Topics in string amplitudes

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Outline: 1. Introduction

Introduction

Two-point amplitude

Crossing symmetry: QFT

Crossing symmetry: string theory

Conclusion
Properties of string theory

String theory = theory of extended objects

▷ consistency? (unitarity, crossing symmetry...)  
▷ differences with local point-particle QFT?  
▷ non-locality?
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- differences with local point-particle QFT?
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Point-particle QFT
- consistency assessed from $S$-matrix
- locality $\sim$ analyticity of $S$-matrix
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  ▶ locality $\sim$ analyticity of $S$-matrix

String theory properties

1. if possible, direct proof
2. otherwise, prove property consequence $\rightarrow$ indirect test
Properties of string theory

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- differences with local point-particle QFT?
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Natural framework: string field theory (off-shell, renormalization...
Properties of (super)string amplitudes:

1. Tree-level 2-point amplitude  
   with: Juan Maldacena, Dimitri Skliros [1906.06051]

2. Analyticity and crossing symmetry at all loops  
   with: Corinne de Lacroix, Ashoke Sen [1810.07197]
Outline: 2. Two-point amplitude

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Crossing symmetry: string theory

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- QFT

\[ A_2(k, k') = 2k^0 (2\pi)^{D-1} \delta^{(D-1)}(k - k') \]

(1-particle state normalization, cluster decomposition)
2-point amplitude

- **QFT**
  
  $$A_2(k, k') = 2k^0(2\pi)^{D-1} \delta^{(D-1)}(k - k')$$
  
  (1-particle state normalization, cluster decomposition)

- **string theory**

  $$A_2 \sim \frac{1}{\text{Vol } SL(2, \mathbb{C})} \int d^2z d^2z' \langle V_k(z, \bar{z}) V_{k'}(z', \bar{z}') \rangle_{S^2}$$
  
  $$\sim \frac{1}{\text{Vol } \mathbb{R}_+} \langle V_k(\infty, \infty) V_{k'}(0, 0) \rangle_{S^2}$$
2-point amplitude

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A_2(k, k') = 2k^0(2\pi)^{D-1} \delta^{(D-1)}(k - k')
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(1-particle state normalization, cluster decomposition)

- **String theory** (standard lore)

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A_2 \sim \frac{1}{\text{Vol} \ SL(2, \mathbb{C})} \int d^2z d^2z' \langle V_k(z, \bar{z})V_{k'}(z', \bar{z}') \rangle_{S^2}
\]

\[
\sim \frac{1}{\text{Vol} \ \mathbb{R}_+} \langle V_k(\infty, \infty)V_{k'}(0, 0) \rangle_{S^2} = 0
\]

BRST point of view: need \( N_{gh} = 6 \) but only 2 operators \( c\bar{c}V \)
2-point amplitude

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QFT result is universal \( \rightarrow \) how to resolve contradiction?
2-point amplitude

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  BRST point of view: need \( N_{gh} = 6 \) but only 2 operators \( c \bar{c} V \)

QFT result is **universal** → how to resolve contradiction?

\[ \langle V_k(\infty, \infty) V_{k'}(0, 0) \rangle_{S^2} \propto \delta(0) \delta^{(D-1)}(k - k') = \infty \]

from on-shell + momentum conservation

→ ambiguous, need regularization / better gauge fixing
Gauge-fixed amplitude

- 2-point amplitude

\[ A_{0,2}(k, k') = \frac{8\pi \alpha'^{-1}}{\text{Vol} \, \mathcal{K}_0} \int d^2 z d^2 z' \langle V_k(z, \bar{z}) V_{k'}(z', \bar{z}') \rangle_{S^2} \]

\[ \mathcal{K}_0 := \text{PSL}(2, \mathbb{C}) \]
Gauge-fixed amplitude

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- simple gauge-fixing

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A_{0,2}(k, k') = \frac{8\pi\alpha'^{-1}}{\text{Vol } \mathcal{K}_2} \langle V_k(\infty, \infty) V_{k'}(0, 0) \rangle_{S^2}
\]

\[\mathcal{K}_2 := \text{U}(1) \times \mathbb{R}_+ = \text{dilatation} \times \text{rotation}\]
Gauge-fixed amplitude

