(b^2 + b\alpha_{23}^1)}{\gamma(b\alpha_{12}^3)}\,, \tag{1.108}$$

where we have used the short-hand notations $\alpha_{123} := \alpha_1 + \alpha_2 + \alpha_3$, and $\alpha_{ij}^k := \alpha_i + \alpha_j - \alpha_k$.

Shift equations

The ratios (1.107) are related to the structure constants by

$$\begin{aligned} x_b(\alpha_1, \alpha_2, \alpha_3) &= \frac{C(\Phi_{12}, \Phi_3, \Phi_{3+})\, C(\Phi_{3+}, \Phi_2, \Phi_1)}{C(\Phi_{12}, \Phi_3, \Phi_{3-})\, C(\Phi_{3-}, \Phi_2, \Phi_1)}\,, \\ y_b(\alpha_1, \alpha_2, \alpha_3) &= \frac{C(\Phi_{12}, \Phi_1, \Phi_{1+})\, C(\Phi_{1+}, \Phi_2, \Phi_3)}{C(\Phi_{12}, \Phi_1, \Phi_{1-})\, C(\Phi_{1-}, \Phi_2, \Phi_3)}\,, \end{aligned} \tag{1.109}$$

where we have used the short-hand notation $\Phi_{j\pm} = \Phi_{h_{\alpha_j \pm b/2}}$. By combining appropriately the ratios, we obtain the shift equation for the structure constants:

$$\frac{C(\Phi_1, \Phi_2, \Phi_{h_{\alpha_3+b}})}{C(\Phi_1, \Phi_2, \Phi_{h_3})} = \frac{x_b(\alpha_1, \alpha_2, \alpha_3 + b/2)}{\sqrt{x_b(\alpha_{12}, \alpha_3 + b/2, \alpha_3 + b/2)}} \tag{1.110}$$

$$= K_b(\alpha_3) \times \frac{\gamma(2b^2 - 1 + b\alpha_{123})\gamma(b^2 + b\alpha_{13}^2)\gamma(b^2 + b\alpha_{23}^1)}{\gamma(b\alpha_{12}^3)}\,, \tag{1.111}$$

where we have gathered the factors depending only on α_3 into the function

$$K_b(\alpha) := \frac{\sqrt{\gamma(2 - b^2 - 2b\alpha)\gamma(2 - 3b^2 - 2b\alpha)}}{\gamma(b^2 + 2b\alpha)}\,. \tag{1.112}$$

1.3.3 Structure constants of minimal models

In the minimal model $\mathcal{M}(p, p')$, the spectrum is discrete, and hence the shift equation (1.111) can be considered as a recursion equation, and solved completely [8]. Let us discuss it in the case of the subalgebra of operators Φ_{1s}:

$$\frac{C_{\min}(\Phi_{1s_1}, \Phi_{1s_2}, \Phi_{1,s_3+2})}{C_{\min}(\Phi_{1s_1}, \Phi_{1s_2}, \Phi_{1s_3})} = \frac{\gamma[\frac{b^2}{2}(s_{123} + 1) - 1]\gamma[\frac{b^2}{2}(s_{13}^2 + 1)]\gamma[\frac{b^2}{2}(s_{23}^1 + 1)]}{\gamma[\frac{b^2}{2}(s_{12}^3 - 1)]} \times K_b(\alpha_{1s_3})\,, \tag{1.113}$$

where $s_{123} = s_1 + s_2 + s_3$ and $s_{ij}^k = s_i + s_j - s_k$.

For example, the coefficients in the OPE:

$$\Phi_{1s} \times \Phi_{1s} \to \sum_{k=1}^{s} \Phi_{1,2k-1} \tag{1.114}$$

are determined by the initial condition $C_{\min}(\Phi_{1s}, \Phi_{1s}, \Phi_{11}) = 1$, and the recursion relation:

$$\frac{C_{\min}(\Phi_{1s}, \Phi_{1s}, \Phi_{1,2k+1})}{C_{\min}(\Phi_{1s}, \Phi_{1s}, \Phi_{1,2k-1})} = \frac{\gamma^2(b^2 k)\,\sqrt{\gamma[2 - b^2(2k-1)]\,\gamma[2 - b^2(2k+1)]}}{\gamma[2 - b^2(s+k)]\,\gamma[b^2(s-k)]\,\gamma[b^2(2k-1)]}\,. \tag{1.115}$$

More general structure constants can be obtained by using also the shift equation associated to Φ_{21}.

1.3.4 Structure constants of the imaginary Liouville CFT

The imaginary Liouville CFT

The imaginary Liouville theory is defined [50] as the CFT with a given central charge $c < 1$, whose spectrum is generated by the Virasoro algebra from all the scalar primary fields $\Phi_{h(\alpha)}$ with real parameter α.

Expression of the structure constants

In the context of the imaginary Liouville CFT, the shift equation (1.111) should be considered as a functional equation for the structure constant. To solve this problem, the main tool is the special function Υ_b, defined for $0 < \mathrm{Re}(x) < b + b^{-1}$ as

$$\Upsilon_b(x) = \exp \int_0^\infty \frac{dt}{t} \left[\left(\frac{b + b^{-1}}{2} - x \right)^2 e^{-t} - \frac{\sinh^2\left(\frac{b+b^{-1}}{2} - x \right) \frac{t}{2}}{\sinh \frac{bt}{2} \sinh \frac{t}{2b}} \right]\,. \tag{1.116}$$

It satisfies the functional relations for any real x:

$$\begin{aligned}
&\Upsilon_b(x+b) = \gamma(bx)\, b^{1-2bx}\, \Upsilon_b(x)\,, \\
&\Upsilon_b(x+b^{-1}) = \gamma(x/b)\, b^{-1+2x/b}\, \Upsilon_b(x)\,, \\
&\Upsilon_b(b+b^{-1}-x) = \Upsilon_b(x)\,.
\end{aligned} \tag{1.117}$$

A solution to (1.111) can be obtained from the following product of Υ functions:

$$U(\alpha_1,\alpha_2,\alpha_3) := \Upsilon_b(2b-b^{-1}+\alpha_{123})\Upsilon_b(b+\alpha_{12}^3)\Upsilon_b(b+\alpha_{13}^2)\Upsilon_b(b+\alpha_{23}^1)\,, \tag{1.118}$$

which obeys the relation:

$$\frac{U(\alpha_1,\alpha_2,\alpha_3+b)}{U(\alpha_1,\alpha_2,\alpha_3)} = b^{4(1-2b^2-2b\alpha_3)}\,\frac{\gamma(2b^2-1+b\alpha_{123})\gamma(b^2+b\alpha_{13}^2)\gamma(b^2+b\alpha_{23}^1)}{\gamma(b\alpha_{12}^3)}\,. \tag{1.119}$$

Note that $U(\alpha_1,\alpha_2,\alpha_3)$ is invariant under any permutation of the α_j's. It is natural to look for a solution to (1.111) of the form

$$C_{\rm IL}(\alpha_1,\alpha_2,\alpha_3) = \frac{U(\alpha_1,\alpha_2,\alpha_3)}{W(\alpha_1)W(\alpha_2)W(\alpha_3)}\,, \tag{1.120}$$

where $W(\alpha)$ is chosen to ensure the correct α_3-dependent normalisation as in (1.111). This is equivalent to the condition:

$$C_{\rm IL}(\alpha,\alpha,0) = 1\,, \tag{1.121}$$

which fixes the function W:

$$W(\alpha) = \frac{\sqrt{\Upsilon_b(2b-b^{-1}+2\alpha)\Upsilon_b(b+2\alpha)}}{A_b^{1/3}}\,, \qquad A_b := \sqrt{\frac{\Upsilon_b(2b-b^{-1})}{\Upsilon_b(b)^3}}\,. \tag{1.122}$$

The final expression for the imaginary Liouville structure constant is [49, 50, 51]:

$$C_{\rm IL}(\alpha_1,\alpha_2,\alpha_3) = \frac{A_b\,\Upsilon_b(2b-b^{-1}+\alpha_{123})\,\prod_{\langle ijk\rangle}\Upsilon_b(b+\alpha_{ij}^k)}{\prod_i \sqrt{\Upsilon_b(b+2\alpha_i)\,\Upsilon_b(2b-b^{-1}+2\alpha_i)}}\,, \tag{1.123}$$

where the product in the numerator is over the cyclic permutations of $\{1,2,3\}$.

Unicity of the solution

The shift equation (1.119) only determines the function $U(\alpha_1,\alpha_2,\alpha_3)$ up to a b-periodic multiplicative factor. The existence of the degenerate operator $\Phi_{21} = V_{-b^{-1}/2}$ produces a second shift equation, obtained from (1.119) by the change $b \mapsto -b^{-1}$:

$$\frac{U(\alpha_1,\alpha_2,\alpha_3-b^{-1})}{U(\alpha_1,\alpha_2,\alpha_3)} = b^{4(-1+2b^{-2}-2\alpha_3/b)}\,\frac{\gamma(2b^{-2}-1-\frac{\alpha_{123}}{b})\gamma(b^{-2}-\frac{\alpha_{13}^2}{b})\gamma(b^{-2}-\frac{\alpha_{23}^1}{b})}{\gamma(-\alpha_{12}^3/b)}\,. \tag{1.124}$$

Using (1.117), one can rewrite (1.118) as:

$$U(\alpha_1,\alpha_2,\alpha_3) = \Upsilon_b(2b^{-1}-b-\alpha_{123})\Upsilon_b(b^{-1}-\alpha_{12}^3)\Upsilon_b(b^{-1}-\alpha_{13}^2)\Upsilon_b(b^{-1}-\alpha_{23}^1)\,, \tag{1.125}$$

and it is straightforward to check that U satisfies both shift equations (1.119) and (1.124). For generic values of the central charge, b^2 is not rational, and the periods b and b^{-1} of the shift equations are incommensurate, which ensures the unicity of the solution (1.118–1.123).

1.4 The W_3 extended conformal algebra

1.4.1 Motivation from Statistical Mechanics

In the description of 2d critical models, the symmetry under the Virasoro algebra expresses the conformal covariance of correlation functions. The series of CFT models $\mathcal{M}(p,p')$, which are rational with respect to the Virasoro algebra, enjoy a $\mathbb{Z}_2$ symmetry for their primary operators $\Phi_{rs} \equiv \Phi_{p'-r,p-s}$, and a $\mathbb{Z}_2$ conservation rule in their operator algebra. From the lattice point of view, this can actually be traced back to the $\mathbb{Z}_2$ symmetry $a \mapsto p-a$ in the Boltzmann weights of the corresponding A_{p-1} RSOS models. (In the simple case of the Ising model A_3, this symmetry corresponds to spin reversal.)

For lattice models with a different symmetry group, in the scaling limit, one naturally expects a CFT description with a spectrum-generating algebra larger than Virasoro, *i.e.* the algebra should include additional holomorphic currents: this is called an extended conformal algebra. In this text, we review one of the simplest examples of such an object, namely the W_3 conformal algebra [81], which relates to lattice models with $\mathbb{Z}_3$ symmetric Boltzmann weights.

1.4.2 Basic representation theory of the $\mathfrak{sl}_3$ Lie algebra

Roots and weights. Let us first fix some conventions for the roots and weights of the $\mathfrak{sl}_3$ Lie algebra. The root vectors $\{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm(\boldsymbol{e}_1+\boldsymbol{e}_2)\}$ are the shifts associated to raising and lowering operators. The simple roots are $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$. The positive roots $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_1+\boldsymbol{e}_2\}$ are obtained by summing one or several distinct simple roots. The dual basis of $(\boldsymbol{e}_1, \boldsymbol{e}_2)$ is given by the fundamental weights $(\boldsymbol{\omega}_1, \boldsymbol{\omega}_2)$. We have the relations:

$$\boldsymbol{\omega}_1^2 = \boldsymbol{\omega}_2^2 = \frac{2}{3}\,, \qquad \boldsymbol{\omega}_1 \cdot \boldsymbol{\omega}_2 = \frac{1}{3}\,, \tag{1.126}$$

$$\boldsymbol{e}_1^2 = \boldsymbol{e}_2^2 = 2\,, \qquad \boldsymbol{e}_1 \cdot \boldsymbol{e}_2 = -1\,, \tag{1.127}$$

$$\boldsymbol{e}_1 = 2\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2\,, \qquad \boldsymbol{e}_2 = 2\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\,, \qquad \boldsymbol{e}_i \cdot \boldsymbol{\omega}_j = \delta_{ij}\,. \tag{1.128}$$

The Weyl vector is $\boldsymbol{\rho} = \boldsymbol{e}_1 + \boldsymbol{e}_2 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2$. We shall denote by $\mathcal{R}$ and $\mathcal{R}^*$ the root and weight lattices, respectively:

$$\mathcal{R} = \mathbb{Z}\boldsymbol{e}_1 + \mathbb{Z}\boldsymbol{e}_2\,, \qquad \mathcal{R}^* = \mathbb{Z}\boldsymbol{\omega}_1 + \mathbb{Z}\boldsymbol{\omega}_2\,. \tag{1.129}$$

Irreducible representations. An irreducible representation (irrep) $[\boldsymbol{\lambda}]$ is specified by a highest weight vector

$$\boldsymbol{\lambda} = (\lambda_1, \lambda_2) = \lambda_1\boldsymbol{\omega}_1 + \lambda_2\boldsymbol{\omega}_2\,, \qquad \lambda_1, \lambda_2 = 0, 1, 2, \ldots \tag{1.130}$$

The set of weight vectors of $[\boldsymbol{\lambda}]$ is constructed recursively, starting from the highest weight $\boldsymbol{\lambda}$, by the algorithm:

$$\forall \boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_1, \boldsymbol{\lambda}'_2) \in [\boldsymbol{\lambda}]\,, \quad \text{if } \lambda_i > 0 \text{ then } \boldsymbol{\lambda}' - \boldsymbol{e}_i, \ldots, \boldsymbol{\lambda}' - \lambda'_i\boldsymbol{e}_i \in [\boldsymbol{\lambda}]\,. \tag{1.131}$$

The multiplicity of the weight $\boldsymbol{\lambda}'$ in $[\boldsymbol{\lambda}]$ is denoted $m_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}')$, and is obtained by the Freudenthal recursion.

