On non-linear sigma models for noncommutative tori

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Non-linear σ models

- Non-linear σ models are field theories, whose configuration space consists of maps X from a compact orientable Riemannian manifold (Σ, g), the source space, to a target space (M, G) another Riemannian manifold.
- From a mathematical point of view these are harmonic maps that describe minimal surfaces Σ embedded in M, which are the critical points of the action functional:

$$S[X] = \frac{1}{2\pi} \int_\Sigma \sqrt{g} g^{\mu\nu} G_{ij}(X) \partial_\mu X^i \partial_\nu X^j$$

  where $g = \det(g_{\mu\nu})$ and $(g^{\mu\nu})$ is the inverse of $(g_{\mu\nu})$, $\mu, \nu = 1, ..., \dim(\Sigma)$ and $i, j = 1, ..., \dim(M)$.
- If $\dim(\Sigma) = 2$, then the action S is conformally invariant. Hence the action only depends only on the conformal class of the metric and can be expressed in terms of a complex structure on Σ

Non-linear σ models

- $S[X] = \frac{1}{2\pi}\int_\Sigma G_{ij}(X)\partial X^i \wedge \overline{\partial} X^j$ where $\partial = \partial_z dz$ and $\overline{\partial} = \partial_{\overline{z}} d\overline{z}$ are suitable local coordinates.
- Noncommutative version due to **Dabrowski, Krajweski and Landi**: Dualize the setting into terms of $\star$-algebras $\mathcal{A}$ and $\mathcal{B}$ of smooth functions on Σ and M. Then embeddings X of Σ into M correspond to $\star$-algebra morphisms from $\mathcal{B}$ to $\mathcal{A}$. Configuration space is the space of all such $\star$-algebra morphism.
- Conformal structures can be understood in terms of Hochschild cohomology.
- Commutative case: the tri-linear map $\Phi : \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A}\to\mathbb{R}$ given by $\Phi(f_0, f_1, f_2) = \frac{i}{\pi}\int_\Sigma f_0 \partial f_1 \wedge \overline{\partial} f_2$ is an extremal element of positive Hochschild cocyles that belong to the positive Hochschild cohomology class of the cyclic cocycle $\Psi(f_0, f_1, f_2) = \frac{i}{2\pi}\int_\Sigma f_0 df_1 \wedge df_2$.

Non-linear σ models

- This formulation also makes sense for a noncommutative algebra $\mathcal{A}$.
- Φ defines a scalar product on 1-forms $a_0 da_1$: $\langle a_0 da_1, b_0 db_1 \rangle = \Phi(b_0^* a_0, a_1, b_1^*)$.
- Ψ allows to integrate 2-forms: $\Psi(a_0, a_1, a_2) = \frac{i}{2\pi} \int a_0 da_1 da_2$
- Let π be a morphism from $\mathcal{B}$ to $\mathcal{A}$. Then $\Phi_\pi = \Phi \circ (\pi \otimes \pi \otimes \pi)$ is a positive cocycle on $\mathcal{B}$.
- Noncommutative analoge of the metric on the target space take a positive element G of universal 2-forms $\Omega^2(\mathcal{B})$.
- The quantity

$$S[\pi] := \Phi_\pi(G)$$

which is well-defined and positive. We consider π as dynamical variable and the critical points of this σ-model are noncommutative harmonic maps or minimally embedded surfaces in the noncommutative space associated to $\mathcal{B}$.

Non-linear σ models

- Take as target space $M = \{1, 2\}$. Then $\mathcal{B} = \mathbb{C}^2$ and we think of it as the unital $\star$-algebra generated by a projection e.
- Any $\star$-algebra morphism from $\mathcal{B}$ to $\mathcal{A}$ is given by $p = \pi(e)$ in $\mathcal{A}$.
- The action functional is $S[p] = \Phi(1, p, p)$ for a Hochschild cocycle representing the conformal structure.
- In this talk we are going to be interested in the Moyal plane and the noncommutative torus as source spaces.

Basic notions

- **Differential structure** is given by a derivation ∂ on $\mathcal{A}$
- **Vector bundles** over a noncommutative manifold $\mathcal{A}$ are **finitely generated projective Hilbert C^*- modules** V over $\mathcal{A}$.
- **Covariant derivation** is a mapping on V that satisfies a **Leibniz rule**:

$$\nabla(A \cdot g) = (\partial A) \cdot g + A \cdot \nabla g.$$

Basic notions

- **Gauge connection** ${}_{\mathcal{A}}V$ is given by two covariant derivatives ∇_1, ∇_2 that are **compatible** with the Hermitian $\mathcal{A}$-module structure and two derivatives ∂_1, ∂_2:

$$\begin{aligned} \partial_1({}_{\mathcal{A}}\langle f, g\rangle) &= {}_{\mathcal{A}}\langle \nabla_1 f, g\rangle + {}_{\mathcal{A}}\langle f, \nabla_1 g\rangle \\ \partial_2({}_{\mathcal{A}}\langle f, g\rangle) &= {}_{\mathcal{A}}\langle \nabla_2 f, g\rangle + {}_{\mathcal{A}}\langle f, \nabla_2 g\rangle. \end{aligned}$$

- **Curvature** of a gauge connection is defined by $F_{12} := [\nabla_1, \nabla_2] - \nabla_{[\partial_1, \partial_2]}$.
- **Constant curvature connection** is a connection with $F_{12} = c\,\mathbb{I}_V$ for some constant c.