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- evaluate CFT correlation function + regularize zero-modes

\[ A_2(k, k') = \lim_{\kappa^0 \to 0} (2\pi)^{D-1} \delta^{(D-1)}(k + k') \frac{16\pi^2 i \delta(\kappa^0)}{\alpha' \text{Vol} \mathcal{K}_2} \]

Normalization: \[ \langle V_k(z, \bar{z}) V_{k'}(z', \bar{z'}) \rangle_{S^2} = \frac{i (2\pi)^D \delta^D(k + k')}{|z - z'|^4}. \]

numerator = zero-modes \( e^{i(k + k') \cdot x} \) for Lorentzian target spacetime
Compute CKV volume (1)

Volume regularization

\[
\text{Vol} \mathcal{K}_2 = \int \frac{d^2 z}{|z|^2} = 2 \int_0^{2\pi} d\theta \int_0^\infty \frac{dr}{r} = 4\pi \int_\infty^{-\infty} d\tau = 4\pi \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} d\tau e^{i\varepsilon \tau}
\]

\[
\text{Vol}_\varepsilon \mathcal{K}_2 = 8\pi^2 \delta(\varepsilon)
\]

(\tau, \varepsilon) Euclidean worldsheet (time, energy) on the cylinder (dimensionless)

problem: Lorentzian spacetime, dimensionful energy

\[\rightarrow\] need Wick rotation and rescaling
Compute CKV volume (2)

1. worldsheet Wick rotation

\[ \tau = it, \quad \varepsilon = -iE \]

2. Lorentzian regularized volume

\[ \text{Vol}_{M,E} \mathcal{K}_2 = 8\pi^2 i \delta(E) \]

3. Lorentzian mode expansion

\[ X^0 = x^0 + \alpha' k^0 t \]

4. scale between spacetime and worldsheet times / energies

\[ t = \frac{\xi^0}{\alpha' k^0} \quad \Rightarrow \quad E = \alpha' k^0 \kappa^0 \]

\((\xi^0, \kappa^0)\) dimensionful worldsheet variables

5. regularized Lorentzian volume

\[ \text{Vol}_{M,\kappa^0} \mathcal{K}_2 = \frac{8\pi^2 i \delta(\kappa^0)}{\alpha' k^0} \]
Result

\[
A_2(k, k') = \lim_{\kappa^0 \to 0} (2\pi)^{D-1} \delta^{(D-1)}(k + k') \frac{16\pi^2 i \delta(\kappa^0)}{\alpha' \text{Vol}_{M,\kappa^0} K_2}
\]

\[
\text{Vol}_{M,\kappa^0} K_2 = \frac{8\pi^2 i \delta(\kappa^0)}{\alpha' k^0}
\]

Recover QFT result:

\[
A_2(k, k') = 2k^0 (2\pi)^{D-1} \delta^{(D-1)}(k + k')
\]
Result

\[ A_2(k, k') = \lim_{\kappa^0 \to 0} (2\pi)^{D-1} \delta^{(D-1)}(k + k') \frac{16\pi^2 i \delta(\kappa^0)}{\alpha' \text{Vol}_{M, \kappa^0} \mathcal{K}_2} \]

\[ \text{Vol}_{M, \kappa^0} \mathcal{K}_2 = \frac{8\pi^2 i \delta(\kappa^0)}{\alpha' k^0} \]

Recover QFT result:

\[ A_2(k, k') = 2k^0 (2\pi)^{D-1} \delta^{(D-1)}(k + k') \]

Remarks:

▶ regularization ambiguous → fixed from unitarity
▶ can **always** insert 6 ghosts, for example (derivation below):

\[ A_2(k, k') = \frac{8\pi \alpha'^{-1}}{\text{Vol} \mathcal{K}_2} \langle c \bar{c} V_k(\infty, \infty) c_0 \bar{c}_0 \ c \bar{c} V_{k'}(0, 0) \rangle_{S^2} \]

using \( \langle 0 | c_{-1} \bar{c}_{-1} c_0 \bar{c}_0 c_{1} \bar{c}_1 | 0 \rangle = 1 \)

▶ operator approach (open string) [1909.03672, Seki-Takahashi]

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Improved gauge fixing (1)

- improved gauge fixing: no need for regularization
  → no ambiguity

- idea: gauge fix $X^0$, i.e. just fix another object transforming under $SL(2, \mathbb{C})$