Conjugation. The conjugate of an irrep is obtained by the reflection around $\boldsymbol{\rho}$, i.e. the exchange of $\boldsymbol{\omega}_1$ and $\boldsymbol{\omega}_2$:

$$(\lambda_1, \lambda_2)^* = (\lambda_2, \lambda_1)\,. \tag{1.132}$$

Some simple representations. The representations associated to the fundamental weights are three-dimensional. One has

$$[\boldsymbol{\omega}_1] = \{\boldsymbol{h}_1, \boldsymbol{h}_2, \boldsymbol{h}_3\}\,, \qquad [\boldsymbol{\omega}_2] = \{-\boldsymbol{h}_1, -\boldsymbol{h}_2, -\boldsymbol{h}_3\}\,, \tag{1.133}$$

with

$$\boldsymbol{h}_1 = \boldsymbol{\omega}_1\,, \quad \boldsymbol{h}_2 = \boldsymbol{\omega}_2 - \boldsymbol{\omega}_1\,, \quad \boldsymbol{h}_3 = -\boldsymbol{\omega}_2\,. \tag{1.134}$$

Let us describe two other simple irreps:

$$[\boldsymbol{\rho}] = \{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \pm\boldsymbol{\rho}, 0\}\,, \tag{1.135}$$

$$[2\boldsymbol{\omega}_1] = \{2\boldsymbol{h}_1, 2\boldsymbol{h}_2, 2\boldsymbol{h}_3, -\boldsymbol{h}_1, -\boldsymbol{h}_2, -\boldsymbol{h}_3\}\,. \tag{1.136}$$

The representation $[\boldsymbol{\rho}]$ has one non-trivial multiplicity: $m_{\boldsymbol{\rho}}(0) = 2$, whereas the weights of $[2\boldsymbol{\omega}_1]$ have no degeneracy.

The Weyl group. The Weyl group W is generated by the reflections about the vectors $\boldsymbol{h}_j$. It preserves the set of root vectors. It acts on the $\boldsymbol{h}_j$'s as the symmetric group $\mathfrak{S}_3$.

Fusion. The tensor product of two irreps can be decomposed as a direct sum of irreps:

$$[\boldsymbol{\lambda}] \otimes [\boldsymbol{\mu}] = \bigoplus_{\boldsymbol{\nu}} N_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\nu}}\,.\,[\boldsymbol{\nu}]\,, \tag{1.137}$$

where the fusion coefficients $N_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\nu}}$ denotes the multiplicity of $[\boldsymbol{\nu}]$ in the decomposition. The $\mathbb{Z}_3$ charge of an irrep is defined as the difference:

$$q_{\boldsymbol{\lambda}} = \lambda_1 - \lambda_2\,. \tag{1.138}$$

The fusion coefficient obey a $\mathbb{Z}_3$ symmetry:

$$\text{if} \quad N_{\boldsymbol{\lambda}\boldsymbol{\mu}}^{\boldsymbol{\nu}} \neq 0 \quad \text{then} \quad q_{\boldsymbol{\lambda}} + q_{\boldsymbol{\mu}} \equiv q_{\boldsymbol{\nu}} \mod 3\,. \tag{1.139}$$

Let us give some fusion rules between simple irreps:

$$\begin{aligned}
&(1,0) \otimes (1,0) = (2,0) \oplus (0,1)\,,\\
&(1,0) \otimes (0,1) = (0,0) \oplus (1,1)\,,\\
&(1,1) \otimes (1,1) = (0,0) \oplus (1,1) \oplus (1,1) \oplus (2,2) \oplus (0,3) \oplus (3,0)\,,\\
&(2,0) \otimes (2,0) = (4,0) \oplus (2,1) \oplus (0,2)\,,\\
&(2,0) \otimes (0,2) = (0,0) \oplus (1,1) \oplus (2,2)\,,\\
&(1,0) \otimes (2,0) = (3,0) \oplus (1,1)\,.
\end{aligned} \tag{1.140}$$

1.4.3 The W_3 algebra

The W_3 algebra is generated by the modes of the stress-energy tensor $T(z)$, together with an additional holomorphic current $W(z)$ of dimension three [82, 81]. The mode decomposition reads

$$T(z) = \sum_{n=-\infty}^{+\infty} L_n \, z^{-n-2} \,, \qquad W(z) = \sum_{n=-\infty}^{+\infty} W_n \, z^{-n-3} \,, \tag{1.141}$$

and the W_3 commutation relations are given by:

$$\begin{aligned}
[L_n, L_m] &= (m-n)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0} \,, \\
[L_n, W_m] &= (2n-m)W_{n+m} \,, \\
[W_n, W_m] &= \frac{c}{3\times 5!}(n^2-4)(n^3-n)\delta_{n+m,0} + \beta^2(n-m)\Lambda_{n+m} \\
&\quad + (n-m)\left[\frac{1}{15}(n+m+2)(n+m+3) - \frac{1}{6}(n+2)(m+2)\right] L_{n+m} \,,
\end{aligned} \tag{1.142}$$

where

$$\begin{aligned}
\beta &= \sqrt{\frac{16}{22+5c}} \,, \\
\Lambda_n &= \sum_{k=-\infty}^{+\infty} \, :L_k L_{n-k}: + \frac{x_n}{5} L_n \,, \qquad :L_n L_m: = \begin{cases} L_n L_m & \text{if } n \leq m \\ L_m L_n & \text{if } n > m \end{cases} \\
x_{2\ell} &= (1+\ell)(1-\ell) \,, \qquad x_{2\ell+1} = (2+\ell)(1-\ell) \,.
\end{aligned} \tag{1.143}$$

A Coulomb-Gas parameterisation of the W_3 algebra, similar to the Virasoro case but based on a two-dimensional bosonic field, is provided by the relations:

$$\boldsymbol{Q} = (b^{-1} - b)\boldsymbol{\rho} \,, \qquad c = 2 - 12\boldsymbol{Q}^2 \,, \qquad \beta = \frac{2}{\sqrt{8 - 15\boldsymbol{Q}^2}} \,. \tag{1.144}$$

1.4.4 Primary fields

The representation theory of the W_3 algebra is expressed in terms of the $\mathfrak{sl}_3$, and hence we shall use many notations from Sec. 1.4.2 in the subsequent discussion.

A primary field $\Phi_{h,w}$ is a highest-weight state for the algebra:

$$L_{n>0}\Phi_{h,w} = W_{n>0}\Phi_{h,w} = 0 \,, \qquad L_0\Phi_{h,w} = h\,\Phi_{h,w} \,, \qquad W_0\Phi_{h,w} = w\,\Phi_{h,w} \,. \tag{1.145}$$

In the CG parameterisation, a primary field is represented as the vertex operator $V_{\boldsymbol{\alpha}}$ defined by two-dimensional vertex charge $\boldsymbol{\alpha}$, with the relations:

$$h_{\boldsymbol{\alpha}} = \frac{1}{2}\boldsymbol{\alpha}\cdot(\boldsymbol{\alpha} - 2\boldsymbol{Q}) \,, \qquad w_{\boldsymbol{\alpha}} = \beta\sqrt{3}\prod_{j=1}^{3}\left[(\boldsymbol{\alpha}-\boldsymbol{Q})\cdot\boldsymbol{h}_j\right] \,. \tag{1.146}$$

A primary field is completely degenerate if it has a two-dimensional space of primary descendants [81]. The corresponding vertex charges are of the form:

$$\boldsymbol{\alpha}\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \left[(1-n_1)b^{-1} - (1-m_1)b\right]\boldsymbol{\omega}_1 + \left[(1-n_2)b^{-1} - (1-m_2)b\right]\boldsymbol{\omega}_2 \,, \tag{1.147}$$

with n_1, n_2, m_1, m_2 positive integers. It will be convenient to introduce the pair $(\boldsymbol{\lambda}, \boldsymbol{\mu})$ of highest-weight vectors of $\mathfrak{sl}_3$ irreducible representations associated to a degenerate primary field, and to denote:

$$\Phi \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}}\,, \qquad \text{where} \quad \lambda_i = n_i - 1, \quad \mu_i = m_i - 1\,. \tag{1.148}$$

The fusion rule of a degenerate primary field with a generic one is given by:

$$\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}} \times V_{\boldsymbol{\alpha}} \to \sum_{\boldsymbol{\lambda}' \in [\boldsymbol{\lambda}],\ \boldsymbol{\mu}' \in [\boldsymbol{\mu}]} m_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}') m_{\boldsymbol{\mu}}(\boldsymbol{\mu}')\,.\, V_{\boldsymbol{\alpha} - b^{-1}\boldsymbol{\lambda}' + b\boldsymbol{\mu}'}\,, \tag{1.149}$$

where $m_{\boldsymbol{\lambda}}(\boldsymbol{\lambda}'), m_{\boldsymbol{\mu}}(\boldsymbol{\mu}')$ are the weight multiplicities. The fusion rule of two completely degenerate fields has the form [81]

$$\Phi_{\boldsymbol{\lambda},\boldsymbol{\mu}} \times \Phi_{\boldsymbol{\lambda}',\boldsymbol{\mu}'} \to \sum_{\boldsymbol{\lambda}'',\boldsymbol{\mu}''} N_{\boldsymbol{\lambda}\boldsymbol{\lambda}'}^{\boldsymbol{\lambda}''} N_{\boldsymbol{\mu}\boldsymbol{\mu}'}^{\boldsymbol{\mu}''}\,.\, \Phi_{\boldsymbol{\lambda}'',\boldsymbol{\mu}''}\,, \tag{1.150}$$

where $N_{\boldsymbol{\lambda}\boldsymbol{\lambda}'}^{\boldsymbol{\lambda}''}$ and $N_{\boldsymbol{\mu}\boldsymbol{\mu}'}^{\boldsymbol{\mu}''}$ are the fusion coefficients of $\mathfrak{sl}_3$ representations.

1.4.5 Correlation functions

In the original paper [81], a Coulomb-Gas formalism similar to the Virasoro one exposed in Sec. 1.2 was developed. It provides an integral representation of the four-point conformal blocks of fully degenerate primary operators, and it allows for the computation of the structure constants of these operators.

For the study of correlation functions in *non-rational* W_3 CFTs, such as the Toda CFT, the bootstrap program involves the knowledge of conformal blocks of one fully degenerate operator, say $\Phi \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, with generic primary operators:

$$F(z) = \langle \Phi_1(\infty)\Phi_2(1)\Phi_3(z)\Phi_{\left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)}(0)\rangle\,. \tag{1.151}$$

In contrast with the Virasoro case, the presence of a single fully degenerate operator is not sufficient to infer an ordinary differential equation (ODE) for $F(z)$. However, such an ODE can be obtained [83] when one of the fields Φ_1, Φ_2, Φ_3 is semi-degenerate, *i.e.* it admits at least one primary descendant under the W-algebra: in this case, the function $F(z)$ satisfies the generalised hypergeometric ODE of order three. This property of conformal blocks was then used to apply the conformal bootstrap program, and obtain a formula for the three-point amplitudes [84, 85]. It was derived in the context of W_3 CFTs with central charge $c > 50$, where the relevant parameterisation is

$$c = 2 + 24(b + b^{-1})$$

and $\boldsymbol{Q} = (b + b^{-1})\rho$ (in contrast with the above discussion for $c < 2$). The result for the three-point amplitude is [85]:

$$C_{\text{Toda}}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \kappa\boldsymbol{\omega}_2) = \frac{A(\boldsymbol{\alpha}_1)A(\boldsymbol{\alpha}_2)B(\kappa)}{\prod_{k,\ell=1}^{3} \Upsilon_b\left[(\boldsymbol{\alpha}_1 - \boldsymbol{Q})\cdot\boldsymbol{h}_k + (\boldsymbol{\alpha}_2 - \boldsymbol{Q})\cdot\boldsymbol{h}_\ell + \frac{\kappa}{3}\right]}\,, \tag{1.152}$$

where $A(\boldsymbol{\alpha})$ and $B(\kappa)$ are some appropriate normalising factors. This formula is the direct higher-rank analog of the DOZZ one [52, 53, 54, 55]. Note that a similar approach was used for the W_4 algebra in [86].