Complex geometry

- Suppose ∂_1, ∂_2 are derivations on $\mathcal{A}$. Then

$$\partial_{(i)} = \tfrac{1}{2}(\partial_1 - i\partial_2) \text{ and } \overline{\partial}_{(i)} = \tfrac{1}{2}(\partial_1 + i\partial_2)$$

  define a complex structure on $\mathcal{A}$.
- Laplacian $\Delta = \partial_1^2 + \partial_2^2$ has a factorization $\Delta = 4\partial_{(i)}\overline{\partial_{(i)}}$
- To the complex structure on $\mathcal{A}$ corresponds a complex structure on the Hilbert C^*-module V:

$$\nabla_{(i)} = \tfrac{1}{2}(\nabla_1 - i\nabla_2) \text{ and } \overline{\nabla_{(i)}} = \frac{1}{2}(\nabla_1 + i\nabla_2)$$

- If $\overline{\nabla_{(i)}}g = 0$ for a $g \in V$, then g is called a **holomorphic vector**.

Action functional

- Define an action functional on the set of projections $\mathcal{P}$ of $\mathcal{A}$:

$$S(p) = \frac{2}{\pi}\mathrm{tr}[\partial_{(i)}(p)\overline{\partial}_{(i)}(p)] = \frac{1}{\pi}\mathrm{tr}[p(\partial_1^2 p + \partial_2^2 p)]$$

- From the properties of a trace we get that $S(p)$ is a non-negative real number.
- Consider the tangent space to $\mathcal{P}$ at the point p.
- An element δp of the tangent space at p must be hermitian, $(\delta p)^* = \delta p$, and $(p + \delta p)^2 = p + \delta p + O(\delta p)$.

Action functional

- $(\delta p)^* = \delta p$ implies that δp has the following form: For some $x = x^*, y = y^*$ and z in $\mathcal{A}$

$$\delta p = pxp + (1-p)y(1-p) + (1-p)zp + pz^*(1-p)$$

- $(p + \delta p)^2 = p + \delta p + O(\delta p)$ yields the following general form of δp: For an abritray element $z \in \mathcal{A}$

$$\delta p = (1-p)zp + pz^*(1-p)$$

Action functional

- $\delta S(p) = 0$ gives equation of motions for the action functional $S(p)$:

$$0 = \delta S(p) = -\frac{1}{2\pi}\mathrm{tr}[(p\Delta(p))z + ((1-p)\Delta(p)p)z^*]$$

- Since z is arbitrary we get the following **field equations**:

$$p\Delta(p) = 0 \quad \text{and} \quad (1-p)\Delta(p)p,$$

- which is equivalent to:

$$p\Delta(p) - \Delta(p)p = 0.$$

- Therefore, the field equation is non-linear and of second order.

Action functional

- The absolute minimum of $S(p)$ fullfill first-order equations.
- For a projection p in $\mathcal{A}$ the **first Connes-Chern number** is defined by

$$\Psi_1(p) = \frac{i}{2\pi}\mathrm{tr}[p(\partial_1(p)\partial_2(p) - \partial_2(p)\partial_1(p))]$$

- $\Psi_1(p)$ is integer-valued and is given by evaluating the cyclic 2-cocycle at $a_0 = a_1 = a_2 = p$:

$$\Psi(a_0, a_1, a_2) = \frac{i}{2\pi}\mathrm{tr}[a_0(\partial_1(a_1)\partial_2(a_2) - \partial_2(a_1)\partial_1(a_2))].$$

Action functional

- **Crucial Fact:** $S(p) \geq 2|\Psi_1(p)|$
- Equality in the preceding inequality occurs when the projection p satisfies:
- **self duality equations:**

$$[(\partial_1 \pm i\partial_2)p]p = 0$$

- **anti-self duality equations:**

$$[(\partial_1 \mp i\partial_2)p]p = 0$$

- self duality equations: $\overline{\partial}_{(i)}(p)p = 0$ and/or $p\partial_{(i)}p = 0$.
- anti-self duality equations: $\partial_{(i)}(p)p = 0$ and/or $p\overline{\partial_{(i)}}p = 0$.

Frames for Hilbert C^*-modules

- Let $\mathcal{A}$ be a unital C^*-algebra. A sequence $\{g_j : j = 1, ..., n\}$ in a (left) Hilbert $\mathcal{A}$-module ${}_{\mathcal{A}}V$ is called a **standard module frame** if there are positive reals C, D such that

$$C \, {}_{\mathcal{A}}\langle f, f\rangle \leq \sum_{j=1}^{n} {}_{\mathcal{A}}\langle f, g_j\rangle {}_{\mathcal{A}}\langle g_j, f\rangle \leq D \, {}_{\mathcal{A}}\langle f, f\rangle$$

  for each $f \in {}_{\mathcal{A}}V$.