- $SL(2, \mathbb{C})$ transformation

\[
\delta z = \beta + \alpha z + \gamma z^2
\]

\[
\delta X(z, \bar{z}) = \delta z \partial X(z, \bar{z}) + \delta \bar{z} \bar{\partial} X(z, \bar{z})
\]
Improved gauge fixing (2)

Procedure (for closed string, unpublished):

1. gauge fix the two vertices \((z^0 \rightarrow \infty)\):

   \[
   f_1 = z - z^0, \quad \bar{f}_1 = \bar{z} - \bar{z}^0, \quad f_2 = z', \quad \bar{f}_2 = \bar{z}'
   \]

   residual group: \(\mathcal{K}_2 := U(1) \times \mathbb{R}_+\)

2. introduce new coordinate system

   \[
   z = r \, e^{i\sigma}, \quad \delta r = \lambda r, \quad \delta \sigma = \theta, \quad \alpha := \lambda e^{i\theta}
   \]

   \[
   \delta X(r, \theta) = \lambda r \, \partial_r X(r, \theta) + \theta \, \partial_\sigma X(r, \theta)
   \]

3. enforce level-matching condition: introduce identity for new coordinate \(\tilde{z} = (0, \tilde{\sigma})\) and gauge fix:

   \[
   1 = \frac{1}{2\pi} \int_0^{2\pi} d\tilde{\sigma}, \quad f_3 = \tilde{\sigma}
   \]
Improved gauge fixing (3)

4. gauge fix $X^0$ (need rotation invariant condition)

$$f_4 = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \ X^0(r, \sigma)$$

5. Faddeev–Popov procedure:

$$\Delta(z^0, \bar{z}^0, X^0) = |z^0|^4 \int \frac{d\sigma}{2\pi} r \partial_r X^0 = \alpha' \hat{p}^\mu |z^0|^4$$

6. insert $\delta(f_1)$, etc., and $\Delta$ in amplitude:

$$A_2(k, k') = \frac{8\pi \alpha'^{-1}}{4\pi} |z^0|^4 \left\langle \delta(x^0) \hat{p}^0 V_k(z^0, \bar{z}^0) V_{k'}(0, 0) \right\rangle_{S^2}$$

$$= 2k^0 (2\pi)^{D-1} \delta^{(D-1)}(k - k')$$
Improved gauge fixing: ghosts

1. BV: introduce ghost-antighost for residual symmetry
to $\forall f_i$, introduce $(\alpha_i, a_i)$ and regulator $R_i$

\[ Q\alpha_i = a_i, \quad Qa_i = 0, \quad R_i := e^{iQ(\alpha_i f_i)} = e^{ia_i f_i - i\alpha_i Qf_i} \]

$\alpha_i$ fermionic, $a_i$ bosonic, amplitude invariant since $QV_k = 0$
[Marnelius-Ogren '91; hep-th/0503038, Craps-Skenderis; Berkovits, unpublished]

2. integrate over $(\alpha_i, a_i)$

\[ \int da_i d\alpha_i R_i = \delta(f_i) \delta_B f_i \]

3. BRST variation

\[ \delta_B \mathcal{O}(z, \bar{z}) = c(z) \delta_z \mathcal{O}(z, \bar{z}) + \bar{c}(\bar{z}) \delta_{\bar{z}} \mathcal{O}(z, \bar{z}) \]
Improved gauge fixing: ghosts (2)

4. vary the gauge-fixing conditions:

\[ \delta(f_1) \delta_B f_1 = c(z^0) \rightarrow c_{-1}, \quad \delta(f_2) \delta_B f_2 = c(z = 0) \rightarrow c_1 \]
\[ \delta(f_3) \delta_B f_3 = c(\tilde{\sigma} = 0) - \bar{c}(\tilde{\sigma} = 0) \rightarrow 2c_0^- \]

\[ \delta(f_4) \delta_B f_4 = \delta(x^0) \int \frac{d\sigma}{2\pi} \left( c(\sigma) + \bar{c}(\sigma) \right) r \partial_r X^0 \rightarrow \alpha' \hat{p}^0 c_0^+ \]

5. plug in the amplitude

\[ A_2(k, k') = 4 \left\langle \delta(x^0) \hat{p}^0 c\bar{c} V_k(z^0, \bar{z}^0) c_{0}^- c_{0}^+ c\bar{c} V_{k'}(0, 0) \right\rangle_{S^2} \]

