ASPECTS OF HIGHER DIMENSIONAL QUANTUM HALL EFFECT: EFFECTIVE ACTIONS, ENTANGLEMENT ENTROPY

DIMITRA KARABALI

City University of New York
Lehman College and Graduate Center

(with V.P. Nair)

Topological Quantum Phases of Matter Beyond Two Dimensions

Sorbonne University

October 20-21, 2022

BASIC FEATURES OF 2D IQHE

Charged particle moving on 2d plane (or S^2) in strong external magnetic field (Landau problem)

- Distinct Landau levels, separated by energy gap ($\sim\ B$)
- Each Landau level is degenerate
- Lowest Landau level (LLL) :

$$\psi_n \sim z^n e^{-|z|^2/2}$$

$$z = x + iy$$

QUANTUM HALL DROPLETS

Many-body problem $\Longrightarrow$ quantum Hall droplets

- Degeneracy of each LL is lifted by confining potential $(V = \frac{1}{2}ur^2)$
- Exclusion principle $\rightarrow$ N-body ground state = incompressible droplet

QUANTUM HALL DROPLETS

Many-body problem $\Longrightarrow$ quantum Hall droplets

- Degeneracy of each LL is lifted by confining potential $(V = \frac{1}{2}ur^2)$
- Exclusion principle → N-body ground state = incompressible droplet
- Low energy excitations of droplets $\Longleftrightarrow$ area preserving boundary fluctuations (edge excitations)

Edge dynamics is collectively described by 1d chiral boson ϕ (WEN, STONE,..)

$$S_{\text{edge}} = \int_{\partial D} \Big(\partial_t \phi + u\, \partial_\theta \phi\Big) \partial_\theta \phi, \qquad u \sim \frac{\partial V}{\partial r^2}\Big]_{\text{boundary}}$$

ELECTROMAGNETIC FLUCTUATIONS

In the presence of electromagnetic fluctuations

- The bulk dynamics is described by an effective action

$$S_{\text{bulk}} = S_{CS} = \frac{\nu}{4\pi} \int_D \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

  S_{CS} is not gauge invariant in presence of boundaries.

ELECTROMAGNETIC FLUCTUATIONS

In the presence of electromagnetic fluctuations

- The bulk dynamics is described by an effective action

$$S_{\text{bulk}} = S_{CS} = \frac{\nu}{4\pi} \int_D \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

  S_{CS} is not gauge invariant in presence of boundaries.

- The edge dynamics is described by

$$S_{\text{edge}} \sim \text{gauged chiral action}$$

ELECTROMAGNETIC FLUCTUATIONS

In the presence of electromagnetic fluctuations

- The bulk dynamics is described by an effective action

$$S_{\text{bulk}} = S_{CS} = \frac{\nu}{4\pi} \int_D \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

  S_{CS} is not gauge invariant in presence of boundaries.

- The edge dynamics is described by

$$S_{\text{edge}} \sim \text{gauged chiral action}$$

Anomaly cancellation between bulk and edge actions,

$$\delta S_{\text{bulk}} + \delta S_{\text{edge}} = 0$$

ELECTROMAGNETIC FLUCTUATIONS

In the presence of electromagnetic fluctuations

- The bulk dynamics is described by an effective action

$$S_{\text{bulk}} = S_{CS} = \frac{\nu}{4\pi} \int_D \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda$$

  S_{CS} is not gauge invariant in presence of boundaries.

- The edge dynamics is described by

$$S_{\text{edge}} \sim \text{gauged chiral action}$$

  Anomaly cancellation between bulk and edge actions,

$$\delta S_{\text{bulk}} + \delta S_{\text{edge}} = 0$$

- The effective action S_{CS} captures the response of the system to electromagnetic fluctuations.

$$J^\mu = \frac{\delta S_{CS}}{\delta A_\mu} = \frac{\nu}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$

EFFECTIVE ACTIONS

What about other transport coefficients?

EFFECTIVE ACTIONS

What about other transport coefficients?

- How does the system respond to stress and strain?

EFFECTIVE ACTIONS

What about other transport coefficients?

- How does the system respond to stress and strain?
- Calculate stress tensor $\Longleftrightarrow$ couple theory to gravity (ABANOV AND GROMOV, 2014)

EFFECTIVE ACTIONS

What about other transport coefficients?

- How does the system respond to stress and strain?
- Calculate stress tensor $\Longleftrightarrow$ couple theory to gravity (ABANOV AND GROMOV, 2014)

$$S_{eff} = \frac{1}{4\pi}\int \left[[A + (s+\tfrac{1}{2})\omega]\, d[A + (s+\tfrac{1}{2})\omega] - \frac{1}{12}\omega d\omega \right] + \cdots$$

$$\omega = \text{spin connection} \qquad s = 0 \rightarrow LLL\;,\; s = 1 \rightarrow \text{1st LL}, \cdots$$

$$\frac{\delta S_{eff}}{\delta \omega_0} \sim n_H = \text{Hall viscosity}$$

KLEVTSOV ET AL; BRADLYN, READ; CAN, LASKIN, WIEGMANN

HIGHER DIMENSIONAL QHE

How do these 2d features extend to higher dimensions?

HIGHER DIMENSIONAL QHE

How do these 2d features extend to higher dimensions?

- QHE on S^4 (HU AND ZHANG, 2001)

Higher dimensional QHE

How do these 2d features extend to higher dimensions?

- QHE on S^4 (Hu and Zhang, 2001)

- Generalization to arbitrary even (spatial) dimensions

  QHE on $\mathbb{CP}^k$ (Karabali and Nair, 2002...)

Higher dimensional QHE

How do these 2d features extend to higher dimensions?

- QHE on S^4 (Hu and Zhang, 2001)

- Generalization to arbitrary even (spatial) dimensions

  QHE on $\mathbb{CP}^k$ (Karabali and Nair, 2002...)

  - higher dimensionality
  - possibility of having both abelian and nonabelian magnetic fields

QHE ON $\mathbb{CP}^k$

$\mathbb{CP}^k$: 2k dim space, locally parametrized by z_i, $i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz \, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2}$$

QHE ON $\mathbb{CP}^k$

$\mathbb{CP}^k$: 2k dim space, locally parametrized by z_i, $i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz \, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2}$$

- Group cosets

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)}$$

QHE ON $\mathbb{CP}^k$

$\mathbb{CP}^k$: 2k dim space, locally parametrized by z_i, $i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz \, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2}$$

- Group cosets

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)}$$

- $U(k) \sim U(1) \times SU(k) \Longrightarrow$ We can have both $U(1)$ and $SU(k)$ background magnetic fields

QHE ON $\mathbb{CP}^k$

$\mathbb{CP}^k$: 2k dim space, locally parametrized by z_i, $i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz\, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2}$$

- Group cosets

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)}$$

- $U(k) \sim U(1) \times SU(k) \Longrightarrow$ We can have both $U(1)$ and $SU(k)$ background magnetic fields
- Landau wavefunctions are functions on $SU(k+1)$ with particular transformation properties under $U(k)$.

QHE ON $\mathbb{CP}^k$

$\mathbb{CP}^k$: 2k dim space, locally parametrized by $z_i, i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz \, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2}$$

- Group cosets

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)}$$

- $U(k) \sim U(1) \times SU(k) \Longrightarrow$ We can have both $U(1)$ and $SU(k)$ background magnetic fields
- Landau wavefunctions are functions on $SU(k+1)$ with particular transformation properties under $U(k)$.
- There are distinct Landau levels, separated by energy gap.

