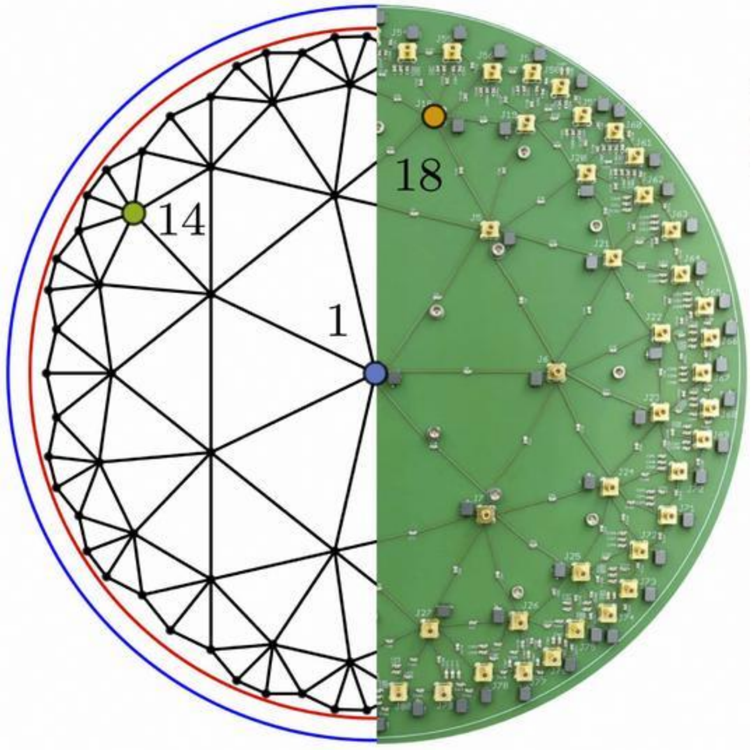
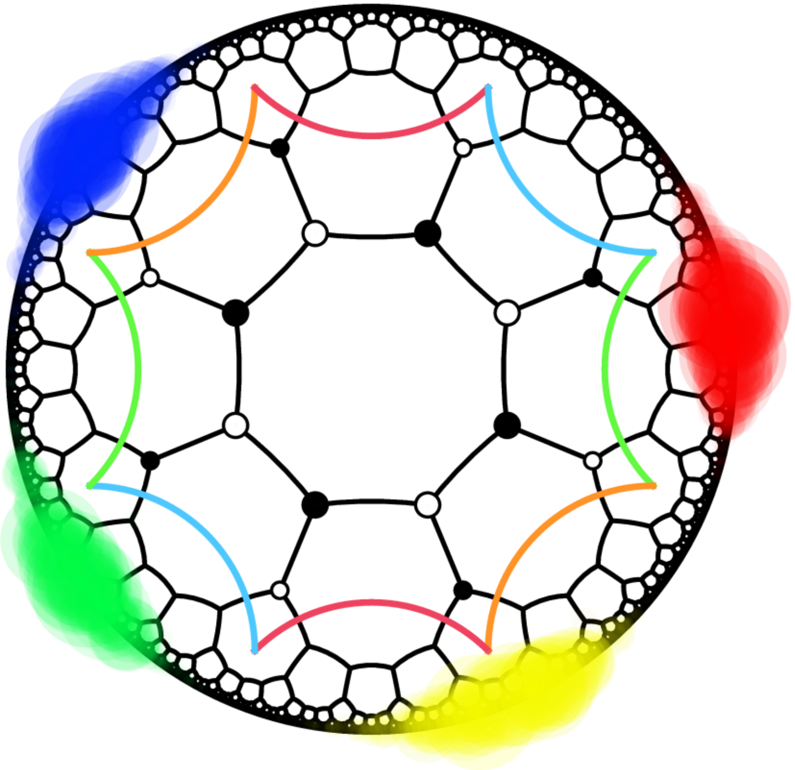
From hyperbolic drum…
… towards hyperbolic topological matter





Tomáš Bzdušek

at Sorbonne Université, Paris

20 October, 2022

PAUL SCHERRER INSTITUT PSI

University of Zurich UZH

Swiss National Science Foundation

Thanks to my collaborators

University of Zurich UZH

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Ching Hua Lee

Julius-Maximilians-UNIVERSITÄT WÜRZBURG

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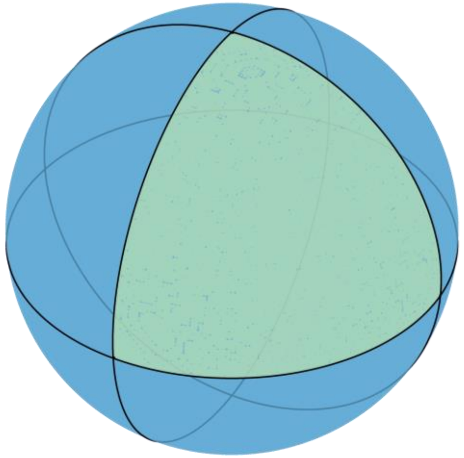
Alex Stegmaier
Lavi Upreti
Martin Greiter
Tobias Hofmann
Tobias Helbig
Tobias Kießling
Stefan Imhof
Hauke Brand

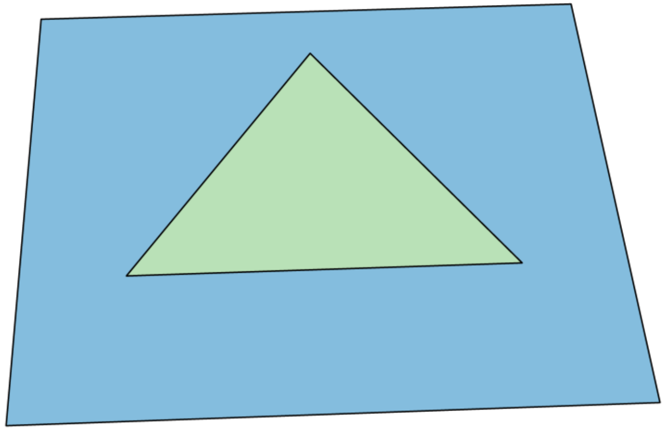
Curved spaces

Sphere, $K > 0$



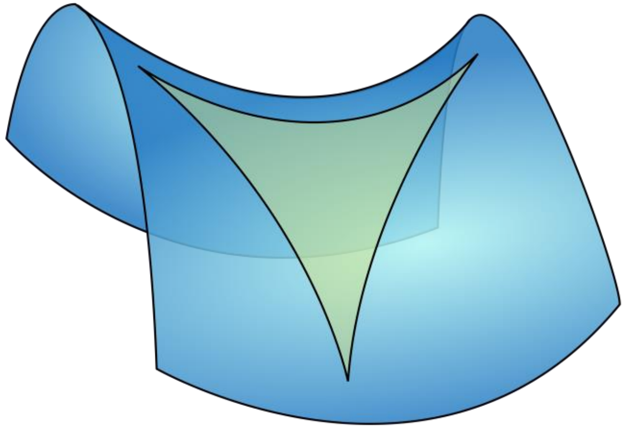
$$\alpha + \beta + \gamma > \pi$$

Euclidean plane, $K = 0$



$$\alpha + \beta + \gamma = \pi$$

Saddle, $K < 0$

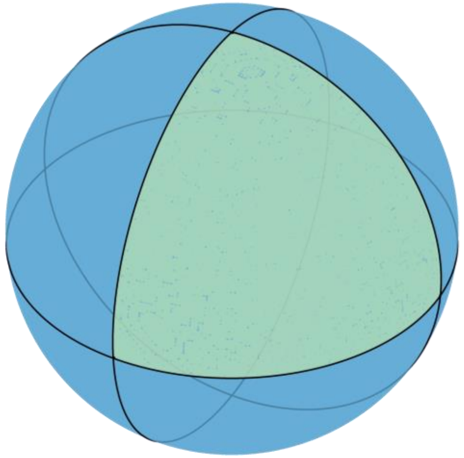


$$\alpha + \beta + \gamma < \pi$$

Curved spaces

Sphere, $K > 0$

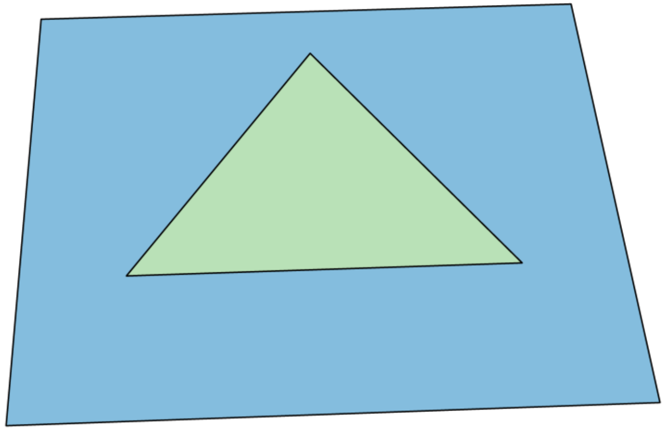
(constant curvature)



$\alpha + \beta + \gamma > \pi$

Euclidean plane, $K = 0$

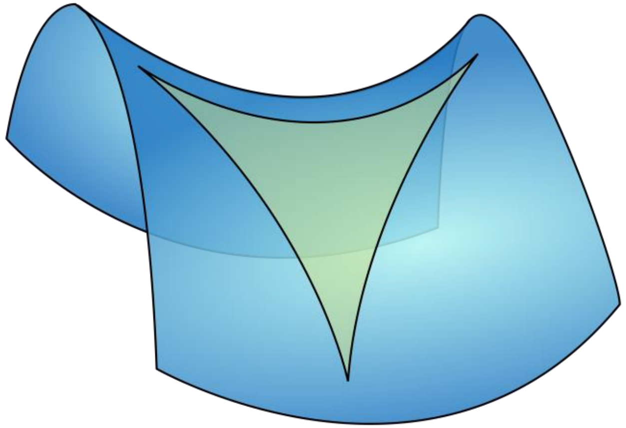
(constant curvature)



$\alpha + \beta + \gamma = \pi$

Saddle, $K < 0$

non-constant curvature

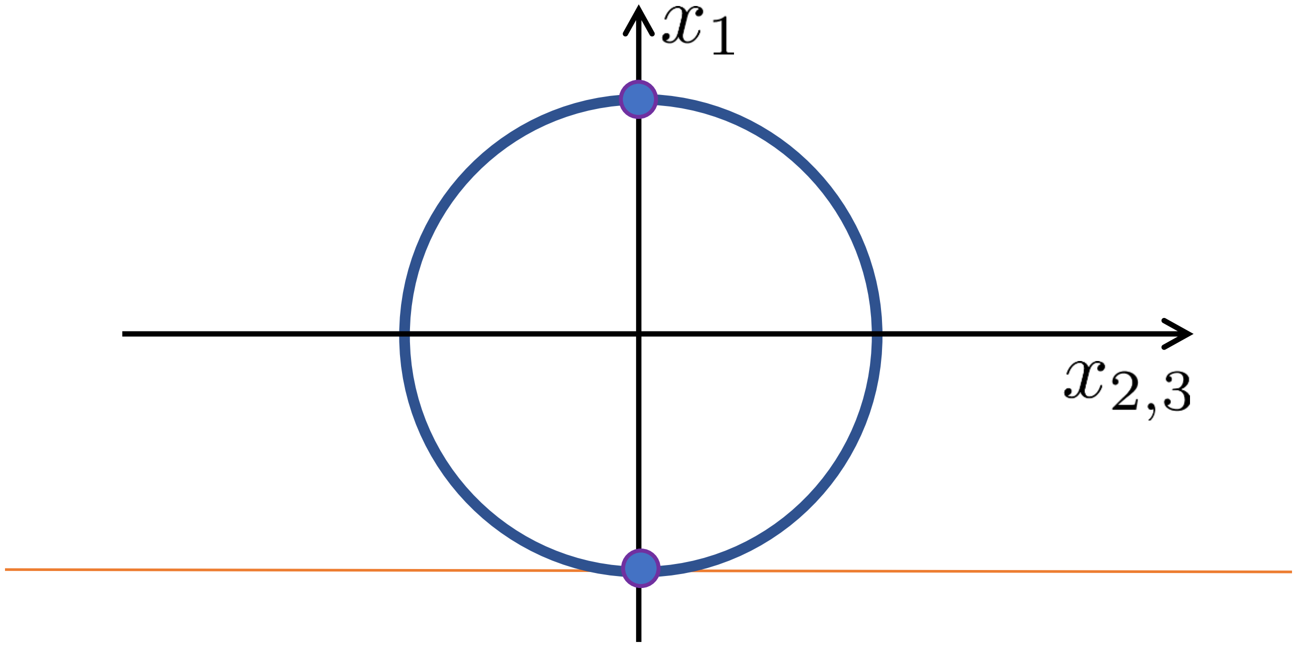


$\alpha + \beta + \gamma < \pi$

Hyperbolic plane – space of *constant* negative curvature

Sphere: points of constant
Euclidean distance from the origin

$$\mathbb{S}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 | + x_1^2 + x_2^2 + x_3^2 = 1\}$$