6. note: natural result from SFT
Zero-point amplitude

Next step

Generalization to 0-point function \( \rightarrow \) compute on-shell action

\[ A_0[\mathcal{M}] \sim \frac{\delta^{(D)}(0)}{\text{Vol SL}(2, \mathbb{C})} = \infty \]

- zero-point amplitude for Minkowski spacetime \( \mathcal{M} \):
Zero-point amplitude

Next step

Generalization to 0-point function $\rightarrow$ compute on-shell action

- zero-point amplitude for Minkowski spacetime $\mathcal{M}$:

$$A_0[\mathcal{M}] \sim \frac{\delta^{(D)}(0)}{\text{Vol} \, \text{SL}(2, \mathbb{C})} \equiv \infty$$

- (curved) background $X$:

$$e^{-(S_{\text{EH}}[X] - S_{\text{EH}}[\mathcal{M}] )} = \frac{A_0[X]}{A_0[\mathcal{M}]} \equiv \text{finite}$$

(à la Gibbons–Hawking–York)

- consider $X = \text{black hole, Rindler space}$?
Outline: 3. Crossing symmetry: QFT

Introduction

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Crossing symmetry: QFT

Crossing symmetry: string theory

Conclusion
Analyticity and crossing symmetry

Analyticity of \( n \)-point amplitude \( A_n(k_1, \ldots, k_n) \)

- starting point for other properties (crossing symmetry, dispersion relations)
- related to locality and causality
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Crossing symmetry:
- relations between amplitudes with exchange of particles/anti-particles in initial/final states
- often assumed or observed (scattering amplitude program...)

Why a general proof?
- ensure observed examples not accident of simple amplitudes
- learn about fundamental properties of QFT
Analyticity and crossing symmetry

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Method

Idea of proof in QFT [Bros-Epstein-Glaser, ’64-65]:

1. prove analyticity of S-matrix in “primitive domain” $\Delta$ from locality
2. analytic extension $\mathcal{H}(\Delta)$
3. show that 2) $\Rightarrow$ crossing symmetry
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Idea of proof in QFT [Bros-Epstein-Glaser, ’64-65]:

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   from locality

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Remarks:

- 1) is non-perturbative (full S-matrix)
- 2) and 3) are **general statements** from theory of several complex variables
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String theory:

- non-local interactions $\rightarrow$ no position space Green functions
- prove 1) perturbatively from Feynman diagrams
Amplitude and Green functions

4-point scattering process

\[ p_a = (E_a, p_a) \in \mathbb{C}, \ a = 1, \ldots, 4: \text{external momenta} \]

\[ \text{momentum conservation: } p_1 + \cdots + p_4 = 0 \]

\[ \text{on-shell condition: } p_a^2 = -m_a^2 \]
Amplitude and Green functions

4-point scattering process

- $p_a = (E_a, \mathbf{p}_a) \in \mathbb{C}$, $a = 1, \ldots, 4$: external momenta
- momentum conservation: $p_1 + \cdots + p_4 = 0$
- on-shell condition: $p_a^2 = -m_a^2$

Green functions:

off-shell $G(p_1, \ldots, p_4) =$

truncated $\tilde{G}(p_1, \ldots, p_4) = G(p_1, \ldots, p_4) \prod_{a=1}^{4} (p_a^2 + m_a^2)$

on-shell $A(p_1, \ldots, p_4) = \lim_{p_a^2 \to -m_a^2} \tilde{G}(p_1, \ldots, p_4)$

QFT: $G =$ sum of Feynman diagrams
Physical amplitudes

Mandelstam variables

\[ s = - (p_1 + p_2)^2, \quad t = - (p_1 + p_3)^2, \quad u = - (p_1 + p_4)^2 \]

mass-shell: \( s + t + u = \sum_a m_a^2 \)
Physical amplitudes

Mandelstam variables

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mass-shell: \( s + t + u = \sum_a m_a^2 \)

Physical regions

- **S (s-channel):** \( s \geq \sum_a m_a^2, \quad t, u \leq 0 \)
- **T (t-channel):** \( t \geq \sum_a m_a^2, \quad s, u \leq 0 \)
- **U (u-channel):** \( u \geq \sum_a m_a^2, \quad s, t \leq 0 \)