QHE ON $\mathbb{CP}^k$

$\mathbb{CP}^k$: 2k dim space, locally parametrized by $z_i, i = 1, \cdots, k$

- Fubini-Study metric

$$ds^2 = \frac{dz \cdot d\bar{z}}{(1 + z \cdot \bar{z})} - \frac{\bar{z} \cdot dz\, z \cdot d\bar{z}}{(1 + z \cdot \bar{z})^2}$$

- Group cosets

$$\mathbb{CP}^k = \frac{SU(k+1)}{U(k)}$$

- $U(k) \sim U(1) \times SU(k) \Longrightarrow$ We can have both $U(1)$ and $SU(k)$ background magnetic fields
- Landau wavefunctions are functions on $SU(k+1)$ with particular transformation properties under $U(k)$.
- There are distinct Landau levels, separated by energy gap.
- Each Landau level forms an irreducible $SU(k+1)$ representation, whose degeneracy and energy is easy to calculate.

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- $\mathbb{CP}^k = SU(k+1)/U(k)$. We can use $(k+1) \times (k+1)$-matrix $g \in SU(k+1)$ as a coordinate.

  $g_{i,k+1} = z_i/\sqrt{1+\bar{z}\cdot z}, \quad g_{k+1,k+1} = 1/\sqrt{1+\bar{z}\cdot z}$

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- $\mathbb{CP}^k = SU(k+1)/U(k)$. We can use $(k+1) \times (k+1)$-matrix $g \in SU(k+1)$ as a coordinate.

  $g_{i,k+1} = z_i/\sqrt{1+\bar{z}\cdot z}, \quad g_{k+1,k+1} = 1/\sqrt{1+\bar{z}\cdot z}$
- Translations correspond to $g \to gg'$ with $g \sim gh$ for $h \in U(k)$. We define right translations: $\quad \hat{R}_A \, g = g \, T_A$

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- $\mathbb{CP}^k = SU(k+1)/U(k)$. We can use $(k+1)\times(k+1)$-matrix $g \in SU(k+1)$ as a coordinate.

  $g_{i,k+1} = z_i/\sqrt{1+\bar{z}\cdot z}, \quad g_{k+1,k+1} = 1/\sqrt{1+\bar{z}\cdot z}$
- Translations correspond to $g \to gg'$ with $g \sim gh$ for $h \in U(k)$. We define right translations: $\quad \hat{R}_A \, g = g \, T_A$
- $\hat{R}_a, \ \hat{R}_{k^2+2k} \to$ gauge transformations $\quad (\, U(k) \,)$
- $\hat{R}_{+i}, \hat{R}_{-i} \to$ covariant derivatives $\quad (i = 1, \cdots, k) \quad [\hat{R}_{+i}, \hat{R}_{-j}] \in U(k)$

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- $\mathbb{CP}^k = SU(k+1)/U(k)$. We can use $(k+1)\times(k+1)$-matrix $g \in SU(k+1)$ as a coordinate.

  $g_{i,k+1} = z_i/\sqrt{1+\bar{z}\cdot z}, \quad g_{k+1,k+1} = 1/\sqrt{1+\bar{z}\cdot z}$
- Translations correspond to $g \to gg'$ with $g \sim gh$ for $h \in U(k)$. We define right translations: $\quad \hat{R}_A\, g = g\, T_A$
- $\hat{R}_a,\ \hat{R}_{k^2+2k} \to$ gauge transformations $\quad (\, U(k)\,)$
- $\hat{R}_{+i}, \hat{R}_{-i} \to$ covariant derivatives $\quad (i = 1, \cdots, k) \quad [\hat{R}_{+i}, \hat{R}_{-j}] \in U(k)$
- Wavefunctions are written in terms of Wigner $\mathcal{D}$-functions

$$\Psi \sim \mathcal{D}^{(J)}_{l,\alpha}(g) = \langle\ l\ |\ \hat{g}\ |\ \alpha\ \rangle$$

quantum numbers of states in J rep.

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- $\mathbb{CP}^k = SU(k+1)/U(k)$. We can use $(k+1)\times(k+1)$-matrix $g \in SU(k+1)$ as a coordinate.

  $g_{i,k+1} = z_i/\sqrt{1+\bar{z}\cdot z}, \quad g_{k+1,k+1} = 1/\sqrt{1+\bar{z}\cdot z}$
- Translations correspond to $g \to gg'$ with $g \sim gh$ for $h \in U(k)$. We define right translations: $\hat{R}_A\, g = g\, T_A$
- $\hat{R}_a,\ \hat{R}_{k^2+2k} \to$ gauge transformations $\quad (\, U(k)\,)$
- $\hat{R}_{+i}, \hat{R}_{-i} \to$ covariant derivatives $\quad (i = 1, \cdots, k) \quad [\hat{R}_{+i}, \hat{R}_{-j}] \in U(k)$
- Wavefunctions are written in terms of Wigner $\mathcal{D}$-functions

$$\Psi \sim \mathcal{D}^{(J)}_{l,\alpha}(g) = \langle\, l \mid \hat{g} \mid \alpha \,\rangle$$

quantum numbers of states in J rep.

- How Ψ transforms under gauge transformations depends on choice of background fields

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- Choose "uniform" $U(1)$ or $U(k)$ background magnetic fields.

$$U(1): \quad \bar{a} \sim in\mathrm{Tr}(t_{k^2+2k} g^{-1} dg) \quad \Rightarrow \quad \bar{F} = d\bar{a} = n\,\Omega, \quad \Omega = \text{Kahler } 2 - \text{form}$$

$$SU(k): \quad \bar{A}^a \sim \mathrm{Tr}(t^a g^{-1} dg) \quad \Rightarrow \quad \bar{F}^a \sim \bar{R}^a \sim f^{aij} e^i \wedge e^j$$

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- Choose "uniform" $U(1)$ or $U(k)$ background magnetic fields.

$$U(1): \quad \bar{a} \sim in\mathrm{Tr}(t_{k^2+2k} g^{-1} dg) \quad \Rightarrow \quad \bar{F} = d\bar{a} = n\,\Omega, \quad \Omega = \text{Kahler } 2 - \text{form}$$

$$SU(k): \quad \bar{A}^a \sim \mathrm{Tr}(t^a g^{-1} dg) \quad \Rightarrow \quad \bar{F}^a \sim \bar{R}^a \sim f^{aij} e^i \wedge e^j$$

- $|\alpha\rangle$ have to obey

$$T^{k^2+2k}\,|\alpha\rangle = -\frac{nk}{\sqrt{2k(k+1)}}\,|\alpha\rangle, \qquad T^a\,|\alpha\rangle = (T^a)_{\alpha\beta}\,|\beta\rangle$$

QHE ON $\mathbb{CP}^k$: SINGLE PARTICLE SPECTRUM

- Choose “uniform” $U(1)$ or $U(k)$ background magnetic fields.

$$U(1): \quad \bar{a} \sim in\mathrm{Tr}(t_{k^2+2k}g^{-1}dg) \quad \Rightarrow \quad \bar{F} = d\bar{a} = n\,\Omega, \quad \Omega = \text{Kahler } 2-\text{form}$$

$$SU(k): \quad \bar{A}^a \sim \ \mathrm{Tr}(t^a g^{-1} dg) \quad \Rightarrow \quad \bar{F}^a \sim \bar{R}^a \sim f^{aij} e^i \wedge e^j$$

