On quantum mean-field models and their quantum annealing

Guilhem Semerjian

LPT-ENS

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joint work with Victor Bapst
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Outline

1. Introduction
   - Introduction to quantum annealing
   - Definition of the model
2. Static properties
   - Thermodynamic properties (and their consequences on the spectrum)
   - Computation of small gaps
   - Density of states
3. Dynamic properties
   - General behaviour
   - Exponentially large times scales
   - Constant times scales
4. Conclusion

Introduction

- Classical optimization problem : quantum 1/2 spins, Hamiltonian $\widehat{H}_P$ diagonal in $\otimes \widehat{\sigma}_z^i$ eigenbasis. One wants to find the ground-state of $\widehat{H}_P$.
- In particular any classical optimization problem on Ising spins can be written in this form. May be very hard to minimize (frustration).
- One can consider a more general operator of the Hilbert space, by adding a "kinetic energy" that induces quantum fluctuations (transverse field for instance)

- Quantum annealing : [Kadowaki, Nishimori 98, Farhi et al 01]

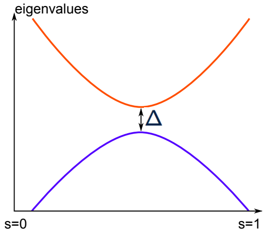
$$\widehat{H}(s) = (1-s)\sum_i \widehat{\sigma}_x^i + s\widehat{H}_P \qquad (\Gamma = (1-s)/s)$$

- Prepare the system in its ground-state at $s = 0$ (easy).
  Slowly increase s up to $s = 1 \rightarrow$ evolution following Schrödinger equation

$$\frac{i}{T}\frac{d\phi_T(s)}{ds} = \widehat{H}(s)|\phi_T(s)\rangle$$

- T is the evolution time. If T is large enough, the system should remain in its instantaneous ground-state at any time.
- How large T should be ?

- Adiabatic theorem (in a nutshell) : the total evolution time T must be large compared to Δ^{-2}, with Δ the minimal gap between the ground-state and the first-excited state during the evolution.



Scaling of T with the system size N ? Roughly:

- Second-order phase transition : $\Delta \propto 1/N^a$, $T = \text{poly}(N)$
- First-order phase transition : $\Delta \propto e^{-aN}$, $T = \exp[O(N)]$

For optimization problems :

- scaling of the adiabatic time (exact algorithm)
- residual energy for non-adiabatic evolutions

(approximation algorithm)

Definition of the model

- Toy model : mean-field fully-connected model for $\widehat{H}_P$.
- Depends only on average magnetizations

$$\widehat{m}^x = \frac{1}{N}\sum_{i=1}^{N} \widehat{\sigma}_i^x , \qquad \widehat{m}^z = \frac{1}{N}\sum_{i=1}^{N} \widehat{\sigma}_i^z$$

- p-spin ferromagnetic model :

$$\widehat{H}(s) = -Ns(\widehat{m}^z)^p - N(1-s)\widehat{m}^x$$

- not a hard optimization problem of course
- yet shares some of their phenomenology
- with much simpler analytical computations

related to Lipkin-Meshkov-Glick model [Ribeiro, Vidal, Mosseri, Filippone, Dusuel], [Jörg, Krzakala, Kurchan, Maggs, Pujos 10]

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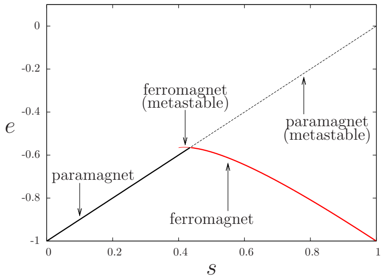
Thermodynamic properties

- Free-energy per spin can be computed exactly at all temperatures
- Groundstate energy per spin:

