# Landau levels for Bochner Laplacian, Topological Quantum Phases of Matter Beyond Two Dimensions, Paris, October 2022

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# Landau level

The Landau Hamiltonian is the operator $\hat{H}$ acting on $L^2(\mathbb{R}^2)$

$$\begin{aligned} \hat{H} &= \tfrac{1}{2}(\pi_x^2 + \pi_y^2) \qquad \text{with } \pi_x = \tfrac{1}{i}\partial_x + \tfrac{B}{2}y,\ \pi_y = \tfrac{1}{i}\partial_y - \tfrac{B}{2}x \\ &= B(\mathfrak{a}^*\mathfrak{a} + \tfrac{1}{2}) \qquad \text{with } \mathfrak{a} = \tfrac{1}{\sqrt{2B}}(-\pi_y + i\pi_x),\ [\mathfrak{a}, \mathfrak{a}^*] = 1 \end{aligned}$$

The spectrum of $\hat{H}$ is $\{B(n+\frac{1}{2}) /\ n \in \mathbb{N}\}$. Its eigenspaces are the Landau levels

$$\operatorname{Ker}(\hat{H} - B(n+\tfrac{1}{2})) = (\mathfrak{a}^*)^n \operatorname{Ker}(H - \tfrac{B}{2})$$

If we restrict $\hat{H}$ to $\operatorname{span}\{(\mathfrak{a}^*)^n|1\rangle,\ n \in \mathbb{N}\}$ with $|1\rangle = e^{-\frac{B}{4}(x^2+y^2)}$, same spectrum but simple eigenvalues.

## Goal

- define Landau levels for Bochner Laplacian of compact manifold, understand influence of topology and geometry
- dimension of Landau levels in terms of characteristic classes, propagation, density of states.

# Bochner Laplacian

Datas:

- $(M, g)$ compact riemannian manifold with $\partial M = \emptyset$
- $L \to M$ hermitian line bundle with a connection $\nabla$

The Bochner Laplacian, or Schrödinger operator with magnetic field $\omega = i\,\mathrm{courb}(\nabla)$ is

$$\Delta = \frac{1}{2}\nabla^*\nabla = -\frac{1}{2\sqrt{g}}\nabla_i(g^{ij}\sqrt{g}\nabla_j)$$

acting on $\mathcal{C}^\infty(M, L)$ with $\nabla_i = \partial_{x_i} + \frac{1}{i}a_i$ where $d(a_i dx_i) = \omega$.

## Semiclassical limit $k = \hbar^{-1}$, $k \to \infty$

take $k \in \mathbb{N}$ and replace $L$ by $L^k = L^{\otimes k}$, $\nabla$ by $\nabla^{L^k}$ and set

$$\hat{H}_k = \frac{k^{-2}}{2}(\nabla^{L^k})^*\nabla^{L^k} = \frac{1}{2}g^{ij}\pi_i\pi_j + b_i k^{-1}\pi_i$$

with $\pi_i = \frac{1}{ik}\partial_{x_i} - a_i$, the dynamical moments, $ik[\pi_i, \pi_j] = \omega_{ij}$.

# Landau level for surfaces

Assume that $\omega$ is non degenerate and $M$ is a surface.
Write $\omega = B \, \mathrm{vol}_g$ with $B \in \mathcal{C}^\infty(M, \mathbb{R})$ positive.

## Iengo-Li (94)

If $B$ and the Gaussian curvature $S$ are constant, then for any $n$, when $k$ is sufficiently large,

1. the $n$-th eigenvalue of $\hat{H}_k$ is $k^{-1}B(n+\frac{1}{2}) + k^{-2}S\frac{n(n+1)}{2}$
2. its multiplicity is $\frac{k}{2\pi}\int_M \omega + (\frac{1}{2}+n)\chi(M)$

$\frac{1}{2\pi}\int_M \omega$ is the degree of $L$.
Proof by Riemann-Roch formula and ladder operators.

What can be said when $B$ is not constant and when we add a potential $k^{-1}V \in \mathcal{C}^\infty(M, \mathbb{R})$ ?
Answer: effective hamiltonian $\lambda_n = B(n+\frac{1}{2}) + V$ on the phase space $(M, \omega)$.

Recall that $\lambda_n = B(n + \frac{1}{2}) + V$ with $\omega = B \operatorname{vol}_g$ and $\hat{H}_k = \frac{k^{-2}}{2}(\nabla^{L^k})^* \nabla^{L^k} + k^{-1} V$.
Choose $E < E'$ and set $d_k(E, E') = \operatorname{rank} 1_{[E,E']}(k\hat{H}_k)$ .
Then

1. if $\max \lambda_n < E < E' < \min \lambda_{n+1}$, then when $k$ is large

$$d_k(E, E') = 0.$$

2. if $\max \lambda_{n-1} < E < \min \lambda_n$ and $\max \lambda_n < E' < \min \lambda_{n+1}$, then when $k$ is large,

$$d_k(E, E') = \frac{k}{2\pi} \int_M \omega + (\frac{1}{2} + n)\chi(M).$$

3. in general except for a countable set of $E$, $E'$,

$$d_k(E, E') = \frac{k}{2\pi} \sum_n \operatorname{vol}(E < \lambda_n < E') + o(k)$$

1. and 2. seem to be new, proof of 2. by a semiclassical construction of ladder operator. 3. is due to Demailly (85).