- $|\alpha\rangle$ have to obey

$$T^{k^2+2k}\,|\alpha\rangle = -\frac{nk}{\sqrt{2k(k+1)}}\,|\alpha\rangle, \qquad T^a\,|\alpha\rangle = (T^a)_{\alpha\beta}\,|\beta\rangle$$

- Wavefunctions for each Landau level form an $SU(k+1)$ representation J

$$\Psi^J_{l;\alpha} \sim \langle l \,|\, \hat{g} \,|\, \underbrace{\alpha}_{\downarrow} \ \rangle$$

fixed $U(1)_R$ charge $\sim$ n and some finite $SU(k)_R$ repr. $\tilde{J}$

$l = 1, \cdots \mathrm{dim}J \Longrightarrow$ counts degeneracy within a Landau level

$\alpha =$ internal index $= 1, \cdots, N' = \mathrm{dim}\tilde{J}$

QHE ON $\mathbb{CP}^k$: HAMILTONIAN

- Hamiltonian

$$H = \frac{1}{4mr^2} \sum_{i=1}^{k} (\hat{R}_{+i}\hat{R}_{-i} + \hat{R}_{-i}\hat{R}_{+i})$$

$$= \frac{1}{2mr^2} [C_2^{SU(k+1)}(J) - C_2^{SU(k)}(\tilde{J}) - \frac{n^2 k}{2(k+1)}]$$

QHE ON $\mathbb{CP}^k$: HAMILTONIAN

- Hamiltonian

$$H = \frac{1}{4mr^2}\sum_{i=1}^{k}(\hat{R}_{+i}\hat{R}_{-i} + \hat{R}_{-i}\hat{R}_{+i})$$
$$= \frac{1}{2mr^2}[C_2^{SU(k+1)}(J) - C_2^{SU(k)}(\tilde{J}) - \frac{n^2k}{2(k+1)}]$$

- Lowest Landau level: $\hat{R}_{-i}\Psi = 0$ Holomorphicity condition

  ($|\,\alpha\,\rangle$ is lowest weight state)

QHE on $\mathbb{CP}^k$: Hamiltonian

- Hamiltonian

$$H = \frac{1}{4mr^2} \sum_{i=1}^{k} (\hat{R}_{+i}\hat{R}_{-i} + \hat{R}_{-i}\hat{R}_{+i})$$

$$= \frac{1}{2mr^2} [C_2^{SU(k+1)}(J) - C_2^{SU(k)}(\tilde{J}) - \frac{n^2 k}{2(k+1)}]$$

- Lowest Landau level: $\hat{R}_{-i}\Psi = 0$ Holomorphicity condition

  ($| \alpha \rangle$ is lowest weight state)

LLL WAVEFUNCTIONS FOR $U(1)$ MAGNETIC FIELD

For a $U(1)$ magnetic field the LLL wavefunctions form a symmetric rank n representation for $SU(k+1)$ of dimension

$$N = \dim J = \frac{(n+k)!}{n!\,k!}$$

They can be written in terms of complex coordinates as

$$\begin{aligned} \Psi_{i_1 i_2 \cdots i_k} &= \sqrt{N}\left[\frac{n!}{i_1! i_2! \ldots i_k! (n-s)!}\right]^{\frac{1}{2}} \frac{z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}}{(1+\bar{z}\cdot z)^{\frac{n}{2}}}\ , \\ s &= i_1 + i_2 + \cdots + i_k\ , \quad 0 \le i_i \le n\ \ , \quad 0 \le s \le n \end{aligned}$$

They are degenerate with energy

$$E = \frac{1}{2mr^2}\frac{nk}{2}$$

MATRIX FORMULATION OF LLL DYNAMICS

- QHE on a compact space $M \Longrightarrow$ LLL defines an N-dim Hilbert space

  In the presence of confining potential $\Longrightarrow$ incompressible QH droplet

- K states are filled, $N - K$ unoccupied

  Density matrix for ground state droplet : $\hat{\rho}_0$

$$\hat{\rho}_0 = \begin{bmatrix} 1 & & & & & & & & \\ & 1 & & & & & & & \\ & & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & 1 & & & & \\ & & & & & 0 & & & \\ & & & & & & 0 & & \\ & & & & & & & \ddots & \\ & & & & & & & & 0 \end{bmatrix} \begin{array}{l} \Big\updownarrow \; K \\ \\ \Big\updownarrow \; N - K \end{array}$$

MATRIX FORMULATION OF LLL DYNAMICS

- QHE on a compact space $M \Longrightarrow$ LLL defines an N-dim Hilbert space

  In the presence of confining potential $\Longrightarrow$ incompressible QH droplet

- K states are filled, $N - K$ unoccupied

  Density matrix for ground state droplet : $\hat{\rho}_0$

$$\hat{\rho}_0 = \begin{array}{cc} \left[\begin{array}{ccccccccc} 1 &&&&&&&& \\ & 1 &&&&&&& \\ && 1 &&&&&& \\ &&& \ddots &&&&& \\ &&&& 1 &&&& \\ &&&&& 0 &&& \\ &&&&&& 0 && \\ &&&&&&& \ddots & \\ &&&&&&&& 0 \end{array}\right] & \begin{array}{l} \updownarrow K \\ \\ \\ \updownarrow N - K \end{array} \end{array}$$

- Under time evolution: $\hat{\rho}_0 \rightarrow \hat{\rho} = \hat{U}\,\hat{\rho}_0\,\hat{U}^{\dagger}$

  $\hat{U} = N \times N$ unitary matrix ; "collective" variable describing excitations within the LLL

MATRIX FORMULATION OF LLL DYNAMICS

The action for $\hat{U}$ is

$$S_0 = \int dt \, \mathrm{Tr} \left[i \hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right]$$

which leads to the evolution equation for density matrix

$$i \frac{d\hat{\rho}}{dt} = [\hat{V}, \hat{\rho}]$$

S_0 : universal matrix action

No explicit dependence on properties of space on which QHE is defined, abelian or nonabelian nature of fermions, etc.

NONCOMMUTATIVE FIELD THEORY

S_0 : action of a noncommutative field theory

$$\begin{aligned} S_0 &= \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right] \\ &= \mathrm{N} \int d\mu \, dt \; \left[i(\rho_0 * U^\dagger * \partial_t U) \; - \; (\rho_0 * U^\dagger * V * U) \right] \end{aligned}$$

$$\underbrace{\hat{\rho}_0, \hat{U}, \hat{V}}_{(N \times N)\text{ matrices}} \quad \Longrightarrow \quad \underbrace{\rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x})}_{\text{symbols}}$$

NONCOMMUTATIVE FIELD THEORY

S_0 : action of a noncommutative field theory

$$\begin{aligned} S_0 &= \int dt \,\mathrm{Tr}\left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \;-\; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U}\right] \\ &= N \int d\mu \, dt \;\left[i(\rho_0 * U^\dagger * \partial_t U) \;-\; (\rho_0 * U^\dagger * V * U)\right] \end{aligned}$$

$$\underbrace{\hat{\rho}_0, \hat{U}, \hat{V}}_{(N \times N)\text{ matrices}} \qquad \Longrightarrow \qquad \underbrace{\rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x})}_{\text{symbols}}$$

- symbol: $O(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}) \hat{O}_{ml}(t) \Psi_l^*(\vec{x})$