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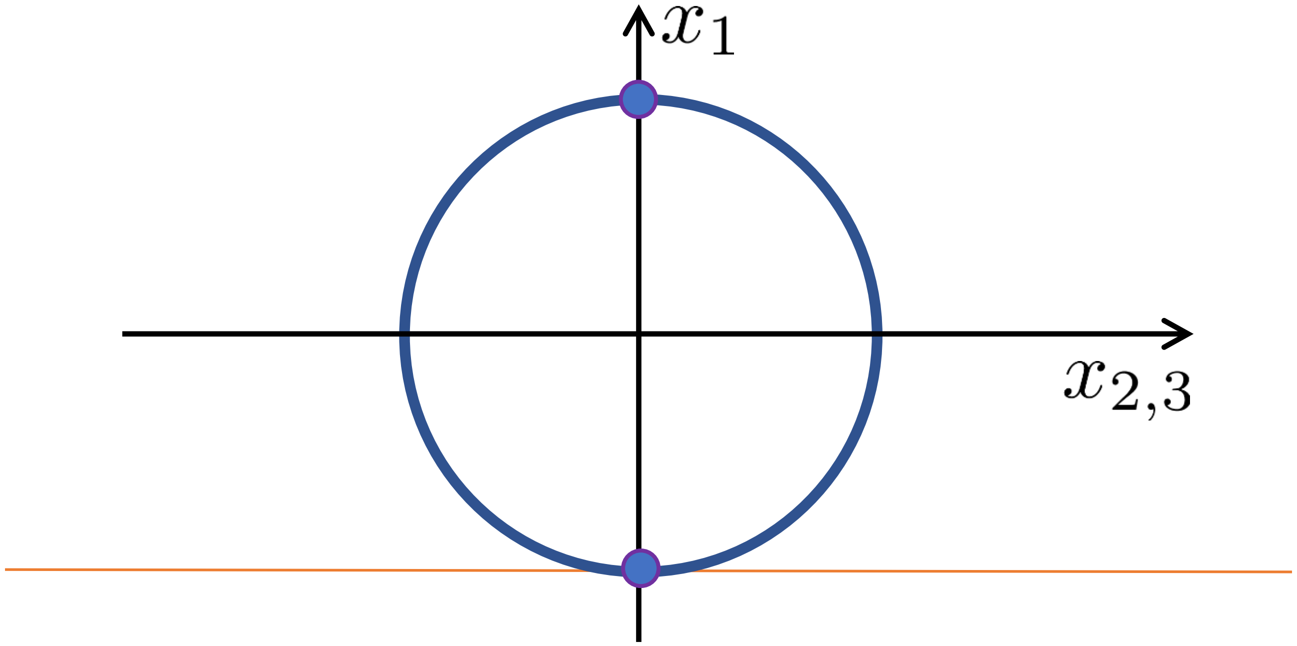
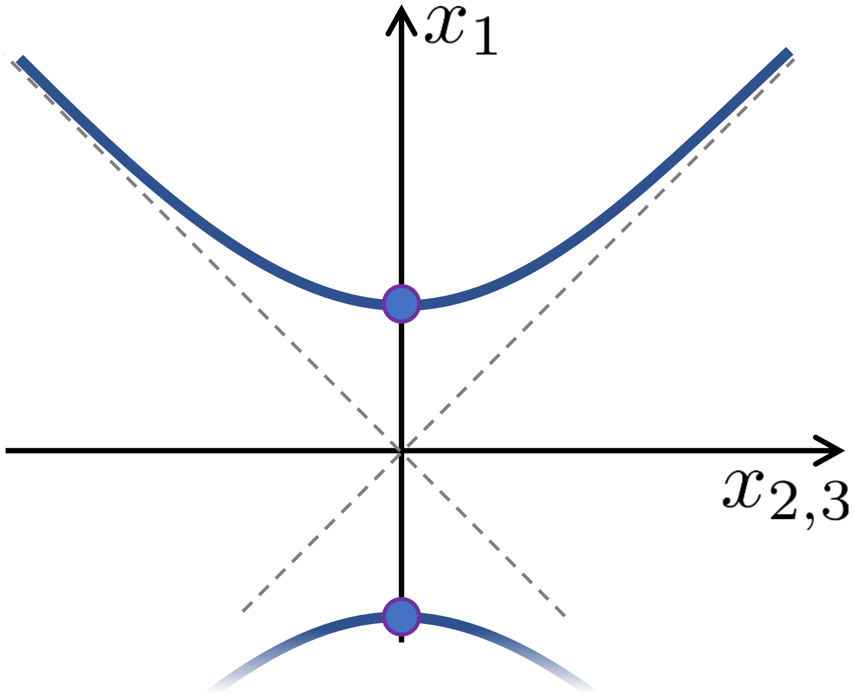


Hyperbolic plane: points of constant
Minkowski distance from the origin

$$\mathbb{H}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 | + x_1^2 - x_2^2 - x_3^2 = 1\}$$



Hyperbolic plane – space of *constant* negative curvature

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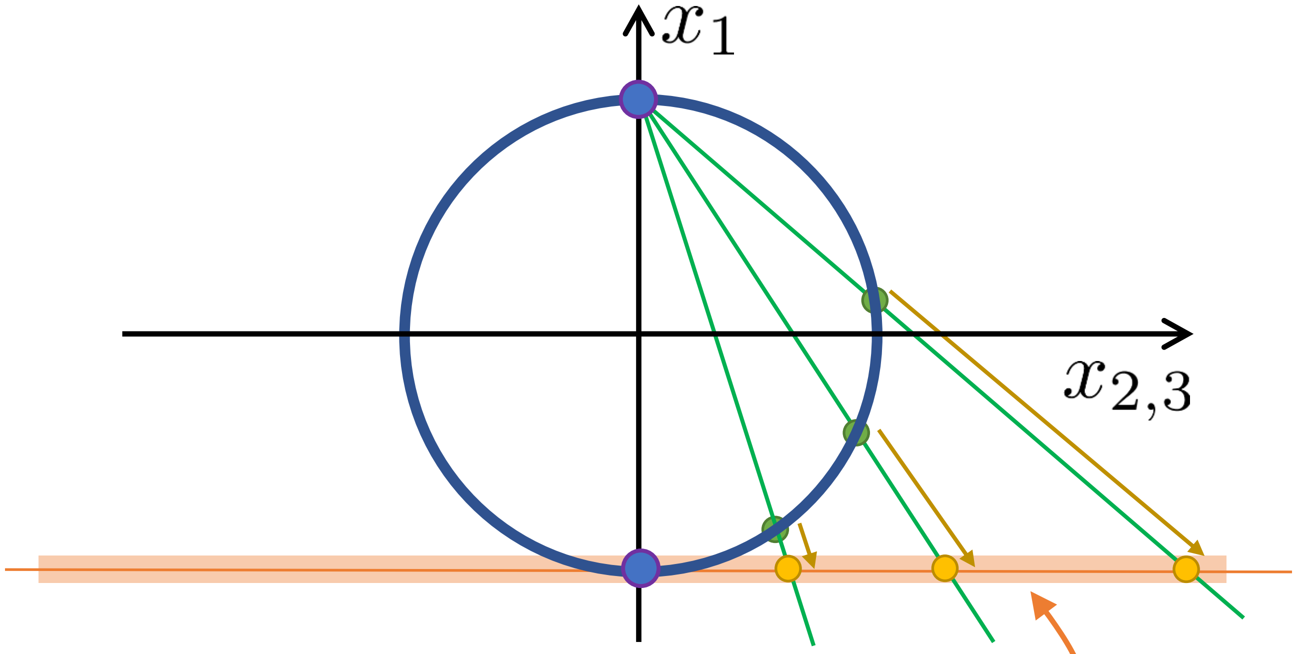
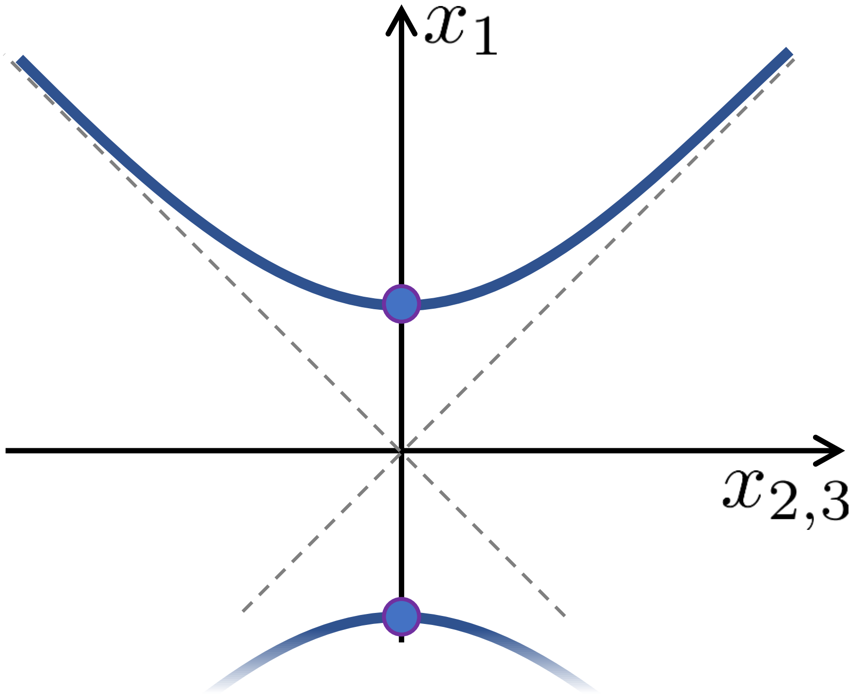


Stereographic projection into the $\mathbb{R}^2$ plane.

Hyperbolic plane: points of constant Minkowski distance from the origin

$$\mathbb{H}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 | + x_1^2 - x_2^2 - x_3^2 = 1\}$$



Hyperbolic plane – space of *constant* negative curvature

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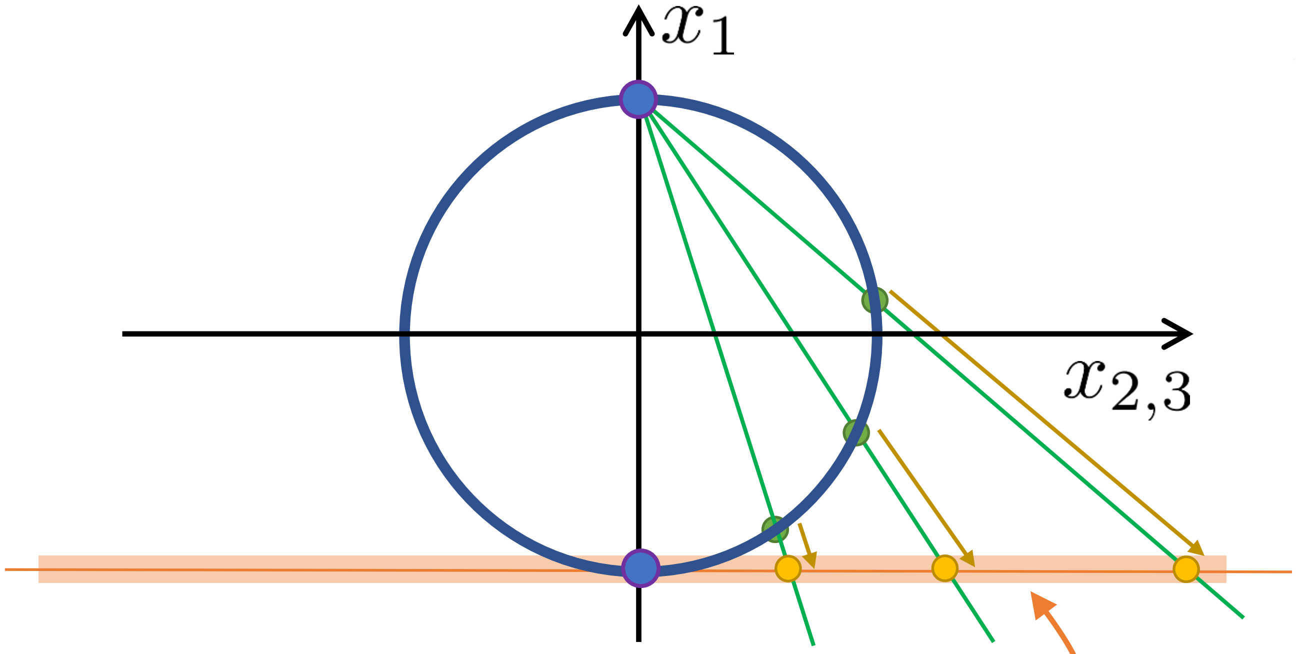

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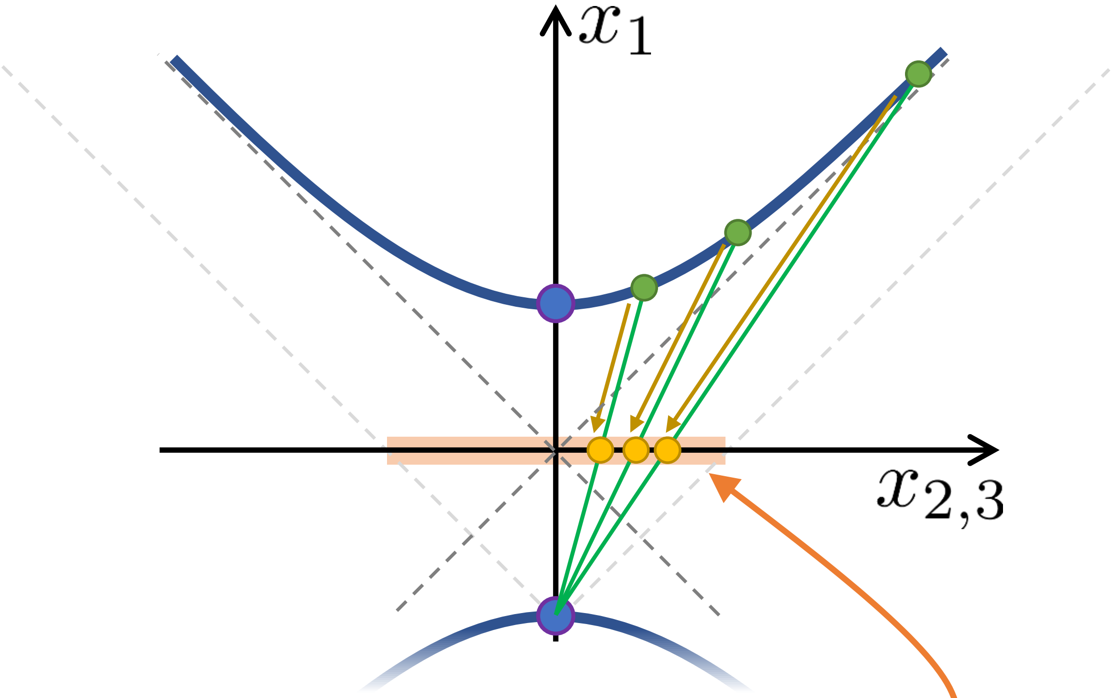