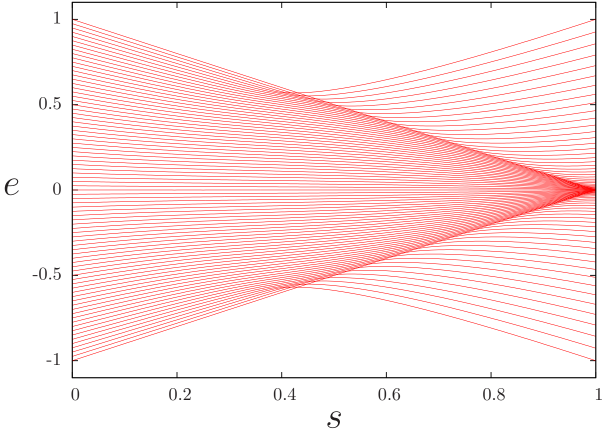
$$e_{gs}(s) = \inf_{m \in [-1,1]} \left[-s\, m^p - (1-s)\sqrt{1-m^2} \right]$$

- First-order phase transition ($p \geq 3$):



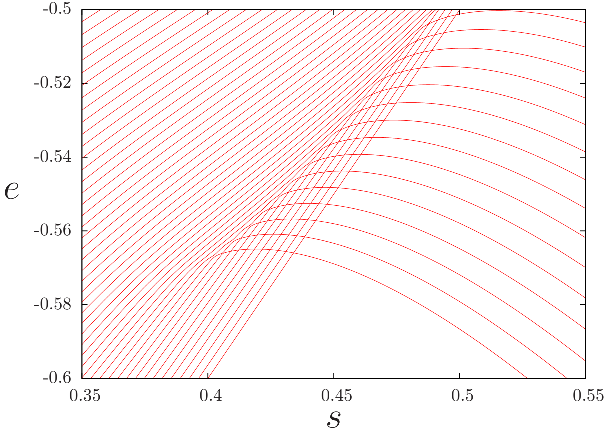
Consequences on the spectrum

- $\widehat{H}(s)$ commute with the total spin operator $\widehat{S}^2 \rightarrow$ block-diagonal

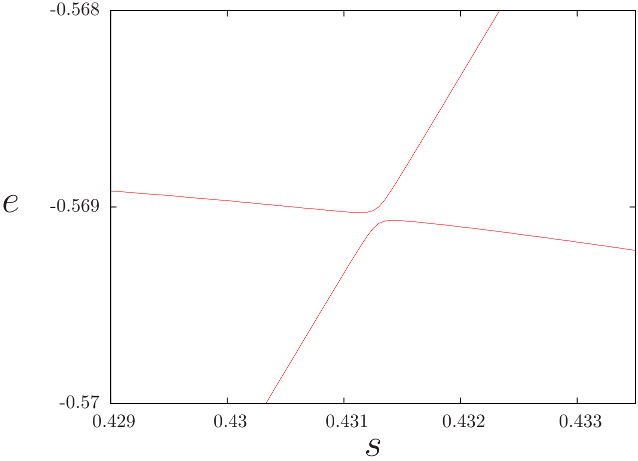


here the $N + 1$ dimensional block of maximal spin
(the relevant one for the dynamics)

- metastable continuations of the groundstates (zooming in)



- avoided crossing at the transition (zooming in more)



Computation of the gap at the transition (1/3)

- At the first-order phase transition,

$$\lim_{N\to\infty} \frac{1}{N} \log(gap) = \lim_{N\to\infty} \frac{1}{N} \log |\langle \phi_f | \phi_p \rangle|$$

  where $|\phi_f\rangle$ (resp $|\phi_p\rangle$) is the analytic continuation of the ferromagnetic (resp. paramagnetic) ground-state eigenvector [Krzakala, Kurchan et al.].
  $\rightarrow$ one needs to compute the eigenvectors.