NONCOMMUTATIVE FIELD THEORY

S_0 : action of a noncommutative field theory

$$\begin{aligned} S_0 &= \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right] \\ &= N \int d\mu \, dt \, \left[i(\rho_0 * U^\dagger * \partial_t U) \; - \; (\rho_0 * U^\dagger * V * U) \right] \end{aligned}$$

$$\underbrace{\hat{\rho}_0, \hat{U}, \hat{V}}_{(N \times N) \text{ matrices}} \qquad \Longrightarrow \qquad \underbrace{\rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x})}_{\text{symbols}}$$

- symbol: $O(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}) \hat{O}_{ml}(t) \Psi_l^*(\vec{x})$
- $\hat{A} \, \hat{B} \quad \Longrightarrow \quad A(x) * B(x)$
- $\mathrm{Tr} \quad \Longrightarrow \quad N \int d\mu$

NONCOMMUTATIVE FIELD THEORY

S_0 : action of a noncommutative field theory

$$\begin{aligned} S_0 &= \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} \right] \\ &= N \int d\mu \, dt \; \left[i(\rho_0 * U^\dagger * \partial_t U) \; - \; (\rho_0 * U^\dagger * V * U) \right] \end{aligned}$$

$$\underbrace{\hat{\rho}_0, \hat{U}, \hat{V}}_{(N \times N)\text{ matrices}} \qquad \Longrightarrow \qquad \underbrace{\rho_0(\vec{x}), U(\vec{x}, t), V(\vec{x})}_{\text{symbols}}$$

- symbol: $O(\vec{x}, t) = \frac{1}{N} \sum_{m,l} \Psi_m(\vec{x}) \hat{O}_{ml}(t) \Psi_l^*(\vec{x})$
- $\hat{A}\;\hat{B} \quad \Longrightarrow \quad A(x) * B(x)$
- $\mathrm{Tr} \quad \Longrightarrow \quad N \int d\mu$

S_0 = exact bosonic action describing the dynamics of LLL fermions

SAKITA, 1993: 2 dim. context

DAS, DHAR, MANDAL, WADIA, 1992

EDGE EFFECTIVE ACTION FOR $\nu = 1$

Large N limit ($n \to \infty$) $\Longrightarrow$ WZW-like chiral edge action

EDGE EFFECTIVE ACTION FOR $\nu = 1$

Large N limit ($n \to \infty$) $\Longrightarrow$ WZW-like chiral edge action

A. Abelian background magnetic field $U(1)$

EDGE EFFECTIVE ACTION FOR $\nu = 1$

Large N limit ($n \to \infty$) $\Longrightarrow$ WZW-like chiral edge action

A. Abelian background magnetic field $U(1)$

- Introduce a boson field: $\hat{U} = \exp i\hat{\phi}$

EDGE EFFECTIVE ACTION FOR $\nu = 1$

Large N limit ($n \to \infty$) $\Longrightarrow$ WZW-like chiral edge action

A. Abelian background magnetic field $U(1)$

- Introduce a boson field: $\hat{U} = \exp i\hat{\phi}$
- $\left([\hat{X},\ \hat{Y}]\right)_{symbol} \to \frac{i}{n}(\Omega^{-1})^{ij}\,\partial_i X(\vec{x},t)\,\partial_j Y(\vec{x},t) + \cdots$

  ρ_0 = constant over the phase volume occupied by droplet

EDGE EFFECTIVE ACTION FOR $\nu = 1$

Large N limit ($n \to \infty$) $\Longrightarrow$ WZW-like chiral edge action

A. Abelian background magnetic field $U(1)$

- Introduce a boson field: $\hat{U} = \exp i\hat{\phi}$
- $\left([\hat{X},\ \hat{Y}]\right)_{symbol} \to \frac{i}{n}(\Omega^{-1})^{ij}\ \partial_i X(\vec{x},t)\ \partial_j Y(\vec{x},t) + \cdots$

  ρ_0 = constant over the phase volume occupied by droplet
- $S_0\ \to$ edge effective action

$$S_0 \sim \int_{\partial D} \left(\partial_t \phi + u\ \mathcal{L}\phi\right) \mathcal{L}\phi$$

$(2k-1)$ (space) dim chiral action defined on droplet boundary

$$\mathcal{L}\phi = (\Omega^{-1})^{ij} \hat{r}_j \partial_i \phi, \qquad \mathcal{L} = \begin{cases} \text{derivative along boundary of droplet} \\ \ \to \partial_\theta \ \text{in 2 dim.} \end{cases}$$

EDGE EFFECTIVE ACTION FOR $\nu = 1$

B. Nonabelian background magnetic field $U(k)$

- Wavefunction is a nontrivial representation of $SU(k) : dim(\tilde{J}) = N'$.
- Symbol = $(N' \times N')$ matrix valued function $\longrightarrow$ action in terms of $G \in U(N')$

EDGE EFFECTIVE ACTION FOR $\nu = 1$

B. Nonabelian background magnetic field $U(k)$

- Wavefunction is a nontrivial representation of $SU(k) : dim(\tilde{J}) = N'$.
- Symbol = $(N' \times N')$ matrix valued function $\longrightarrow$ action in terms of $G \in U(N')$
- The effective edge action is a gauged WZW action in $(2k-1, 1)$ dimensions.

$$
\begin{aligned}
S_0 =& \frac{1}{4\pi}\int_{\partial D} \mathrm{tr}\left[\left(G^\dagger \dot{G} + u\, G^\dagger \mathcal{L} G\right) G^\dagger \mathcal{L} G\right] \\
&+ \frac{1}{4\pi}\int_{D} \mathrm{tr}\left[-d\left(i\bar{A} dG G^\dagger + i\bar{A} G^\dagger dG\right) + \frac{1}{3}\left(G^\dagger dG\right)^3\right] \wedge \left(\frac{\Omega}{2\pi}\right)^{k-1} \frac{1}{(k-1)!} \\
\equiv& S_{\mathrm{WZW}}(A^L = A^R = \bar{A})
\end{aligned}
$$

$\mathcal{L} = (\Omega^{-1})^{ij}\hat{r}_j D_i$ = covariant derivative along the boundary of droplet

EFFECTIVE ACTION IN PRESENCE OF GAUGE FLUCTUATIONS

- In the presence of gauge fluctuations one starts with a gauged matrix action.

$$\partial_t \to \hat{D}_t = \partial_t + i\hat{\mathcal{A}}$$

$$S = \int dt \, \mathrm{Tr} \left[i\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U} \; - \; \hat{\rho}_0 \hat{U}^\dagger \hat{V} \hat{U} - \underbrace{\hat{\rho}_0 \, \hat{U}^\dagger \hat{\mathcal{A}} \hat{U}}_{\text{gauge interactions}} \right]$$

In terms of bosonic fields

$$S = N \int dt \, d\mu \, \mathrm{tr} \, \left[i\rho_0 * U^\dagger * \partial_t U \; - \; \rho_0 * U^\dagger * (V + \mathcal{A}) * U \right]$$

QUESTION: How is $\mathcal{A}$ related to the gauge fields coupled to the original fermions?

