Geometric test for topological states of matter

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Workshop on Topological quantum phases beyond two dimensions, Jussieu

October 20-21, 2022

based on recent work with Dimitri Zvonkine (CNRS and Paris-Saclay U.)

When a state of quantum matter, in dimension n, many-body or not, interacting or free,

is topological?

Topological order is a kind of order in the zero-temperature phase of quantum matter, [that] is defined and described by robust ground state degeneracy and quantized non-Abelian geometric phases of degenerate ground states. Microscopically, topological orders correspond to patterns of long-range quantum entanglement. (Wikipedia)

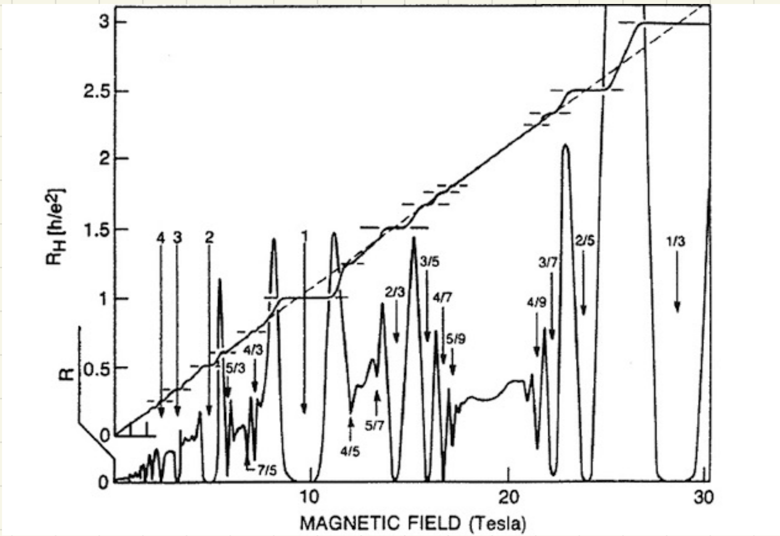
A phase of matter (of a quantum material) is called topological, within a given energy bound Δ (the “gap”) if external deformations of the system that are gentle enough – namely “adiabatic” – not to excite modes above the ground state leave the main properties of the system invariant. (nLab)

In this talk we propose a (theoretical) test to decide if the state of matter is topological or not.

The test is of a geometric kind, it involves Chern classes.

Quantum Hall effect

Fractional quantum Hall states provide historically one of the first examples of a topological state of matter.



Hall conductance is quantized

$$\sigma_H = p/q$$

(Chern number?)

Laughlin state:

$$\Psi = \prod_{n<m}^{N} (z_n - z_m)^q \cdot e^{-\frac{B}{4}\sum_{n=1}^{N} |z_n|^2}$$

Laughlin 1983: the following is the good Hilbert space on the plateaux of QHE (Nobel prize)

$$\mathcal{H}_{N,d,q} = \left\{ \Psi \;\middle|\; \Psi = P(z_1, \dots, z_N)\, e^{-\frac{B}{2}\sum_{n=1}^{N} |z_n|^2} \right\}$$

$*$ $P \in \mathbb{C}[z_1, \dots, z_N]$ are symmetric or anti-symmetric degree d polynomials in N letters z_n.

For the degree d the notation d= N_ϕ for the flux of magnetic field through the sample.

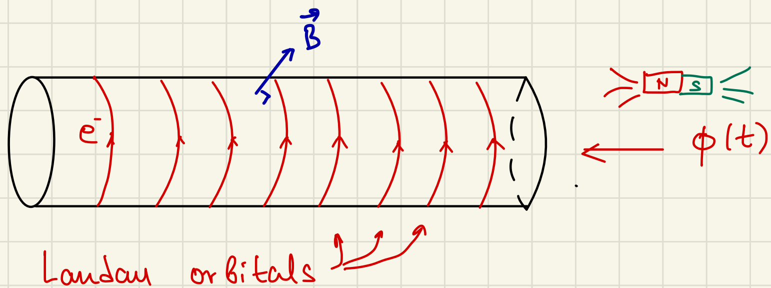
$*$ For the plateau 1/q polynomials vanish to the order q when any two particles meet, i.e.