Physical amplitudes

\[ A_{S,T,U}(p_1, \ldots, p_4) = \lim_{p_a \in S, T, U} A(p_1, \ldots, p_4) \]
Mandelstam plane

\[ p_a \in \mathbb{R} \text{ on-shell} \]
Statement of crossing symmetry

**Crossing symmetry**

\[
S: \ 1 + 2 \rightarrow 3 + 4
\]

The processes

\[
T: \ 1 + 3 \rightarrow \bar{2} + 4 \quad \text{(and CPT conjugates) are}
\]

\[
U: \ 1 + \bar{4} \rightarrow 3 + \bar{2}
\]
equivalent under analytic continuation on the complex mass-shell

\[
A_S(s, t) = A_T(t, s), \quad A_S(s, u) = A_U(u, s)
\]
Statement of crossing symmetry

Crossing symmetry

\[ S : 1 + 2 \rightarrow 3 + 4 \]

The processes

\[ T : 1 + \bar{3} \rightarrow \bar{2} + 4 \quad \text{(and CPT conjugates)} \]

\[ U : 1 + \bar{4} \rightarrow 3 + \bar{2} \]

equivalent under analytic continuation on the complex mass-shell

\[ A_S(s, t) = A_T(t, s), \quad A_S(s, u) = A_U(u, s) \]

- looks natural from LSZ: \( A_S, T, U \) all come from a single function \( A \)
- but: **not guaranteed** that \( A \) is analytic in a domain with paths between \( S, T, U \)
QFT proof (1)

Outline of proof [Bros-Epstein-Glaser ’64-65][Bros ’86]:

1. assumptions: $m^2_a > 0$, asymptotic states = stable particles
2. define the “primitive domains”

$$
\Delta_k = \bigcap_{A_\alpha} \left[ \left\{ \text{Im } P(\alpha) \neq 0, (\text{Im } P(\alpha))^2 \leq 0 \right\} \cup \left\{ \text{Im } P(\alpha) = 0, -P^2(\alpha) < M^2_\alpha \right\} \right.
\left. \cap \left\{ \text{Im } p^i_a = 0, i = k, \ldots, D - 1 \right\} \right]
$$

$A_\alpha \subset \{1, \ldots, n\}$, $P(\alpha) = \sum_{a \in A_\alpha} p_a$, $M_\alpha$: production threshold

In words: $p_a$ with $k$ possible complex components s.t. all $P_\alpha$ have:
1) non-zero imaginary timelike part, or 2) real momentum squared below multi-particle threshold in channel $A_\alpha$
QFT proof (2)

3. prove analyticity inside $\Delta_D$ of S-matrix from locality / micro-causality (fields commute at spacelike separations) [Araki, Burgoyne, Ruelle, Steimann, ’60-61]

problem: $\Delta_D \cap \text{mass-shell} = \emptyset$

4. compute the “envelope of holomorphy” $\mathcal{H}(\Delta_2)$ (= analytic extension)

$\rightarrow \mathcal{H}(\Delta_2) \cap \text{mass-shell} \neq \emptyset$

5. show $\exists$ a path in $\mathcal{H}(\Delta_2) \cap \text{mass-shell}$ between all pairs of $i\epsilon$-neighbourhoods of physical regions

Notes:
- $\mathcal{H}(\Delta_2)$ is necessary
- 4) and 5) $\Leftarrow$ theory of several complex variables only
- work with the complete S-matrix
QFT proof (2)

3. prove analyticity inside $\Delta_D$ of S-matrix from locality / micro-causality (fields commute at spacelike separations)  
   [Araki, Burgoyne, Ruelle, Steimann, '60-61]  
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5. show $\exists$ a path in $\mathcal{H}(\Delta_2) \cap \text{mass-shell}$ between all pairs of $i\epsilon$-neighbourhoods of physical regions

Notes:

- only $\mathcal{H}(\Delta_2)$ is necessary
- 4) and 5) $\Leftarrow$ theory of several complex variables only
- work with the complete S-matrix
Analyticity from locality: example

▶ micro-causality: fields commute at spacelike separation

\[ [\phi(x), \phi(x')] = 0, \quad (x - x')^2 > 0 \]