EFFECTIVE ACTION IN PRESENCE OF GAUGE FLUCTUATIONS

- S is invariant under

$$\begin{aligned} \delta U &= -i\lambda * U \\ \delta\mathcal{A}(\vec{x},t) &= \partial_t\lambda(\vec{x},t) - i\left(\lambda * (V+\mathcal{A}) - (V+\mathcal{A}) * \lambda\right) \end{aligned} \tag{1}$$

- Since S describes gauge interactions it has to be invariant under usual gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda + i[\ \bar{A}_\mu\ +\ A_\mu\ ,\ \Lambda]\ , \qquad \delta\bar{A}_\mu = 0 \tag{2}$$

Background

Perturbation

The strategy is to choose

$$\begin{aligned} \mathcal{A} &= \text{function}(A_\mu,\ \bar{A}_\mu, V) \\ \lambda &= \text{function}(\Lambda, A_\mu, \bar{A}_\mu) \end{aligned}$$

such that the gauge transformation (2) induces $\delta\mathcal{A}$ in (1) (generalized Seiberg-Witten map) (KARABALI, 2005)

EFFECTIVE ACTION IN PRESENCE OF GAUGE FLUCTUATIONS

- In the large N limit the result is $S = S_{\text{edge}} + S_{\text{bulk}}$

$S_{\text{edge}} \sim S_{\text{WZW}}\left(A^L = A + \bar{A}\,, A^R = \bar{A}\right)$ = Chirally gauged WZW action in $2k$ dim

$S_{\text{bulk}} \sim S_{CS}^{2k+1}(\tilde{A}) + \cdots$ = $(2k+1)$ dim CS action

$\tilde{A} = (A_0 + V, \bar{a}_i + \bar{A}_i + A_i)$ = background + fluctuations

- Gauge Invariance $\Longrightarrow$ Anomaly Cancellation

$$\delta S_{\text{edge}} \neq 0, \qquad \delta S_{\text{bulk}} \neq 0$$

$$\delta S_{\text{edge}} + \delta S_{\text{bulk}} = 0$$

BULK EFFECTIVE ACTION INCLUDING GAUGE AND METRIC FLUCTUATIONS

- What about metric fluctuations? There is another way to construct the bulk action including both gauge and metric fluctuations.

BULK EFFECTIVE ACTION INCLUDING GAUGE AND METRIC FLUCTUATIONS

- What about metric fluctuations? There is another way to construct the bulk action including both gauge and metric fluctuations.
- The lowest Landau level obeys the holomorphicity condition $\hat{R}_{-i}\Psi = 0$
- The number of normalizable solutions is given by the Dolbeault index.

$$\text{Index} = \int_M \text{td}(T_{C}M) \wedge \text{ch}(V)$$

BULK EFFECTIVE ACTION INCLUDING GAUGE AND METRIC FLUCTUATIONS

- What about metric fluctuations? There is another way to construct the bulk action including both gauge and metric fluctuations.
- The lowest Landau level obeys the holomorphicity condition $\hat{R}_{-i}\Psi = 0$
- The number of normalizable solutions is given by the Dolbeault index.

$$\text{Index} = \int_M \text{td}(T_{C}M) \wedge \text{ch}(V)$$

- For a fully filled LLL (each particle carries unit charge $e = 1$):

  degeneracy = Dolbeault index = charge

  $\Longrightarrow$ Dolbeault index density = charge density $\equiv J_0$

BULK EFFECTIVE ACTION INCLUDING GAUGE AND METRIC FLUCTUATIONS

- What about metric fluctuations? There is another way to construct the bulk action including both gauge and metric fluctuations.
- The lowest Landau level obeys the holomorphicity condition $\hat{R}_{-i}\Psi = 0$
- The number of normalizable solutions is given by the Dolbeault index.

$$\text{Index} = \int_M \text{td}(T_C M) \wedge \text{ch}(V)$$

- For a fully filled LLL (each particle carries unit charge $e = 1$):

  degeneracy = Dolbeault index = charge

  $\Longrightarrow$ Dolbeault index density = charge density $\equiv J_0$

- So we can use

$$\frac{\delta S_{eff}}{\delta A_0} = J_0 = \text{Dolbeault index density}$$

BULK TOPOLOGICAL EFFECTIVE ACTION: EXAMPLES

- $\mathbb{CP}^1 = SU(2)/U(1)$; s-th LL

$$S_{3d}^{(LLL)} = \frac{i^2}{4\pi} \int \left\{ \Big(A + (s + \frac{1}{2})\omega \Big) d \Big(A + (s + \frac{1}{2})\omega \Big) - \frac{1}{12} \omega \, d\omega \right\}$$

Agrees with ABANOV, GROMOV; KLEVTSOV ET AL; BRADLYN, READ; CAN, LASKIN, WIEGMANN

BULK TOPOLOGICAL EFFECTIVE ACTION: EXAMPLES

- $\mathbb{CP}^1 = SU(2)/U(1)$; s-th LL

$$S_{3d}^{(LLL)} = \frac{i^2}{4\pi} \int \left\{ \left(A + (s + \frac{1}{2})\omega \right) d \left(A + (s + \frac{1}{2})\omega \right) - \frac{1}{12} \omega \, d\omega \right\}$$

Agrees with ABANOV, GROMOV; KLEVTSOV ET AL; BRADLYN, READ; CAN, LASKIN, WIEGMANN

- We have general results for arbitrary dimensions, higher Landau levels and nonabelian magnetic fields (KARABALI AND NAIR, 2016)

BULK TOPOLOGICAL EFFECTIVE ACTION: EXAMPLES

- $\mathbb{CP}^1 = SU(2)/U(1)$; s-th LL

$$S_{3d}^{(LLL)} = \frac{i^2}{4\pi}\int\left\{\Big(A+(s+\frac{1}{2})\omega\Big)d\Big(A+(s+\frac{1}{2})\omega\Big)-\frac{1}{12}\omega\, d\omega\right\}$$

Agrees with ABANOV, GROMOV; KLEVTSOV ET AL; BRADLYN, READ; CAN, LASKIN, WIEGMANN

- We have general results for arbitrary dimensions, higher Landau levels and nonabelian magnetic fields (KARABALI AND NAIR, 2016)

- $\mathbb{CP}^2 = SU(3)/U(2)$; LLL, Abelian gauge field

$$\begin{aligned} S_{5d}^{(s)} &= \frac{i^3}{(2\pi)^2}\int\left\{\frac{1}{3!}\Big(A+\omega^0\Big)\Big(dA+d\omega^0\Big)^2\right. \\ &\qquad \left.-\frac{1}{12}\Big(A+\omega^0\Big)\left[(d\omega^0)^2+\frac{1}{2}\mathrm{Tr}(\tilde{R}\wedge\tilde{R})\right]\right\} \end{aligned}$$

$\omega^0 \sim U(1)$ part of spin connection; $\tilde{R} \sim SU(2)$ nonabelian part of the curvature.

ENTANGLEMENT ENTROPY FOR QHE

- We divide the system into two regions, D and its complementary D^C, and define the reduced density matrix

  $$\rho_D = \mathrm{Tr}_{D^C} \left|GS\right\rangle \left\langle GS\right|$$

  where $\left|GS\right\rangle = \prod_m c_m^\dagger \left|0\right\rangle$.

ENTANGLEMENT ENTROPY FOR QHE

- We divide the system into two regions, D and its complementary D^C, and define the reduced density matrix

  $$\rho_D = \mathrm{Tr}_{D^C} \left|GS\right\rangle \left\langle GS\right|$$

  where $\left|GS\right\rangle = \prod_m c_m^\dagger \left|0\right\rangle$.
- The entanglement entropy is defined as

  $$S = -\mathrm{Tr}\rho_D \log \rho_D$$

- We choose D to be the spherically symmetric region of $\mathbb{CP}^k$ satisfying $z \cdot \bar{z} \leq R^2$. For $\mathbb{CP}^1 \sim S^2$, D is a polar cap around the north pole with latitude angle θ. $R = \tan\theta/2$ via stereographic projection.