$$P \in \bigcap_{n<m} (z_n - z_m)^q \quad \Rightarrow \quad P = \underset{\uparrow \text{ symmetric}}{S(z_1, \dots, z_n)} \cdot \Delta^q(z_1, \dots, z_n) \qquad q=1 \quad \text{IQHE}$$

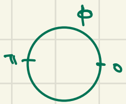
$*$ different plateaux p/q correspond to other « interaction ideals » — higher FQHE states: Pfaffian, RR, Jain etc

$$P \in I_{p/q} = \bigcap_{n<m<k} (z_n - z_m, z_m - z_k) \qquad \text{etc.}$$

Laughlin (1981): To make electrons move, let's wrap the sample into a cylinder and thread a magnetic field through the hole. This brings the parameter space M of AB (solenoid) fluxes into the game.

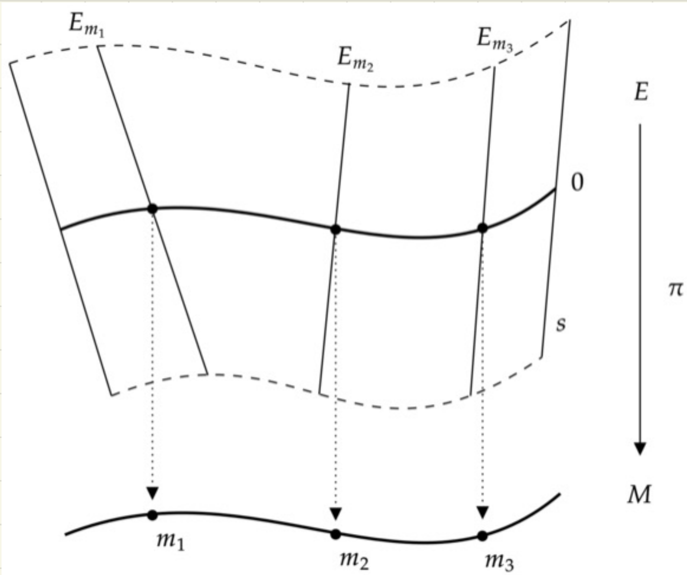


"Riemann surface"



"parameter space" S^1

As ϕ changes from 0 to 2π, and due to the Faraday's law and Lorentz force, one electrons drops from the left edge and another one appears on the right edge. Hence Hall conductance equals one.



We are in the setting of a rank-r complex vector bundle E of finite dimensional Hilbert spaces of quantum states, over a parameter space M. In the previous example M a circle and there is only one quantum state (Integer QHE state) so rank r=1.

Now we move to a more nontrivial example.

$$\Psi_i(x_1, \dots, x_N \mid m)$$

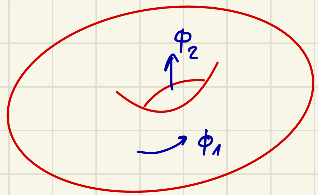
$i = 1, \dots, r$

Thouless et al, Avron et al, 80's —

Hall conductance is quantized because it's belongs to

the first Chern class of the bundle E of QH states over the

torus of AB fluxes:



$$\sigma_H = 1 \cdot d\varphi_1 \wedge d\varphi_2 \in c_1(E)$$

$$c_1 \in H^2(T^2_\varphi, \mathbb{Z})$$

$$M = T^2_\varphi = \{ \phi_1, \phi_2 \in [0,1] \}$$

"Flux torus"

Haldane-Rezayi'85 — actually, there are q degenerate Laughlin states on the torus.

$$\Psi_i(z_1, \dots, z_N) = \Theta\begin{bmatrix} \frac{i}{q} \\ 0 \end{bmatrix}\left(q \sum_n z_n + \phi, q\tau\right) \prod_{n<m}^{N} \theta_1^q(z_n - z_m), \quad i = 1, \dots, q$$

Indeed, the computation of the Berry curvature gives

$$\frac{1}{2\pi} \operatorname{tr} d_\varphi \langle \Psi_i, d_\varphi \Psi_j \rangle_{L^2} = d\varphi_1 \wedge d\varphi_2$$

and the Hall conductance is

$$\sigma_H = \frac{1}{q} d\varphi_1 \wedge d\phi_2 \notin H^2(T_\varphi, \mathbb{Z})$$