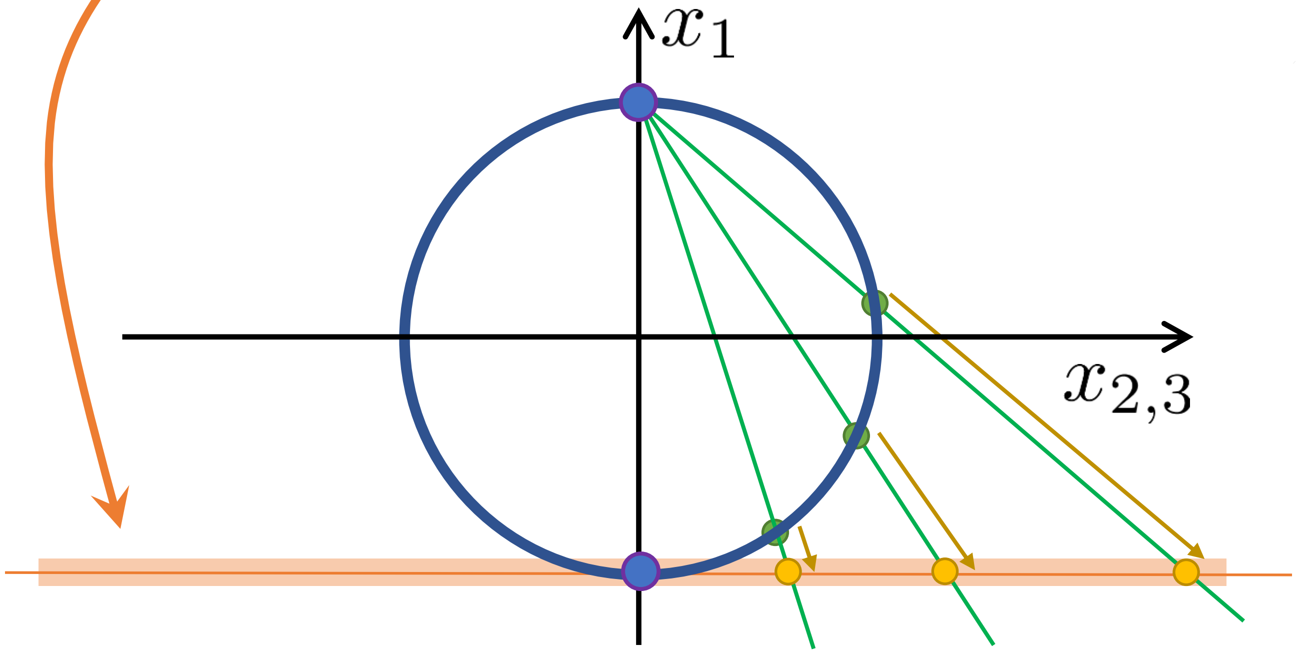
Stereographic projection into the “Poincaré disk”.

Hyperbolic plane – space of *constant* negative curvature

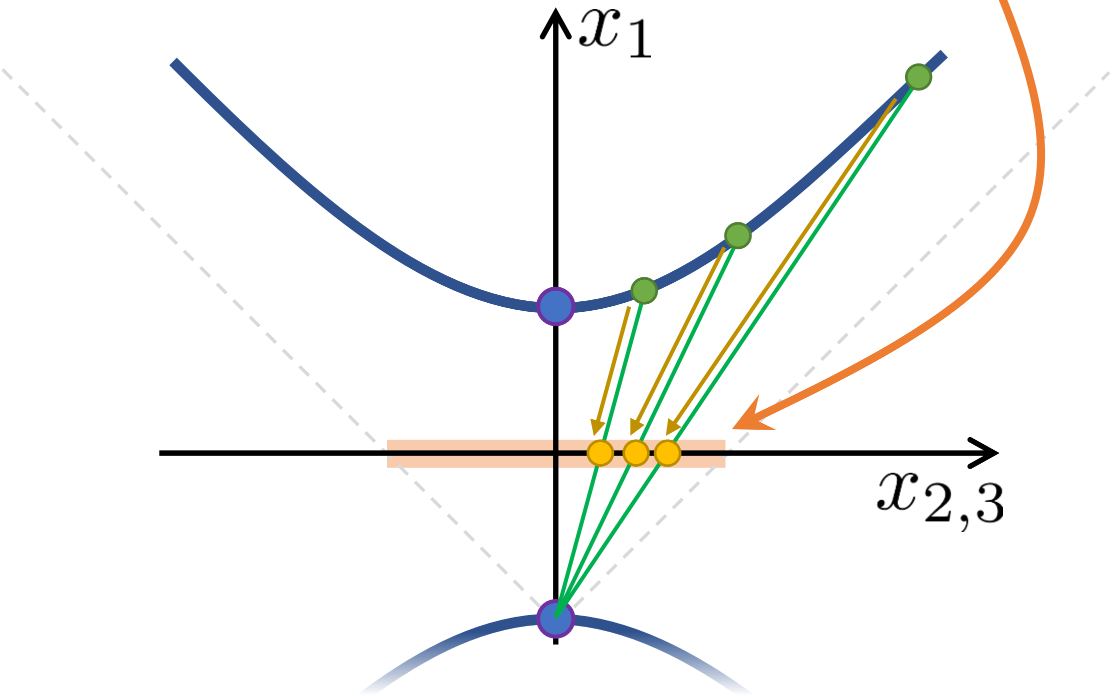
$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left(1 + r^2\right)^2}$$

Metric tensor inside the stereographic projection

$$\mathrm{d}s^2 = \frac{\mathrm{d}x^2 + \mathrm{d}y^2}{\left(1 - r^2\right)^2}$$



Stereographic projection
into the $\mathbb{R}^2$ plane.



Stereographic projection
into the “Poincaré disk”.

Hilbert’s theorem

There exists no complete regular surface of constant negative Gaussian curvature immersed in $\mathbb{R}^3$.

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⬇

The hyperbolic plane cannot be “realized” in laboratory space.

Hilbert’s theorem

There exists no complete regular surface of constant negative Gaussian curvature immersed in $\mathbb{R}^3$.

↓

The hyperbolic plane cannot be “realized” in laboratory space.

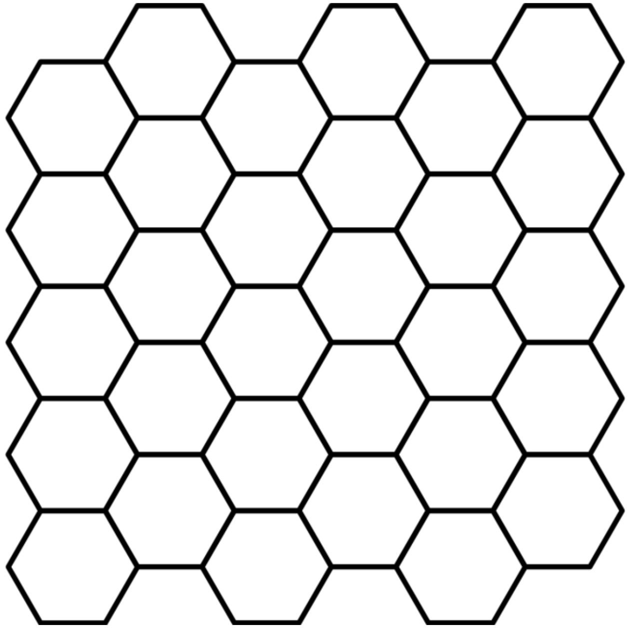
↓

Solution: **discretize it and realize the lattice!**

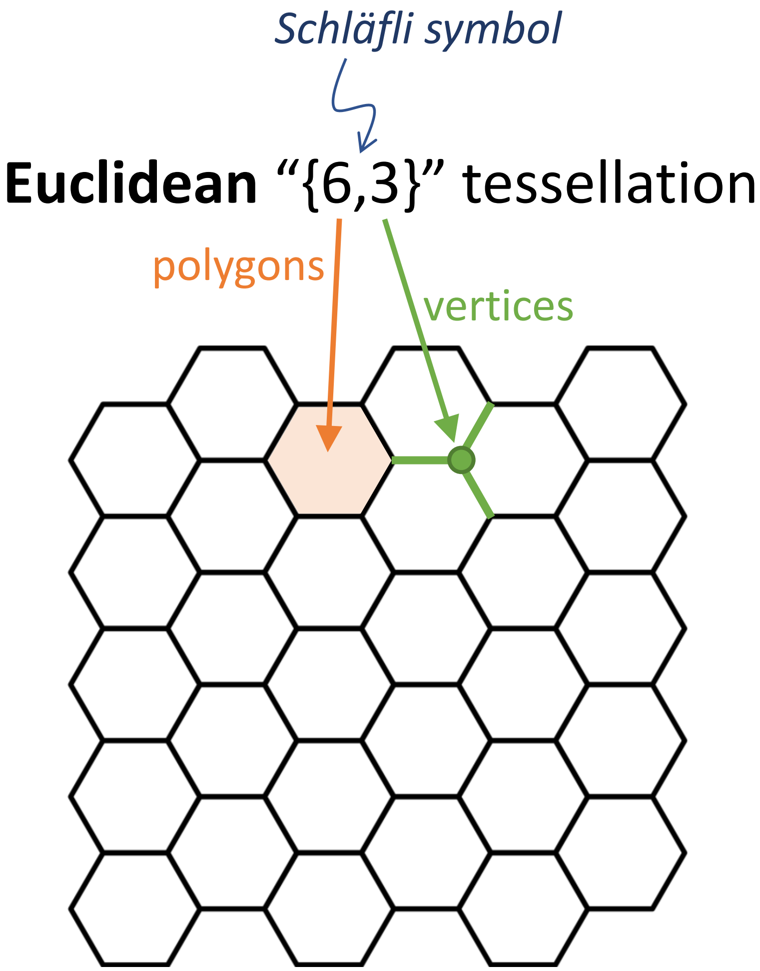
Regular “{p,q}” lattices

Regular “{*p*,*q*}” lattices

Euclidean “{6,3}” tessellation



Regular "{*p*,*q*}" lattices



Schläfli symbol

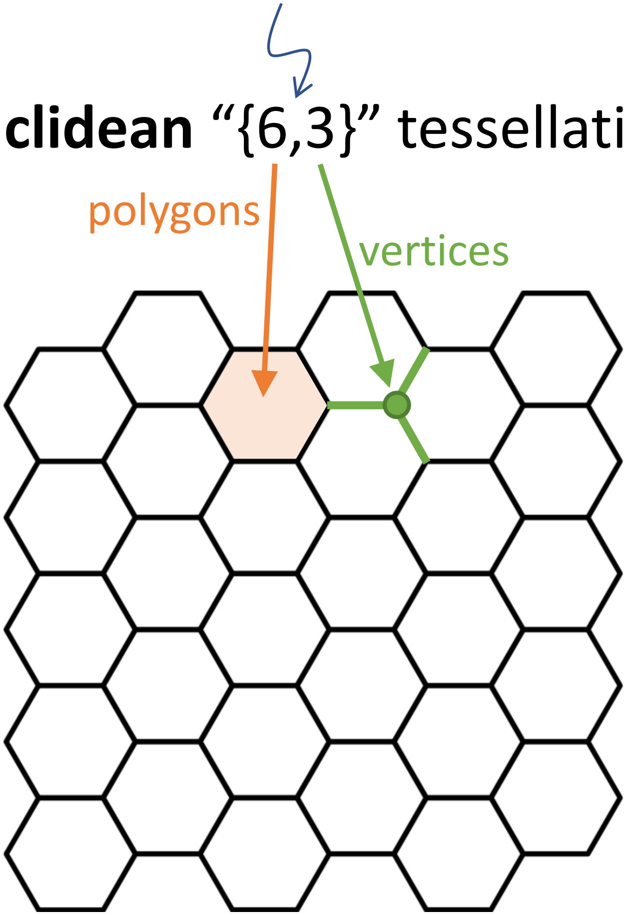
Euclidean "{6,3}" tessellation

polygons

vertices

Regular "{*p*,*q*}" lattices

Spherical "{5,3}" tessellation



Schläfli symbol

Euclidean "{6,3}" tessellation

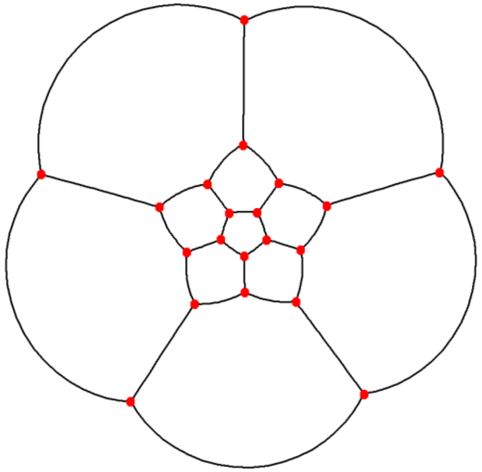
polygons

vertices

Regular “$\{p,q\}$” lattices

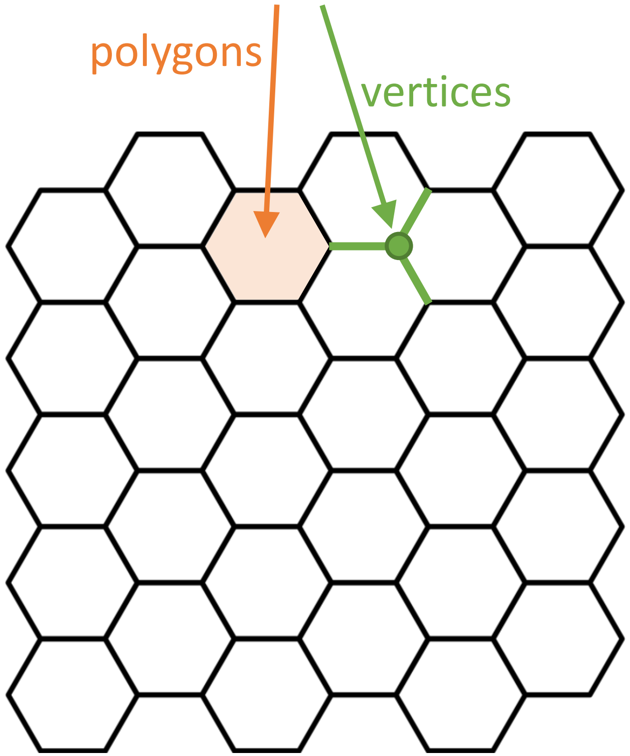
Spherical “{5,3}” tessellation

Stereogr. proj.