ENTANGLEMENT ENTROPY FOR INTEGER QHE

- The entanglement entropy can also be written as

$$S = -\mathrm{Tr}\rho_D \log \rho_D = -\sum_{m=1}^{N} \left[\lambda_m \log \lambda_m + (1-\lambda_m)\log(1-\lambda_m)\right]$$

ENTANGLEMENT ENTROPY FOR INTEGER QHE

- The entanglement entropy can also be written as

$$S = -\mathrm{Tr}\rho_D \log \rho_D = -\sum_{m=1}^{N} \left[\lambda_m \log \lambda_m + (1-\lambda_m)\log(1-\lambda_m)\right]$$

- λ's are eigenvalues of the two-point correlator (PESCHEL, 2003)

$$C(r,r') = \sum_{m=1}^{N} \Psi_m^*(z)\,\Psi_m(z') \ , \qquad z, z' \in D$$

$$\int_D C(r,r')\Psi_l^*(z')d\mu(z') = \lambda_l\,\Psi_l^*(z)$$

where

$$\lambda_l = \int_D |\Psi_l|^2 d\mu$$

2D RESULTS

- For 2d gapped systems

$$S = c\,L - \gamma + \mathcal{O}(1/L)$$

L: perimeter of boundary

c: non-universal constant

γ: universal, topological entanglement entropy ; $\gamma = 0$ for IQHE

2D RESULTS

- For 2d gapped systems

$$S = c\,L - \gamma + \mathcal{O}(1/L)$$

L: perimeter of boundary

c: non-universal constant

γ: universal, topological entanglement entropy ; $\gamma = 0$ for IQHE

- For integer QHE on $S^2 = \mathbb{CP}^1$ RODRIGUEZ AND SIERRA, 2009

For $\nu = 1$: $c = 0.204$

General results on Kähler manifolds CHARLES AND ESTIENNE, 2019

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND ABELIAN MAGNETIC FIELD

A. QHE on $\mathbb{CP}^k$ with $U(1)$ magnetic field

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND ABELIAN MAGNETIC FIELD

A. QHE on $\mathbb{CP}^k$ with $U(1)$ magnetic field

The LLL wavefunctions are essentially the coherent states of $\mathbb{CP}^k$.

$$\begin{aligned}\Psi_{i_1 i_2 \cdots i_k} &= \sqrt{N}\left[\frac{n!}{i_1! i_2! \ldots i_k! (n-s)!}\right]^{\frac{1}{2}} \frac{z_1^{i_1} z_2^{i_2} \cdots z_k^{i_k}}{(1+\bar{z}\cdot z)^{\frac{n}{2}}} \,, \\ s &= i_1 + i_2 + \cdots + i_k \,, \quad 0 \leq i_i \leq n \;\;, \quad 0 \leq s \leq n\end{aligned}$$

They form an $SU(k+1)$ representation of dimension

$$N = \mathrm{dim} J = \frac{(n+k)!}{n!\, k!}$$

The volume element for $\mathbb{CP}^k$ is

$$d\mu = \frac{k!}{\pi^k} \frac{d^2 z_1 \cdots d^2 z_k}{(1+\bar{z}\cdot z)^{k+1}} \quad , \quad \int d\mu = 1$$

ENTANGLEMENT ENTROPY FOR QHE ON $\mathbb{CP}^k$ AND ABELIAN MAGNETIC FIELD

- The eigenvalues $\lambda = \int_D \Psi^* \Psi$ are given by

$$\lambda_{i_1 i_2 \cdots i_k} \equiv \lambda_s = \frac{(n+k)!}{(n-s)!(s+k-1)!} \int_0^{t_0} dt\, t^{s+k-1}\,(1-t)^{n-s}$$

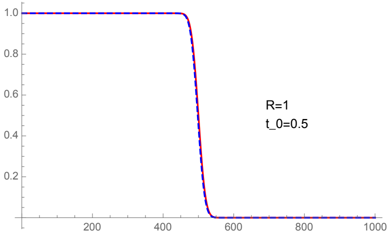
  where $t_0 = R^2/(1+R^2)$.

- The entanglement entropy is

$$\begin{aligned} S &= \sum_{s=0}^{n} \overbrace{\frac{(s+k-1)!}{s!(k-1)!}}^{degeneracy} H_s \\ H_s &= \left[-\lambda_s \log \lambda_s - (1-\lambda_s)\log(1-\lambda_s)\right] \end{aligned}$$

- For large n, this is amenable to an analytical semiclassical calculation for all $k \ll n$.

SEMICLASSICAL TREATMENT FOR LARGE n

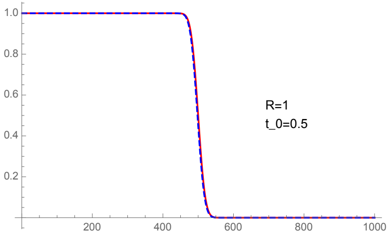


Graph of λ_s vs s

Transition ($\lambda = \frac{1}{2}$) at $s^* \sim n\, t_0$

$k = 1$, $k = 5$

SEMICLASSICAL TREATMENT FOR LARGE n

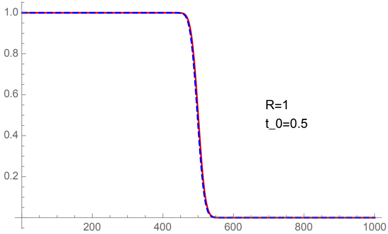


Graph of λ_s vs s

Transition ($\lambda = \frac{1}{2}$) at $s^* \sim n\, t_0$

$k = 1, k = 5$

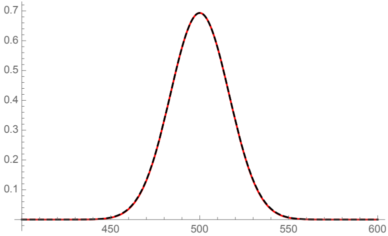
SEMICLASSICAL TREATMENT FOR LARGE n



Graph of λ_s vs s

Transition ($\lambda = \frac{1}{2}$) at $s^* \sim n\, t_0$

$k = 1, k = 5$

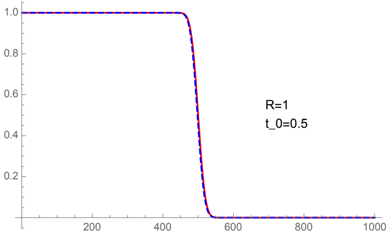


Graph of H_s vs s

—- exact

- - - - Gaussian approximation

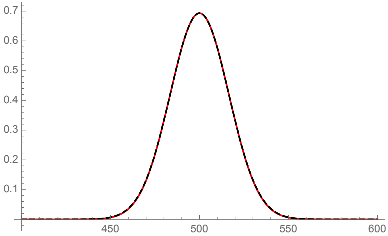
SEMICLASSICAL TREATMENT FOR LARGE n



Graph of λ_s vs s

Transition ($\lambda = \frac{1}{2}$) at $s^* \sim n\, t_0$

$k = 1$, $k = 5$



Graph of H_s vs s

—- exact

- - - - Gaussian approximation

Only wavefunctions localized around the boundary of the entangling surface contribute to entropy.