Thouless et al, Avron et al, 80's —

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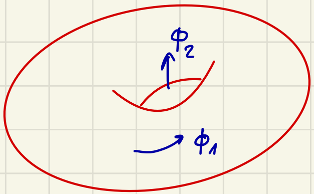

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and the Hall conductance is

$$\sigma_H = \frac{1}{q} d\varphi_1 \wedge d\phi_2 \in \frac{1}{rk(E)} c_1(E)$$

Higher genus surfaces

In the completely filled state on a genus-g surface, the number of electrons a Laughlin state can contain is bounded

$$N \leq N_{max} = \frac{1}{q} d + 1 - g$$ (Wen-Zee formula)

$\frac{1}{q}$ is called filling fraction

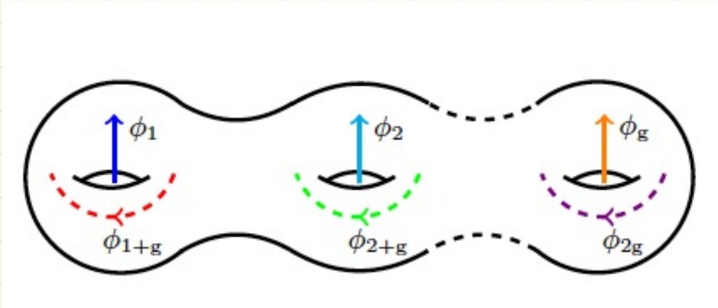
Then Laughlin states are q^g degenerate

$$\dim \mathcal{H}_{N,d,g} = q^g$$ (Wen-Niu conjecture)

"topological degeneracy"

Avron-Seiler-Zograf'94 — in the IQHE, the Hall conductance is the first Chern class

$$\sigma_H = \sum_{a=1}^{g} d\varphi_a \wedge d\varphi_{a+g} =: \Theta$$



Laughlin states

$$\sigma_H = \frac{c_1(E)}{q^g} \quad ?$$

$$M = T_\varphi^{2g}$$

puzzle

Once we pick a connection on a rank-r (r=q^g in the previous example) complex vector bundle and compute the curvature R, its trace is a closed differential two-form on M, which defines a cohomology class on M called first Chern class.

$$\left[\frac{i}{2\pi}\,\mathrm{tr}\,R\right] = c_1(E) \in H^2(M,\mathbb{Z})$$

For example, in physics the Berry (also called Chern conection) is often used.

$$R = d\langle \Psi, d\Psi\rangle_{L^2}$$

Now, the traces of higher powers of the curvature also define closed differential forms on M, they are called Chern characters

$$ch_m(E) = \left[\left(\frac{i}{2\pi}\right)^m \frac{1}{m!}\,\mathrm{tr}\,R^m\right] \in H^{2m}(M,\mathbb{Z}) \qquad 0 < m < \tfrac{1}{2}\dim M$$

The first Chern class c_1=ch_1 and the higher Chern classes c_m(E) are related to Ch_m(E) by polynomial identities, but Chern characters are more convenient objects.

Main result-1. (Zvonkine, SK) Degeneracy of Laughlin states on a genus g surface —

$$\boxed{\dim \mathcal{H}_{N,d,q,g} = \sum_{k=0}^{g} \binom{g}{k} \binom{N-g+p}{k-g+p} q^k}$$

Corollary: there are no Laughlin states with p<0, in other words the for given d, q, g the maximal number of particles (completely filled state) is given by

$$N_{max} = \left[\frac{d}{q} \right] + 1 - g$$

Wen-Zee formula

if moreover q divides d, $\dim \mathcal{H}_{N_{max},d,q,g} = q^g$ Wen-Niu topological degeneracy

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Notations: $p = d - q(N+g-1)$ — how many quasiholes the L. state can fit

$\binom{a}{b} = \frac{a!}{b!(a-b)!}$ — binomial coefficient

(main result-2, cont'd)

$$\boxed{ch_m(E) = \sum_{k=m}^{g} \binom{g-m}{k-m} \binom{N-g+p}{k-g+p} q^{k-m} \frac{\Theta^m}{m!}} \qquad (m \leq g)$$

The 2-form $\Theta = \sum_{a=1}^{g} d\phi_a \wedge d\phi_{a+g}$ defines the so-called theta class.