Chapter 2

Lattice parafermions

2.1 Introduction: parafermions from spin and disorder operators

Parafermions are operators which obey a non-trivial monodromy with each other:

$$\psi(r)\psi(0) \to e^{-2i\pi s}\,\psi(r)\psi(0) \tag{2.1}$$

when r is taken along a continuous anti-clockwise circuit around 0, and where s is called the conformal spin. A simple way to construct such operators in a lattice model is by using spin and "disorder" operators. Let us describe this construction in the context of $\mathbb{Z}_N$ clock models (see Sec. 1.1.5 for notations). The spin operators are simply given by:

$$\sigma_q(r_j) = \exp\left(\frac{2i\pi q n_j}{N}\right), \tag{2.2}$$

where r_j is a lattice site, and n_j is the $\mathbb{Z}_N$ variable assigned to this site. The disorder operators μ_p are defined [32] through their correlation functions:

$$\langle \mu_p(\widetilde{r}_1)\mu_{-p}(\widetilde{r}_2)X\rangle = \frac{1}{Z}\sum_{\{n_i\}}\left(\prod_{\langle ij\rangle\notin\gamma^\perp} W(n_i-n_j)\right)\left(\prod_{\langle ij\rangle\in\gamma^\perp} W(n_i-n_j+p)\right)X[\{n_i\}]\,, \tag{2.3}$$

where $\widetilde{r}_1$ and $\widetilde{r}_2$ are two points on the dual lattice, γ is an oriented path from $\widetilde{r}_1$ to $\widetilde{r}_2$ on the dual lattice, and X is any function of the spin variables. By convention, in (2.3) the condition $\langle ij\rangle \in \gamma^\perp$ means that the edge $\langle ij\rangle$ crosses γ, and n_i is the spin sitting on the left of γ.

Let us consider the typical case when X is a product of spin operators:

$$X = \sigma_{q_1}(r_1)\ldots\sigma_{q_m}(r_m)\,. \tag{2.4}$$

From the $\mathbb{Z}_N$ symmetry of Boltzmann weights, we see that (2.3) is invariant under a deformation of γ, unless one of the r_j's is crossed. So the correlation function (2.3) is a well-defined function of $r_1,\ldots,r_m$ on the N-sheeted covering of the lattice, with branch points at $\widetilde{r}_1$ and $\widetilde{r}_2$. The n-point functions of the operators $\{\mu_p\}$ are defined in an analogous way, and the correlation function $\langle \mu_{p_1}\ldots\mu_{p_n}X\rangle$ is nonzero only if $p_1+\cdots+p_n = q_1+\cdots+q_m = 0$ mod N.

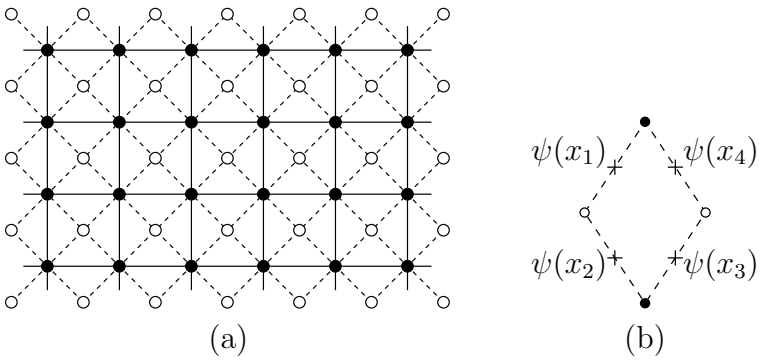


Figure 2.1: (a) The spin lattice (full lines) and its covering lattice (dotted lines). The spin variables (resp.) are represented as black (resp. white) dots. (b) A face of the covering lattice, and the corresponding locations (indicated by crosses) of parafermionic operators $\psi_p(x_1), \dots \psi_p(x_4)$ in discrete Cauchy-Riemann equations (2.7–2.8).

Discrete parafermions are defined on the covering lattice, defined in Fig. 2.1a. For any spin and disorder variables $\sigma(r_i)$ and $\mu(\widetilde{r}_j)$ adjacent to a given edge of the covering lattice, we denote by x_{ij} the midpoint of this edge and θ_{ij} its angle with the horizontal axis, and we define the operator:

$$\psi_p(x_{ij}) = e^{-is_p\theta_{ij}}\, \sigma_p(r_i)\mu_p(\widetilde{r}_j)\,, \qquad \bar{\psi}_p(x_{ij}) = e^{+is_p\theta_{ij}}\, \sigma_p(r_i)\mu_{-p}(\widetilde{r}_j)\,. \tag{2.5}$$

From the definition of the spin and disorder operators, the conformal spin s_p appearing in the monodromy (2.1) is always of the form $s_p \in \mathbb{Z} - p^2/N$.

Let us now specialise the discussion to the case of the critical, integrable $\mathbb{Z}_N$ clock model: the Fateev-Zamolodchikov model [77]. The analysis of the scaling limit [33] shows that the spin of ψ_p is

$$s_p = \frac{p(N-p)}{N} \tag{2.6}$$

In [38] it was shown that for $p = 1$, if one inserts the value $s = 1 - 1/N$ in (2.5), then the lattice parafermions defined above satisfy simple linear relations around any face of the covering lattice (see Fig. 2.1):

$$\psi_1(x_1)\,\delta z_1 + \psi_1(x_2)\,\delta z_2 + \psi_1(x_3)\,\delta z_3 + \psi_1(x_4)\,\delta z_4 = 0\,, \tag{2.7}$$
$$\bar{\psi}_1(x_1)\,\delta \bar{z}_1 + \bar{\psi}_1(x_2)\,\delta \bar{z}_2 + \bar{\psi}_1(x_3)\,\delta \bar{z}_3 + \bar{\psi}_1(x_4)\,\delta \bar{z}_4 = 0\,, \tag{2.8}$$

where $\delta z_1, \dots \delta z_4$ are the anticlockwise complex displacements along the edges of the face. These equations are a lattice version of the Cauchy-Riemann equations, and hence $\psi_1(x)$ and $\bar{\psi}_1(x)$ can be identified as the lattice analogs of the (anti-)holomorphic currents $\psi_1(z)$ and $\bar{\psi}_1(\bar{z})$ in the $\mathbb{Z}_N$-parafermion CFT.

In the following we will show how to use the quantum affine symmetry underlying integrable lattice models (see Sec. 1.1.1), to construct systematically this type of discrete holomorphic parafermions.

2.2 Bernard-Felder conserved currents

2.2.1 Basic assumptions and graphical notations

The fundamental ingredient for the construction of discrete holomorphic parafermions is given by the Bernard-Felder (BF) currents associated to a quantum algebra [46]. One considers a vertex model defined by the R-matrix between representations of a Hopf algebra $\mathcal{A}$, and one assumes that the generators of $\mathcal{A}$ are $\{J_a, \theta_{ab}, \widehat{\theta}_{ab}\}$, with the following specific form of the coproduct, antipode and co-unit:

$$\begin{aligned}
&\Delta(J_a) = J_a \otimes \mathbf{1} + \sum_b \theta_{ab} \otimes J_b \,, && S(J_a) = -\sum_b \widehat{\theta}_{ab} J_b \,, && \epsilon(J_a) = 0 \,, \\
&\Delta(\theta_{ab}) = \sum_c \theta_{ac} \otimes \theta_{cb} \,, && S(\theta_{ab}) = \widehat{\theta}_{ab} \,, && \epsilon(\theta_{ab}) = \delta_{ab} \,, \\
&\Delta(\widehat{\theta}_{ab}) = \sum_c \widehat{\theta}_{cb} \otimes \widehat{\theta}_{ac} \,, && S(\widehat{\theta}_{ab}) = \theta_{ab} \,, && \epsilon(\widehat{\theta}_{ab}) = \delta_{ab} \,.
\end{aligned} \tag{2.9}$$

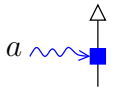
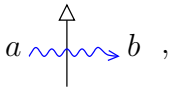
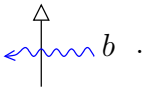
As a consequence of (2.9) and the defining relations of a Hopf algebra, one gets the inversion relations

$$\sum_c \theta_{ac}\, \widehat{\theta}_{cb} = \sum_c \widehat{\theta}_{ca}\, \theta_{bc} = \delta_{ab}\, \mathbf{1} \,. \tag{2.10}$$

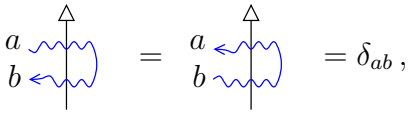
Moreover, one assumes the commutation rules[1]

$$\sum_b \widehat{\theta}_{ab}\, J_b = \sum_{b,c} \alpha_{ab}\, J_c\, \widehat{\theta}_{bc} \,, \qquad \sum_c \theta_{ac}\, \alpha_{cb} = \sum_c \alpha_{ac}\, \theta_{cb} \,, \tag{2.11}$$

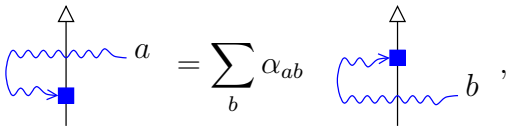
where α is some given matrix with complex entries. In the following, we represent the generators as blue objects, while a black oriented line carries a representation of $\mathcal{A}$:

$J_a =$  , $\theta_{ab} =$  , $\widehat{\theta}_{ab} =$  .

For instance, the inversion relations (2.10) are drawn as:



and the commutation rules:

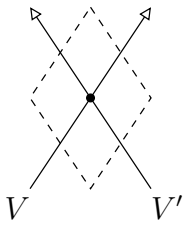
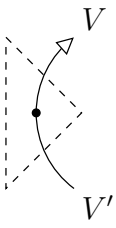
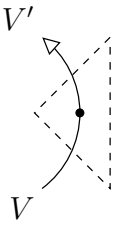


where the internal indices are summed over, and not depicted.

[1] The relations (2.11) do not appear in [46]. We have added this assumption in order to ensure the parafermionic nature of the currents in a general way.

2.2.2 Intertwiners

To simplify the discussion, we consider the situation of a lattice where the lines can carry one of the two representations V, V' of the algebra $\mathcal{A}$, with corresponding homomorphisms $\pi_V, \pi_{V'}$. We represent the associated $\check{R}$- and K-matrices as:

$\check{R}_{VV'} =$  , $K_{VV'} =$  , $K_{V'V} =$  . (2.12)

The $\check{R}$-matrix intertwines between tensor-product representations of $\mathcal{A}$:

$$\forall x \in \mathcal{A}\,, \quad \check{R}_{VV'}\,(\pi_V \otimes \pi_{V'})\Delta(x) = (\pi_{V'} \otimes \pi_V)\Delta(x)\,\check{R}_{VV'}\,, \tag{2.13}$$

whereas the K-matrices are the intertwiners of a left and a right coideal subalgebra ($\mathcal{B}_\ell$ and $\mathcal{B}_r$, respectively) – see Sec. 1.1.1 :

$$\begin{aligned} &\forall x \in \mathcal{B}_\ell\,, \quad K_{VV'}\,\pi_{V'}(x) = \pi_V(x)\,K_{VV'}\,, \\ &\forall x \in \mathcal{B}_r\,, \quad K_{V'V}\,\pi_V(x) = \pi_{V'}(x)\,K_{V'V}\,. \end{aligned} \tag{2.14}$$

In the following, we shall choose boundary conditions so that the "tail operators" θ_{ab} and $\widehat{\theta}_{ab}$ belong respectively to the left and right coideal subalgebra $\mathcal{B}_{\ell,r}$. In contrast, the J_a's do not belong to $\mathcal{B}_{\ell,r}$ in general.

In the following, to ligthen the notations, we shall sometimes write $\check{R} := \check{R}_{VV'}$, $K_\ell := K_{VV'}$ and $K_r := K_{V'V}$. Similarly, for any element x of the algebra $\mathcal{A}$, we shall write x instead of $\pi_V(x)$ or $\pi_{V'}(x)$ whenever the space on which x acts is clearly specified by the context.

From (2.13–2.14), we get the commutation relations in the bulk:

$$\check{R}(J_a \otimes \mathbf{1} + \theta_{ab} \otimes J_b) = (J_a \otimes \mathbf{1} + \theta_{ab} \otimes J_b)\check{R}\,, \tag{2.15}$$

$$\check{R}(\theta_{ab} \otimes \theta_{bc}) = (\theta_{ab} \otimes \theta_{bc})\check{R}\,, \tag{2.16}$$

At the boundary, only the tail operators commute with K-matrices:

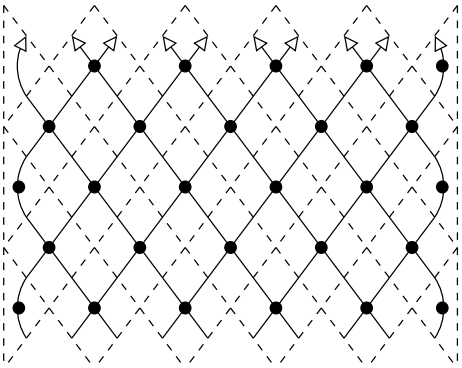
$$K_\ell \theta_{ab} = \theta_{ab} K_\ell\,, \qquad K_r \widehat{\theta}_{ab} = \widehat{\theta}_{ab} K_r\,. \tag{2.17}$$

The J_a's do not commute in general with the K-matrices, so we introduce two families of matrices $K_{\ell,a}$ and $K_{r,a}$, defined as:

$$K_{\ell,a} := [K_\ell, J_a]\,, \qquad K_{r,a} := [K_r, J_a]\,. \tag{2.18}$$

2.2.3 "Conserved" charges

The simplest formulation of the associated conserved currents appears in the context of the diagonal-to-diagonal transfer matrix, which builds the lattice for a vertex model:



. (2.19)

The diagonal-to-diagonal transfer matrix is

$$t(V,V') = \left[K_{VV'} \otimes (\check{R}_{VV'})^{\otimes(L-1)} \otimes K_{V'V}\right] \, (\check{R}_{VV'})^{\otimes L} \,, \tag{2.20}$$

acting on the space $(V \otimes V')^{\otimes L}$.