- Let $\mathcal{A}$ be a unital C^*-algebra. A sequence $\{g_j : j = 1, ..., n\}$ in a (left) Hilbert $\mathcal{A}$-module ${}_{\mathcal{A}}V$ is a **standard module frame** if the **reconstruction formula**

$$f = \sum_{j=1}^{n} {}_{\mathcal{A}}\langle f, g_j\rangle \cdot g_j \quad \text{for all} \quad f \in {}_{\mathcal{A}}V.$$

Here is a very useful description of finitely generated projective $\mathcal{A}$-modules:

Rieffel

Suppose ${}_{\mathcal{A}}V$ is a finitely generated projective $\mathcal{A}$-module with an $\mathcal{A}$-valued innerproduct. Then every projection $p \in M_n(\mathcal{A})$ such that ${}_{\mathcal{A}}V \cong p\mathcal{A}^n$ is of the form $p = (p_{jk})$ with

$$p_{jk} = {}_{\mathcal{A}}\langle g_j, g_k \rangle$$

for some standard module frame $\{g_1, .., g_n\}$ for ${}_{\mathcal{A}}V$.

Morita equivalence

- If $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then ${}_{\mathcal{A}}V_{\mathcal{B}}$ is a projective left $\mathcal{A}$-module, and a projective right $\mathcal{B}$-module.
- There exist $g_1, ..., g_n \in {}_{\mathcal{A}}V_{\mathcal{B}}$ such that

$$f = \sum_{j=1}^{n} {}_{\mathcal{A}}\langle f, g_j \rangle \cdot g_j = \sum_{j=1}^{n} f \cdot \langle g_j, g_j \rangle_{\mathcal{B}}$$

  for all $f \in {}_{\mathcal{A}}V_{\mathcal{B}}$.
- $S^{\mathcal{A}}_{g,h} f = {}_{\mathcal{A}}\langle f, g \rangle h$ is a rank-one Hilbert $\mathcal{A}$-modules,
- $S^{\mathcal{B}}_{g,h} f = f \cdot \langle g, h \rangle_{\mathcal{B}}$ is a rank one $\mathcal{B}$-module operator.

Projections in matrix algebras over C^*-algebras

Suppose V is an $\mathcal{A}-\mathcal{B}$ equivalence bimodule between the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$. Then there exist $g_1, ..., g_n$ in V such that $\langle g_1, g_1\rangle_{\mathcal{B}} + \cdots \langle g_n, g_n\rangle_{\mathcal{B}} = I_{\mathcal{B}}$.

- Reconstruction formula:

$$f = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle f, g_i\rangle \cdot g_i = \sum_{i=1}^{n} f \cdot \langle g_i, g_i\rangle_{\mathcal{B}}.$$

- Idempotents P in $M_n(\mathcal{A})$ such that $P^2 = P$ and $P\mathcal{A}^n = V$: $P = (p_{i,j})$, where $p_{ij} = {}_{\mathcal{A}}\langle g_i, g_j\rangle$.

Projection in C^*-algebras

- Suppose V is an $\mathcal{A}-\mathcal{B}$ equivalence bimodule between the C^*-algebras $\mathcal{A}$ and $\mathcal{B}$.
- Then for every p in $\mathcal{A}$ there exists a standard module frame g of ${}_{\mathcal{A}}V_{\mathcal{B}}$ $\langle g, g\rangle_{\mathcal{B}} = I_{\mathcal{B}}$ such that $p = {}_{\mathcal{A}}\langle g, g\rangle$.
- Every $f \in {}_{\mathcal{A}}V_{\mathcal{B}}$ can be expressed in the form

$$f = {}_{\mathcal{A}}\langle f, g\rangle \cdot g = f \cdot \langle g, h\rangle_{\mathcal{B}}. \tag{1}$$

- If we have a spectrally invariant subalgebras $\mathcal{A}_0$ of $\mathcal{A}$ and $\mathcal{B}_0 \subseteq \mathcal{B}$ and a subspace V_0 of V such that the ${}_{\mathcal{A}_0}\langle V_0, V_0\rangle \subseteq \mathcal{A}_0$ and $\langle V_0, V_0\rangle_{\mathcal{B}_0} \subseteq \mathcal{B}_0$.
- Then there exists a $g \in V_0$ such that $V_0 = p\mathcal{A}_0$ for $p = {}_{\mathcal{A}_0}\langle g, g\rangle$ equivalently $\langle V_0, V_0\rangle_{\mathcal{B}_0} = \mathbb{I}$.