In particular, in the completely filled state (p=0) the first Chern class is $c_1(E) = q^{g-1} \cdot \Theta$

which means that the Hall conductance is $\sigma_H = \frac{c_1(E)}{q^g} = \frac{1}{q}\Theta$

Now, if p>0 with the extra condition that the quasiholes are localized at some points $w_1, w_2, \ldots, w_p$

then the rank of the bundle is again q^g and same formulas for the Chern classes apply as if p=0.

Idea of the proof: use Hirzebruch-Riemann-Roch theorem to compute the rank

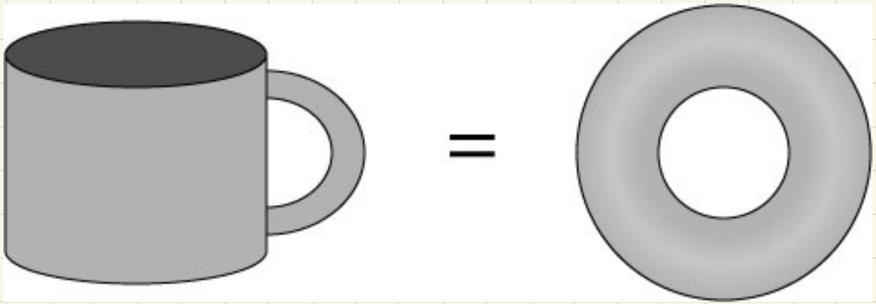
and Grothendieck-Riemann-Roch formula to compute the Chern classes

Application: geometric test for topological states of matter

It is often said that FQHE states are first examples of “topological states of matter”.

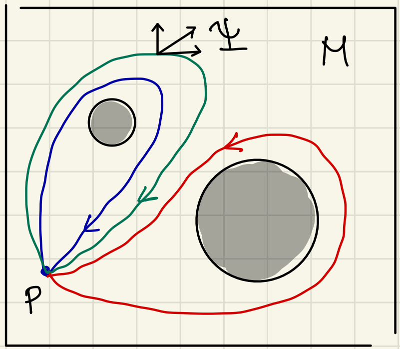
What does it mean for the state of matter to be “topological”?

Topology is concerned with geometric properties that are preserved under continuous deformations.



Definition: in a topological state, adiabatic transport along a path in the parameter space is independent under continuous deformations of the path. This means the bundle is flat — locally on M the bundle looks like a product space of M and C^r with constant transition functions between the local charts.

Remark: usually all this assumes existence of a gap.



For a complex vector bundle E —> M. the following is equivalent:

1. E is flat

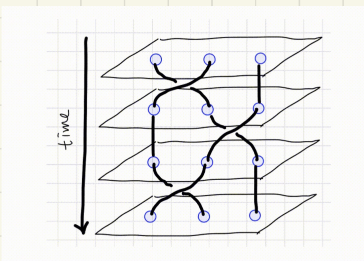
2. E admits a flat connection $\nabla : \Gamma(E) \to \Omega^1(M, \Gamma(E))$

$R = \nabla^2 = 0$

3. E is defined by a representation

$\rho : \pi_1(M) \to GL(r, \mathbb{C})$

(Kobayashi, *Complex vector bundles)*



Fundamental group $\pi_1(M)$ is the set of all loops modulo continuous deformations between them, i.e. on the picture above it would not distinguish between the blue and green loops, but distinguish them from the red one.

Since we are in quantum mechanics, we will allow for transport independent of the continuous deformations of the path “up to a phase factor” — i. e. projectively flat bundles. For projectively flat bundles there exists a connection ∇ with curvature being a scalar matrix

$$R = \alpha I_r \quad , \quad \alpha \in \Omega^2(M)$$

The key fact for us is the following — if E is projectively flat,

then this relation holds in cohomology, i.e. up to a total derivative

$$\boxed{ch_m(E) = \frac{[c_1(E)]^m}{m!\, r^{m-1}}}$$

Geometric test

Consider any degenerate qua'n''tum mechanical system with a parameter space M of dim M >3.