▶ relation Feynman propagator and commutator

\[ \text{Re} \ G_F(x, x') = 2i \text{sign}(t - t') \langle 0 | [\phi(x), \phi(x')] | 0 \rangle \]

▶ 2-point Green function

\[ G(k) = \int d^d x \ e^{-i k x} G_F(x, 0) \neq 0 \quad \text{if} \ x^2 \leq 0 \ (\text{timelike}) \]

▶ exponential damping → analyticity in the primitive tube

\[ k \in \mathbb{R}^d + iV^+ \]

\[ V^+ : \text{future light-cone} \]
Analyticity from locality: general case

[Araki, Burgoyne, Ruelle, Steimann, ’60-61; Bros-Epstein-Glaser ’64]

▶ main idea: distribution support in $x \Leftrightarrow$ analyticity in $k$
▶ consider generalized advanced/retarded Green functions
▶ locality and micro-causality $\Rightarrow$ analyticity in primitive tubes

$$\text{Im } P(\alpha) \neq 0, \quad (\text{Im } P(\alpha))^2 \leq 0$$

▶ coincidence of some Green functions below mass threshold: no singularity $\rightarrow$ analyticity

$$\text{Im } P(\alpha) = 0, \quad -P^2(\alpha) < M^2_\alpha$$

▶ edge-of-the-wedge theorem: generalized advanced/retarded Green functions are boundary values from a unique function, analytic in the primitive domain $\Delta_D$
Proof that $\Delta_D \cap \text{mass-shell} = \emptyset$:

1. **complex mass-shell:**

   \[
   \text{Re } p_a \cdot \text{Im } p_a = 0, \quad (\text{Re } p_a)^2 - (\text{Im } p_a)^2 + m_a^2 = 0
   \]

2. if $\text{Im } p_a$ timelike, $(\text{Im } p_a)^2 \leq 0$, then need $\text{Re } p_a$ timelike, $(\text{Re } p_a)^2 < 0$, for 2nd condition, but violates 1st condition

3. if $\text{Im } p_a = 0$, then $-P_{(\alpha)}^2 \geq M_\alpha^2$
More on the envelope of holomorphy:

- consider \( f(z_1, \ldots, z_n) \) analytic in \( \Delta \)
- analyticity in several variables \( \Rightarrow \) constrain shape of \( \Delta \)
- if shape not arbitrary: analyticity in \( \Delta \) \( \Rightarrow \) analyticity in \( \mathcal{H}(\Delta) \)
- given \( \Delta \), \( \mathcal{H}(\Delta) \) is independent of \( f \)
- typically: use edge-of-the-wedge theorem (Bogoliubov)
Outline: 4. Crossing symmetry: string theory

Introduction

Two-point amplitude

Crossing symmetry: QFT

Crossing symmetry: string theory

Conclusion
String field theory

- field theory (second-quantization)
- rigorous, constructive formulation [hep-th/9206084, Zwiebach]
- make gauge invariance explicit ($\mathcal{L}_\infty$ algebras et al.)
- use standard QFT techniques (renormalization, analyticity...) → prove consistency (Cutkosky rules, unitarity, soft theorems, background independence...) [Sen ’14-19]
- study backgrounds (= classical solutions), marginal and RR fluxes deformations, instantons [1811.00032, Cho-Collier-Yin; Sen ’19-21]
- access collective, non-perturbative, thermal, dynamical effects (dream goal)
Shameless advertisement
SFT in a nutshell

\[ \text{SFT} = \text{standard QFT s.t.:} \]

- infinite number of fields (of all spins)
- infinite number of interactions
- non-local interactions \( \propto e^{-\#k^2} \)
- reproduce worldsheet amplitudes (if well-defined)

review: [1703.06410, De Lacroix-HE-Kashyap-Sen-Verma]
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- cannot use position representation
- cannot use assumptions from local QFT (micro-causality. . .)
- cannot derive analyticity like in QFT
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→ study Green function singularities from Feynman diagrams in momentum space
Action and Feynman diagrams

- gauge-fixed action

\[ S = \frac{1}{2} \langle \psi | c_0^- c_0^+ L_0^+ | \psi \rangle + \sum_{g,n \geq 0} \frac{\hbar g g_s^{2g-2+n}}{n!} \mathcal{V}_{g,n}(\Psi^n) \]
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▶ gauge-fixed action

\[ S = \frac{1}{2} \langle \psi | c_0^- c_0^+ L_0^+ | \psi \rangle + \sum_{g,n \geq 0} \frac{\hbar g g_s^{2g-2+n}}{n!} V_{g,n}(\Psi^n) \]