By iterating the coproduct (2.9), one constructs the representation of J_a, θ_{ab} and $\widehat{\theta}_{ab}$ on the tensor-product space:

$$\begin{aligned} Q_a &= \Delta^{2L-1}(J_a) = \sum_{k=1}^{2L} \theta_{ab_1} \otimes \theta_{b_1 b_2} \otimes \ldots \theta_{b_{j-2} b_{k-1}} \otimes J_{b_{k-1}} \otimes \mathbf{1} \otimes \ldots \mathbf{1} \,, \\ \Theta_{ab} &= \Delta^{2L-1}(\theta_{ab}) = \theta_{ac_1} \otimes \theta_{c_1 c_2} \otimes \ldots \theta_{c_{2L-1} b} \,, \\ \widehat{\Theta}_{ab} &= \Delta^{2L-1}(\widehat{\theta}_{ab}) = \widehat{\theta}_{ac_1} \otimes \widehat{\theta}_{c_1 c_2} \otimes \ldots \widehat{\theta}_{c_{2L-1} b} \,, \end{aligned} \tag{2.21}$$

where the summation of $\{b_k\}$ and $\{c_k\}$ is implied.

The local commutation relations yield:

$$t(V,V')\, Q_a = Q_a\, t(V,V') + t_{\ell,a}(V,V') + t_{r,a}(V,V') \,, \tag{2.22}$$

$$t(V,V')\, \Theta_{ab} = \Theta_{ab}\, t(V,V') \,, \tag{2.23}$$

$$t(V,V')\, \widehat{\Theta}_{ab} = \widehat{\Theta}_{ab}\, t(V,V') \,, \tag{2.24}$$

where

$$\begin{aligned} t_{\ell,a}(V,V') &:= \left[K_{\ell,a} \otimes \check{R}^{\otimes(L-1)} \otimes K_r\right] \, \check{R}^{\otimes L} \,, \\ t_{r,a}(V,V') &:= \left[K_{\ell} \otimes \check{R}^{\otimes(L-1)} \otimes K_{r,a}\right] \, \check{R}^{\otimes L} \,. \end{aligned} \tag{2.25}$$

In the scaling limit $L \to \infty$, the terms $t_{\ell,a}(V,V')$ and $t_{r,a}(V,V')$ in (2.23) may be considered as boundary contributions. In this sense, Q_a can be considered as a conserved charge, up to boundary terms.

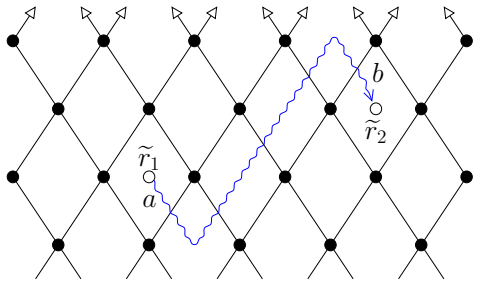
2.2.4 Disorder operators

The BF analog of the $\mathbb{Z}_N$ disorder operators of Sec. 2.1 is given by the family of operators $\{\mu_{ab}\}$ (resp. $\{\widehat{\mu}_{ab}\}$) which insert a "tail"of θ operators:

$$\begin{aligned}\mu_{ab}(k) &= \sum_{\{c_j\}} \theta_{ac_1} \otimes \theta_{c_1c_2} \otimes \cdots \otimes \theta_{c_{k-1}b} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,,\\ \widehat{\mu}_{ab}(k) &= \sum_{\{c_j\}} \widehat{\theta}_{c_1b} \otimes \widehat{\theta}_{c_2c_1} \otimes \cdots \otimes \widehat{\theta}_{ac_{k-1}} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,.\end{aligned} \tag{2.26}$$

Extending the notation, we may write $\mu_{ab}(\widetilde{r})$ [resp. $\widetilde{\mu}_{ab}(\widetilde{r})$] for the action of μ_{ab} (resp. $\widetilde{\mu}_{ab}$) on the face at position $\widetilde{r}$. Using the inversion and intertwining relations for the θ's, we see that the two-point function of the disorder operators consists in inserting an arbitrary path between the two points:

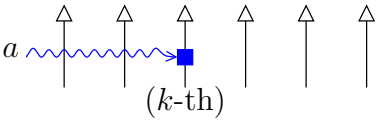
$$\sum_c \langle \widehat{\mu}_{ac}(\widetilde{r}_1)\mu_{cb}(\widetilde{r}_2)\rangle = \quad .$$



2.2.5 Discrete parafermions

Now we notice that each term in the sum for Q_a (2.21), which we depict as

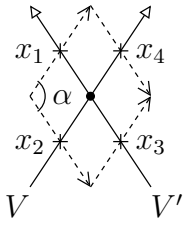
$$Q_a = \sum_k \quad ,$$



has a form very similar to the lattice $\mathbb{Z}_N$ parafermions constructed as $\psi = \mu \times \sigma$ in Sec. 2.1. This suggests the following construction of discrete parafermions in the vertex model associated to $\mathcal{A}$. The local representation of the J_a's play the role of spin operators:

$$\sigma_a(k) = \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes \underset{(k\text{-th})}{J_a} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,. \tag{2.27}$$

We consider a rhombic embedding of the square lattice, defined by the angle α as follows:



. (2.28)

We denote by $\alpha(x_j)$ the angle between the horizontal axis and the dual edge (dotted lines) passing through x_j, and oriented to the right. In (2.28), we have:

$$\alpha(x_1) = \alpha(x_3) = \frac{\alpha}{2}\,, \qquad \alpha(x_2) = \alpha(x_4) = -\frac{\alpha}{2}\,. \tag{2.29}$$

With these conventions, the lattice parafermions operators $\{\psi_a\}$ are defined on the edges of the lattice as

$$\psi_a(x_j) = e^{-i\alpha(x_j)} \sum_b \mu_{ab}(\widetilde{r}_j)\, \sigma_b(r_j)\,, \tag{2.30}$$

where r_j and $\widetilde{r}_j$ are site and dual site adjacent to x_j. The conserved charges Q_a are simply recovered by "integrating" ψ_a along a horizontal line:

$$Q_a = \sum_{x \in \text{ horizontal row}} \psi_a(x)\, \delta\ell_x\,,$$

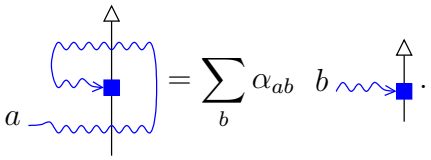
where $\delta\ell_x$ stands for the complex displacement along the dual edges, oriented from left to right [the dotted lines in (2.28)].

The intertwining relation (2.15) for $x = J_a$ yields the discrete Cauchy-Riemann equation around a face of the dual lattice:

$$\psi_a(x_1)\,\delta z_1 + \psi_a(x_2)\,\delta z_2 + \psi_a(x_3)\,\delta z_3 + \psi_a(x_4)\,\delta z_4 = 0\,, \tag{2.31}$$

where the δz_j's are the anti-clockwise displacements along the dual edges.

The parafermionic nature of ψ_a is a consequence of the commutation rules (2.11), which yield:



(2.32)

Hence the matrix α in (2.11) is nothing but the monodromy matrix of the "disorder" operators μ_{ab}'s around a "spin" operator σ_b.

Thus, the Bernard-Felder construction [46] naturally produces discretely holomorphic operators ψ_a for integrable vertex models. Due to the identity (2.32), these operators are parafermionic, *i.e.* they transform with a monodromy matrix α_{ab} when one operator is moved around another one within a correlation function.

2.2.6 Application to the six-vertex model

Let us describe explicitly the BF construction in the simple example of the six-vertex model. The six-vertex model (1.8) with $q = e^{i\eta}$ is based on the $U_q(\widehat{\mathfrak{sl}}_2)$ algebra (1.2) and its two-dimensional representation (1.6). On the lattice (2.19), we take the representations:

$$V = V_{e^{-iu/2}}\,, \qquad V' = V_{e^{+iu/2}}\,. \tag{2.33}$$

The $U_q(\widehat{\mathfrak{sl}}_2)$ algebra satisfies the BF requirements (2.9–2.11) if we organise the generators as:

$$J_i = e_i\,, \qquad \bar{J}_i = \bar{e}_i := q t_i f_i\,, \tag{2.34}$$

$$\theta_i = \bar{\theta}_i = t_i\,, \qquad \widehat{\theta}_i = \widehat{\bar{\theta}}_i = t_i^{-1}\,, \tag{2.35}$$

for $i = 0, 1$. The "tail operators" take the simple form:

$$\mu_i(k) = t_i \otimes \cdots \otimes \underset{k-\text{th}}{t_i} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,. \tag{2.36}$$

The BF commutation rules (2.11) are given by (1.2):

$$t_i^{-1} e_i = q^{-2}\, e_i t_i^{-1}\,, \qquad t_i^{-1} \bar{e}_i = q^{2}\, \bar{e}_i t_i^{-1}\,. \tag{2.37}$$

It turns out that it is convenient to associate the generators e_i to holomorphic operators, and the $\bar{e}_i$'s to anti-holomorphic operators[2]. Hence, we write:

$$\psi_i(x) = e^{-i\alpha_r}\, \mu_i(\widetilde{r})\, e_i(r)\,, \qquad \bar{\psi}_i(x) = e^{+i\alpha_r}\, \mu_i(\widetilde{r})\, \bar{e}_i(r)\,, \tag{2.38}$$

with the same conventions as in (2.30). From the commutation rules (2.37), their respective internal spins satisfy:

$$\exp(2i\pi s_i) = \exp(2i\eta)\,, \qquad \exp(2i\pi \bar{s}_i) = \exp(-2i\eta)\,. \tag{2.39}$$

Let us describe the explicit form of ψ_0:

$$\psi_0(k) = \begin{cases} e^{+i\alpha} \times q^{-\sigma^z} \otimes \cdots \otimes q^{-\sigma^z} \otimes \underset{(k-\mathrm{th})}{e^{-iu/2}\sigma^-} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \\ e^{-i\alpha} \times q^{-\sigma^z} \otimes \cdots \otimes q^{-\sigma^z} \otimes \underset{(k-\mathrm{th})}{e^{+iu/2}\sigma^-} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1} \end{cases} \tag{2.40}$$

if the edge k carries the representation $V = V_{e^{-iu/2}}$ or $V' = V_{e^{+iu/2}}$, respectively. We can use the relation between the spectral parameter and the opening angle: $\alpha = \pi u/\lambda$. Hence we can write

$$\psi_0(k) = e^{-is_0\alpha_k} \times q^{-\sigma^z} \otimes \cdots \otimes q^{-\sigma^z} \otimes \underset{(k-\mathrm{th})}{\sigma^-} \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}\,, \tag{2.41}$$

where $s_0 = 1 - \frac{\lambda}{\pi} = \frac{\eta}{\pi}$. This is compatible with the monodromy property (2.39). Similarly, we have

$$s_0 = s_1 = \frac{\eta}{\pi}\,, \qquad \bar{s}_0 = \bar{s}_1 = -\frac{\eta}{\pi}\,. \tag{2.42}$$

Finally, let us discuss the interpretation of the parafermionic operators in the scaling limit of the 6V model, *i.e.* the compactified boson theory (1.63). Since $e_0, \bar{e}_1$ (resp. $e_1, \bar{e}_0$) lower (resp. raise) the magnetisation by one unit, the parafermions $\psi_0, \bar{\psi}_1$ (resp. $\psi_1, \bar{\psi}_0$) carry a vortex charge $m = -1$ (resp. $m = +1$). Due to the non-local nature of the μ_i operators, the vertex charge e for these parafermions is allowed to be non-integer. The internal spin is given by $s = em$, which yields the value of the "electric charge" $e_0 = -\eta/\pi$ and $e_1 = +\eta/\pi$ for the operators $W_{e,m}$ (1.66) corresponding to the parafermions:

$$\begin{aligned} \psi_0 &\propto W_{e_0,-1}\,, \qquad & \bar{\psi}_0 &\propto W_{e_0,+1}\,, \\ \psi_1 &\propto W_{e_1,+1}\,, \qquad & \bar{\psi}_1 &\propto W_{e_1,-1}\,. \end{aligned} \tag{2.43}$$

In the subsequent sections, we shall use the 6V conserved currents (2.38) to construct discretely holomorphic parafermions in loop and face models, by applying the exact mappings of Sec. 1.1.4.

2.3 Discrete parafermions in integrable lattice models

2.3.1 Loop models

The Temperley-Lieb (TL) loop model with loop weight $n = -2\cos\eta$ is related to the 6V model by the exact mapping described in Sec. 1.1.2. Let us use this mapping to

[2]Note that this is a matter of convention, since the discrete Cauchy-Riemann relations (2.31) actually form an underdetermined linear system for the ψ_a's on the full lattice.

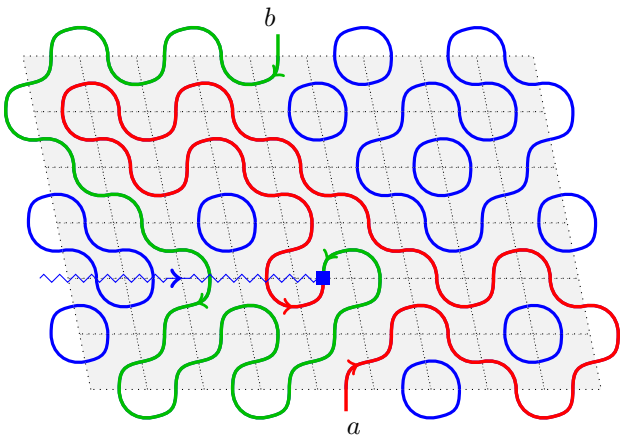


Figure 2.2: The one-point function of the parafermionic current ψ_0 in the TL loop model with reflecting boundary conditions.

translate the 6V conserved currents (2.38) into discrete parafermions for the TL loop model. For simplicity, we describe the one-point function of parafermions in a simply-connected domain, but the construction extends naturally to n-point correlation functions, and more general surfaces.