Schläfli symbol

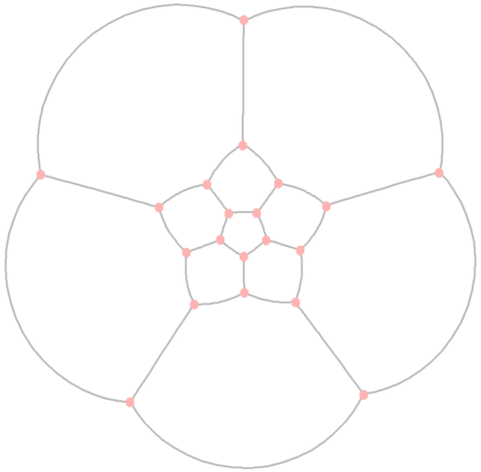
Euclidean “{6,3}” tessellation



Regular "{*p*,*q*}" lattices

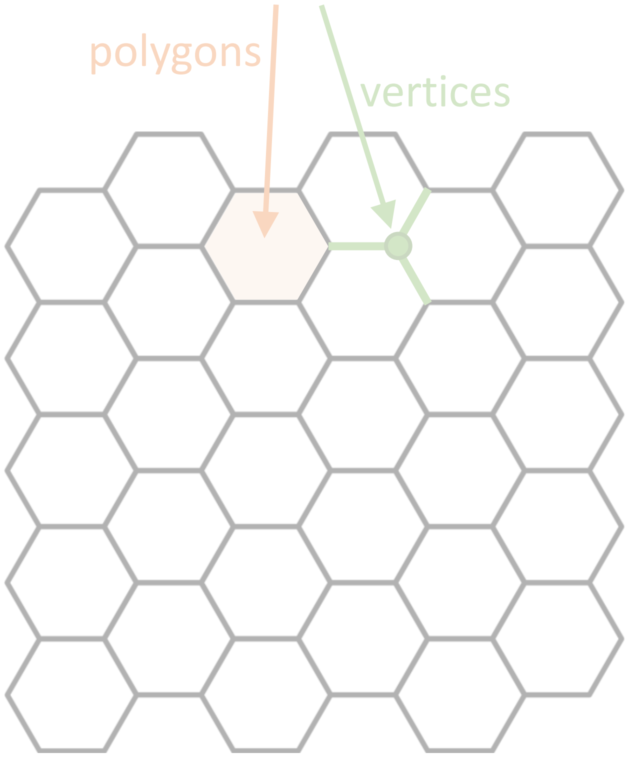
Spherical "{5,3}" tessellation

Stereogr. proj.



Schläfli symbol

Euclidean "{6,3}" tessellation

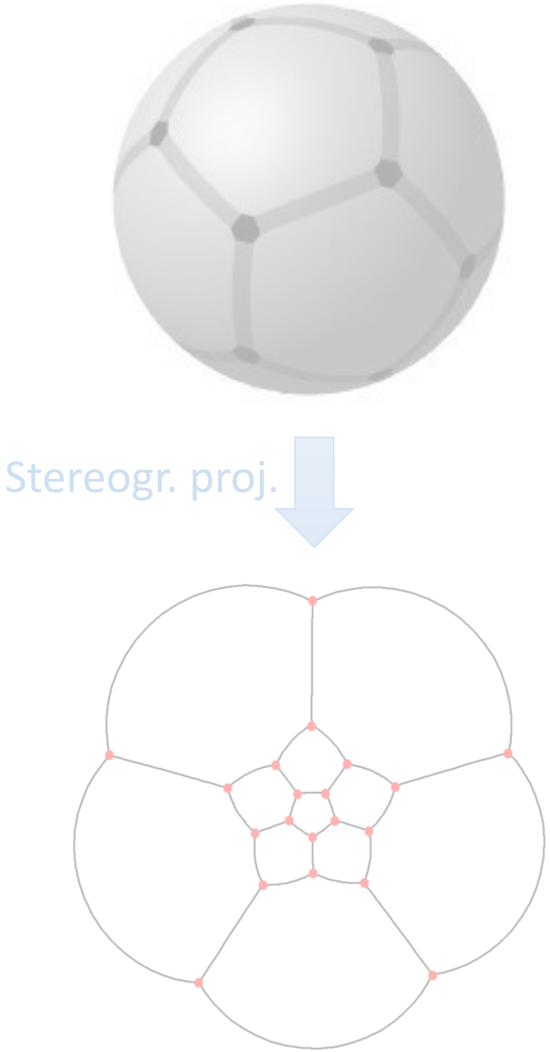


Hyperbolic "{7,3}" tessellation

[...]

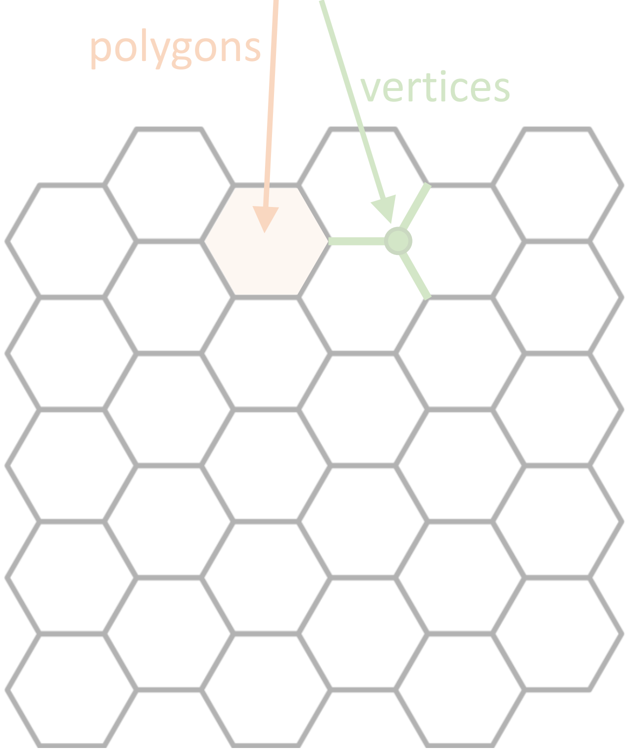
Regular "{*p*,*q*}" lattices

Spherical "{5,3}" tessellation



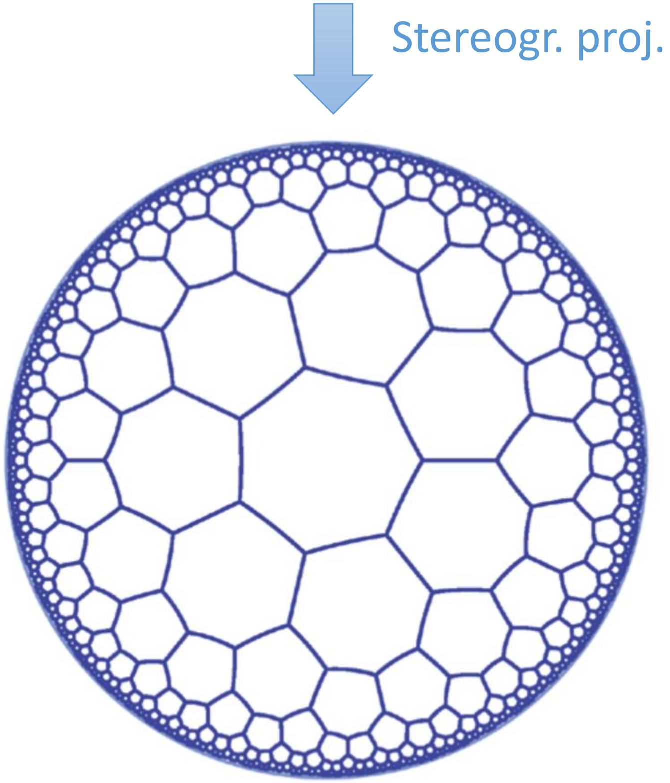
Schläfli symbol

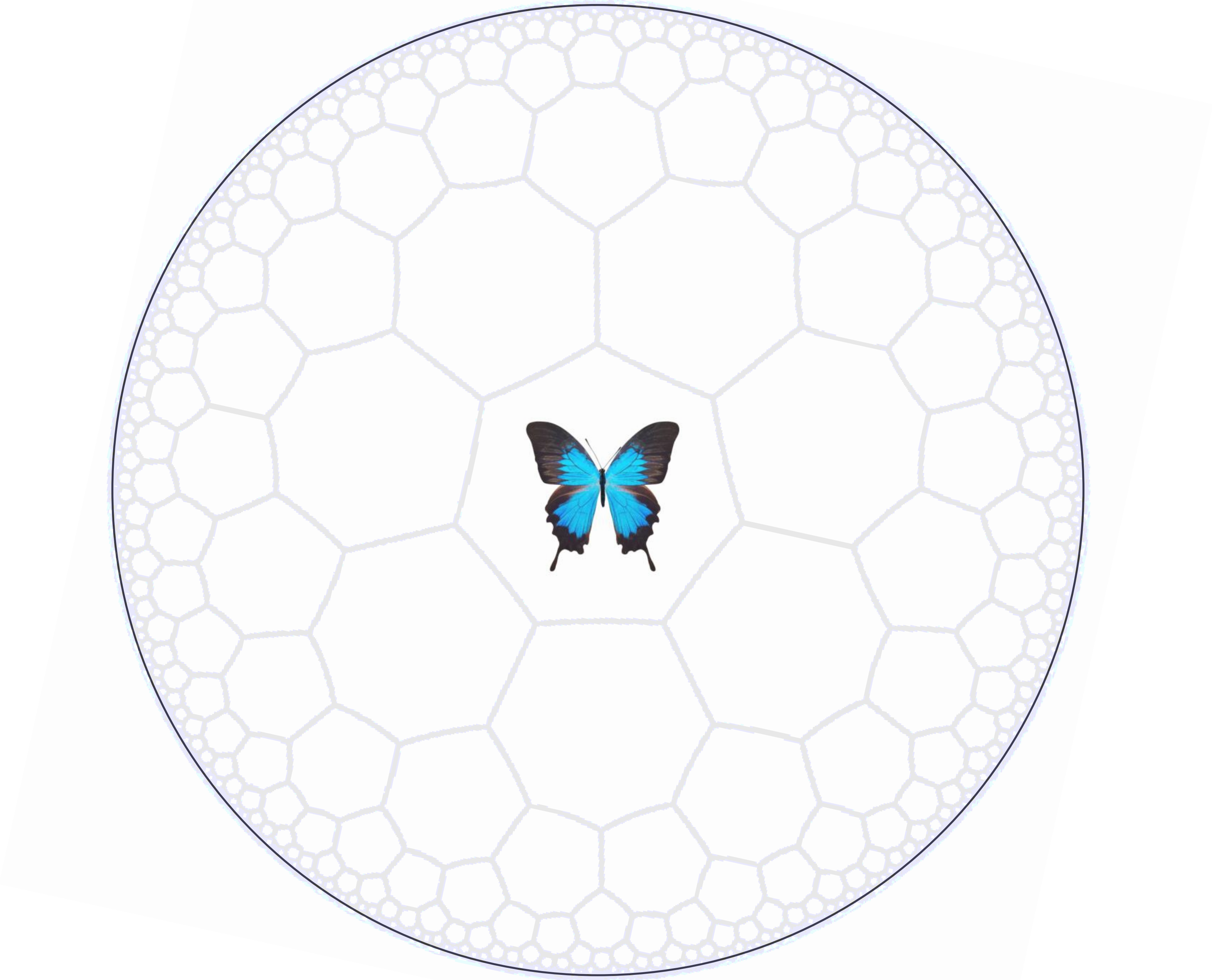
Euclidean "{6,3}" tessellation



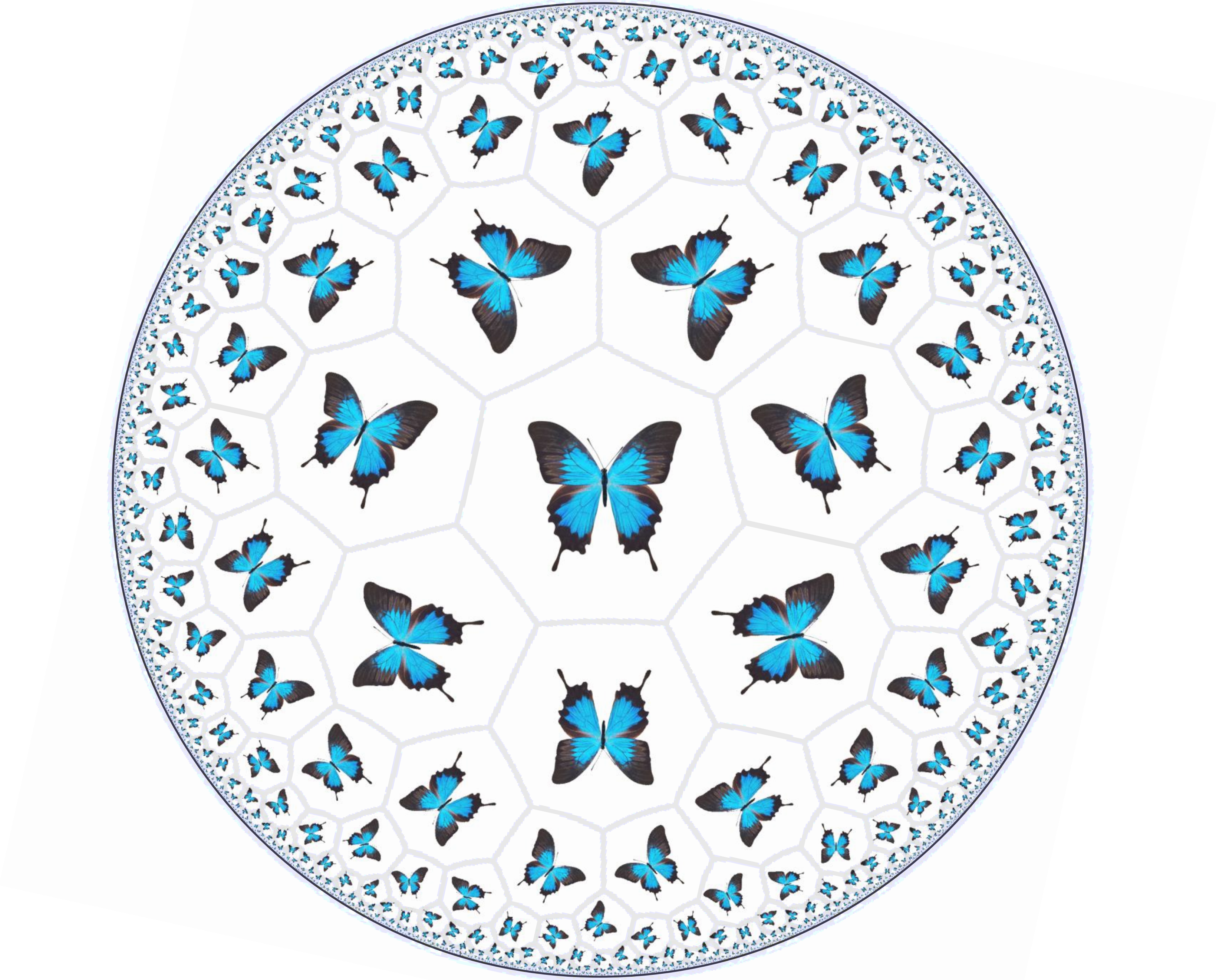
Hyperbolic "{7,3}" tessellation

[...]