- Action of $\widehat{S}^x$ on an eigenvector $|S, M\rangle_z$ of $\widehat{S}^2, \widehat{S}^z$:

$$\widehat{S}^x |S, M\rangle_z = \frac{1}{2}\sqrt{S(S+1) - M(M+1)}\, |S, M+1\rangle_z + \frac{1}{2}\sqrt{S(S+1) - M(M-1))}\, |S, M-1\rangle_z$$

Computation of the gap at the transition (2/3)

- Eigenvalue equation on $|\phi\rangle = \sum_m \phi(m)|m\rangle_z$:

$$
\begin{aligned}
e\,\phi(m) = -s\,m^p\,\phi(m) \quad &- \quad \frac{(1-s)}{2}\sqrt{1-m^2+\frac{2}{N}(1-m)}\;\phi\left(m+\frac{2}{N}\right) \\
&- \quad \frac{(1-s)}{2}\sqrt{1-m^2+\frac{2}{N}(1+m)}\;\phi\left(m-\frac{2}{N}\right)
\end{aligned}
$$

  One dimensional equation with m playing the role of a space coordinate.

- In the large N limit, semi-classical (WKB like) Ansatz:

$$
\phi(m) = e^{-N\varphi(m)}
$$

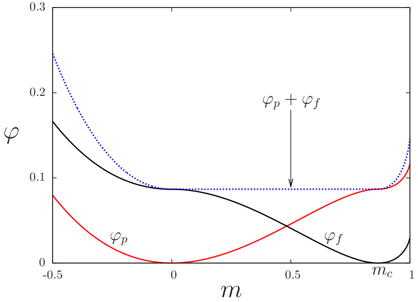
  (semi-classical was done before with instantons, and coherent states)

- The eigenvalue equation on $|\phi\rangle$ becomes a differential equation on φ:

$$
e = -sm^p - (1-s)\sqrt{1-m^2}\cosh\left(2\varphi'(m)\right)
$$

Computation of the gap at the transition (3/3)

- At the transition, two quasi-degenerate groundstates



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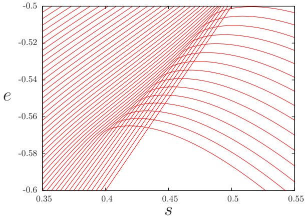
$$|\langle \phi_f | \phi_p \rangle| = \sum_m \langle \phi_f | m \rangle \langle m | \phi_p \rangle = \sum_m e^{-N(\varphi_p(m)+\varphi_f(m))}$$

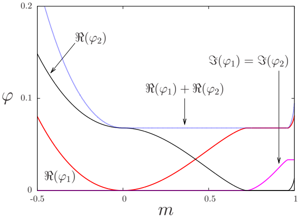
$$\lim_{N\to\infty} \frac{1}{N} \log(|\phi_f|\phi_p\rangle|) = \int_0^{m_c} |\varphi'(m, s_c, e_c)| dm$$

explicit formula for the exponential rate of the gap

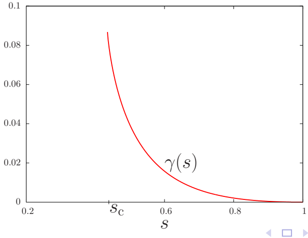
Computation of the gap along one metastable line

- This construction can be repeated to compute the gap along the metastable continuation of the paramagnetic ground-state.





- Gap on this line as a function of the interpolation parameter.



Computation of the density of states

- for excited states the large deviation function $\varphi(m)$ takes complex values
- this is indeed a "one-dimensional" problem: the ground-state wave function is real, the n-th excited state has n nodes
- this translates into [Ribeiro, Vidal, Mosseri 08]:

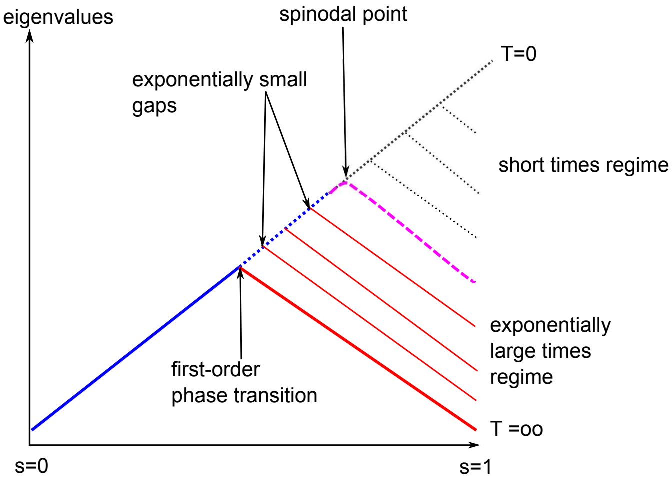
$$\mathcal{D}(s,e) = \frac{1}{\pi}\int_{-1}^{1} |\Im \varphi'(m,s,e)| dm$$

  for the integrated density of states
- eigenenergy $E_n(s)$ becomes an iso-$\mathcal{D}$ line
- allows also to compute finite gaps, and polynomially small gaps at second-order (or spinodal) transition points through matching arguments [Botet, Jullien 83]

Outline

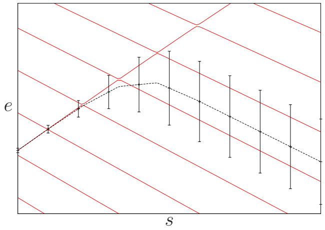
Behaviour under quantum annealing: general properties

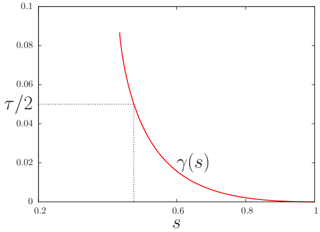
- A schematic view of annealing through a first-order phase transition:



Evolution on exponentially large times scale ($T = e^{\tau N}$)

- In the thermodynamic limit, exponentially small gaps can be seen as independant and the probability of excitation at a given crossing becomes 0 or 1.
- The condition $T \gg gap^{-2}$ is equivalent to $\tau = \frac{1}{N} \log T > -\frac{2}{N} \log gap$.
  This selects a turning point on the curve $\gamma(s)$.

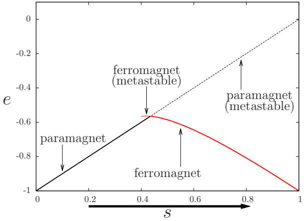


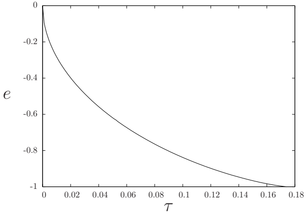


- After the turn, no more level crossings → conservation of the excitation (integrated density of states).

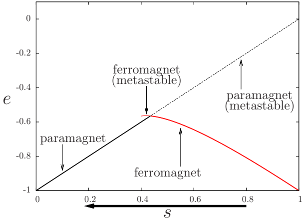
Evolution on exponentially large times scale ($T = e^{\tau N}$)

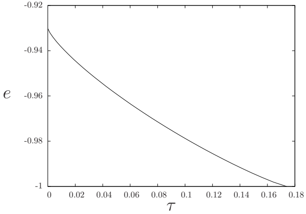
- Allows to compute exactly the final energy $e_{\text{fin}}(\tau)$ on this time-scale.