# Dynamics in Landau levels

Set $\mathcal{H}_n = \operatorname{Im} 1_{[E_-,E_+]}(k\hat{H}_k)$ with $\lambda_{n-1} < E_- < \lambda_n < E_+ < \lambda_{n+1}$.

Let $\Psi \in \mathcal{H}_n$ and define

$$\Psi(t) = \exp(itk\hat{H}_k)\Psi, \qquad t \in \mathbb{R}$$

The $L^2$-norm of $\Psi(t)$ is $(\int_M |\Psi(t)|^2(x) d\,\mathrm{vol}_g(x))^{1/2}$, so if $\|\Psi\| = 1$, $|\Psi(t)|^2$ is the probability density function of the particle's position.

## Theorem (C, 21)

$$|\Psi(kt)|^2 \,\mathrm{vol}_g = (\Phi_t)_*(|\Psi|^2 \,\mathrm{vol}_g) + \mathcal{O}(k^{-1})$$

*where $(\Phi_t)$ is the Hamiltonian flow of $\lambda_n$ in $(M, \omega)$.*

More precisely, for any $f \in \mathcal{C}^\infty(M, \mathbb{R})$,
$\int |\Psi(kt)|^2 f \,\mathrm{vol}_g = \int |\Psi|^2 (f \circ \Phi_t)\ \mathrm{vol}_g + \mathcal{O}_f(k^{-1})$ and the $\mathcal{O}$ is uniform when $t$ remains bounded.

# Toeplitz quantization

Recall that in the geometric quantization of $(M, \omega)$ :

1. the space $H^0(M, L^k)$ of holomorphic sections is the quantum space
2. to any $f \in \mathcal{C}^\infty(M)$ is associated the Toeplitz operator $T_k(f)$ acting on $H^0(M, L^k)$ such that $\langle T_k(f)\Psi, \Psi' \rangle = \langle f\Psi, \Psi' \rangle$.

We have the usual semi-classical properties:

$$T_k(f)T_k(g) \equiv T_k(fg), \qquad ik[T_k(f), T_k(g)] \equiv T_k(\{f, g\})$$

modulo $\mathcal{O}(k^{-1})$.

## Theorem (C, 21)

*There exists unitary isomorphisms*

$$U_k : \mathcal{H}_n = \operatorname{Im} 1_{[E_-, E_+]}(k\hat{H}_k) \to H^0(M, L^k \otimes K^{-n}), \; k \in \mathbb{N}$$

*with $K$ the canonical bundle of $M$ such that*

$$U_k(k\hat{H}_k)U_k^* \equiv T_k(\lambda_n), \qquad U_k\, f\, U_k^* \equiv T_k(f), \, \forall f \in \mathcal{C}^\infty(M)$$

*modulo $\mathcal{O}(k^{-1})$.*

# Generalization in higher dimension

There exists a semi-classical theory where

1. $\hat{H}_k$ is a semi-classical differential operator with symbol $H \in \mathcal{C}^\infty(T^*M)$ equal to $H(x,\xi) = \frac{1}{2} g^{ij}(x)\xi_i\xi_j$.
2. the phase space is $T^*M$ with the symplectic form $\Omega = \sum d\xi_i \wedge dx_i - \omega$ (minimal coupling).

In particular the Weyl law holds:

$$\operatorname{rank} 1_{]-\infty,E]}(\hat{H}_k) = \left(\frac{k}{2\pi}\right)^{m'} (\operatorname{vol}(\{H \leqslant E\}) + o(1))$$

with $m' = \dim M$.

The spectrum of $\hat{H}_k$ in a window $[E - Ck^{-1}, E + Ck^{-1}]$ can be described in terms of the closed hamiltonian orbits of $H$ in the level set $\{H = E\}$, for instance with

1. the Bohr-Sommerfeld conditions when $H$ is integrable
2. Gutzwiller trace formula when the orbits are non-degenerate.

Here, the Landau levels are in windows $[-Ck^{-1}, Ck^{-1}]$, $\{H = 0\}$ is the null-section of $T^*M$, the flow is stationnary...

From now on, assume $\omega$ is non degenerate, $\dim M = 2m$ and let $0 < B_1(y) \leqslant \ldots \leqslant B_m(y)$ be the $g$-eigenvalues of $\omega$ at $y \in M$. [1] These functions are continuous.

In his paper on holomorphic Morse inequalities, Demailly proved that for almost any $E$,

$$\operatorname{rank} 1_{(-\infty,E)}(k\hat{H}_k) = \left(\frac{k}{2\pi}\right)^m \sum_\alpha \operatorname{vol}(\lambda_\alpha \leqslant E) + \mathrm{o}(k^m)$$

where vol is the volume in $M$ for $\mu_L = \omega^{\wedge m}/m!$ and for any $\alpha \in \mathbb{N}^m$, $\lambda_\alpha = \sum_i B_i(\frac{1}{2} + \alpha(i)) + V$.