Generate your own hyperbolic tiling! – http://www.malinc.se/m/ImageTiling.php



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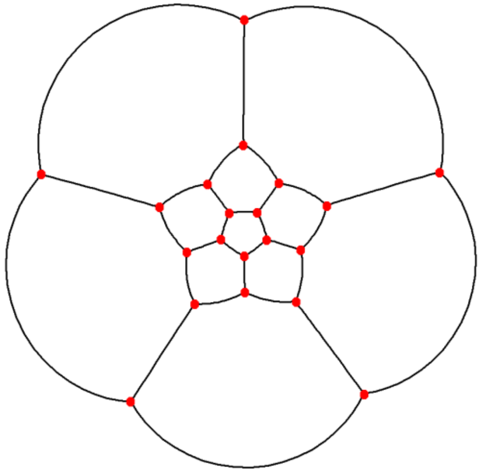
Regular hyperbolic lattices

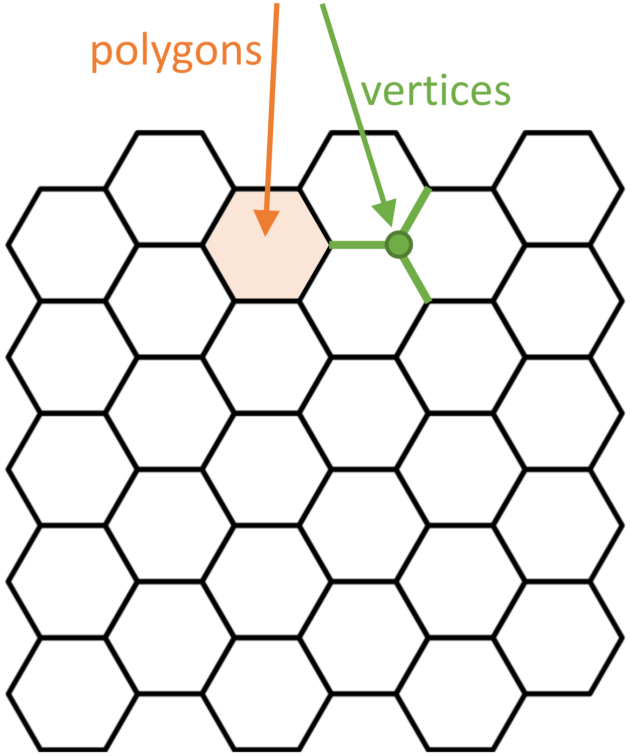
Spherical “{5,3}” tessellation

Stereogr. proj.



Schläfli symbol

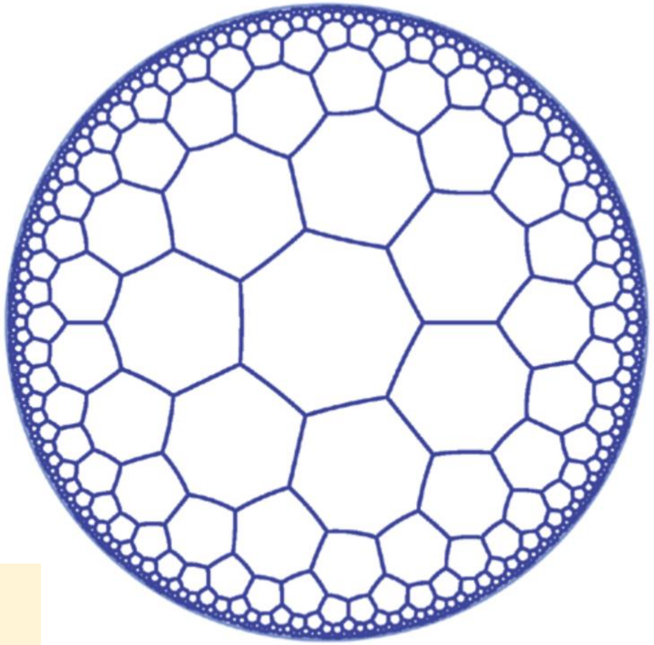
Euclidean “{6,3}” tessellation



Hyperbolic “{7,3}” tessellation

[...]

Stereogr. proj.



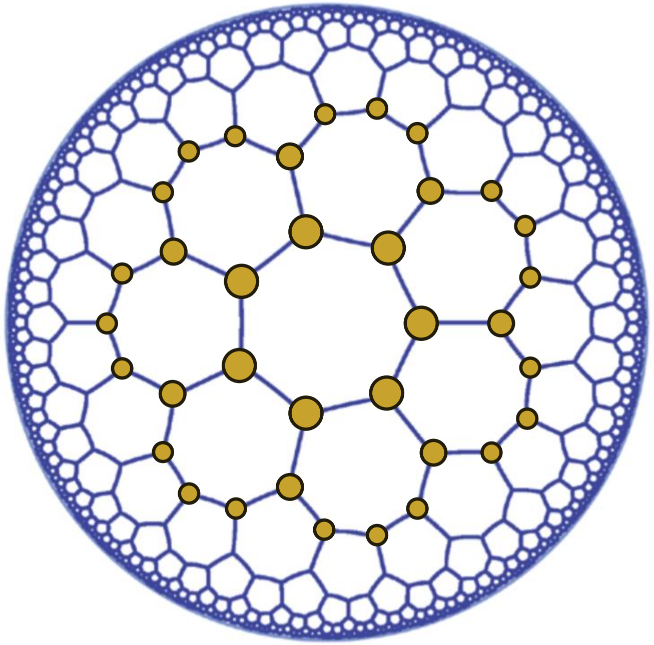
Sign of curvature determined by

Euler class per vertex $\Delta\chi = 1 - \frac{q}{2} + \frac{q}{p}$

Regular hyperbolic lattices

Hyperbolic “{7,3}” tessellation

[…]

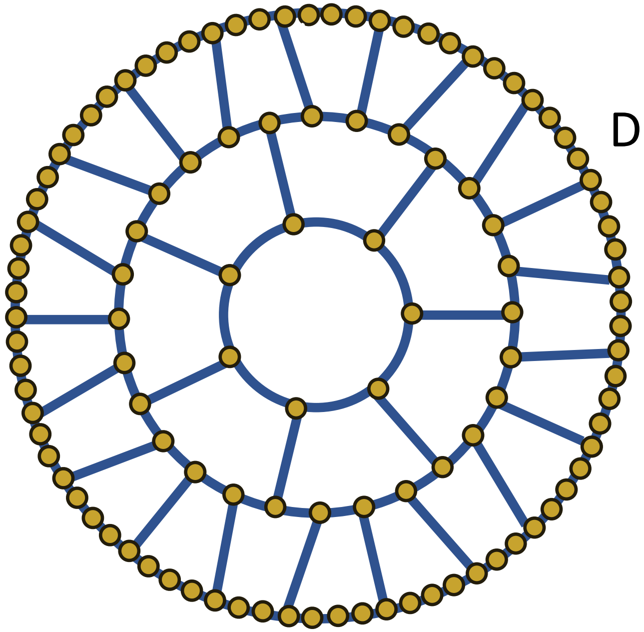


Regular hyperbolic lattices

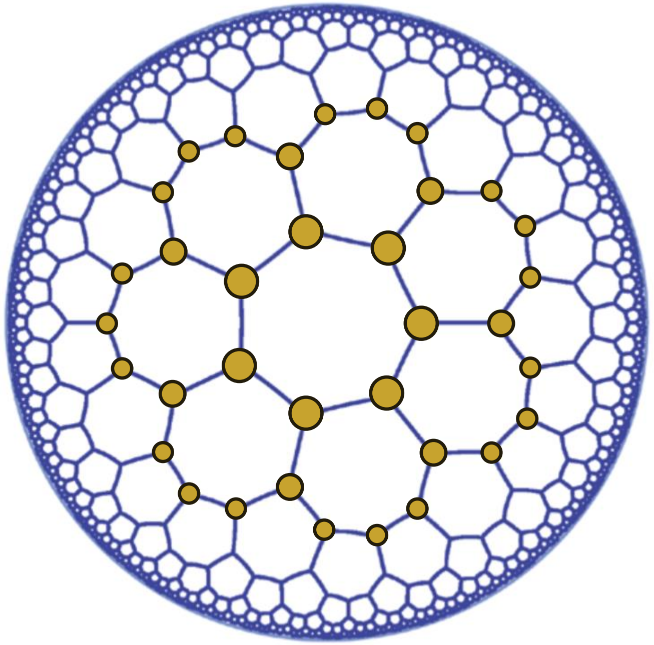
Hyperbolic “{7,3}” tessellation

[…]

Stereogr. proj.



Deform the graph *while respecting the coupling strength* on each bond



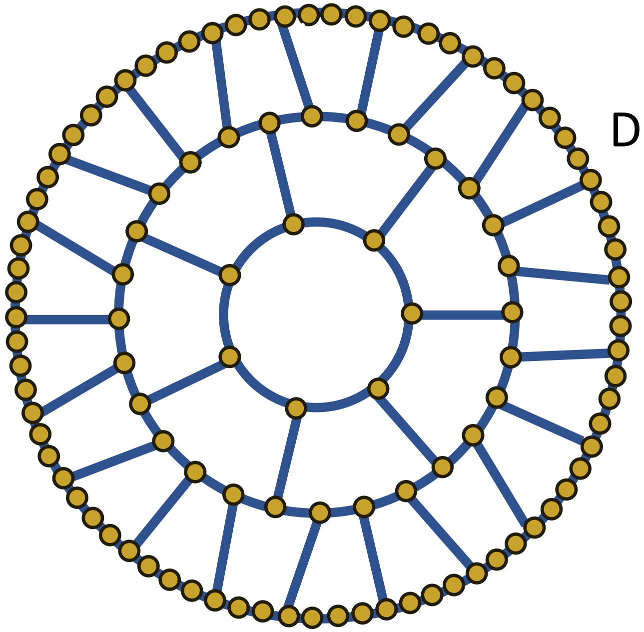
Regular hyperbolic lattices

Realization in ‘metamaterials’ (such as circuit QED): Coupling strength on bonds engineered to be *the same irrespective of the bond length* – only the *graph* matters!

Hyperbolic “{7,3}” tessellation

[...]

Stereogr. proj.