- One can also consider an annealing starting from the ferromagnet:





Evolution on exponentially large times scale ($T = e^{\tau N}$)

Variaous asymptotics can be computed for the extreme cases of this regime:

- Close to adiabaticity ($\tau \to \tau_{\rm adb}$),

$$e_{\rm fin}(\tau - \tau_{\rm adb}) - e_{\rm GS} \propto \frac{\tau_{\rm adb} - \tau}{|\ln(\tau_{\rm adb} - \tau)|}$$

- In the opposite limit, in the presence of a spinodal point, $e_{\rm fin}(\tau)$ has a non-trivial limit $\hat{e}_{\rm fin}$ when $\tau \to 0$, and one has:

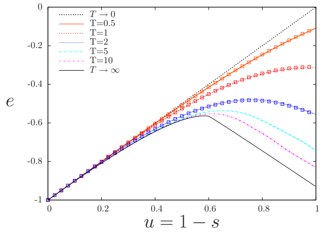
$$e_{\rm fin}(\tau) - \hat{e}_{\rm fin} \propto -\tau^{\frac{4}{5}}$$

Evolution on constant times scale ($N \to \infty$, T finite)

- As for the eigenvalue equation, semi-classical approximation:

$$|\phi_T(m,s)\rangle = e^{-N\varphi_T(m,s)}$$

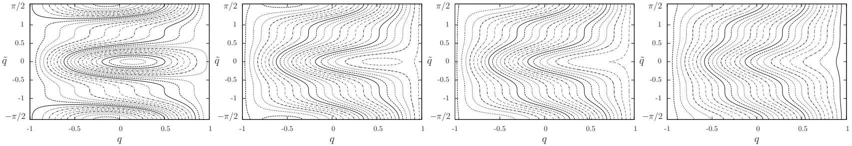
- Leads to classical Hamilton equations on $q_T(s) := \arg\min_m \varphi_T(m,s)$ and its conjugated momentum.
- The classical Hamiltonian is obtained from the quantum one by the canonical substitution [Biroli, Sciolla 11]
- Allows to compute the residual energy on this time scale:



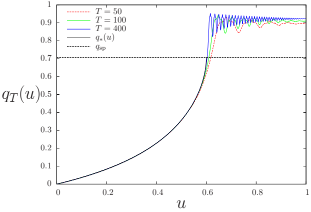
$$\lim_{T\to\infty} \lim_{N\to\infty} e_{\rm fin}(T,N) = \lim_{\tau\to 0} \lim_{N\to\infty} e_{\rm fin}(e^{\tau N}, N) = \hat{e}_{\rm fin}$$

Large (finite) times

- Particle evolving in a time dependent potential.
- Large evolution time limit: classical adiabatic theory; conservation of the (classical) adiabatic invariant.

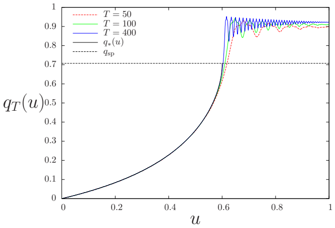


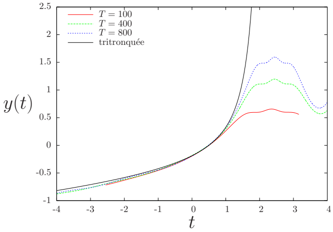
- Formally equivalent to the excitation conservation. Breaks down when crossing a separatrix $\Leftrightarrow$ at the the spinodal point.



Large (finite) times

- A more refined analysis is mandatory to understand what happens near the separatrix crossing.
- Matching with a Painleve equation ($y = T^{2/5}(q - q_{\rm sp})$, $t = T^{4/5}(u - u_{\rm sp})$) .





- Allows to characterize the decay of the residual energy on large (constant) times:

$$e_{\rm fin}(T) - \hat{e}_{\rm fin} \propto T^{-\frac{4}{5}}$$

Conclusion

- Simplified model with no disorder
- methods remain valid for any Hamiltonian with site permutation symmetry
- Link between static and dynamic properties can be made explicit
- Common features with more realistic mean-field random optimization problems : first-order transition, spinodals
- $$e_{\rm fin}(\tau) - \hat{e}_{\rm fin} \propto -\tau^{\frac{4}{5}} \ , \quad e_{\rm fin}(T) - \hat{e}_{\rm fin} \propto T^{-\frac{4}{5}}$$

  could be "universal" results for mean-field models crossing a spinodal