▶ propagator

\[ \langle A_1 | b_0^+ L_0^- b_0^- | A_2 \rangle = A_1 \quad \text{---} \quad A_2 \]

▶ fundamental $g$-loop $n$-point vertex

\[ V_{g,n}(A_1, \ldots, A_n) = A_1 \quad \text{---}^g \quad \ldots \quad A_n \]

defined s.t. sum of all graphs $\Rightarrow$ recover worldsheet amplitudes
Momentum representation (1)

- string field Fourier expansion

\[ |\psi\rangle = \sum_A \int \frac{d^D k}{(2\pi)^D} \phi_A(k) |A, k\rangle \]

\( k \): \( D \)-dimensional momentum
\( A \): discrete labels (Lorentz indices, group repr., KK modes. . .)

- 1PI action

\[ S = \frac{1}{2} \int d^D k \phi_A(k) K_{AB}(k) \phi_B(-k) \]

\[ + \sum_n \int d^D k_1 \cdots d^D k_n V_{A_1,\ldots,A_n}^{(n)}(k_1, \ldots, k_n) \phi_{A_1}(k_1) \cdots \phi_{A_n}(k_n) \]
Momentum representation (2)

Propagator

\[ K_{AB}(k)^{-1} = \frac{-i M_{AB}}{k^2 + m_A^2} Q_A(k) \]

- \( M_{AB} \) mixing matrix for states of equal mass
- \( Q_A \) polynomial
Momentum representation (3)

Vertices

\[-iV_{A_1,\ldots,A_n}^{(n)}(k_1,\ldots,k_n) = -i \int dt e^{-g_{ij}^{\{A_a\}}(t) k_i \cdot k_j - c \sum_{a=1}^{n} m_a^2} \times P_{A_1,\ldots,A_n}(k_1,\ldots,k_n; t)\]

- \( t \) moduli parameters
- \( P_{\{A_a\}} \) polynomial
- \( c > 0 \rightarrow \) damping in sum over states
- \( g_{ij} \) positive definite

- no singularity for \( k_i \in \mathbb{C} \) (finite)
- \( \lim_{k^0 \to \pm i\infty} V^{(n)} = 0 \)
- \( \lim_{k^0 \to \pm \infty} V^{(n)} = \infty \)
Green function

Truncated Green function = sum of Feynman diagrams of the form

\[
\mathcal{F}(p_1, \ldots, p_n) \sim \int dT \prod_s d^D \ell_s e^{-G_{rs}(T) \ell_r \cdot \ell_s - 2H_{ra}(T) \ell_r \cdot p_a - F_{ab}(T) p_a \cdot p_b} \\
\times \prod_i \frac{1}{k_i^2 + m_i^2} \mathcal{P}(p_a, \ell_r; T)
\]

\(T\), moduli parameters, \(\mathcal{P}\), polynomial in \((p_a, \ell_r)\)

▷ momenta:
  ▷ external \(\{p_a\}\)  
  ▷ internal \(\{k_i\}\)  
  ▷ loop \(\{\ell_s\}\)

\(k_i = \) linear combination of \(\{p_a, \ell_s\}\)

▷ \(G_{rs}\) positive definite
  ▷ integrations over spatial loop momenta \(\ell_r\) converge
  ▷ integrations over loop energies \(\ell_r^0\) diverge
Momentum integration

Prescription = generalized Wick rotation [1604.01783, Pius-Sen]:

1. define Green function for Euclidean internal/external momenta
2. analytic continuation of external energies + integration contour s.t.
   ▶ keep poles on the same side
   ▶ keep ends at ±i∞

→ analyticity for $p_a \in \mathbb{R}$, $p_a^0$ in first quadrant \( \text{Im } p_a^0 > 0, \text{Re } p_a^0 \geq 0 \)
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→ analyticity for \( p_a \in \mathbb{R}, \ p_0^a \) in first quadrant \( \text{Im} \ p_0^a > 0, \text{Re} \ p_0^a \geq 0 \)