We consider the TL loop model on a domain of the square lattice, with reflecting boundary conditions everywhere, except at two given boundary points a and b, where a single path γ is inserted: see Fig. 2.2. By applying the mapping of Sec. 1.1.2, the following correlation function can be shown [43] to correspond to the one-point function $\langle \psi_0(x) \rangle$ in the 6V model with appropriate boundary conditions:

$$\langle \psi_0(x) \rangle_{\text{loop}} := \frac{1}{Z} \sum_{C | x \in \gamma} W(C)\, e^{i(1-2\eta/\pi)\theta_x(C)}\,, \tag{2.44}$$

where the sum is over every loop configuration C such that the path γ (in red and green in Fig. 2.2) passes through x, $W(C)$ is the Boltzmann weight (1.31), and $\theta_x(C)$ is the winding angle of the path γ from the boundary point a to x. Similarly, the average value of the discrete parafermions associated to $\bar{e}_0, e_1, \bar{e}_1$ are:

$$\begin{aligned}
\langle \bar{\psi}_0(x) \rangle_{\text{loop}} &:= \frac{1}{Z} \sum_{C | x \in \gamma} W(C)\, e^{-i(1-2\eta/\pi)\theta_x(C)}\,, \\
\langle \psi_1(x) \rangle_{\text{loop}} &:= \frac{1}{Z} \sum_{C | x \in \gamma} W(C)\, e^{-i\theta_x(C)}\,, \\
\langle \bar{\psi}_1(x) \rangle_{\text{loop}} &:= \frac{1}{Z} \sum_{C | x \in \gamma} W(C)\, e^{+i\theta_x(C)}\,.
\end{aligned} \tag{2.45}$$

In a general correlation function, one may define $\langle \psi_i(x) \dots \rangle$ and $\langle \bar{\psi}_i(x) \dots \rangle$ as the insertion of two open paths starting at x, with a phase factor associated to the winding of these paths. These objects satisfy the discrete Cauchy-Riemann equations (2.7–2.8) as operators. The internal spins of these operators are, respectively:

$$\sigma_0 = 1 - \frac{2\eta}{\pi}\,, \qquad \sigma_1 = -1 + \frac{2\eta}{\pi}\,, \qquad \bar{\sigma}_0 = +1\,, \qquad \bar{\sigma}_1 = -1\,. \tag{2.46}$$

In terms of the CG action (1.70) associated to the loop model in the scaling limit, one can identify the parafermionic operators with the chiral degenerate operators and screening charges:

$$\begin{aligned} \psi_0 &\propto \phi_{13}(z)\,, & \bar\psi_0 &\propto \bar\phi_{13}(\bar z)\,, \\ \psi_1 &\propto \phi_{1,-1}(z)\,, & \bar\psi_1 &\propto \bar\phi_{1,-1}(\bar z)\,. \end{aligned} \tag{2.47}$$

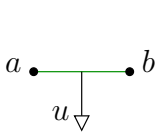
Interestingly, the parafermions ψ_0 and $\bar\psi_0$ have the conformal dimension of the *boundary* two-leg operator, *i.e.* the operator inserting two open paths at a boundary point. This is easy to interpret, from the basic principles of boundary CFT [30]. Let us consider the half-plane geometry to simplify this argument. The correlation functions of the bulk operator $\psi_0(z,\bar z) = \phi_{13}(z) \times \mathbf{1}(\bar z)$ on the upper half-plane $\mathbb{H}$ are obtained through the method of images:

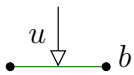
$$\langle \psi_0(z,\bar z)\dots\rangle_{\mathbb{H}} = \langle \phi_{13}(z)\mathbf{1}(\bar z)\dots\rangle_{\mathbb{C}} = \langle \phi_{13}(z)\dots\rangle_{\mathbb{C}}\,.$$

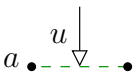
As we approach z to a boundary point $z \to z_B \in \mathbb{R}$, the operator $\psi_0(z,\bar z)$ thus behaves like $\phi_{13}(z_B)$. Consistently, on the lattice, as x approaches the boundary, the phase factors in (2.44) are all fixed to the same value, and $\psi_0(x)$ reduces to the boundary two-leg operator.

2.3.2 Face models

In this section, we explain the construction [45] of discrete parafermions in unrestricted SOS models, by applying the vertex-IRF correspondence (1.39–1.40) to the BF conserved currents of the 6V model. At this point it is convenient to introduce a graphical convention for the Baxter's intertwiners [70, 71]:

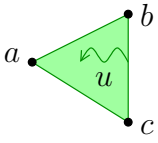
$\phi(a,b|u) =$ 

, $\phi^-(a,b|u) =$ 

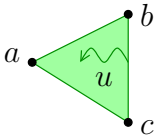
, $\phi^+(a,b|u) =$ 

.

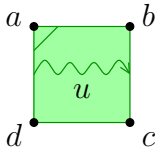
The building blocks for discrete parafermions in the SOS model will then be the "dressed" Chevalley generators defined as:



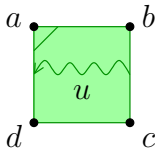
$= F_i\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| u\right) = \phi^-(a,c|u)\, f_i\, \phi(a,b|u)\,,$ (2.48)



$= \bar F_i\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| u\right) = \phi^-(a,c|u)\, \bar f_i\, \phi(a,b|u)\,,$ (2.49)



$= T_i^+\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) = \phi^+(d,c|u)\, t_i\, \phi(a,b|u)\,,$ (2.50)

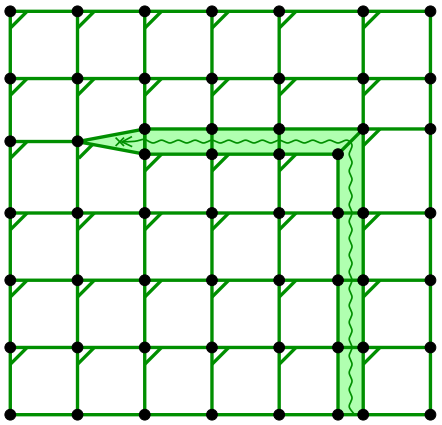


$= T_i^-\left(\begin{matrix} a & b \\ d & c \end{matrix}\middle| u\right) = \phi^-(d,c|u)\, t_i^{-1}\, \phi(a,b|u)\,,$ (2.51)

where $\bar{f}_i = e_i\, t_i^{-1}$.

Consider the SOS model on a simply-connected domain with fixed boundary conditions. The one-point function of lattice parafermions $\psi_i(r)$ (resp. $\bar{\psi}_i(r)$) are defined by the insertion of an F_i (resp. $\bar{F}_i$) plaquette at position r, together with a path of $T_i^{\pm}$ connecting r to a reference point on the boundary:

$$\langle \psi_0(x) \rangle_{\text{SOS}} = \quad . \tag{2.52}$$



Through the vertex-IRF correspondence, this one-point function can be shown to correspond to $\langle \psi_0 \rangle$ in the 6V model. Analogous arguments can be made to construct multi-point correlation functions of the ψ_i's and $\bar{\psi}_i$'s in the unrestricted SOS model. The internal spins for $\psi_0, \bar{\psi}_0, \psi_1, \bar{\psi}_1$ are, respectively:

$$\tau_0 = 1\,, \qquad \bar{\tau}_0 = -1\,, \qquad \tau_1 = 1 - \frac{2\eta}{\pi}\,, \qquad \bar{\tau}_0 = -1 + \frac{2\eta}{\pi}\,. \tag{2.53}$$

The general CFT interpretation of these operators is quite a subtle issue, in particular because the CFT describing the scaling limit of *unrestricted* SOS models is not well understood. In the rational case when $\eta = \pi p'/p$, where p and p' are coprime integers, the Boltzmann weights become periodic under $a \mapsto a + p$, and one obtains the Cyclic Solid-On-Solid (CSOS) model: see Sec. 1.1.3. The latter has a finite number of height configurations, and a well-defined scaling limit, with the same central charge as the RSOS model (see Sec. 1.2.3), but a different operator content. In this context, the operators ψ_1 and $\bar{\psi}_1$ have conformal dimensions corresponding to the degenerate operators ϕ_{13} and $\bar{\phi}_{13}$, similarly to $\psi_0, \bar{\psi}_0$ in the loop model.

N.B. : The above construction does not extend directly to the case of RSOS models, because the vertex-IRF local relations (1.39–1.40) are only valid for the unrestricted SOS models.

2.3.3 Clock models

Unlike the Temperley-Lieb loop and the SOS models, the chiral Potts model is not based on the evaluation representation of the $U_q(\widehat{\mathfrak{sl}}_2)$ algebra, but on the cyclic representation (1.51), which only exists at roots of unity $q = -e^{i\pi/N}$. The construction of lattice conserved currents associated to the Chevalley generators in this family of representations, even though it is more technical, is very similar to that of 6V conserved currents explained in Sec. 2.2.6. Let us outline the main results of this construction [44].

Due to the form of the operators e_i, f_i, t_i in the cyclic representation (1.51), when applying the BF construction, the "tail" operators associated to the t_i's insert a $\mathbb{Z}_N$ defect

path of matrices Z (1.52), precisely like the $\mathbb{Z}_N$ disorder operators described in Sec. 2.1, whereas the e_i's and f_i's measure the local spin through the matrix X, and correspond to spin operators (and they also include a defect contribution Z corresponding to the end of the "tail").

At the isotropic critical point, where the chiral Potts model reduces to the critical FZ lattice model [77], the parafermions obtained by the BF procedure from the generators e_i and $\bar{e}_i$ are exactly the ψ_p's and $\bar{\psi}_p$'s (2.5) with $p = 1$. Hence, the discrete Cauchy-Riemann relations (2.7–2.8), which were observed empirically in [38], are in fact a consequence of the intertwining relations of the underlying $U_q(\widehat{\mathfrak{sl}}_2)$ algebra.

An interesting feature of the chiral Potts model is that it remains integrable and $\mathbb{Z}_N$ invariant outside the critical point $k = 0$. However, outside criticality, the embedding of star-triangle relations into adjacent rhombi, which relies on the additivity of spectral parameters, does not hold anymore. Let us extend the rhombic embedding from the critical FZ model to a generic value of k: a plaquette with incoming spectral parameters a and b has an opening angle $\alpha = u_a - u_b$. In [44], we proposed the following definition of off-critical parafermions:

$$\psi_i(x) = e^{-i\alpha_x - i\epsilon_x \phi/N}\, \mu_i(\widetilde{r})\, e_i(r)\,, \qquad \bar{\psi}_i(x) = e^{+i\alpha_x + i\phi/N}\, \mu_i(\widetilde{r})\, \bar{e}_i(r)\,, \tag{2.54}$$

for $i = 0, 1$, and where $\epsilon_x = \pm 1$, depending on the parity of the edge x. With this definition, the overall dependence on spectral parameters reduces to a factor $e^{\pm is\alpha_x}$, where $s = 1 - 1/N$ is the internal spin of the parafermion (2.5) with $\mathbb{Z}_N$ charge $p = 1$. These off-critical lattice parafermions then obey *twisted* discrete Cauchy Riemann relations:

$$e^{+\frac{i\phi_a}{N}}\, \psi_i(x_1)\, \delta z_1 + e^{+\frac{i\phi_b}{N}}\, \psi_i(x_2)\, \delta z_2 + e^{-\frac{i\phi_a}{N}}\, \psi_i(x_3)\, \delta z_3 + e^{-\frac{i\phi_b}{N}}\, \psi_i(x_4)\, \delta z_4 = 0\,, \tag{2.55}$$

$$e^{-\frac{i\phi_a}{N}}\, \bar{\psi}_i(x_1)\, \delta \bar{z}_1 + e^{-\frac{i\phi_b}{N}}\, \bar{\psi}_i(x_2)\, \delta \bar{z}_2 + e^{+\frac{i\phi_a}{N}}\, \bar{\psi}_i(x_3)\, \delta \bar{z}_3 + e^{+\frac{i\phi_b}{N}}\, \bar{\psi}_i(x_4)\, \delta \bar{z}_4 = 0\,. \tag{2.56}$$

At the critical, isotropic point $k = 0$, we have $\phi = \bar{\phi} = 0$, and we recover the discrete Cauchy-Riemann equations (2.7–2.8).

For small deviations from criticality, in the scaling limit, the chiral Potts model can be shown [87] to correspond to a perturbation of the FZ parafermionic CFT by the energy operator ε with conformal dimension $h_\varepsilon = 2/(N+2)$ and its descendants $W_{-1}\varepsilon$ and $\bar{W}_{-1}\varepsilon$ under the W_N algebra :

$$S = S_{\mathrm{FZ}} + \int d^2 r \left(g_0\, \varepsilon + g_1\, W_{-1}\varepsilon + \bar{g}_1\, \bar{W}_{-1}\varepsilon \right)\,, \tag{2.57}$$

where $g_0, g_1, \bar{g}_1$ are some effective coupling constants.