Action functional

- Suppose V is an equivalence bimodule between $\mathcal{A}$ and $\mathcal{B}$. Let $g \in V$ be such that $\langle g, g\rangle_{\mathcal{B}} = I$. Then $p_g = \langle g, g\rangle_{\mathcal{A}}$ is a projection in $\mathcal{A}$.
- If there exists an element $\lambda \in \mathcal{B}$ such that g is a solution of

$$\overline{\nabla} g - g\lambda = 0,$$

then p_g is a solution of the self-duality equation:

$$\overline{\partial}_{(i)}(p_g)p_g = 0.$$

- If p_g is a solution of the self-duality equation, then g satisfies this equation for

$$\lambda = (\langle g, g\rangle_{\mathcal{A}})^{-1}\langle g, \overline{\nabla} g\rangle_{\mathcal{A}}.$$

Frames in a Hilbert space

- Duffin and Schaeffer introduced the notion of **frames** for a Hilbert space in their investigation of non-harmonic Fourier series in 1952.
- Frames are a generalization of bases for Hilbert (Banach) spaces.
- A family of elements $\{g_i\}_{i\in I}$ in a Hilbert space $\mathcal{H}$ is a **frame** for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{i\in I} |\langle f, g_i\rangle|^2 \leq B\|f\|^2$$

  holds for all $f \in \mathcal{H}$.
- Frames $\{g_i\}_{i\in I}$ are in general **linearly dependent** systems!

Frames in a Hilbert space

- The linear dependence of $\{g_i\}_{i\in I}$ is in many cases an advantage, e.g. coding theory, wireless communication, etc.
- The dependence yields some problems in the determination of the coefficients in

$$f = \sum_{i\in I} a_i g_i .$$

- If $\{g_i\}_{i\in I}$ is a frame in $\mathcal{H}$, then there exists a **dual frame** $\{h_i\}_{i\in I}$ such that for all $f \in \mathcal{H}$ we have

$$f = \sum_{i\in I} \langle f, h_i \rangle g_i = \sum_{i\in I} \langle f, g_i \rangle h_i$$

Frames in a Hilbert space

- The **analysis mapping** C maps f to $(\langle f, g_i\rangle)_{i\in I}$.
- The **synthesis mapping** D maps a sequence $\{a_i\}_{i\in I}$ to

$$f = \sum_{i\in I} a_i g_i.$$

- The **frame operator** is $S = D \circ C$:

$$Sf = \sum_{i\in I} \langle f, g_i\rangle g_i.$$

- If $\{g_i\}_{i\in I}$ is a frame in $\mathcal{H}$, then S is invertible on $\mathcal{H}$.

Frames in a Hilbert space

- If $\{g_i\}_{i\in I}$ is a frame for $\mathcal{H}$, then $\{S^{-1}g_i\}_{i\in I}$ is a dual frame, so-called **canonical dual frame**.
- The factorizations $I = S^{-1}S = SS^{-1}$ provide reconstruction formulas for elements in $\mathcal{H}$. The factorization $I = S^{-1/2}SS^{-1/2}$ yields a "base-like" representation in terms of the so-called **canonical tight frame** $\{S^{-1/2}g_i\}_{i\in I}$:

$$f = \sum_{i\in I} \langle f, S^{-1/2}g_i\rangle S^{-1/2}g_i.$$

- A frame $\{g_i\}_{i\in I}$ is tight if $\sum_{i\in I} |\langle f, g_i\rangle|^2 = A\|f\|^2$

Schrödinger representation

- **translation** $T_x f(t) = f(t-x)$ for $x \in \mathbb{R}$, **modulation** $M_\omega f(t) = e^{2\pi i t\cdot\omega} f(t)$ for $\omega \in \widehat{\mathbb{R}}$
- **time-frequency shift** $\pi(x,\omega)f(t) = M_\omega T_x f(t)$ for $(x,\omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$

$$M_\omega T_x = e^{2\pi i x\cdot\omega} T_x M_\omega$$
$$\pi(x+y,\omega+\eta) = e^{2\pi i x\cdot\eta}\pi(x,\omega)\pi(y,\eta)$$
$$\pi(x,\omega)\pi(y,\eta) = e^{2\pi i(y\cdot\omega - x\cdot\eta)}\pi(y,\eta)\pi(x,\omega)$$

- The projective representation of $\mathbb{R}^2$ has as 2-cocycle $c\big((x,\omega),(y,\eta)\big) = e^{2\pi i x\cdot\eta}$.
- matrix coefficient aka Short-Time-Fourier Transform:

$$V_g f(x,\omega) = \langle f, \pi(x,\omega)g\rangle = \int_{\mathbb{R}} f(t)\overline{g(t-x)}e^{-2\pi i t\omega}\,dt$$

Moyal plane

- Moyal plane $\mathcal{A}(\mathbb{R}^2, c)$: norm closure of $\{\pi(z) : z \in \mathbb{R}^2\}$

$$A = \iint_{\mathbb{R}^2} a(z)\pi(z)dz$$

  for $a \in L^1(\mathbb{R}^2)$
- Trace on Moyal plane $\mathrm{Tr}(A) = a(0)$
- If $a \in \mathscr{S}(\mathbb{R}^2)$, then A is a trace-class operator.
- Smooth structure $\mathcal{A}^\infty(\mathbb{R}^2, c)$: $A \in \mathcal{A}(\mathbb{R}^2, c)$ with $a \in \mathcal{S}(\mathbb{R}^2)$