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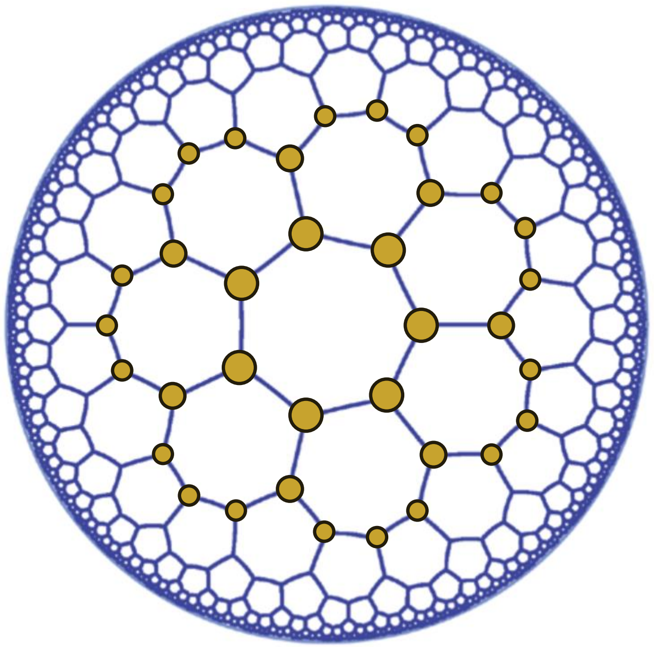
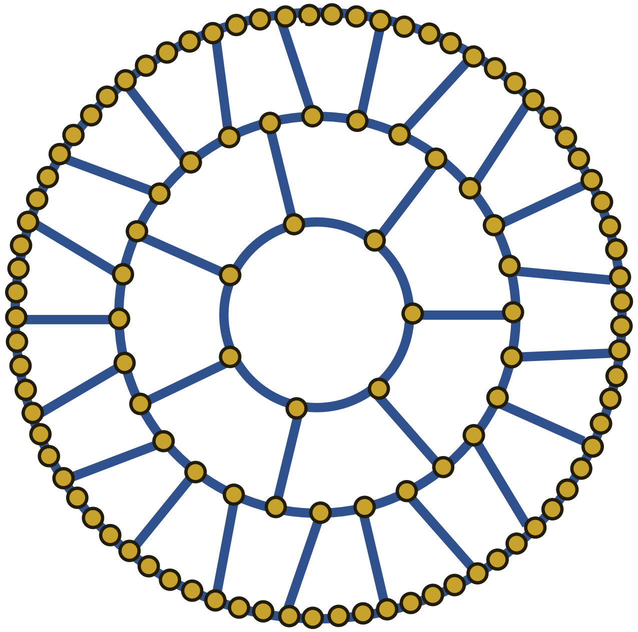
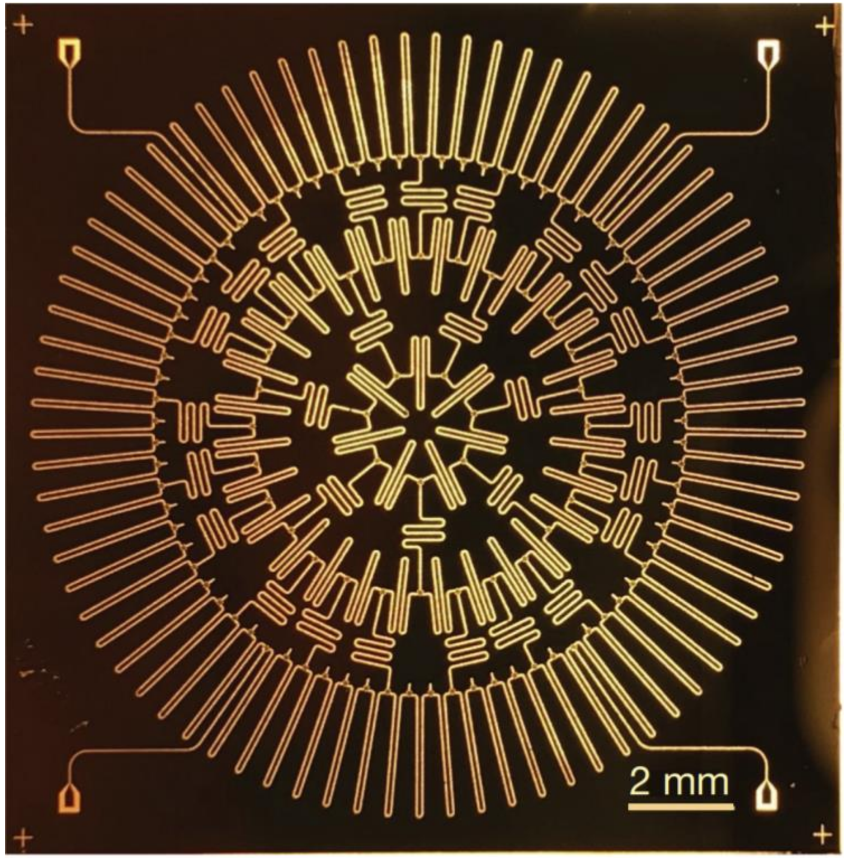


Hyperbolic lattice in circuit QED

$$\mathcal{H}_{\mathrm{TB}} = \omega_0 \sum_i a_i^\dagger a_i - t \sum_{\langle i,j \rangle} \left(a_i^\dagger a_j + a_j^\dagger a_i \right)$$





A. J. Kollár, M. Fitzpatrick, and A. A. Houck, *Nature* **571**, 45—50 (2019)

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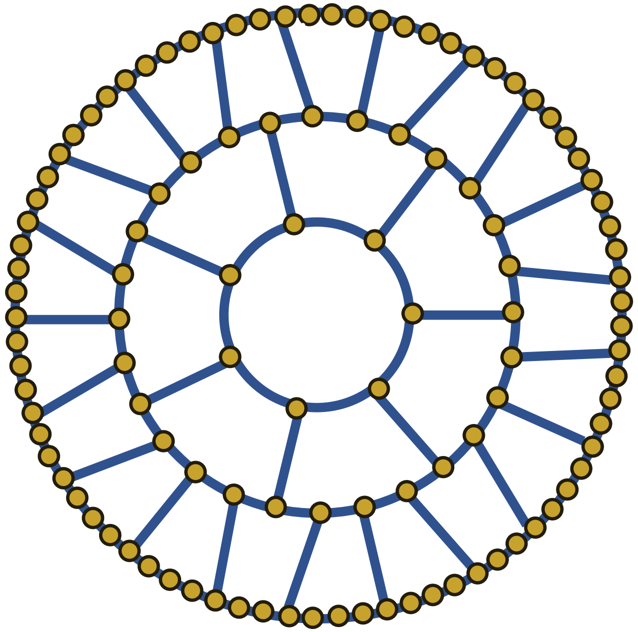

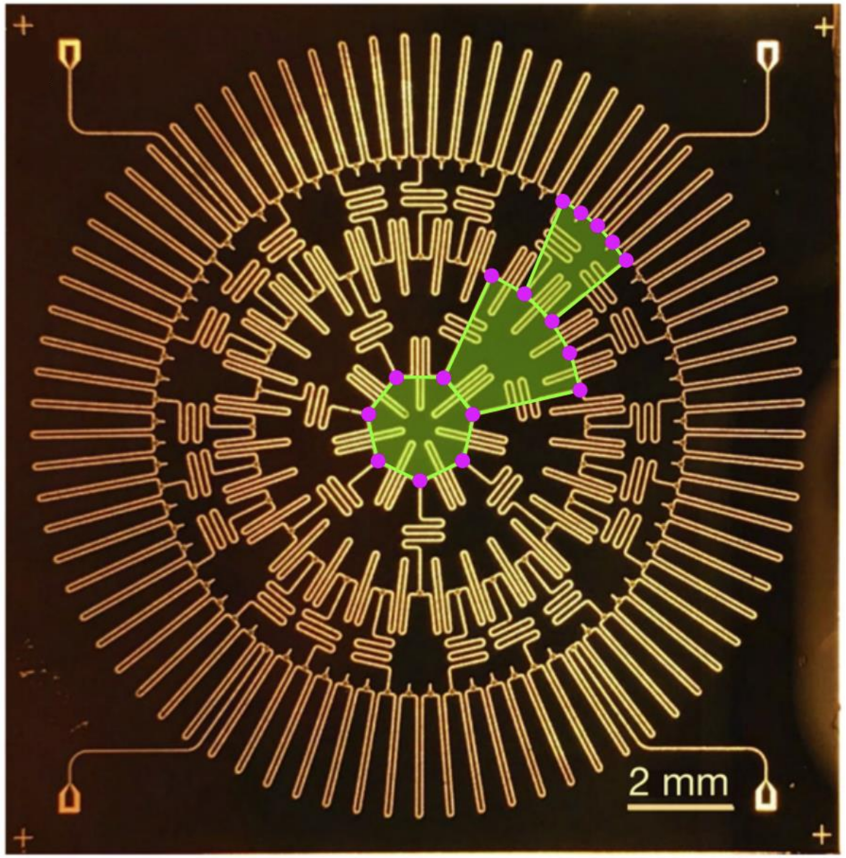

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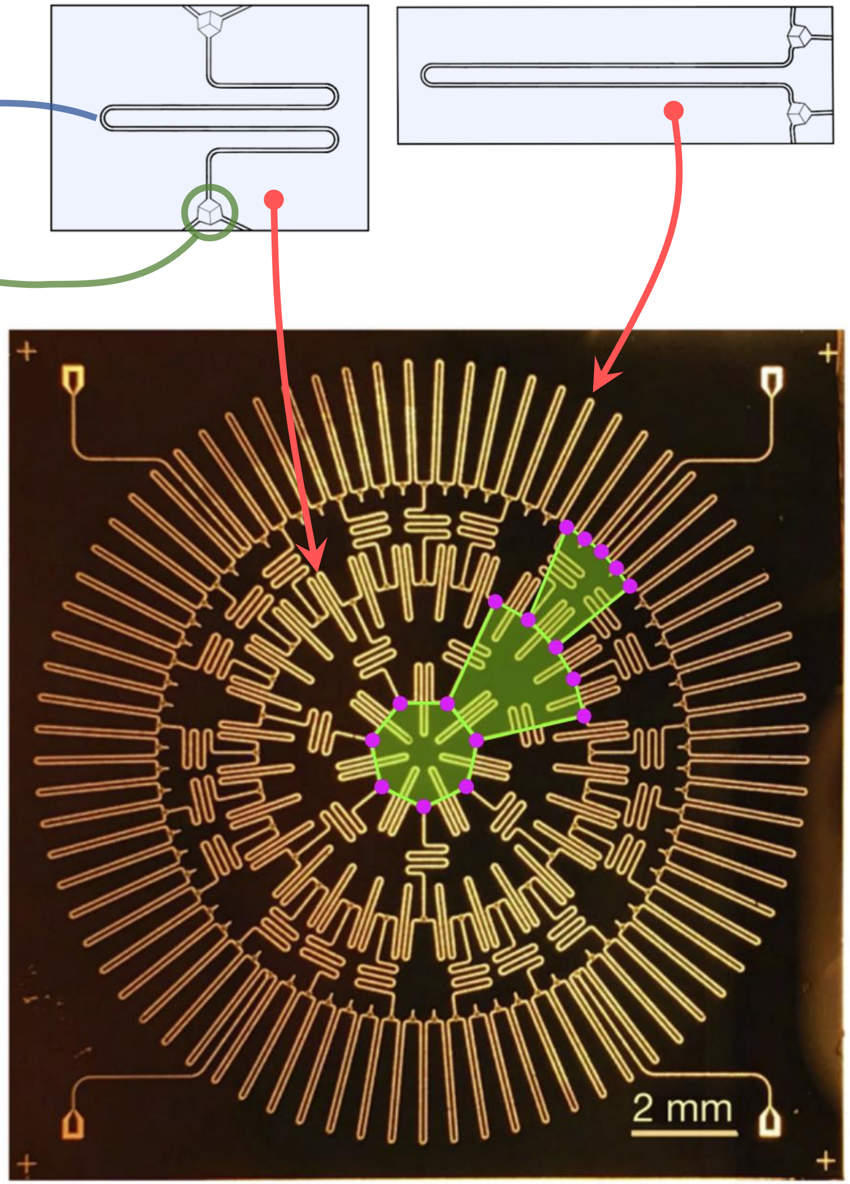

Degrees of freedom on edges,

coupled by tri-junctions.

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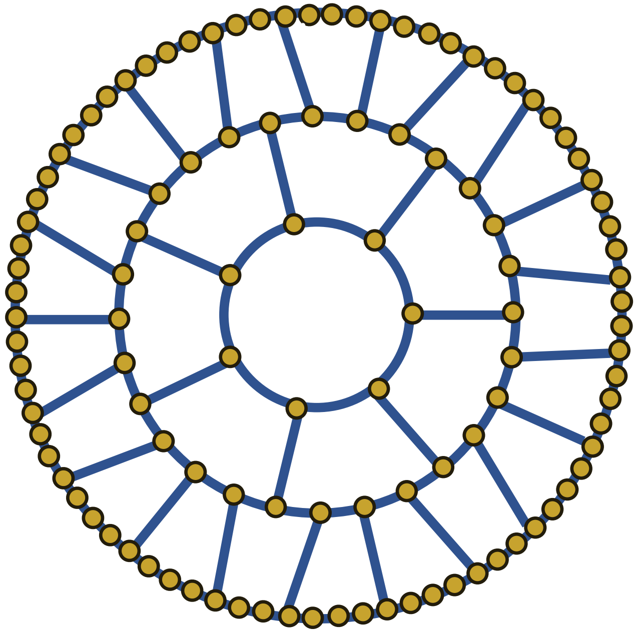

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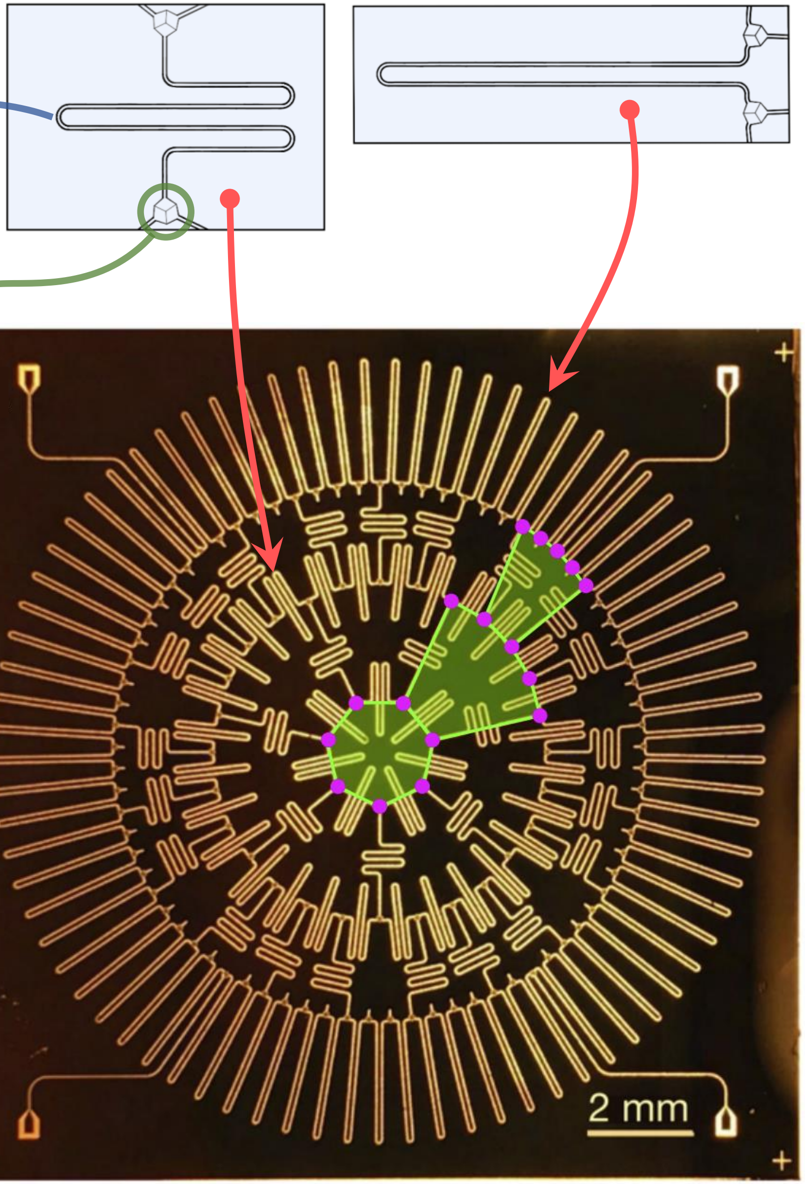