UNIVERSAL FORM FOR ENTANGLEMENT ENTROPY FOR $\nu = 1$

From semiclassical analysis

$$S \sim n^{k - \frac{1}{2}} \frac{\pi\,(\log 2)^{3/2}}{2\,k!} \underbrace{2k\frac{R^{2k-1}}{(1+R^2)^k}}_{\textit{geometric area}} \sim c_k \text{ Area}$$

In agreement with $k = 1$ result by RODRIGUEZ AND SIERRA

UNIVERSAL FORM FOR ENTANGLEMENT ENTROPY FOR $\nu = 1$

From semiclassical analysis

$$S \sim n^{k-\frac{1}{2}} \frac{\pi\,(\log 2)^{3/2}}{2\,k!}\ \underbrace{2k\frac{R^{2k-1}}{(1+R^2)^k}}_{\textit{geometric area}} \sim c_k \text{ Area}$$

In agreement with $k = 1$ result by RODRIGUEZ AND SIERRA

- Formula for entropy becomes universal if expressed in terms of a ”phase space” area instead of a geometric area.
- $V_{\text{phase space}} \to \frac{n^k}{k!} \int \Omega^k = \frac{n^k}{k!} \int d\mu$

$$A_{\text{phase space}} = \frac{n^{k-\frac{1}{2}}}{k!} A_{\text{geom}} = \frac{n^{k-\frac{1}{2}}}{k!}\ 2k\frac{R^{2k-1}}{(1+R^2)^k}$$

$$S \sim \frac{\pi}{2} (\log 2)^{3/2} A_{\text{phase space}}$$

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND NONABELIAN MAGNETIC FIELD

B. QHE on $\mathbb{CP}^k$ with $U(1) \times SU(k)$ magnetic field

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND NONABELIAN MAGNETIC FIELD

B. QHE on $\mathbb{CP}^k$ with $U(1) \times SU(k)$ magnetic field

- Wavefunctions carry $SU(k)$ charge : $\Psi_\alpha \,, \alpha = 1, \cdots \dim \tilde{J} = N'$. There are N' distinct classes of λ_s^α. Calculations long and tedious....

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND NONABELIAN MAGNETIC FIELD

B. QHE on $\mathbb{CP}^k$ with $U(1) \times SU(k)$ magnetic field

- Wavefunctions carry $SU(k)$ charge : Ψ_α , $\alpha = 1, \cdots \dim\tilde{J} = N'$. There are N' distinct classes of λ_s^α. Calculations long and tedious....

- Simplifications at large n
  - $S \to \dim\tilde{J}\; n^{k-\frac{1}{2}}\, \frac{\pi\ (\log 2)^{3/2}}{2\ k!} A_{geom}$

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND NONABELIAN MAGNETIC FIELD

B. QHE on $\mathbb{CP}^k$ with $U(1) \times SU(k)$ magnetic field

- Wavefunctions carry $SU(k)$ charge : Ψ_α , $\alpha = 1, \cdots \dim\tilde{J} = N'$. There are N' distinct classes of λ_s^α. Calculations long and tedious....

- Simplifications at large n
  - $S \to \dim\tilde{J}\; n^{k-\,\frac{1}{2}}\; \frac{\pi\;(\log 2)^{3/2}}{2\;k!} A_{geom}$
  - Degeneracy of LLL : $N \to \dim\tilde{J}\; \frac{n^k}{k!}$

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND NONABELIAN MAGNETIC FIELD

B. QHE on $\mathbb{CP}^k$ with $U(1) \times SU(k)$ magnetic field

- Wavefunctions carry $SU(k)$ charge : Ψ_α , $\alpha = 1, \cdots \dim \tilde{J} = N'$. There are N' distinct classes of λ_s^α. Calculations long and tedious....

- Simplifications at large n
  - $S \to \dim \tilde{J}\; n^{k-\frac{1}{2}}\; \frac{\pi\; (\log 2)^{3/2}}{2\; k!} A_{geom}$
  - Degeneracy of LLL : $N \to \dim \tilde{J}\; \frac{n^k}{k!}$

- The corresponding phase-space volume in this case is $V_{\text{phase space}} = \dim \tilde{J}\; \frac{n^k}{k!} \int d\mu$

$$S \sim \frac{\pi}{2} (\log 2)^{3/2}\, A_{\text{phase space}}$$

for any dimension and Abelian or non-Abelian background. (KARABALI, 2020)

ENTANGLEMENT ENTROPY FOR $\nu = 1$ ON $\mathbb{CP}^k$ AND NONABELIAN MAGNETIC FIELD

B. QHE on $\mathbb{CP}^k$ with $U(1) \times SU(k)$ magnetic field

- Wavefunctions carry $SU(k)$ charge : Ψ_α , $\alpha = 1, \cdots \dim \tilde{J} = N'$. There are N' distinct classes of λ_s^α. Calculations long and tedious....

- Simplifications at large n
  - $S \to \dim\tilde{J}\; n^{k-\frac{1}{2}}\; \frac{\pi\;(\log 2)^{3/2}}{2\;k!} A_{geom}$
  - Degeneracy of LLL : $N \to \dim\tilde{J}\; \frac{n^k}{k!}$

- The corresponding phase-space volume in this case is $V_{\text{phase space}} = \dim\tilde{J}\; \frac{n^k}{k!} \int d\mu$

$$S \sim \frac{\pi}{2} (\log 2)^{3/2} A_{\text{phase space}}$$

for any dimension and Abelian or non-Abelian background. (KARABALI, 2020)

- What about higher Landau levels?

1ST EXCITED LANDAU LEVEL

QHE on $S^2 = \mathbb{CP}^1$; 1st excited Landau level

1ST EXCITED LANDAU LEVEL

QHE on $S^2 = \mathbb{CP}^1$; 1st excited Landau level

- Degeneracy of q-th excited level = $n + 2q + 1$

1ST EXCITED LANDAU LEVEL

QHE on $S^2 = \mathbb{CP}^1$; 1st excited Landau level

- Degeneracy of q-th excited level = $n + 2q + 1$

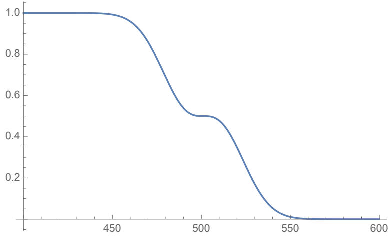
$$\lambda_s^{(q=1)} = \frac{(n+3)!(n+2)}{s!(n+2-s)!} \int_0^{t_0} dt\, t^{s-1}(1-t)^{n-s+1} \, [t - \frac{s}{n+2}]^2$$

1ST EXCITED LANDAU LEVEL

QHE on $S^2 = \mathbb{CP}^1$; 1st excited Landau level

- Degeneracy of q-th excited level = $n + 2q + 1$

$$\lambda_s^{(q=1)} = \frac{(n+3)!(n+2)}{s!(n+2-s)!} \int_0^{t_0} dt\, t^{s-1}(1-t)^{n-s+1} \, [t - \frac{s}{n+2}]^2$$



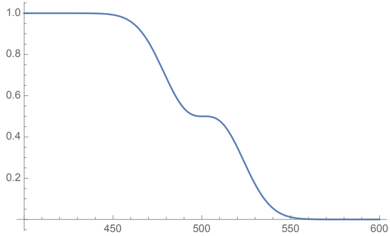
- Step-like pattern around the transition point.