From a general perturbation theory argument (see [88]), one expects the off-critical currents to satisfy the *massive* Cauchy-Riemann equations:

$$\bar{\partial}\psi = \pi g_0 \chi + \pi g_1 W_{-1}\chi + \pi \bar{g}_1 \bar{W}_{-1}\chi\,, \tag{2.58}$$

where χ is the operator in the $1/z$ term of the OPE $\psi(z)\varepsilon(0)$ in the unperturbed FZ CFT:

$$\psi(z)\varepsilon(0) = \cdots + \frac{\chi(0)}{z} + \ldots \tag{2.59}$$

By a simple dimensional analysis, we see that this operator must have conformal dimensions:

$$h_\chi = h_\psi + h_\varepsilon - 1\,, \qquad \bar{h}_\chi = h_\varepsilon\,. \tag{2.60}$$

In particular, χ should have conformal spin $s_\chi = h_\chi - \bar{h}_\chi = h_\psi - 1 = 1/N$. Hence, in the RHS of (2.58), we expect primary operators with conformal spins $1/N$, $1/N+1$ and $1/N-1$. It can be checked explicitly that the twisted CR relations (2.55), when expanded around $k = \phi = \bar{\phi} = 0$, produce lattice operators with exactly these three values of the internal spin. Hence, (2.55) can be considered as a lattice analog of the massive CR equation (2.58).

Chapter 3

Operator algebra of non-rational CFTs

3.1 The operator algebra for loop models

3.1.1 The conformal bootstrap for non-scalar operators

Loop models such as the Temperley-Lieb and O(n) model have a discrete, infinite spectrum of conformal dimensions: see Sec. 1.2.2. This spectrum includes non-scalar operators, namely the mixed operators W_{em} with dimensions $(h, \bar{h}) = (h_{em}, h_{e,-m})$, where m is a positive integer, and $e \in \mathbb{Z}/m$. Let us examine the consistency of the bootstrap approach in this context [61].

From the chiral fusion rules of the degenerate field Φ_{21}, with the Coulomb Gas parameterisation $h(\alpha) = \alpha(\alpha - 2Q)$, we can *a priori* expect four terms in the OPE:

$$\Phi_{21} \times \Phi_{h(\alpha),h(\bar{\alpha})} \to \sum_{(\epsilon,\bar{\epsilon})\in\{-1,1\}^2} \Phi_{h(\alpha-\epsilon\frac{b^{-1}}{2}),\ h(\bar{\alpha}-\bar{\epsilon}\frac{b^{-1}}{2})} \,. \tag{3.1}$$

In the OPE $\Phi_{21}(0)\Phi_{h(\alpha),h(\bar{\alpha})}(z,\bar{z})$, as $z \to 0$, the term $(\epsilon, \bar{\epsilon})$ comes with a factor:

$$z^{-h_{21}-h(\alpha)+h(\alpha-\epsilon b^{-1}/2)}\, \bar{z}^{\,-h_{21}-h(\bar{\alpha})+h(\bar{\alpha}-\bar{\epsilon}b^{-1}/2)} \,. \tag{3.2}$$

The factor picked by this term under the monodromy $(z, \bar{z}) \to (e^{2i\pi}z, e^{-2i\pi}\bar{z})$ is $e^{2i\pi\eta(\epsilon,\bar{\epsilon})}$, where the monodromy exponent η is given by:

$$\eta(\epsilon,\bar{\epsilon}) := [h(\alpha - \epsilon b^{-1}/2) - h(\alpha)] - [h(\bar{\alpha} - \bar{\epsilon}b^{-1}/2) - h(\bar{\alpha})] \tag{3.3}$$

$$= -b^{-1}[\epsilon(\alpha - Q) - \bar{\epsilon}(\bar{\alpha} - Q)] \,. \tag{3.4}$$

Inside a correlation function $\langle \Phi_{21}(0)\Phi_{h,\bar{h}}(z,\bar{z})\dots\rangle$, if several terms from the RHS of (3.1) are included, they should have the same monodromy factor $e^{2i\pi\eta}$, so that the correlation function has a well-defined monodromy around $z = 0$.

In the loop model, we shall parameterise the primary fields as:

$$\Phi_{r1} = \Phi_{h(\alpha_{r1}),h(\alpha_{r1})} \,, \qquad W_{em} = \Phi_{h(\alpha_{em}),h(\alpha_{e,-m})} \,. \tag{3.5}$$

We obtain the monodromy exponents:

$$\eta(\epsilon,\bar{\epsilon}) = \begin{cases} (\epsilon - \bar{\epsilon})\frac{rb^{-2}-1}{2} & \text{for } \Phi_{r1} \,, \\ (\epsilon - \bar{\epsilon})\frac{eb^{-2}}{2} - (\epsilon + \bar{\epsilon})\frac{m}{2} & \text{for } W_{em} \,. \end{cases} \tag{3.6}$$

We restrict to the case of a generic central charge, *i.e.* when b^2 is not a rational. For the "energy-like" operators Φ_{r1} with $r = 2, 3, 4 \dots$ and for mixed operators W_{em} with both e and m nonzero, only the terms with $\epsilon = \bar{\epsilon}$ can coexist in a correlation function, corresponding to the fusion rules[1]:

$$\Phi_{21} \times \Phi_{r1} \to \Phi_{r-1,1} + \Phi_{r+1,1} \,, \tag{3.7}$$

$$\Phi_{21} \times W_{em} \to W_{e-1,m} + W_{e+1,m} \qquad (e \neq 0, m \neq 0) \,. \tag{3.8}$$

3.1.2 Non-scalar shift equation

We consider the four-point function:

$$G(z, \bar{z}) = \langle \Phi_1(\infty) \Phi_2(1) \Phi_3(z, \bar{z}) \Phi_{21}(0) \rangle \,, \tag{3.9}$$

where Φ_1, Φ_2, Φ_3 can be any (scalar or non-scalar) primary fields of the loop model, except those of the form W_{0m}. The analysis of (3.9) is similar to the case of scalar operators developed in Sec. 1.3.2, except that the holomorphic and anti-holomorphic parts may carry distinct exponents [61]. First, the null-vector equations for Φ_{21} are:

$$(L_{-2} - b^2 \, L_{-1}^2) \Phi_{21} = 0 \,, \qquad (\bar{L}_{-2} - b^2 \, \bar{L}_{-1}^2) \Phi_{21} = 0 \,. \tag{3.10}$$

After the rescaling:

$$G(z, \bar{z}) = z^{-\frac{\alpha_3}{b}} \, \bar{z}^{-\frac{\bar{\alpha}_3}{b}} (1-z)^{h_{21} - h_2 - h_3 - \frac{\alpha_1}{b}} (1-\bar{z})^{h_{21} - \bar{h}_2 - \bar{h}_3 - \frac{\bar{\alpha}_1}{b}} \times g(z, \bar{z}) \,, \tag{3.11}$$

we obtain the hypergeometric equations:

$$\begin{aligned} &z(1-z) \partial_z^2 g + [C - (A + B + 1) z] \partial_z g - AB \, g = 0 \,, \\ &\bar{z}(1-\bar{z}) \partial_{\bar{z}}^2 g + [\bar{C} - (\bar{A} + \bar{B} + 1) \bar{z}] \partial_{\bar{z}} g - \bar{A} \bar{B} \, g = 0 \,, \end{aligned} \tag{3.12}$$

with parameters

$$\begin{aligned} A &= 2h_{21} - (\alpha_1 + \alpha_2 + \alpha_3)/b \,, & \bar{A} &= 2h_{21} - (\bar{\alpha}_1 + \bar{\alpha}_2 + \bar{\alpha}_3)/b \,, \\ B &= 2h_{21} - [\alpha_1 + (2Q - \alpha_2) + \alpha_3]/b \,, & \bar{B} &= 2h_{21} - [\bar{\alpha}_1 + (2Q - \bar{\alpha}_2) + \bar{\alpha}_3]/b \,, \\ C &= b^{-2} - 2\alpha_3/b \,, & \bar{C} &= b^{-2} - 2\bar{\alpha}_3/b \,. \end{aligned} \tag{3.13}$$

We denote by $\{I_1, I_2\}$ and $\{\bar{I}_1, \bar{I}_2\}$ the holomorphic and anti-holomorphic bases of solutions converging on the disc $|z| < 1$, and $\{J_1, J_2\}$ and $\{\bar{J}_1, \bar{J}_2\}$ the analogous solutions for $|z - 1| < 1$: see (1.98–1.100). The physical correlation is given by the bilinear forms:

$$g(z, \bar{z}) = \sum_{i,j} X_{ij} \, \bar{I}_i(\bar{z}) \, I_j(z) = \sum_{k,\ell} Y_{k\ell} \, \bar{J}_k(\bar{z}) \, J_\ell(z) \,. \tag{3.14}$$

[1] In the limiting case of mixed operators W_{0m} with $m = 1, 2, 3 \dots$ all the terms $(\epsilon, \bar{\epsilon})$ in the OPE $\Phi_{21} \times W_{0m}$ have integer monodromy exponent η. This suggests the presence of unconventional terms in the fusion rule:

$$\Phi_{21} \times W_{0m} \to W_{1m} + W_{-1,m} + \dots$$

The logarithmic nature (*i.e.* the non-diagonalisable nature of the evolution operator $L_0 + \bar{L}_0$) should be taken into account to properly analyse this particular fusion rule, which goes beyond the scope of the present study.

Denoting by P (resp. $\bar{P}$) the change of basis from $\{I_1, I_2\}$ to $\{J_1, J_2\}$ (resp. from $\{\bar{I}_1, \bar{I}_2\}$ to $\{\bar{J}_1, \bar{J}_2\}$), we have:

$$Y = \bar{P}^t \, X \, P \,. \tag{3.15}$$

From the above analysis of OPEs (3.7–3.8), only the diagonal terms $i = j$ and $k = \ell$ are allowed. This yields the linear system for $X_1 := X_{11}$ and $X_2 := X_{22}$:

$$\begin{aligned} &\bar{p}_{11} p_{12} \, X_1 + \bar{p}_{21} p_{22} X_2 = 0 \,, \\ &\bar{p}_{12} p_{11} \, X_1 + \bar{p}_{22} p_{21} X_2 = 0 \,. \end{aligned} \tag{3.16}$$

The determinant of this system is

$$\det = g \, \bar{g} \left[s(\bar{C} - \bar{A}) s(\bar{C} - \bar{B}) s(A) s(B) - s(\bar{A}) s(\bar{B}) s(C - A) s(C - B) \right] \,, \tag{3.17}$$

where $s(x) := \sin(\pi x)/\pi$ and

$$g := \Gamma(C)\Gamma(2 - C)\Gamma(D)\Gamma(-D) \,, \qquad \bar{g} := \Gamma(\bar{C})\Gamma(2 - \bar{C})\Gamma(\bar{D})\Gamma(-\bar{D}) \,. \tag{3.18}$$

For any $j = 1, 2, 3$, we introduce the integer number

$$m_j = \begin{cases} 0 & \text{if } \Phi_j = \Phi_{r1} \,, \\ m & \text{if } \Phi_j = W_{em} \,. \end{cases} \tag{3.19}$$

We get:

$$\bar{A} = A + m_1 + m_2 + m_3 \,, \qquad \bar{B} = B + m_1 - m_2 + m_3 \,, \qquad \bar{C} = C + 2m_3 \,, \tag{3.20}$$

and hence the determinant vanishes, and we can write the solution as:

$$\frac{X_1}{X_2} = -\frac{\bar{p}_{21} p_{22}}{\bar{p}_{11} p_{12}} = -\frac{p_{21} \bar{p}_{22}}{p_{11} \bar{p}_{12}} = \sqrt{x_{-1/b}(\alpha_1, \alpha_2, \alpha_3) x_{-1/b}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3)} \,, \tag{3.21}$$

where

$$\begin{aligned} x_{-1/b}(\alpha_1, \alpha_2, \alpha_3) &= \frac{\gamma(2 - C)}{\gamma(C)} \times \gamma(A)\gamma(B)\gamma(C - A)\gamma(C - B) \,, \\ x_{-1/b}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) &= \frac{\gamma(2 - \bar{C})}{\gamma(\bar{C})} \times \gamma(\bar{A})\gamma(\bar{B})\gamma(\bar{C} - \bar{A})\gamma(\bar{C} - \bar{B}) \,. \end{aligned} \tag{3.22}$$

This leads to shift equations where the RHS is the geometric mean of the holomorphic and anti-holomorphic contributions:

$$\frac{C(\Phi_1, \Phi_2, \Phi_3')}{C(\Phi_1, \Phi_2, \Phi_3)} = \sqrt{u_{-1/b}(\alpha_1, \alpha_2, \alpha_3) K_{-1/b}(\alpha_3) u_{-1/b}(\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3) K_{-1/b}(\bar{\alpha}_3)} \,, \tag{3.23}$$

where we have used the CG parameterisation:

$$\Phi_3 = \Phi_{h(\alpha_3), h(\bar{\alpha}_3)} \,, \qquad \Phi_3' = \Phi_{h(\alpha_3 - b^{-1}), h(\bar{\alpha}_3 - b^{-1})} = \begin{cases} \Phi_{r+2,1} & \text{if } \Phi_3 = \Phi_{r1} \,, \\ W_{e+2,m} & \text{if } \Phi_3 = W_{em} \,, \ e \neq -1 \,, \end{cases} \tag{3.24}$$

and the functions $u_{-1/b}$ and $K_{-1/b}$ are obtained from Sec. 1.3.2 by the change $b \to -1/b$. We have

$$u_{-1/b}(\alpha_1, \alpha_2, \alpha_3) = \frac{\gamma(2b^{-2} - 1 - \alpha_{123}/b)\gamma(b^{-2} - \alpha_{13}^2/b)\gamma(b^{-2} - \alpha_{23}^1/b)}{\gamma(-\alpha_{12}^3/b)} \,, \tag{3.25}$$

$$K_{-1/b} = \frac{\sqrt{\gamma(2 - b^{-2} + 2\alpha/b)\, \gamma(2 - 3b^{-2} + 2\alpha/b)}}{\gamma(b^{-2} - 2\alpha/b)} \,, \tag{3.26}$$

where $\alpha_{123} := \alpha_1 + \alpha_2 + \alpha_3$, and $\alpha_{ij}^k := \alpha_i + \alpha_j - \alpha_k$.