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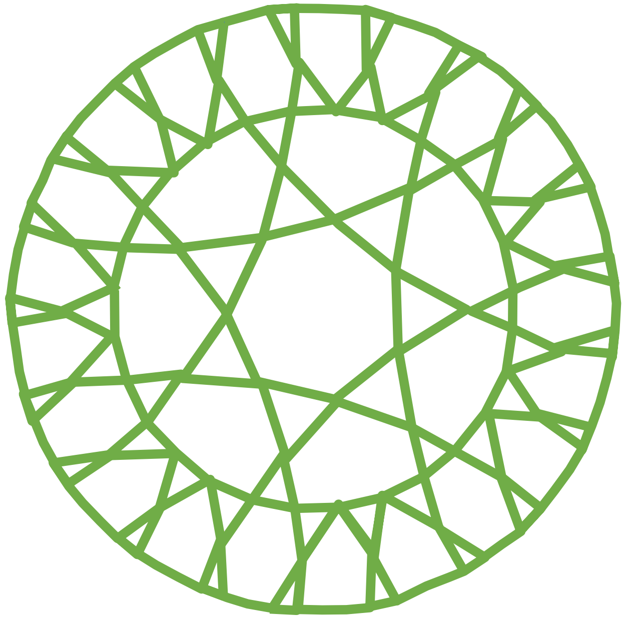

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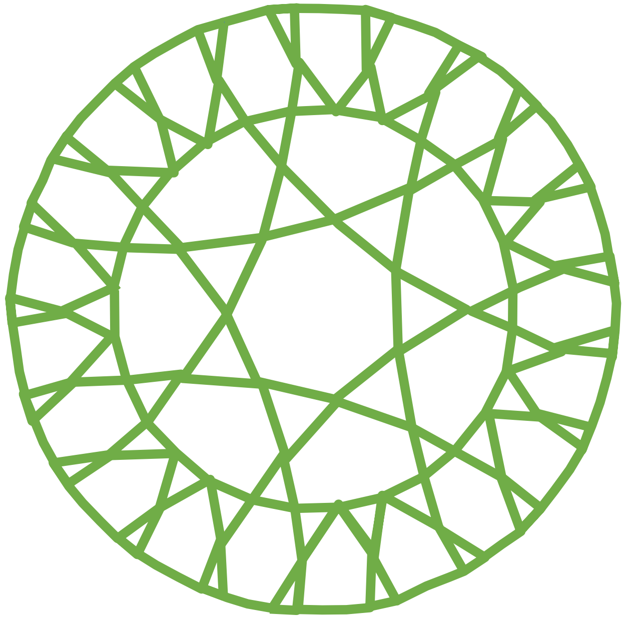

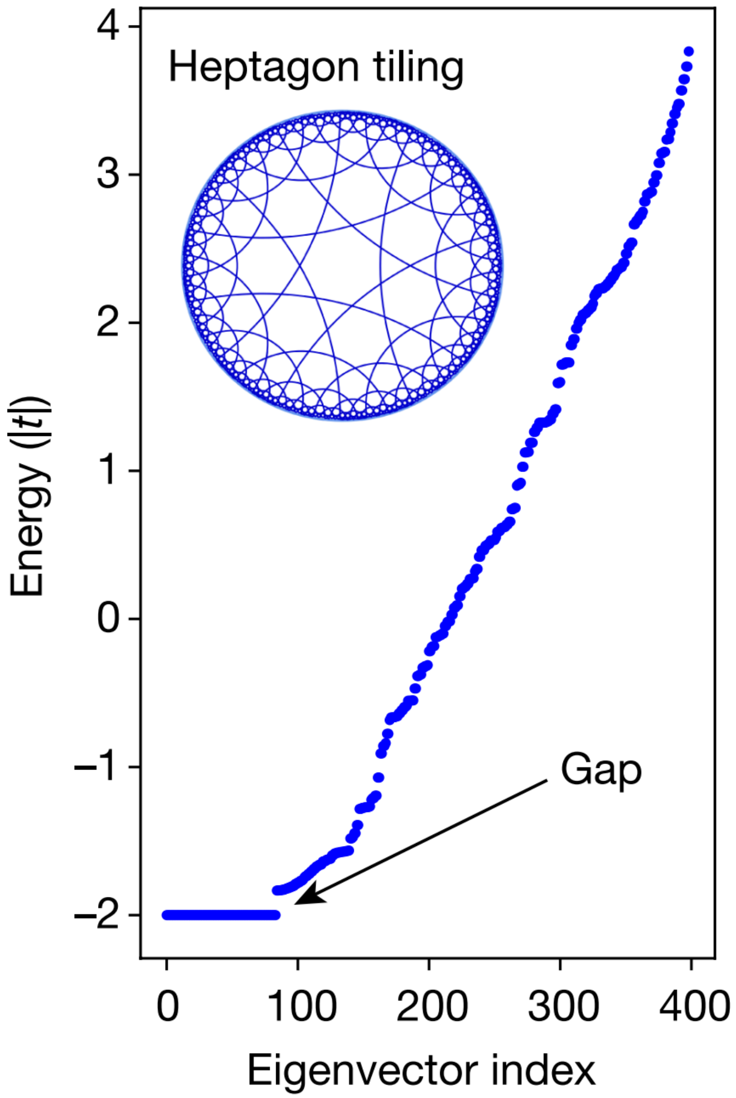

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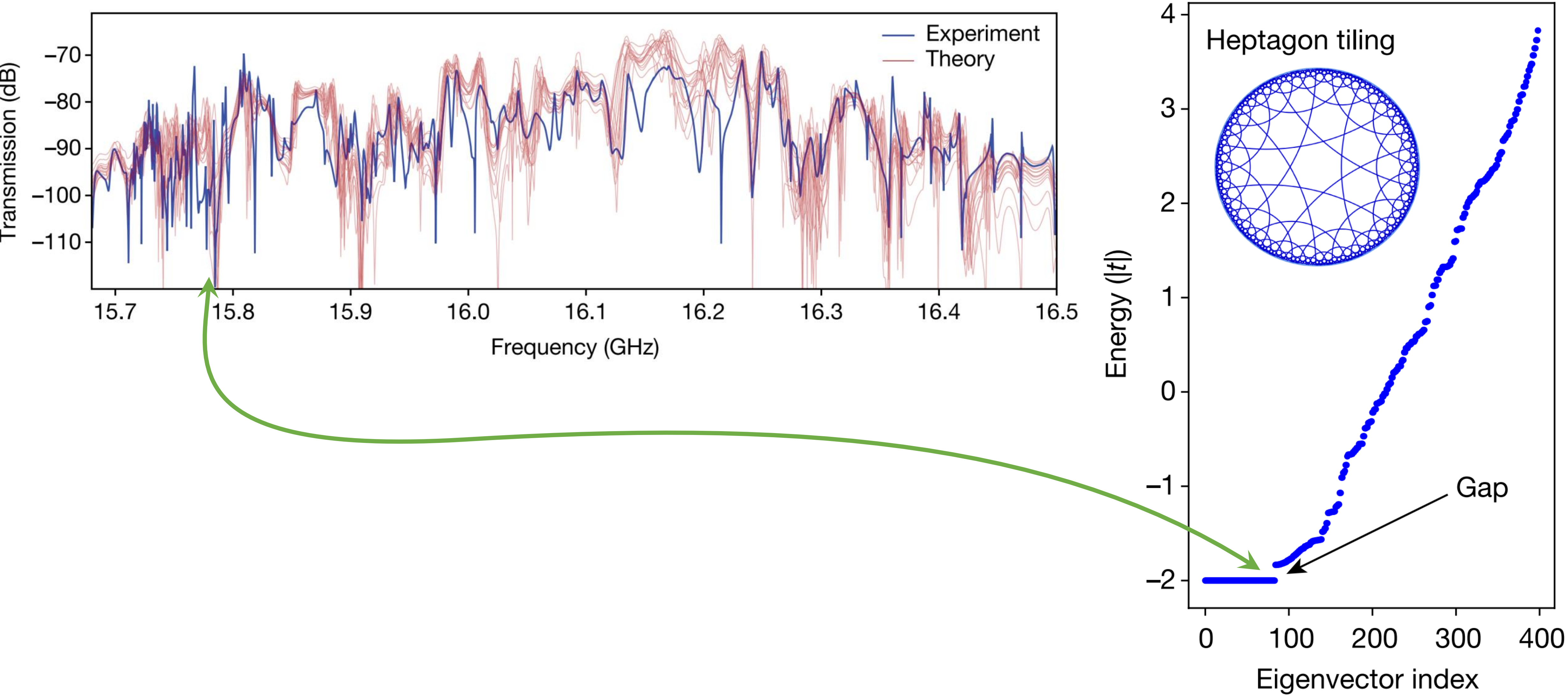
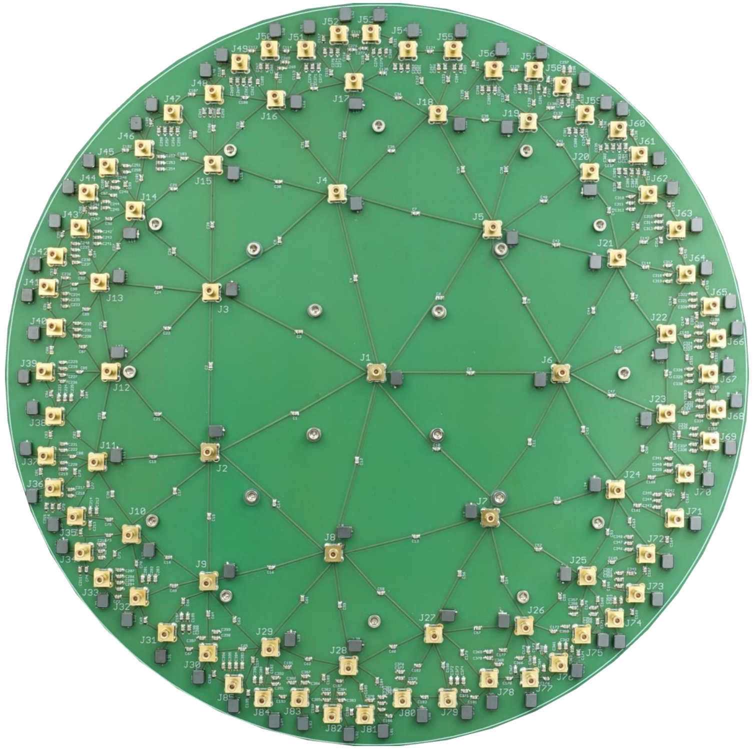
Hyperbolic lattice in circuit QED



A. J. Kollár, M. Fitzpatrick, and A. A. Houck, *Nature* **571**, 45—50 (2019)

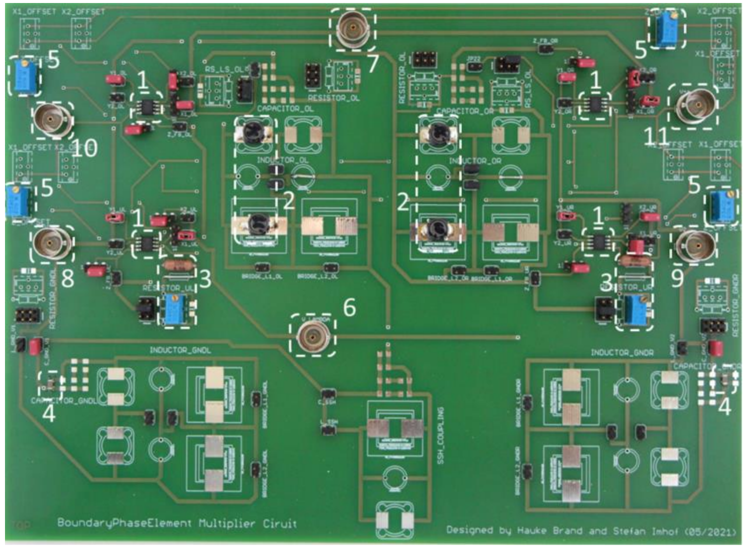
Electric-circuit simulations of hyperbolic lattices

hyperbolic continuum



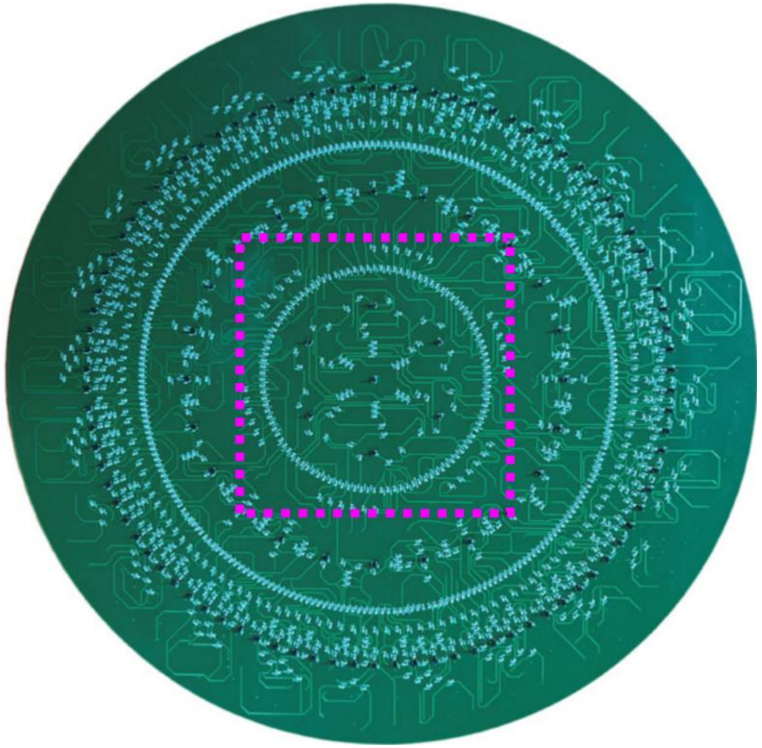
Lenggenhager, TB, et al.,
Nat. Commun. **13**, 4373 (2022)