$\{\lambda_\alpha(y), \alpha \in \mathbb{N}^m\}$ is the spectrum of the Landau Hamiltonian

$$L_H(y) := \tfrac{1}{2} g^{ij}(y)\pi_i(y)\pi_j(y) + V(y)$$

acting on $T_y M \simeq \mathbb{R}^{2m}$ with $\pi_i(y) = \frac{1}{i}\partial_{x_i} + \frac{1}{2}\omega_{ij}(y)x_j$.

[1] With good coordinates $g_{ij}(y) = \delta_{ij}$ and $\omega|_y = B_1(y)dx_1 \wedge dx_2 + \ldots + B_m(y)dx_{2m-1} \wedge dx_{2m}$

# Beyond Demailly Weyl law

Ground state: $|1\rangle_y = \exp(-\frac{1}{2}\tilde{g}_{ij}(y)x_i x_j)$ with $\tilde{g}$ the normalised metric [2].
The restriction $\tilde{L}_H(y)$ of $L_H(y)$ to $\mathrm{span}(\pi_{i_1}(y)\dots\pi_{i_\ell}(y)|1\rangle_y)$ has the same spectrum with finite multiplicities.

## Theorem (C 21)

*Assume that $[E, E'] \cap \mathrm{spec}(L_H(y)) = \emptyset$ for every $y \in M$. Then when $k$ is large, $\mathrm{spec}(k\hat{H}_k) \cap [E, E'] = \emptyset$ and*

$$\begin{aligned}\mathrm{rank}\, 1_{]-\infty,E]}(k\hat{H}_k) &= \int_M \mathrm{Ch}(L^k \otimes F)\,\mathrm{Todd}(M) \\ &= \left(\frac{k}{2\pi}\right)^m \mathrm{vol}(M)\,\mathrm{rank}(F) + \mathcal{O}(k^{m-1})\end{aligned}$$

*with $F \to M$ the vector bundle with $F_y = \mathrm{Im}\, 1_{]-\infty,E]}(\tilde{L}_H(y))$.*

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[2] With good coordinates $g_{ij}(y) = \delta_{ij}$ and
$\omega|_y = B_1(y)dx_1 \wedge dx_2 + \dots + B_m(y)dx_{2m-1} \wedge dx_{2m}$,
$\tilde{g}|_y = B_1(y)(dx_1^2 + dx_2^2) + \dots + B_m(y)(dx_{2m-1}^2 + dx_{2m}^2)$

## Corollary ($n$-th Landau level)

*When $B_1 = \ldots = B_m = B$ and $E_-$, $E_+$ are such that $\lambda_{n-1} < E_- < \lambda_n < E_+ < \lambda_{n+1}$ with $\lambda_n = B(n + \frac{m}{2}) + V$, we have for large $k$*

$$\operatorname{rank}(1_{(E_-,E_+)}(k\hat{H}_k)) = \int_M \operatorname{Ch}(L^k \otimes \operatorname{Sym}^n(T^{1,0}M)) \operatorname{Todd} M$$

Earlier results for $B_1 = \ldots = B_m = 1$ and $V = 0$ so $\operatorname{spec} L_H(x) = \frac{m}{2} + \mathbb{N}$:

1. Lowest Landau level ($n = 0$): when $\omega$ is Kähler, this follows from Riemann-Roch-Hirzebruch theorem and Kodaira vanishing theorem. In the symplectic case, this is a theorem of Guillemin-Uribe (88) and Borthwick-Uribe (96).
2. Higher levels: the existence of gaps was proved by Faure-Tsuji (15)

# Density of states

Let $(\psi_{k,i})$ be an onb of eigenvectors, $\hat{H}_k \psi_{k,i} = E_{k,i}\psi_{k,i}$.
For any $E \in \mathbb{R}$ and $x \in M$, set

$$N(x, E, k) = \sum_{i,\ kE_{k,i} \leqslant E} |\psi_{k,i}(x)|^2$$

## Theorem (C. 21)

*if* $[E_-, E_+] \cap \operatorname{spec} L_H(x) = \emptyset$, *then*

$$\begin{aligned} N(x, E_-, k) =& N(x, E_+, k) + \mathcal{O}(k^{-\infty}) \\ =& \left(\frac{k}{2\pi}\right)^m \sum_{\ell=0}^{\infty} a_\ell k^{-\ell} + \mathcal{O}(k^{-\infty}) \end{aligned}$$

*with* $a_0 = \sharp((-\infty, E_-] \cap \operatorname{spec} \tilde{L}_H(x))$.

This is proved only for $E_+ < E$ with $E \in \mathbb{R} \setminus \bigcup_{x \in M} \operatorname{spec}(L_H(x))$.

# References

## My own work

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