Moyal plane

$$A = \iint_{\mathbb{R}^2} a(z)\pi(z)dz \quad a \in L^1(\mathbb{R}^2)$$

$$B = \iint_{\mathbb{R}^2} b(z)\pi(z)dz \quad b \in L^1(\mathbb{R}^2)$$

Then we have

$$AB = \iint_{\mathbb{R}^2} (a\natural b)(z)\pi(z)dz$$

where $(a\natural b)(z)$ is the **twisted convolution**

$$(a\natural b)(z) = \iint a(y,\eta)b(x-y,\omega-\eta)e^{-2\pi iy(\omega-\eta)}dyd\eta.$$

Line bundles over Moyal plane

Hermitian structure on $\mathcal{A}^\infty(\mathbb{R}^2, c)$

$$_{\mathbb{R}^2}\langle f, g\rangle = \iint_{\mathbb{R}^2} \langle f, \pi(z)g\rangle \pi(z) dz \quad f, g \in \mathscr{S}(\mathbb{R})$$

and as left action:

$$A \cdot g = \iint_{\mathbb{R}^2} a(x, \omega)\pi(x, \omega) g dx d\omega.$$

Compitability condition: $_{\mathbb{R}^2}\langle A \cdot f, g\rangle = \iint_{\mathbb{R}^2} (a \natural V_g f)(z)\pi(z) dz$

Hermitian structure on $\mathbb{C}$

$$\langle f, g\rangle_{\mathbb{C}} = \langle g, f\rangle_{L^2} \quad f, g \in \mathscr{S}(\mathbb{R})$$

and as right action:

$$g \cdot \lambda := g\overline{\lambda} \quad \lambda \in \mathbb{C}$$

Associativity condition

- For $f, g, h \in \mathscr{S}(\mathbb{R})$

$$ {}_{\mathbb{R}^2}\langle f, g \rangle \cdot h = f \cdot \langle g, h \rangle_{\mathbb{C}} $$

- For all $k \in \mathscr{S}(\mathbb{R})$ we have that

$$ \langle {}_{\mathbb{R}^2}\langle f, g \rangle \cdot h, k \rangle_{L^2} = \langle f \cdot \langle g, h \rangle_{\mathbb{C}}, k \rangle_{L^2}, $$

which is just Moyal's Identity:

$$ \langle V_g f, V_h k \rangle_{L^2} = \langle f, h \rangle_{L^2} \langle k, g \rangle_{L^2} $$

Line bundles over Moyal plane

- $\mathcal{A}^\infty(\mathbb{R}^2, c)$ is Morita equivalent to $\mathbb{C}$.
- In other words, there exists a function $g \in \mathscr{S}(\mathbb{R})$ with $\|g\|_{L^2} = 1$ such that

$$f = \iint_{\mathbb{R}^2} \langle f, \pi(z)g\rangle \pi(z)g dz \quad \text{for all} \quad f \in \mathscr{S}(\mathbb{R}).$$

- The last equation is aka **Resolution of Identity**.
- In other words, $\mathscr{S}(\mathbb{R})$ is a line bundle over $\mathcal{A}^\infty(\mathbb{R}^2, c)$.
- $\mathscr{S}(\mathbb{R})$ is isomorphic to $p\mathcal{A}^\infty(\mathbb{R}^2, c)$ for $p = {}_{\mathbb{R}^2}\langle g, g\rangle$ with $g \in \mathscr{S}(\mathbb{R})$ and $\|g\|_{L^2} = 1$.

Covariant derivatives – Moyal plane

- Derivations ∂_1 and ∂_2 on $\mathcal{A}^\infty(\mathbb{R}^2, c)$:

$$\partial_1 A = 2\pi i \iint_{\mathbb{R}^2} x a(x,\omega)\pi(x,\omega)dxd\omega$$

$$\partial_2 A = 2\pi i \iint_{\mathbb{R}^2} \omega a(x,\omega)\pi(x,\omega)dxd\omega$$

- Covariant derivatives ∇_1 and ∇_2 on $\mathcal{S}(R)$:

$$(\nabla_1 g)(t) = 2\pi i t g(t),$$

$$(\nabla_2 g)(t) = g'(t).$$

Connections – Moyal plane

- ∇_1, ∇_2 are compatible connections:

$$\nabla_1(A \cdot g) = (\partial_1 A) \cdot g + A \cdot (\nabla_1 g)$$

$$\nabla_2(A \cdot g) = (\partial_2 A) \cdot g + A \cdot (\nabla_2 g)$$

$$\partial_i({}_{\mathbb{R}^2}\langle f, g\rangle) = {}_{\mathbb{R}^2}\langle \nabla_i f, g\rangle + {}_{\mathbb{R}^2}\langle f, \nabla_i g\rangle$$

$$2\pi i \omega V_g f(x,\omega) = V_g f'(x,\omega) + V_{g'} f(x,\omega)$$

- Gauge connection is of constant curvature: $F_{12} = 2\pi i \mathbb{I}$.
- Gauge theortic formulation of Canonical Commutation Relation.