3.1.3 Expression of the structure constants

First, the energy-like operators Φ_{r1} are all scalar and form a closed subalgebra. Their structure constants can be treated like for the case of minimal models: see Sec. 1.3.3. One has the fusion rules

$$\Phi_{r1} \times \Phi_{r'1} \to \sum_{\substack{k=|r-r'|+1 \\ k\equiv r+r-1\,[2]}}^{r+r'-1} \Phi_{k1}\,. \tag{3.27}$$

In particular, for $r = r'$, the structure constants are given by:

$$C(\Phi_{r1}, \Phi_{r1}, \Phi_{2k+1,1}) = \prod_{\ell=1}^{k} \frac{\gamma^2(\varrho\ell)\,\sqrt{\gamma[2-\varrho(2\ell-1)]\,\gamma[2-\varrho(2\ell+1)]}}{\gamma[2-\varrho(r+\ell)]\,\gamma[\varrho(r-\ell)]\,\gamma[\varrho(2\ell-1)]}\,, \tag{3.28}$$

where $\varrho := b^{-2}$.

Through a similar argument, by solving the recursion equation (3.23), one gets the structure constants of the form:

$$C(W_{em}, W_{-e,m}, \Phi_{2k+1,1}) = \prod_{\ell=1}^{k} \sqrt{\mu_{e,m,\ell}\,\mu_{-e,m,\ell}}\,, \tag{3.29}$$

$$\mu_{e,m,\ell} := \frac{\gamma(\ell\varrho - m)\,\gamma(\ell\varrho + m)\sqrt{\gamma[2-\varrho(2\ell-1)]\,\gamma[2-\varrho(2\ell+1)]}}{\gamma[2-(e+\ell)\varrho]\,\gamma[(e-\ell)\varrho]\,\gamma[\varrho(2\ell-1)]}\,. \tag{3.30}$$

3.1.4 Generalised three-point amplitudes

Although the vertex operators in the CG description of the loop model are restricted to the set of "energy-like" operators Φ_{r1} by the compactification condition in (1.70), it is possible to define two- and three-point correlation functions in the lattice model, which relate to vertex operators with generic vertex charge.

As explained in Sec. 1.2.1 for the case of the 6V model, the two-point function of vertex operators corresponds to the insertion of a twist line. For the loop model on the sphere, the two-point function:

$$\langle V_{\kappa/2b}(r_1) V_{2Q-\kappa/2b}(r_2)\rangle_{\mathrm{CG}} \tag{3.31}$$

changes the fugacity of non-contractible loops on the sphere punctured at r_1 and r_2 to the value

$$n' = 2\cos(\lambda - \pi\kappa)\,. \tag{3.32}$$

We denote by $Z_{n'}(r_1, r_2)$ the corresponding modified loop partition function. In particular, the charge $\alpha_{1/2,0}$ corresponds to $\pi\kappa = \lambda - \pi/2$ and $n' = 0$. In this case, the two-point function only counts the configurations with contractible loops on the punctured sphere. In terms of the related Fortuin-Kasteleyn cluster model, these are the cluster configurations where r_1 and r_2 sit on the same connected component.

In [56] it was suggested to extend these ideas to three-point functions, in the case of critical percolation. Consider the three-point connectivity, *i.e.* the probability that three given points r_1, r_2, r_3 sit on the same percolation cluster. Naively, this should correspond

to the CG three-point function of $V_{\alpha_{1/2,0}}$, but, since it is impossible to obtain any neutral combination of three charges $\alpha_{\pm 1/2,0}$ (even by introducing screening charges $V_{1/b}$), this three-point function does not make sense in the CG formalism. Nevertheless, there is strong numerical evidence that the three-point amplitude of connectivities coincides with the *imaginary Liouville* OPE coefficient:

$$\frac{P_c(r_1,r_2,r_3)}{\sqrt{P_c(r_1,r_2)P_c(r_2,r_3)P_c(r_1,r_3)}} = C_{\rm IL}(\alpha_{1/2,0},\alpha_{1/2,0},\alpha_{1/2,0})\,, \tag{3.33}$$

where $P_c(r_1,\dots,r_n)$ is the probability that $r_1,\dots,r_n$ sit on the same percolation cluster, and $C_{\rm IL}$ is the $c=0$ imaginary Liouville OPE coefficient (1.123).

Using the transfer-matrix approach, we have given further numerical evidence [58] that this result generalises to the O(n) and TL loop models with any loop fugacity in the critical regime $-2<n<2$, and for any triplet of modified loop fugacities. We introduce

$$n_j = 2\cos(\lambda-\pi\kappa_j)\,, \qquad j=1,2,3\,, \tag{3.34}$$

and we define $Z_{n_1,n_2,n_3}(r_1,r_2,r_3)$ as the partition function where the loops which separate one point r_j from the other two gets a modified fugacity n_j. If we denote

$$\alpha_j = \frac{\kappa_j}{2b}\,, \qquad h_j = \alpha_j(\alpha_j-2Q)\,, \tag{3.35}$$

our numerical studies on infinite cylinders show that, in the scaling limit

$$\frac{Z_{n_j}(r_1,r_2)}{Z} = \frac{A(n_j)}{|r_1-r_2|^{4h_j}}\,, \tag{3.36}$$

$$\frac{Z_{n_1,n_2,n_3}(r_1,r_2,r_3)}{Z} = \frac{A(n_1)A(n_2)A(n_3)\times C_{\rm IL}(\alpha_1,\alpha_2,\alpha_3)}{|r_1-r_2|^{2(h_1+h_2-h_3)}\,|r_2-r_3|^{2(h_2+h_3-h_1)}\,|r_1-r_3|^{2(h_1+h_3-h_2)}}\,, \tag{3.37}$$

where $A(n_j)$ is the non-universal two-point amplitude associated to Z_{n_j}.

3.2 Non-diagonal imaginary Liouville and Toda models

3.2.1 The Virasoro case

Consistency conditions on OPEs

We consider the non-diagonal imaginary Liouville CFT, *i.e.* the CFT with central charge $c\le 1$, where the spectrum is generated by the Virasoro algebra acting on all possible scalar or non-scalar primary operators, which we parameterise as vertex operators. It contains, by assumption, the two degenerate fields Φ_{12} and Φ_{21}. The central charge is written as $c=1-6(1/b-b)^2$, with b^2 irrational. We want to examine the consistency of OPEs of these degenerate fields with generic non-scalar vertex operators (see [62]). For OPEs with Φ_{12}, the possible terms are

$$\Phi_{12}\times V_{\alpha,\bar\alpha} \to \sum_{\epsilon,\bar\epsilon\in\{-1,1\}^2} V_{\alpha+\frac{\epsilon b}{2},\bar\alpha+\frac{\bar\epsilon b}{2}}\,. \tag{3.38}$$

The monodromy exponent associated to the term $(\epsilon, \bar{\epsilon})$ is

$$\eta(\epsilon, \bar{\epsilon}) = b\left[\epsilon(\alpha - Q) - \bar{\epsilon}(\bar{\alpha} - Q)\right] . \tag{3.39}$$

For a given value $\epsilon \in \{-1, 1\}$, we have

$$\eta(\epsilon, 1) - \eta(\epsilon, -1) = -2b(\bar{\alpha} - Q) . \tag{3.40}$$

If the two terms $(\epsilon, 1)$ and $(\epsilon, -1)$ coexist in an OPE, then the difference of their monodromy exponents should be an integer, which forces $\bar{\alpha}$ to be of the form $\bar{\alpha} = \alpha_{0s}$ with $s \in \mathbb{Z}$. Similarly, the terms $(1, \bar{\epsilon})$ and $(-1, \bar{\epsilon})$ can only coexist if $\alpha = \alpha_{r0}$. In the following, we shall consider *generic* vertex operators, whose vertex charges $\alpha, \bar{\alpha}$ are not of the form α_{0s} or α_{r0}.

Under this condition, the only possible coexisting terms in the RHS of (3.38) are $V_{\alpha+b/2,\bar{\alpha}+b/2} + V_{\alpha-b/2,\bar{\alpha}-b/2}$ and $V_{\alpha+b/2,\bar{\alpha}-b/2} + V_{\alpha-b/2,\bar{\alpha}+b/2}$. The first possibility implies that the difference $(\eta(1,1) - \eta(-1,-1))$ is an integer, which yields the condition $\alpha - \bar{\alpha} \in \mathbb{Z}/2b$. The second possibility can be reparameterised by using the identity $h(2Q - \alpha) = h(\alpha)$:

$$\Phi_{12} \times V_{\alpha,\bar{\alpha}} \to V_{\alpha+b/2,\bar{\alpha}-b/2} + V_{\alpha-b/2,\bar{\alpha}+b/2} \equiv V_{2Q-\alpha-b/2,\bar{\alpha}-b/2} + V_{2Q-\alpha+b/2,\bar{\alpha}+b/2} , \tag{3.41}$$

which amounts to the previous case, with α replaced by $\widehat{\alpha} = 2Q - \alpha$. Hence, without loss of generality, we shall always consider the case:

$$\begin{aligned} &\Phi_{12} \times V_{\alpha,\bar{\alpha}} \to V_{\alpha+b/2,\bar{\alpha}+b/2} + V_{\alpha-b/2,\bar{\alpha}-b/2} , \\ &\alpha - \bar{\alpha} \in \frac{\mathbb{Z}}{2b} . \end{aligned} \tag{3.42}$$

Once the conventional choice between α and $2Q - \alpha$ has been made, the OPE with the second degenerate field Φ_{21} has two genuinely different possible forms:

$$\Phi_{21} \times V_{\alpha,\bar{\alpha}} \to \begin{cases} V_{\alpha-b^{-1}/2,\bar{\alpha}-b^{-1}/2} + V_{\alpha+b^{-1}/2,\bar{\alpha}+b^{-1}/2} & \text{if } \alpha - \bar{\alpha} \in \mathbb{Z}b/2 \\ V_{\alpha-b^{-1}/2,\bar{\alpha}+b^{-1}/2} + V_{\alpha+b^{-1}/2,\bar{\alpha}-b^{-1}/2} & \text{if } 2Q - \alpha - \bar{\alpha} \in \mathbb{Z}b/2 \end{cases} \tag{3.43}$$

Spectrum of primary fields

The above argument leads to a classification of the primary fields of the model into two classes $V^{(\pm)}_{\alpha,\bar{\alpha}}$, according to their OPEs with Φ_{12} and Φ_{21}:

$$\begin{aligned} &\Phi_{12} \times V^{(\epsilon)}_{\alpha,\bar{\alpha}} \to V^{(\epsilon)}_{\alpha+b/2,\bar{\alpha}+b/2} + V^{(\epsilon)}_{\alpha-b/2,\bar{\alpha}-b/2} , \\ &\Phi_{21} \times V^{(\epsilon)}_{\alpha,\bar{\alpha}} \to V^{(\epsilon)}_{\alpha-b^{-1}/2,\bar{\alpha}-\epsilon b^{-1}/2} + V^{(\epsilon)}_{\alpha+b^{-1}/2,\bar{\alpha}+\epsilon b^{-1}/2} . \end{aligned} \tag{3.44}$$

The charges for $V^{(+)}_{\alpha,\bar{\alpha}}$ satisfy $\alpha - \bar{\alpha} \in \mathbb{Z}/2b \cap \mathbb{Z}b/2$. Since b^2 is irrational, this means that $\alpha = \bar{\alpha}$, and hence the operator is scalar. The charges for $V^{(-)}_{\alpha,\bar{\alpha}}$ satisfy

$$\alpha - \bar{\alpha} = -\frac{e}{b} , \qquad 2Q - \alpha - \bar{\alpha} = -mb , \qquad (e, m) \in (\mathbb{Z}/2)^2 , \tag{3.45}$$

which gives, in Kac notations:

$$\alpha = \alpha_{em} , \qquad \bar{\alpha} = \alpha_{-e,m} . \tag{3.46}$$

The dimensions of operators $V^{(-)}_{\alpha,\bar{\alpha}}$ have a form similar to the "mixed" operators W_{em} in the loop model. However, the range of indices is quite different. In particular, for $m \neq 0$, the fractional values $e \in \mathbb{Z}/m$ are not allowed in the non-diagonal imaginary Liouville model.

Structure constants

The structure constants of vertex operators satisfy two shift equations:

$$\frac{C(V_1,V_2,V^{(\epsilon_3)}_{\alpha_3+b,\bar\alpha_3+b})}{C(V_1,V_2,V^{(\epsilon_3)}_{\alpha_3,\bar\alpha_3})} = \sqrt{u_b(\alpha_1,\alpha_2,\alpha_3)K_b(\alpha_3)u_b(\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)K_b(\bar\alpha_3)}\,, \tag{3.47}$$

$$\frac{C(V_1,V_2,V^{(\epsilon_3)}_{\alpha_3-\epsilon_3/b,\bar\alpha_3-\epsilon_3/b})}{C(V_1,V_2,V^{(\epsilon_3)}_{\alpha_3,\bar\alpha_3})} = \sqrt{u_{-1/b}(\alpha_1,\alpha_2,\alpha_3)K_{-1/b}(\alpha_3)u_{-1/b}(\bar\alpha_1,\bar\alpha_2,\bar\alpha_3)K_{-1/b}(\bar\alpha_3)}\,, \tag{3.48}$$

where the functions u_b, K_b are defined in Sec. 1.3.2.