▶ Cutkosky rules, unitarity, spacetime and moduli space
  \( i\epsilon \)-prescriptions [Pius, Sen]
▶ timelike Liouville theory [1905.12689, Bautista-Dabholkar-HE]
Analyticity for string theory (1)

Result

Analyticity inside $\Delta_2$ of $n$-point superstring Green functions at all loop orders:

- implies crossing symmetry for $n = 4$
- identical analyticity properties for QFT and string theory

Comments:

- Feynman graphs $\rightarrow$ perturbative computations
- valid for states with any spin
- technical assumptions: mass gap, stable external states
- regularization of massless states: removes IR non-analyticity (identical to QFT)

$[2009.03375$, Bhattacharya-Mahanta$]$: analytic extension to $\Delta^D$ for 3- and 4-point functions
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- [2009.03375, Bhattacharya-Mahanta]: analytic extension to $\Delta_D$ for 3- and 4-point functions
Method to study singularity:

1. start with some $p_a = p_a^{(1)}$, $\ell_r^0 \in i\mathbb{R}$, $\ell_r \in \mathbb{R}$ s.t. no singularity
2. find a path $p_a = p_a^{(1)} \rightarrow$ desired $p_a = p_a^{(2)}$
3. deform the integral contour as the poles move
4. assume $\exists$ singularity $=$ on-shell internal propagator
   pinching $=$ collision of two poles from opposite sides
5. analyze reduced diagram, display an inconsistency
Analyticity for string theory (2)

Method to study singularity:

1. start with some \( p_a = p_a^{(1)} \), \( \ell^0_r \in i\mathbb{R} \), \( \ell_r \in \mathbb{R} \) s.t. no singularity
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4. assume \( \exists \) singularity = on-shell internal propagator
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Proceed by steps:

1. analyticity in \( \Delta_1 \): go from \( p_a = 0 \) to desired \( \text{Re } p_a \) and \( \text{Im } p_a^0 \)
   (keep \( \text{Im } p_a = 0 \))
2. analyticity in \( \Delta_2 \): go from \( p_a \in \Delta_1 \) to desired \( \text{Im } p_a^1 \) (keep
   \( \text{Im } p_a^i = 0 \) \( \forall i \geq 2 \))
First step

- \( p_a^0 \in \mathbb{C}, \ p_a \in \mathbb{R} \)
- pinching implies reduced graph:

\[ k_i^2 = -m_i^2, \text{ arrow = sign of } k_i^0 \]
- \( p_a, \ell_r \in \mathbb{R} \Rightarrow k_i \in \mathbb{R}, \text{ then } k_i^2 = -m_i^2 \Rightarrow k_i \in \mathbb{R} \)
- one can prove \( \forall i : k_i^0 > 0 \)
- implies

\[ P(\alpha) = \sum_{i} k_i \in \mathbb{R}, \quad k_i^2 = -m_i^2 \quad \Rightarrow \quad -P^2(\alpha) \geq M^2_{\alpha} \]

\( \rightarrow \) contradiction – one must have \( -P^2(\alpha) < M^2_{\alpha} \)
Second step

- \( p_a^\parallel = (p_0^a, p_1^a) \in \mathbb{C}, \ p_a^\perp \in \mathbb{R} \)
- pinching implies reduced graph:

\[
\begin{align*}
\text{arrow} &= \text{sign of } \text{Im} \ k_i^1 \\
\text{one can prove } \forall i : \text{Im} \ k_i^1 > 0, \ \text{and} \\
k_i^2 = -m_i^2 &\Rightarrow \text{Im} \ k_i^\parallel \in W^+ \Rightarrow \text{Im} \ P(\alpha) = \sum_i \text{Im} \ k_i \in W^+ \\
\text{→ contradiction – one must have } \text{Im} \ P(\alpha) \text{ timelike}
\end{align*}
\]
Outline: 5. Conclusion

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Two-point amplitude

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Conclusion
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Results:
- tree-level 2-point amplitude computation consistent with QFT
- analyticity of superstring $n$-point amplitudes in $\Delta_2$
- proof of crossing symmetry for 4-point superstring amplitudes at the same level as in QFT
- show that, in some sense, string theory behaves like local QFT
- new proof of analyticity valid for more general QFTs
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▶ tree-level 2-point amplitude computation consistent with QFT
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Outlook:
▶ tree-level 0-point function for generic background
▶ CPT theorem
▶ explore non-locality from SFT