Complex structures on Moyal plane

- $\partial_{(i)} = \partial_1 - i\partial_2$ and $\overline{\partial_i} = \partial_1 + i\partial_2$
- $\nabla_{(i)} = \nabla_1 - i\nabla_2$ and $\overline{\nabla_i} = \nabla_1 + i\nabla_2$
- $\nabla_{(i)}g(t) = 2\pi i t g(t) - ig'(t)$ and $\overline{\nabla_{(i)}}g(t) = 2\pi i t g(t) + ig'(t)$
- $\nabla_{(i)}$ is the **creation operator** and $\overline{\nabla_{(i)}}$ the **annihilation operator**
- $H = -(\nabla_1^2 + \nabla_2^2)$ Hamiltonian of quantum harmonic oscillator
- $\nabla_{(i)}\overline{\nabla_{(i)}} = 2\pi I - \nabla_1^2 + \nabla_2^2$...Hermite operator.
- **holomorphic vector** for Moyal plane: Gaussians $g(t) = e^{-\pi t^2}$

Connes-Chern number of line bundles over Moyal plane

Since every projetion in $\mathcal{A}^\infty(\mathbb{R}^2, c)$ is given by some $p_g = {}_{\mathbb{R}^2}\langle g, g\rangle$ for some $g \in \mathscr{S}(\mathbb{R})$ with $\|g\|_{L^2} = 1$, we have that the Connes-Chern character of a line bundle over $\mathcal{A}^\infty(\mathbb{R}^2, c)$ is given by

$$\Psi_1(p_g) = -\frac{1}{2\pi i}\mathrm{tr}[p_g(\partial_1(p_g)\partial_2(p_g) - \partial_2(p_g)\partial_1(p_g))] = 1.$$

The expression $p_g(\partial_1(p_g)\partial_2(p_g) - \partial_2(p_g)\partial_1(p_g))$ is a sum of various terms of the form

$$V_{g_3} f_3 \natural V_{g_2} f_2 \natural V_{g_1} f_1.$$

which can be evaluated
$V_{g_3} f_3 \natural V_{g_2} f_2 \natural V_{g_1} f_1(z) = \langle f_1, g_2\rangle\langle f_2, g_3\rangle V_{g_3} f_3(z)$

Connes-Chern number of line bundles over Moyal plane

- Compatibility of ∇_1 and ∇_2 allows to replace $\delta_i({}_{\mathbb{R}^2}\langle g, g\rangle)$ by ${}_{\mathbb{R}^2}\langle \nabla_i g, g\rangle + {}_{\mathbb{R}^2}\langle g, \nabla_i g\rangle$ in $\Psi_1(p_g)$.
- Repeated use of the relation between triple twisted convolutions of matrix coefficients for the Schroedinger representation yields the following expression for $\Psi_1(p_g)$:

$$\Psi_1(p_g) = \frac{1}{2\pi i}\langle(\nabla_1\nabla_2 - \nabla_2\nabla_1)g, g\rangle |\langle g, g\rangle|^2 = 1,$$

where we have used that the gauge connection ∇_1, ∇_2 have constant curvature, i.e. canoncial commutation relation!

Action functional – Moyal plane

Let $g \in \mathscr{S}(\mathbb{R})$ be such that $\|g\|_{L^2} = 1$. Then $p_g = {}_{\mathbb{R}^2}\langle g, g\rangle$ is a projection in $\mathcal{A}^\infty(\mathbb{R}^2, c)$. Let $\overline{\nabla}_{(i)} = \nabla_1 + i\nabla_2$ be the anti-holomorphic connection on $\mathscr{S}(\mathbb{R})$. Then p_g is a solution of the self duality equations:

$$\overline{\partial}_{(i)}(p)p = 0$$

if and only if g satisfies

$$\overline{\nabla} g - \lambda g = 0$$

for some $\lambda \in \mathbb{C}$.

Action functional – Moyal plane

$$\overline{\nabla} g - \lambda g = 0$$

for some $\lambda \in \mathbb{C}$. The solutions to this equations are Gaussians $g(t) = Ce^{-\pi(t^2+2i\lambda t)}$.

Action functional – Moyal plane

If p_g is a solution of the self-duality equation: $\overline{\partial}_{(i)}(p_g)p_g = 0$, then g solves the equation

$$\overline{\nabla} g - \lambda g = 0$$

for $\lambda = \langle g, \overline{\nabla} g \rangle_{L^2}^{-1}$.

Noncommutative torus

For $\alpha\mathbb{Z} \times \beta\mathbb{Z}$ and the 2-cocycle $c((\alpha k, \beta l), (\alpha m, \beta n)) = e^{-2\pi i m(l-n)\alpha\beta}$ we consider the **twisted group algebra** $\ell^1(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$

- **Twisted convolution** of **a** and **b** is defined by

$$\mathbf{a} \natural \mathbf{b}(k,l) = \sum_{m,n} a(m,n) b(k-m, l-n) e^{-2\pi i m(l-n)\alpha\beta}$$

- **Twisted involution** of **a** given by

$$a^*(k,l) = e^{2\pi i k l \alpha\beta} \overline{a(-k,-l)}.$$

- smooth noncommutative torus $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ consists of all elements in $\mathcal{A}_\theta$ with rapidly decaying coefficients.