Choose a connection ∇ that defines a parallel transport along M of the bundle of states E.

(1) Compute its curvature $R = \nabla^2$ (non-abelian Berry curvature)

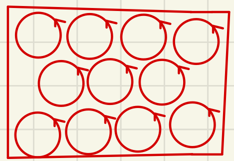
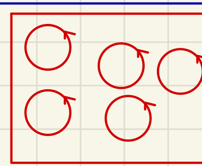
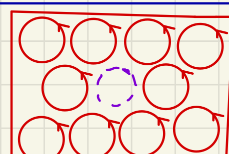
(2) compute traces of all its powers. $\frac{1}{2\pi} Tr\, R^m$ (obtain 2m form on M.)

(3) check whether $Tr\, R^m = \frac{(Tr\, R)^m}{m!\, r^{m-1}}$ up to total derivative.

Candidates for topological states of matter will pass the test

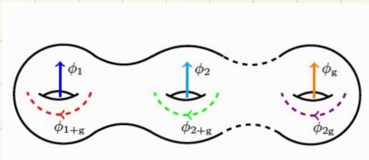
Summary

$$\boxed{ch_m(E) = \sum_{k=m}^{g} \binom{g-m}{k-m} \binom{N-g+p}{k-g+p} q^{k-m} \frac{\theta^m}{m!}}$$

Laughlin state		geometric test $ch_m(E) = \dfrac{[c_1(E)]^m}{m!\, r^{m-1}}$?	
Completely filled		✓	
Partially filled		✗	independent 2nd and higher Chern classes
Localized quasi-holes		✓	

Physics interpretation of first Chern class (Avron-Seiler-Simon-Zograf 1990's)

Changing AB-flux $\varphi_b = V_b t$ through the cycle b of the surface adiabatically with time t creates the Hall current I_a through the cycle a, given by $I_a = (\sigma_H)_{ab} V_b$

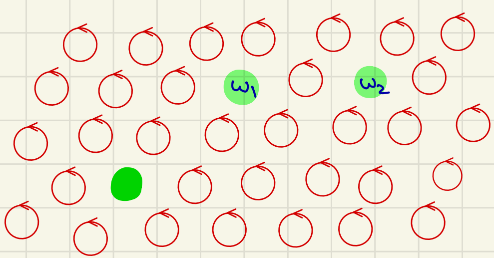


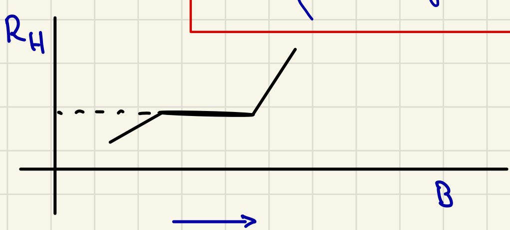
$$\sigma_H = \frac{c_1(E)}{rk(E)} = \sum_{a,b=1}^{g} \nu \, \delta_{b,a+g} \, d\varphi_a \wedge d\varphi_b$$

$\nu \in \mathbb{Q}$

* In a completely filled state the Hall conductance is quantized $\boxed{\sigma_H = \frac{1}{q} \, \Theta}$

* Once p quasiholes become non-localized (e.g. no impurities left), the Hall conductance starts to deviate from the quantized value $\boxed{\sigma_H = \left(\frac{1}{q} - \frac{p}{g\, q^2 N} + o\left(\frac{1}{N}\right) \right) \Theta}$





increasing degree d

Laughlin state on a Riemann surface

The definition of the Hilbert space of Laughlin states on a Riemann surface mimics the one given before for the complex plane, except that we replace the notion of holomorphic polynomial of degree d by the notion of holomorphic line bundle of degree d (flux of magnetic field B).