In the case $\epsilon_3 = +1$, $\alpha_3 = \bar\alpha_3$, the operator $V^{(+)}_{\alpha_3,\alpha_3}$ is a scalar vertex operator with unconstrained real vertex charge. The unique solution of the shift equations is then [62]

$$C(V_1,V_2,V_3^{(+)}) = \sqrt{C_{\rm IL}(\alpha_1,\alpha_2,\alpha_3)\,C_{\rm IL}(\bar\alpha_1,\bar\alpha_2,\alpha_3)}\,, \tag{3.49}$$

where $C_{\rm IL}$ is the imaginary Liouville OPE coefficient (1.123).

3.2.2 The W_3 case

The imaginary Toda CFT

The imaginary Toda CFT is defined by the action:

$$\mathcal{A}[\boldsymbol{\phi}] = \int \frac{d^2x}{8\pi}\sqrt{|g|}\left[\partial_\mu\boldsymbol{\phi}\cdot\partial^\mu\boldsymbol{\phi} + 2iR(x)\boldsymbol{Q}\cdot\boldsymbol{\phi} + {:}e^{i\boldsymbol{e}_1\cdot\boldsymbol{\phi}/b}{:} + {:}e^{i\boldsymbol{e}_2\cdot\boldsymbol{\phi}/b}{:}\right], \tag{3.50}$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2$ are the $\mathfrak{sl}_3$ roots, $\rho = \boldsymbol{e}_1 + \boldsymbol{e}_2$, and $\boldsymbol{Q} = (1/b - b)\boldsymbol{\rho}$. It was shown in [81] that the conserved currents for this action generate the W_3 algebra.

The central charge is given by

$$c = 2 - 12\boldsymbol{Q}^2\,. \tag{3.51}$$

The vertex operator $V_{\boldsymbol{\alpha}} = {:}e^{i\boldsymbol{\alpha}\cdot\boldsymbol{\phi}}{:}$ has eigenvalues for L_0 and W_0:

$$h_{\boldsymbol{\alpha}} = \frac{1}{2}\boldsymbol{\alpha}\cdot(\boldsymbol{\alpha} - 2\boldsymbol{Q})\,, \tag{3.52}$$

$$w_{\boldsymbol{\alpha}} = \sqrt{\frac{48}{22+5c}}\prod_{j=1}^{3}\left[(\boldsymbol{\alpha}-\boldsymbol{Q})\cdot\boldsymbol{h}_j\right]. \tag{3.53}$$

We let the Weyl group W act on vertex charges as

$$\forall x \in W\,, \qquad x\star\boldsymbol{\alpha} := \boldsymbol{Q} + x(\boldsymbol{\alpha}-\boldsymbol{Q})\,. \tag{3.54}$$

The eigenvalues are then invariant under this action of W:

$$\forall x \in W\,, \qquad (h_{x\star\boldsymbol{\alpha}}, w_{x\star\boldsymbol{\alpha}}) = (h_{\boldsymbol{\alpha}}, w_{\boldsymbol{\alpha}})\,. \tag{3.55}$$

The vertex charges associated to degenerate fields are of the form:

$$\boldsymbol{\alpha}\begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix} = \left[(1-n_1)b^{-1} - (1-m_1)b\right]\boldsymbol{\omega}_1 + \left[(1-n_2)b^{-1} - (1-m_2)b\right]\boldsymbol{\omega}_2\,, \tag{3.56}$$

with n_1, n_2, m_1, m_2 positive integers.

Structure constants for scalar operators

Consider the correlation function of scalar operators

$$\langle V_{\boldsymbol{\alpha}_1}(\infty)V_{\boldsymbol{\alpha}_2}(1)V_{\boldsymbol{\alpha}_3}(z,\bar{z})\Phi_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}(0)\rangle\,. \tag{3.57}$$

If one of the vertex charges, say $\boldsymbol{\alpha}_3$, is of the form $\boldsymbol{\alpha}_3 = \kappa\boldsymbol{\omega}_1$, then the null-vector condition on $\Phi_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}$ translates into a generalised hypergeometric differential equation on (3.57), and the conformal bootstrap approach can be applied [83, 84, 85]. The same can be done with $\Phi_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$. Combining the resulting shift equations, one gets the unique solution [63]:

$$C_{\mathrm{IT}}(\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_2,\kappa\boldsymbol{\omega}_1) = M(\kappa)\times\frac{\prod_{k,\ell=1}^{3}\Upsilon_b[b-(\boldsymbol{\alpha}_1-\boldsymbol{Q})\cdot\boldsymbol{h}_k-(\boldsymbol{\alpha}_2-\boldsymbol{Q})\cdot\boldsymbol{h}_\ell+\kappa/3]}{\sqrt{\prod_{i=1}^{2}\prod_{\boldsymbol{e}>0}\Upsilon_b[b+(\boldsymbol{\alpha}_i-\boldsymbol{Q})\cdot\boldsymbol{e}]\Upsilon_b[b-(\boldsymbol{\alpha}_i-\boldsymbol{Q})\cdot\boldsymbol{e}]}}\,, \tag{3.58}$$

where the product in the denominator is over positive roots $\boldsymbol{e}\in\{\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{\rho}\}$, and the normalising factor is

$$M(\kappa) = \frac{1}{\Upsilon_b(b)^3}\sqrt{\frac{\Upsilon_b(b)\Upsilon_b(3b^{-1}-2b)}{\Upsilon_b(b+\kappa)\Upsilon_b(3b^{-1}-2b-\kappa)}}\,. \tag{3.59}$$

Consistency of non-scalar OPEs

Like in the case of imaginary Liouville, one may consider non-scalar vertex operators $V_{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}}$, and examine the consistency of their OPEs with the degenerate operators $\Phi_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}$ and $\Phi_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}$. Following an analogous argument to Sec. 3.2.1, we find that the vertex operators are labelled by a pair of vertex charges $(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}})$, together with a permutation $\sigma\in\mathfrak{S}_3$, subject to the constraints:

$$\boldsymbol{\alpha}-\bar{\boldsymbol{\alpha}}\in\mathcal{R}^*/b\,,\qquad \boldsymbol{\alpha}-\sigma\star\bar{\boldsymbol{\alpha}}\in b\mathcal{R}^*\,, \tag{3.60}$$

where $\mathcal{R}^*$ is the $\mathfrak{sl}_3$ weight lattice defined above. The permutation σ characterises the fusion rules as follows:

$$\Phi_{\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)}\times V^{(\sigma)}_{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}} = \sum_{j=1}^{3} V^{(\sigma)}_{\boldsymbol{\alpha}+b\boldsymbol{h}_j,\bar{\boldsymbol{\alpha}}+b\boldsymbol{h}_j}\,, \tag{3.61}$$

$$\Phi_{\left(\begin{smallmatrix}2&1\\1&1\end{smallmatrix}\right)}\times V^{(\sigma)}_{\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}}} = \sum_{j=1}^{3} V^{(\sigma)}_{\boldsymbol{\alpha}-\boldsymbol{h}_{\sigma(j)}/b,\bar{\boldsymbol{\alpha}}-\boldsymbol{h}_j/b}\,. \tag{3.62}$$

The reparameterisation $(\boldsymbol{\alpha},\bar{\boldsymbol{\alpha}},\sigma)\to(\mu\star\boldsymbol{\alpha},\mu\star\bar{\boldsymbol{\alpha}},\mu\sigma\mu^{-1})$ leaves the constraint (3.60) and the rules (3.61–3.62) invariant, so the vertex operators are classified by the conjugacy classes of σ in $\mathfrak{S}_3$:

- The case $\sigma=\mathbf{1}$ corresponds to scalar operators $\boldsymbol{\alpha}=\bar{\boldsymbol{\alpha}}\in\mathbb{R}^2$.

- When σ is a cyclic permutation, say $\sigma = (123)$, we get vertex charges of the form:

$$\boldsymbol{\alpha} = \boldsymbol{\alpha} \begin{pmatrix} n_1 & m_1 \\ n_2 & m_2 \end{pmatrix}, \qquad \bar{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \begin{pmatrix} n_2 & m_1 \\ -n_1 - n_2 & m_2 \end{pmatrix}, \tag{3.63}$$

  where $(n_1, n_2, m_1, m_2) \in (\mathbb{Z}/3)^4$, with $n_1 - m_1 \in \mathbb{Z}$ and $n_2 - m_2 \in \mathbb{Z}$.

- When σ is a transposition, say $\sigma = (12)$, the constraints (3.60) translate into

$$\boldsymbol{\alpha} = \boldsymbol{Q} + \beta\, \boldsymbol{h}_3 + (-r/b + sb)\, \boldsymbol{e}_1\,, \tag{3.64}$$
$$\bar{\boldsymbol{\alpha}} = \boldsymbol{Q} + \beta\, \boldsymbol{h}_3 + (+r/b + sb)\, \boldsymbol{e}_1\,, \tag{3.65}$$

  with $\beta \in \mathbb{R}$ and $(r, s) \in (\mathbb{Z}/2)^2$. This is a mixed situation, where the $\boldsymbol{h}_3$ component of the charge is unconstrained, whereas the $\boldsymbol{e}_1$ component is quantised.

Structure constants of non-scalar operators

From the analysis of the four-point functions

$$\langle V_{\boldsymbol{\alpha}_1,\bar{\boldsymbol{\alpha}}_1} V_{\boldsymbol{\alpha}_2,\bar{\boldsymbol{\alpha}}_2} V_{\boldsymbol{\alpha}_3,\bar{\boldsymbol{\alpha}}_3} \Phi \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \rangle \qquad \text{and} \qquad \langle V_{\boldsymbol{\alpha}_1,\bar{\boldsymbol{\alpha}}_1} V_{\boldsymbol{\alpha}_2,\bar{\boldsymbol{\alpha}}_2} V_{\boldsymbol{\alpha}_3,\bar{\boldsymbol{\alpha}}_3} \Phi \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \rangle\,, \tag{3.66}$$

with $\boldsymbol{\alpha}_3 = \kappa\boldsymbol{\omega}_1, \bar{\boldsymbol{\alpha}}_3 = \bar{\kappa}\boldsymbol{\omega}_1$, one gets shift equations on the structure constants

$$C(V_{\boldsymbol{\alpha}_1,\bar{\boldsymbol{\alpha}}_1}, V_{\boldsymbol{\alpha}_2,\bar{\boldsymbol{\alpha}}_2}, V_{\kappa\boldsymbol{\omega}_1,\bar{\kappa}\boldsymbol{\omega}_1})\,. \tag{3.67}$$

Like in the non-diagonal imaginary Liouville case, if we further impose that one of the fields is scalar, we get the simple result:

$$C(V_{\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_1}, V_{\boldsymbol{\alpha}_2,\bar{\boldsymbol{\alpha}}_2}, V_{\kappa\boldsymbol{\omega}_1,\bar{\kappa}\boldsymbol{\omega}_1}) = \sqrt{C_{\mathrm{IT}}(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \kappa\boldsymbol{\omega}_1) C_{\mathrm{IT}}(\boldsymbol{\alpha}_1, \bar{\boldsymbol{\alpha}}_2, \bar{\kappa}\boldsymbol{\omega}_1)}\,. \tag{3.68}$$

Conclusion

On the lattice, as far as discrete holomorphic parafermions are concerned, the series of papers [43, 44, 45] introduced a generic method for constructing these objects in many types of integrable lattice models, using the underlying quantum group symmetry. This was an important improvement over previous studies, where the lattice holomorphic parafermions were found empirically by several authors [39, 36, 37, 38, 42, 89, 90]. It is a also a potential starting point for the systematic construction of more general discrete parafermions in integrable lattice models. However, despite some substantial effort, the "missing half" of the Cauchy-Riemann equations (see discussion in the Introduction), which would open the way to mathematical proofs of convergence and conformal invariance in a variety of lattice models, has not yet been found. One promising direction for solving this problem could be to generalise the notion of S-embedding [41] to other models than Ising.

For the problem of scaling correlation functions in critical models, with the works [61, 58, 91, 63, 92], we have contributed to a better understanding of the operator algebra in non-rational CFTs. Some major aspects of these works are: the adaptation of the analytical conformal bootstrap approach [55] to the case of non-scalar primary operators. This applies particularly to the O(n) loop model, which has an infinity of such operators. The major remaining challenge is to take into account the indecomposability of Virasoro modules associated to some sectors of the O(n) model – leading to Logarithmic Conformal Field Theory (LCFT): see [93] for a review. Some work in this direction has been initiated in [94, 95] for Ising and critical percolation in the case of boundary correlation functions; the logarithmic behaviour of bulk correlation functions has also been addressed recently in [96, 97].

Another interesting perspective would be to describe systematically the operator algebra and the related structure constants for the replicated CFTs [98] arising in the study of entanglement entropies [99]. Indeed, in this context we have shown [100] how to exploit the null-vector equations to apply the conformal bootstrap and derive the four-point functions related to Rényi entropies in minimal models. I think a more thorough analysis of the operator algebra for these theories is possible, and would yield interesting results on these entropies.

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