Hermitian structure over $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$

- Left action of $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ on $\mathscr{S}(\mathbb{R})$

$$A \cdot g = \Big[\sum_{k,l \in \mathbb{Z}} a(k,l)\pi(\alpha k, \beta l) \Big] g \quad \text{for} \quad \mathbf{a} \in \mathscr{S}(\mathbb{Z}^2)$$

- For f, g in $\mathscr{S}(\mathbb{R})$

$${}_\bullet\langle f, g\rangle = \sum_{k,l \in \mathbb{Z}} \langle f, \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l)$$

Hilbert $C^*(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$-module

The completion of $\mathscr{S}(\mathbb{R})$ with respect to $\|g\|_\bullet = \|{}_\bullet\langle g, g\rangle\|_{\mathrm{op}}^{1/2}$ is a full left Hilbert $C^*(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$-module.

Right module structure

- right action for $\mathcal{A}^\infty(\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},\overline{c})$ on $\mathscr{S}(\mathbb{R})$:

$$g\cdot B=\sum_{k,l\in\mathbb{Z}}\pi(\tfrac{k}{\beta},\tfrac{l}{\alpha})^* g\overline{b}(\tfrac{k}{\beta},\tfrac{l}{\alpha})\quad \mathbf{b}\in\mathscr{S}(\mathbb{Z}^2).$$

- For f,g in $\mathscr{S}(\mathbb{R})$:

$$\langle f,g\rangle_\bullet=\sum\pi(\tfrac{k}{\beta},\tfrac{l}{\alpha})^*\langle g,\pi(\tfrac{k}{\beta},\tfrac{l}{\alpha}f\rangle.$$

Hilbert $C^*(\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},\overline{c})$-module

The completion of $\mathscr{S}(\mathbb{R})$ with respect to $\|g\|_\bullet=\|\langle g,g\rangle_\bullet\|_{\mathrm{op}}^{1/2}$ is a full right Hilbert $C^*(\frac{1}{\beta}\mathbb{Z}\times\frac{1}{\alpha}\mathbb{Z},\overline{c})$-module.

Vector bundles over noncommutative tori

- $\mathscr{S}(\mathbb{R})$ is an equivalence bimodule between $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ and $\mathcal{A}^\infty(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$, i.e. there exist $g_1, ..., g_n \in \mathscr{S}(\mathbb{R})$ such that
$$f = \sum_{i=1}^{n} {}_\bullet\langle f, g_i\rangle \cdot g_i = \sum_{i=1}^{n} f \cdot \langle g_i, g_i\rangle_\bullet.$$
- The last expression is an example of an atomic decomposition and is known in applied harmonic analysis as multi-window Gabor frame expansion of a funciton f.
- Associativity conditions is known as Janssen's representation of the Gabor frame operator.

Gabor frames

$\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z}) = \{\pi(\alpha k, \beta l)g : k, l \in \mathbb{Z}\}$ is a **Gabor system**.

- **analysis operator**: $C_g f = (\langle f, \pi(\alpha k, \beta l)g\rangle)_{k,l\in\mathbb{Z}}$
- **synthesis operator**: $D_{\mathbf{a}} f = \sum_{k,l\in\mathbb{Z}} a_{kl}\pi(\alpha k, \beta l)f$
- **frame operator**: $S_{g,g} f = \sum_{k,l\in\mathbb{Z}}\langle f, \pi(\alpha k, \beta l)g\rangle\pi(\alpha k, \beta l)g$
- $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a **frame** for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that

$$A\|f\|_2^2 \leq \sum_{k,l\in\mathbb{Z}} |\langle f, \pi(\alpha k, \beta l)g\rangle|^2 \leq B\|f\|_2^2$$

- Gabor frame type operators $S_{g,h} f = \sum_{k,l\in\mathbb{Z}}\langle f, \pi(\alpha k, \beta l)g\rangle\pi(\alpha k, \beta l)h$ are rank-one operators on ${}_{\bullet}V$

Totally positive functions

- A function g is **totally positive** if for every two sets of increasing real numbers $x_1 < \cdots < x_N$ and $y_1 < \cdots < y_N$ the determinant of the matrix $(g(x_j - y_k))_{1 \leq j,k \leq N}$ is non-negative.
- A totally positive function is of finite type if the Fourier transform $\widehat{g}$ of g has the following form:

$$\widehat{g}(\omega) = \prod_{j=1}^{M} (1 + 2\pi i \delta_j \omega)^{-1}$$

for real non-zero parameters δ_j and $M \in \mathbb{N}$ with $M \geq 2$.

Totally positive functions and projections

Gröchenig-Stöckler

Assume that g is a totally positive function of finite type M. Then $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Gabor frame if and only if $\alpha\beta < 1$.

Consequence

Assume that g is a totally positive function of finite type M in $\mathscr{S}(\mathbb{R})$. Then ${}_\bullet\langle g, g\rangle$ is a projection in $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ if and only if $\alpha\beta < 1$.