Def. Let $L \to \Sigma$ be a holomorphic line bundle of degree $d \geq 2g-1$. Take $N \geq g$ and fix a positive integer q. Consider Nth power Σ^N and let $z_1, \dots, z_N$ be N coordinates. Denote by $\pi_1, \dots, \pi_N$ the N projections from Σ^N to Σ. Consider the following line bundle on Σ^N

$$L^{\boxtimes N} = \pi_1^* L \otimes \cdots \otimes \pi_N^* L$$

The Laughlin state of weight q is a section Ψ of $L^{\boxtimes N}$, which is

* completely symmetric or anti-symmetric for q even, resp. odd
* vanishes to the order q on all the diagonals $\Delta_{nm} = \{ z_n = z_m \}$

We have a vector space of Laughlin states for a given line bundle L, $\mathcal{H}_{N,d,q,g}(L)$

The parameter space M will be the moduli space of degree-d line bundles L, the Picard variety $\mathrm{Pic}^d(\Sigma)$ which is a g-dimensional complex torus (isomorphic to Jacobian $\mathrm{Jac}(\Sigma)$). This is the space of inequivalent configurations of the magnetic field of given flux d piercing the surface, parametrized by the solenoid phases through the 2g holes on the surface

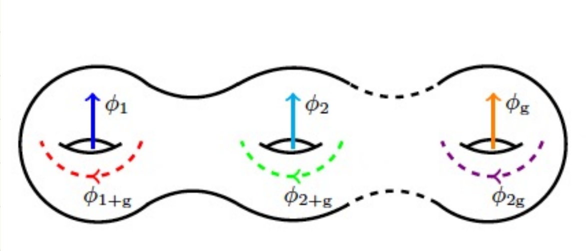
$$\{\phi_a\}_{a=1,\dots,2g} \in [0,2\pi]^{2g}$$

Varying L, by varying the solenoid fluxes we obtain the vector bundle of Laughlin states

$$\pi : E \to M$$

$$M = \mathrm{Jac}(\Sigma)$$

$$\pi^{-1}(\{\phi\}) = \mathcal{H}_{N,d,q,g}$$



Laughlin states for g>0, in a completely filled state

$$|\Psi_i|^2 = \left|\Theta\begin{bmatrix} i/q \\ 0 \end{bmatrix}\left(q\sum_{n=1}^{N} z_n - D + \phi\right)\right|^2 \cdot \prod_{n<m}^{N} E^q(z_n - z_m)$$

$i = (1, \dots, q)^g$

$\phi \in M = \mathrm{Jac}(\Sigma)$

D is a divisor (vanishing set of Ψ),

encodes magnetic line bundle L

Θ is Riemann theta function on the Jacobian of the Riemann surface

When we know the sections of the bundle, we can take the Chern connection (known in physics as Berry connection) and compute its curvature

$$R_{ij} = d_\phi \, \mathrm{Im} \langle \Psi_i , d_\phi \Psi_j \rangle_{L^2}$$

genuine Colomb-gas type N-fold integral over Σ^N

This is hard, but there is a way around this computation — Hirzebruch-Grothendieck-Riemann-Roch formulas compute the corresponding Chern classes.

Idea of the derivation: use Hirzebruch-Riemann-Roch and Grothendieck-Riemann-Roch formulas.

Laughlin state is a section of $L^{\boxtimes N}$ over N copies of the Riemann surface Σ^N

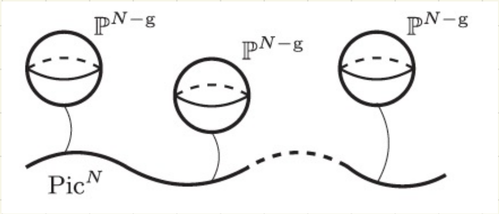
Twisting this line bundle by the q times the divisor of the diagonals $q\Delta = q \bigcup_{n<m}^{N} \{z_n = z_m\}$

we reinterpret Laughlin states as completely symmetric sections of $S = L^{\boxtimes N}(-q\Delta)$

over the Nth symmetric power of the Riemann surface Σ^N / S_N

$X = \Sigma^N / S_N$ has a useful description as a bundle of projective spaces $\mathbb{P}^{N-g}$

fibered over $\mathrm{Pic}^N(\Sigma)$ isomorphic to Jacobian.

$\Sigma^N / S_N \simeq$ 

The first Chern class of line bundle $L^{\boxtimes N}(-q\Delta)$ over $X = \Sigma^N / S_N$

$$c_1(S) = \beta\, \Theta_{\mathrm{conf}} + p\,\zeta \qquad (p = d - q(N+g-1))$$

where $\Theta_{\mathrm{conf}}, \zeta \in H^2(X, \mathbb{Z})$, $\Theta_{\mathrm{conf}} = \sum_{a=1}^{g} d\varphi'_a \wedge d\varphi'_{a+g}$

in terms of coordinates on Jacobian.