1ST EXCITED LANDAU LEVEL

QHE on $S^2 = \mathbb{CP}^1$; 1st excited Landau level

- Degeneracy of q-th excited level = $n + 2q + 1$

$$\lambda_s^{(q=1)} = \frac{(n+3)!(n+2)}{s!(n+2-s)!} \int_0^{t_0} dt\, t^{s-1}(1-t)^{n-s+1} \left[t - \frac{s}{n+2}\right]^2$$

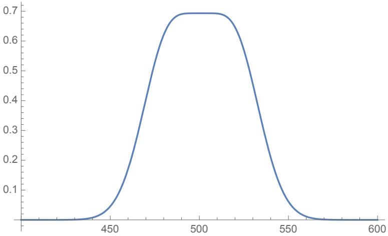


- Step-like pattern around the transition point.

  1st excited level wavefunctions have a node.

1st excited Landau level

- The step-like plateau of λ causes the broadening of the entropy H_s around $\lambda = 1/2$. H_s cannot be approximated with a simple Gaussian.



- Previous analysis does not work.

$$S^{(q=1)} = 1.65\, S^{(q=0)}$$

$\nu = 2$ CASE

What happens when both $q = 0$ and $q = 1$ Landau levels are full, namely $\nu = 2$?

$\nu = 2$ CASE

What happens when both $q = 0$ and $q = 1$ Landau levels are full, namely $\nu = 2$?

The two-point correlator now is given by

$$C(r, r') = \sum_{s=0}^{n} \Psi_s^{*0}(r)\Psi_s^0(r') + \sum_{s=0}^{n+2} \Psi_s^{*1}(r)\Psi_s^1(r')$$

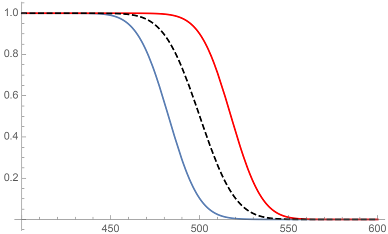
There are $2n + 4$ eigenvalues: λ_0^1 , $\tilde{\lambda}_s^{\pm}$, λ_{n+2}^1, $s = 0, \cdots, n$ and

$$\tilde{\lambda}_s^{\pm} = \frac{\lambda_s^0 + \lambda_{s+1}^1 \pm \sqrt{(\lambda_s^0 - \lambda_{s+1}^1)^2 + 4(\delta\lambda)_{s,s+1}^2}}{2}$$

where

$$\delta\lambda_{s,s+1} = \int_D \Psi_s^{*(q=0)}(r)\ \Psi_{s+1}^{(q=1)}(r)\ d\mu$$

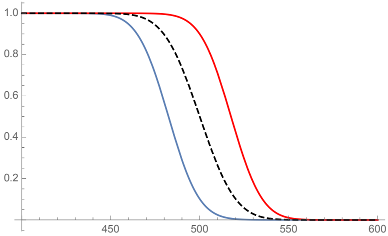
$\nu = 2$ CASE



$\tilde{\lambda}_s^+$, $\tilde{\lambda}_s^-$

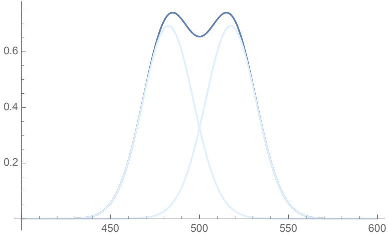
- - - for $\nu = 1$

$\nu = 2$ CASE



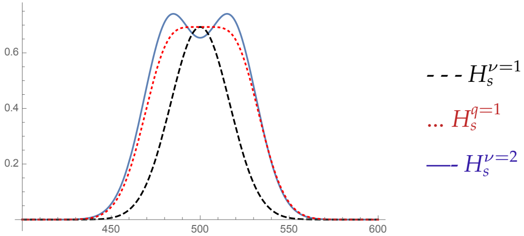
$\tilde{\lambda}_s^+$, $\tilde{\lambda}_s^-$

- - - for $\nu = 1$



$\tilde{H}_s{}^+ + \tilde{H}_s{}^-$

COMPARISON BETWEEN $q = 0$, $q = 1$, $\nu = 2$



$$S = \sum H_s$$

$$S^{(\nu=2)} > S^{(q=1)} > S^{(\nu=1)}$$

$$S^{(q=1)} = 1.65\, S^{(\nu=1)}$$

$$S^{(\nu=2)} = 1.76\, S^{(\nu=1)}$$

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action $\rightarrow$ noncommutative bosonic field theory

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action → noncommutative bosonic field theory
- At large N limit → anomaly free bulk/edge dynamics

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action → noncommutative bosonic field theory
- At large N limit → anomaly free bulk/edge dynamics
- Use index theorems to include gauge and metric perturbations : New response functions associated with non-Abelian gauge/gravitational fluctuations

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action → noncommutative bosonic field theory
- At large N limit → anomaly free bulk/edge dynamics
- Use index theorems to include gauge and metric perturbations : New response functions associated with non-Abelian gauge/gravitational fluctuations
- Entanglement entropy for higher dim QHE on $\mathbb{CP}^k$: For $\nu = 1$ there is a universal formula valid for any k, Abelian or non-Abelian background if area is expressed in terms of phase-space area.

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action → noncommutative bosonic field theory
- At large N limit → anomaly free bulk/edge dynamics
- Use index theorems to include gauge and metric perturbations : New response functions associated with non-Abelian gauge/gravitational fluctuations
- Entanglement entropy for higher dim QHE on $\mathbb{CP}^k$: For $\nu = 1$ there is a universal formula valid for any k, Abelian or non-Abelian background if area is expressed in terms of phase-space area.
- When the boundary of the entangling surface intersects the edge boundary there is additional log contribution in 2d, $S_{edge} \sim \frac{c}{6}\log(l)$.

  ESTIENNE AND STEPHAN, 2019; ROZON, BOLTEAU AND WITZAK-KREMPA, 2019

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action → noncommutative bosonic field theory
- At large N limit → anomaly free bulk/edge dynamics
- Use index theorems to include gauge and metric perturbations : New response functions associated with non-Abelian gauge/gravitational fluctuations
- Entanglement entropy for higher dim QHE on $\mathbb{CP}^k$: For $\nu = 1$ there is a universal formula valid for any k, Abelian or non-Abelian background if area is expressed in terms of phase-space area.
- When the boundary of the entangling surface intersects the edge boundary there is additional log contribution in 2d, $S_{edge} \sim \frac{c}{6}\log(l)$.

  Estienne and Stephan, 2019; Rozon, Bolteau and Witzak-Krempa, 2019

  This was extended to 4d by Estienne, Oblak and Stephan, 2021

SUMMARY, COMMENTS

- QHE on $\mathbb{CP}^k$: platform for arbitrary even dimensions
- LLL dynamics: Universal matrix action → noncommutative bosonic field theory
- At large N limit → anomaly free bulk/edge dynamics
- Use index theorems to include gauge and metric perturbations : New response functions associated with non-Abelian gauge/gravitational fluctuations
- Entanglement entropy for higher dim QHE on $\mathbb{CP}^k$: For $\nu = 1$ there is a universal formula valid for any k, Abelian or non-Abelian background if area is expressed in terms of phase-space area.
- When the boundary of the entangling surface intersects the edge boundary there is additional log contribution in 2d, $S_{edge} \sim \frac{c}{6} \log(l)$.

  ESTIENNE AND STEPHAN, 2019; ROZON, BOLTEAU AND WITZAK-KREMPA, 2019

  This was extended to 4d by ESTIENNE, OBLAK AND STEPHAN, 2021

  What are the contributions from non-Abelian droplets?