Connections on noncommutative tori

- **derivations** on the noncommutative torus $C^*(\alpha\mathbb{Z}\times\beta\mathbb{Z},c)$:

$$\partial_1(A)=2\pi i\alpha\sum_{k,l}ka_{kl}\pi(\alpha k,\beta l)$$

$$\partial_2(A)=2\pi i\beta\sum_{k,l}la_{kl}\pi(\alpha k,\beta l)$$

- **Connections** on $C^*(\alpha\mathbb{Z}\times\beta\mathbb{Z},c)$

$$\nabla_1 g(t)=2\pi i\alpha t g(t)\quad\text{and}\quad\nabla_2 g(t)=\beta g'(t)$$

- Compatibility conditions follow from elementary facts about $V_g f$.

Gabor frames and projections

Suppose $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a tight frame for $L^2(\mathbb{R})$ for $g \in \mathscr{S}(\mathbb{R})$. Then the first Connes-Chern number of the line bundle $p_g(\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c))$ is

$$\Psi_1({}_\bullet\langle g, g\rangle) = \alpha\beta.$$

Curvature of $\mathscr{S}(\mathbb{R})$

The curvature F_{12} of the line bundle $\mathscr{S}(\mathbb{R})$ over $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ is $\alpha\beta$.

Balian-Low

- Let $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ be a Riesz basis for its closed span $\mathcal{H}$ in $L^2(\mathbb{R})$ and $\alpha\beta = 1$. Then $\nabla_i g$ or $\nabla_i h$ is not in $\mathscr{S}(\mathbb{R})$, where h denotes the canonical dual Gabor atom $h = S_{g,g}^{-1}$.
- There does not exist a line bundle over $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z})$ for $\alpha\beta = 1$ due to the existence of a constant curvature connection on $\mathscr{S}(\mathbb{R})$.

Proof is based on an observation of G. Battle, which uses the left Leibniz property for $A = \pi(\alpha k, \beta l)$ implies:

$$\langle \nabla_i g, \pi(\alpha k, \beta l)h \rangle = \langle \pi(-\alpha k, -\beta l)g, \nabla_i h \rangle$$

Balian-Low theorem

Suppose $\nabla_i g$ and $\nabla_i h$ are in $\mathscr{S}(\mathbb{R})$ for $i = 1, 2$. Then using the frame expansion we have that

$$\begin{aligned} \langle \nabla_1 g, \nabla_2 h \rangle &= \langle \sum_{k,l} \langle \nabla_1 g, \pi(\alpha k, \beta l) h \rangle \pi(\alpha k, \beta l) g, \nabla_2 h \rangle \\ &= \sum_{k,l} \langle \pi(-\alpha k, -\beta l) g, \nabla_1 h \rangle \langle \nabla_2 g, \pi(-\alpha k, -\beta l) h \rangle \\ &= \langle \nabla_2 g, \sum_{k,l} \langle \nabla_1 h, \pi(\alpha k, \beta l) g \rangle \pi(\alpha k, \beta l) h \rangle \\ &= \langle \nabla_2 g, \nabla_1 h \rangle \end{aligned}$$

However, $\nabla_1 \nabla_2 - \nabla_2 \nabla_1 = 2\pi i I$, constant curvature connection, gives

$$1 = \langle g, h \rangle = \langle \nabla_2 g, \nabla_1 h \rangle - \langle \nabla_1 g, \nabla_2 h \rangle = 0.$$

Self-duality

Let $g \in \mathscr{S}(\mathbb{R})$ be such that $\langle g, g \rangle_\bullet = \mathbb{I}$. Then $p_g = {}_\bullet\langle g, g \rangle$ is a projection in $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$. Let $\overline{\nabla}_{(i)} = \nabla_1 + i\nabla_2$ be the anti-holomorphic connection on $\mathscr{S}(\mathbb{R})$. Then p_g is a solution of the self duality equations:

$$\overline{\partial}_{(i)}(p_g)p_g = 0$$

if and only if g satisfies

$$\overline{\nabla} g - \lambda g = 0$$

for some $\lambda \in \mathcal{A}^\infty(\frac{1}{\beta}\mathbb{Z} \times \frac{1}{\alpha}\mathbb{Z}, \overline{c})$.

Self-duality

If λ is a multiple of the identity, $\lambda = c\mathbb{I}$ for some $c \in \mathbb{C}$, then

$$\overline{\nabla} g - cg = 0$$

for some $c \in \mathbb{C}$. The solutions to this equations are Gaussians $g(t) = Ce^{-\pi(t^2+2ict)}$.

Self-duality

If p_g is a solution of the self-duality equation: $\overline{\partial}_{(i)}(p)p = 0$, then g solves the equation

$$\overline{\nabla} g - \lambda g = 0$$

for $\lambda = \langle g, g \rangle_{\bullet}^{-1} \langle g, \overline{\nabla} g \rangle_{\bullet}$.

Gabor frames and non-linear σ-models

- Let $g \in \mathscr{S}(\mathbb{R})$ be such that $\langle g, g \rangle_\bullet = \mathbb{I}$. Then $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a tight Gabor frame for $L^2(\mathbb{R})$.
- Gabor atoms satisfy the self-duality equations! Solutions to self-duality equations are noncommutative analogs of solitons.
- Different approach to Manin's quantum theta function as solitons over noncommutative tori, based on the interpretation of quantum theta functions as Gabor systems for Gaussians as obtained in joint work with Manin.