The Hirzebruch-Riemann-Roch formula gives the dimension of the space of sections of S

(degeneracy of ground states)

$$\sum_{i=1}^{N} (-1)^i\, h^i(X, S) = \int_X e^{c_1(S)}\, td(X) = h^0 = r = \sum_{k=0}^{g} \binom{g}{k} \binom{N-g+p}{k-g+p} q^k$$

* $td(X)$ -Todd class is a mixed degree cohomology class which is known

* higher cohomology groups vanish by Kodaira vanishing since S is positive line bundle.

Turning on AB solenoid phases, the line bundle $S = L^{\otimes N}(-q\Delta)$ can be viewed as line bundle over $M \times X$ where $M = \mathrm{Pic}^d(\Sigma) \simeq \mathrm{Jac}(\Sigma)$ is our parameter space (2g dimensional torus)

Electromagnetic connection changes as $D_z \to D_z + \sum_{a=1}^{g} (\varphi_a \alpha_a + \varphi_{a+g} \beta_g)$

where (α_a, β_a) is the basis of harmonic one forms on Σ,

and we take a trivial connection component along the jacobian $\nabla_\varphi = \sum_a (d\varphi_a \partial_{\varphi_a} + d\varphi_{a+g} \partial_{\varphi_{a+g}})$

leads to the first Chern class

$$c_1(S) = \beta \Theta_{conf} + p\zeta + \sum_{a=1}^{g} (d\varphi_a \wedge d\varphi'_{a+g} + d\varphi'_a \wedge d\varphi_{a+g})$$

and Grothendieck-Riemann-Rich formula gives all Chern characters of Laughlin bundle $E \to M = \mathrm{Jac}(\Sigma)$

$$ch(E) = ch_0(E) + ch_1(E) + \dots = \int_X e^{c_1(S)} td(X)$$

$$ch_m(E) = \sum_{k=m}^{g} \binom{g-m}{k-m} \binom{N-g+p}{k-g+p} q^{k-m} \frac{\Theta^m}{m!}$$

* We computed rank and Chern classes of the bundle of Laughlin states over the Jacobian, proving Wen-Zee formula and Wen-Niu topological degeneracy as a corollary.

* In order to prove projective flatness we still need to find a connection $\nabla : \Gamma(E) \to \Omega^1(M, \Gamma(E))$ whose curvature is a scalar matrix

$$R = \alpha \, I_E \qquad \text{for some} \qquad \alpha \in H^2(M)$$

* First Chern class is the quantized Hall conductance. What are the signatures of the second Chern class, if any?

* Apply geometric test in other situations — other FQHE plateaux, Brillouin zone, etc. Prerequisites: degenerate states, gap, parameter space of complex dimension at least two.

Thank you!

Editors' Suggestion

Geometric Test for Topological States of Matter

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(Received 31 May 2021; revised 1 September 2021; accepted 21 December 2021)

We generalize the flux insertion argument due to Laughlin, Niu-Thouless-Tao-Wu, and Avron-Seiler-Zograf to the case of fractional quantum Hall states on a higher-genus surface. We propose this setting as a test to characterize the robustness, or topologicity, of the quantum state of matter and apply our test to the Laughlin states. Laughlin states form a vector bundle, the Laughlin bundle, over the Jacobian—the space of Aharonov-Bohm fluxes through the holes of the surface. The rank of the Laughlin bundle is the degeneracy of Laughlin states or, in the presence of quasiholes, the dimension of the corresponding full many-body Hilbert space; its slope, which is the first Chern class divided by the rank, is the Hall conductance. We compute the rank and all the Chern classes of Laughlin bundles for any genus and any number of quasiholes, settling, in particular, the Wen-Niu conjecture. Then we show that Laughlin bundles with nonlocalized quasiholes are not projectively flat and that the Hall current is precisely quantized only for the states with localized quasiholes. Hence our test distinguishes these states from the full many-body Hilbert space.