hyperbolic momentum space



Chen, TB, et al.,
arXiv:2205.05106 (2022)

hyperbolic Haldane model

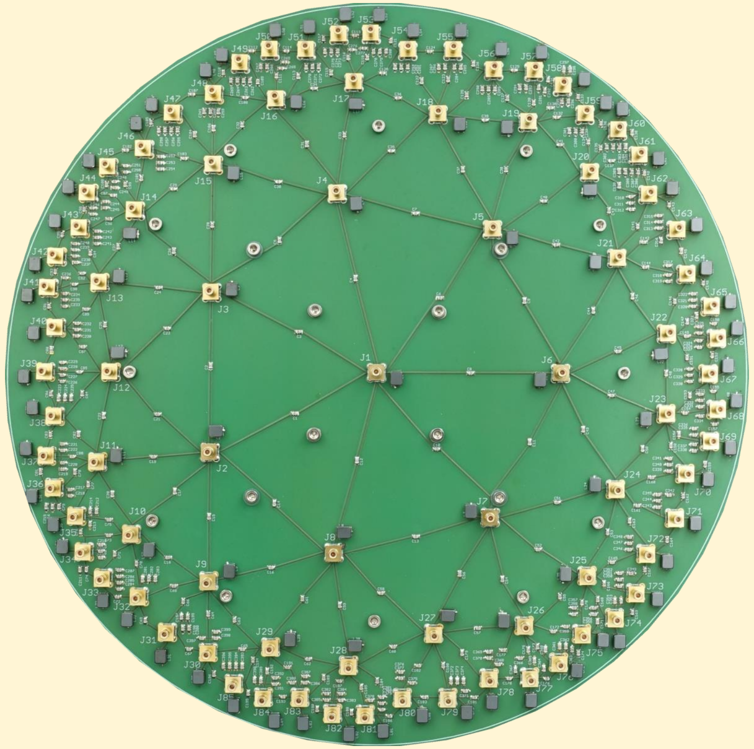


Zhang, et al.,
Nat. Commun. **13**, 2937 (2022)

Outline of the remainder of the talk

1. **exp.**

hyperbolic
continuum



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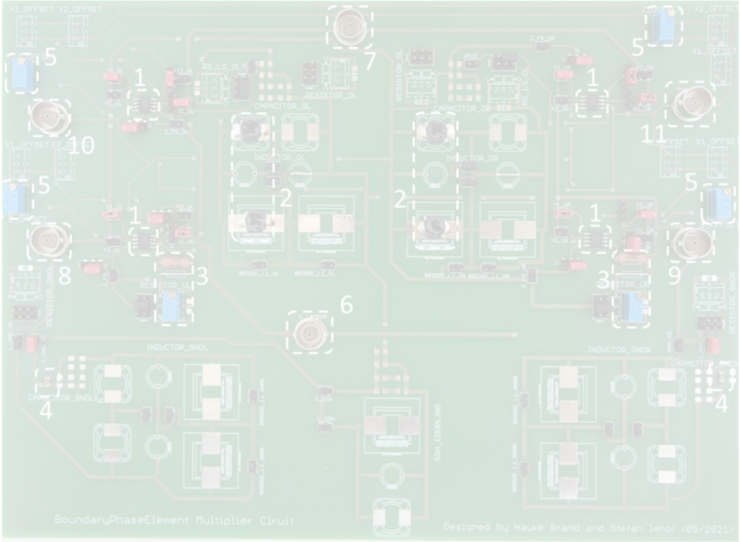
Nat. Commun. **13**, 4373 (2022)

hyperbolic
momentum space



Chen, TB, et al.,
arXiv:2205.05106 (2022)

hyperbolic
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Outline of the remainder of the talk

1. **exp.**

hyperbolic
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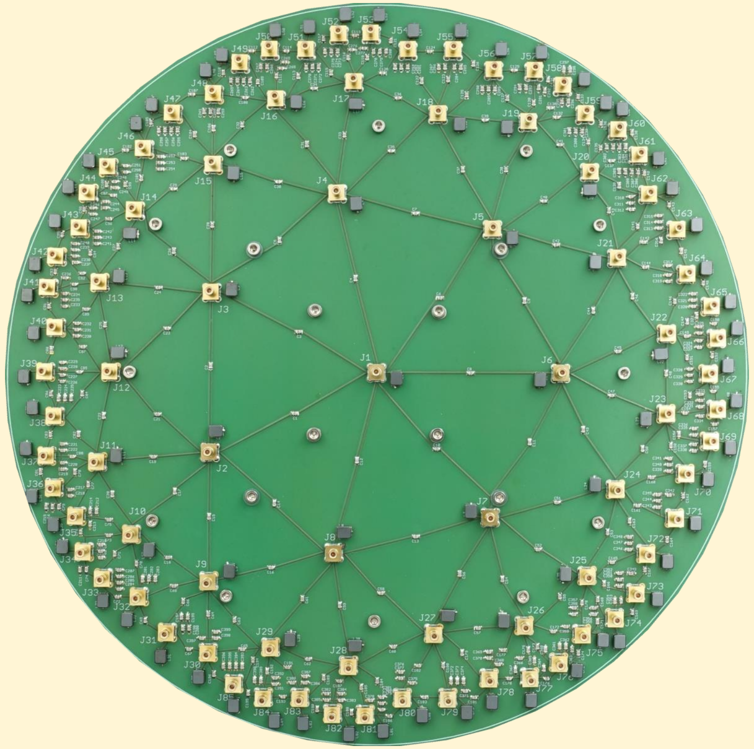

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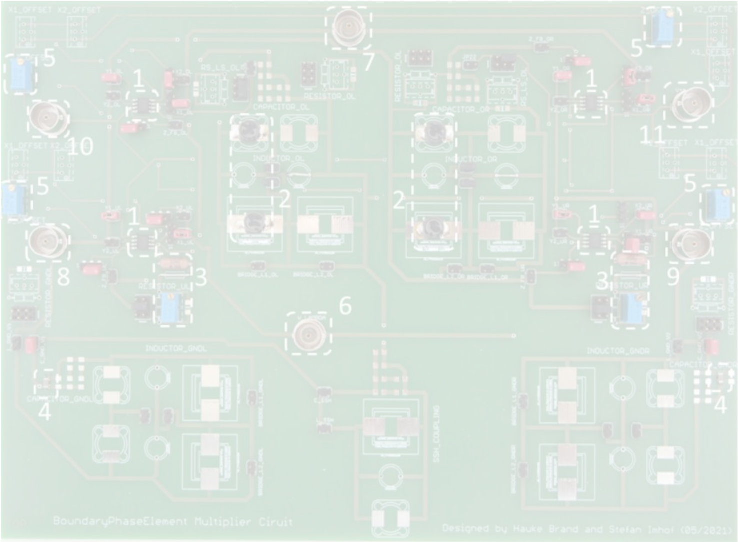

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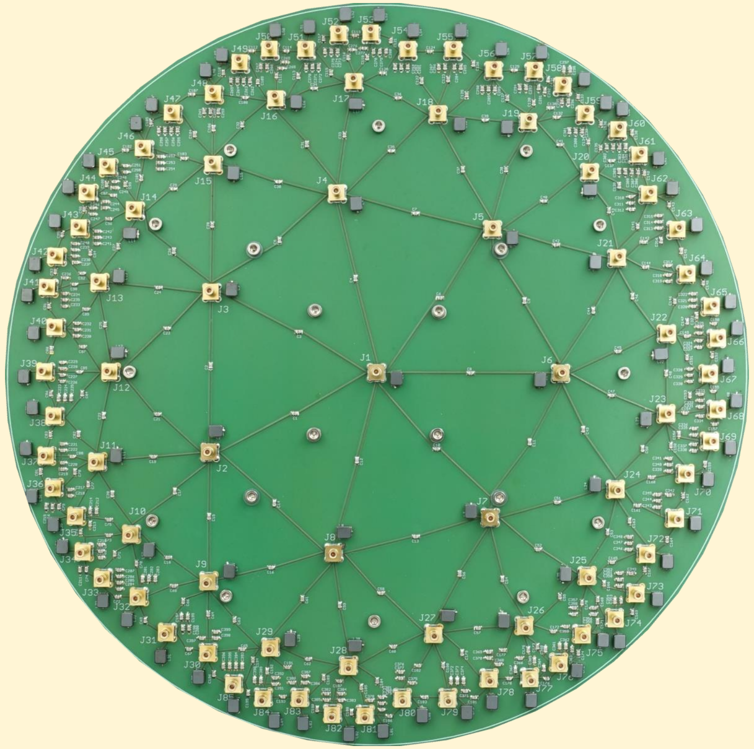

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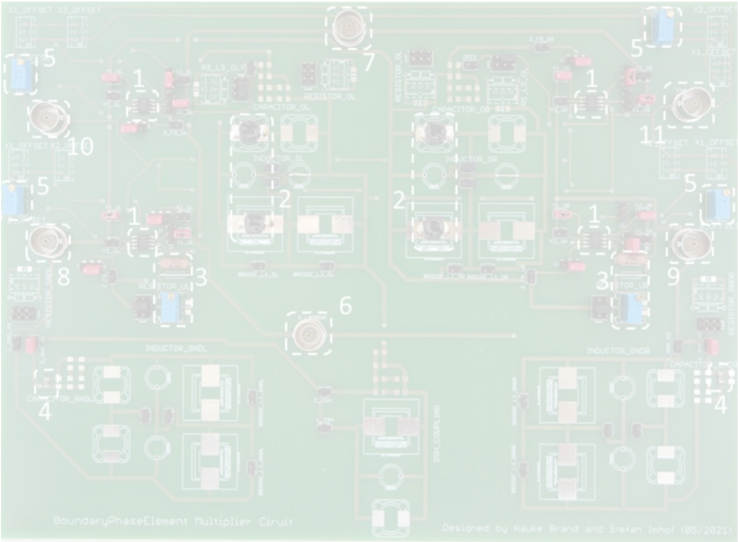

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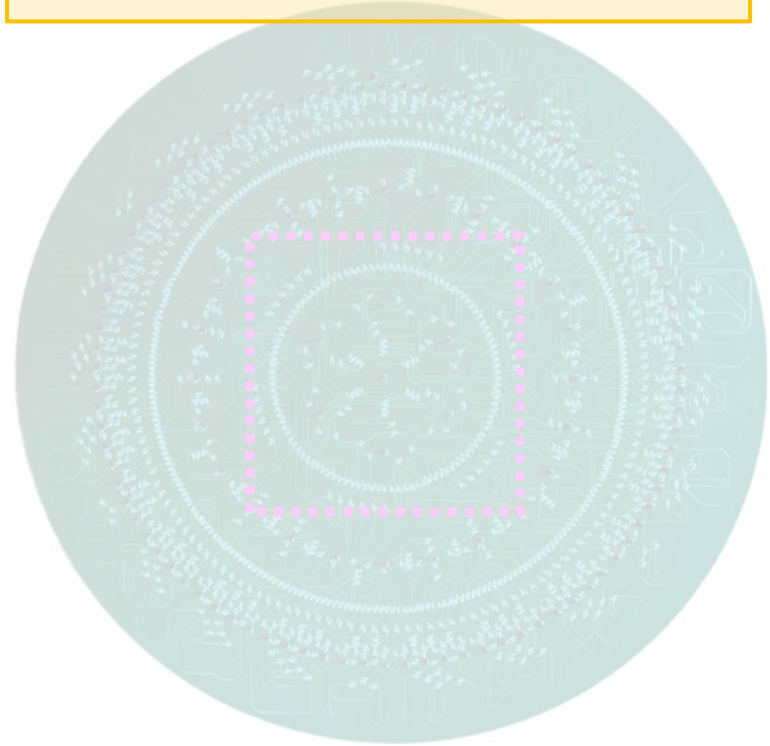

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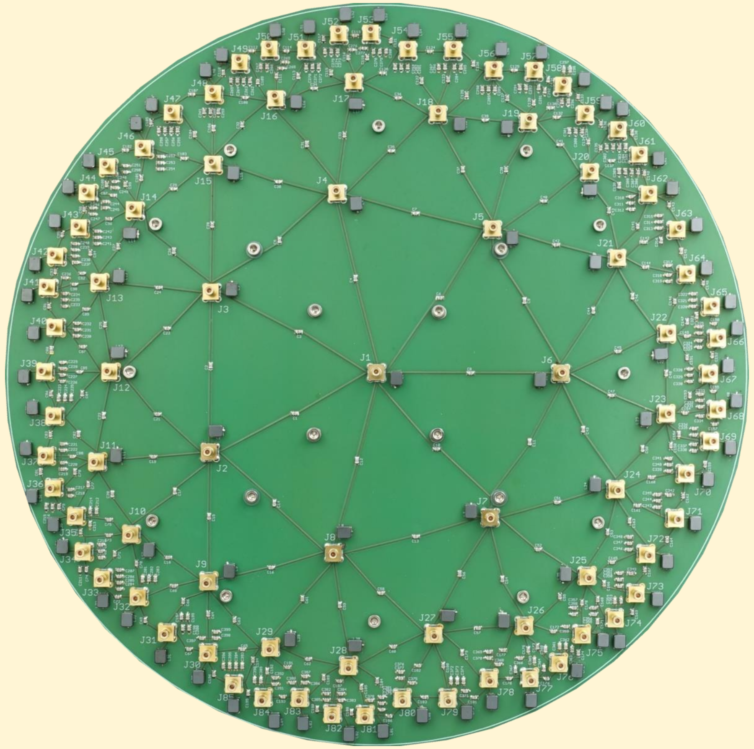

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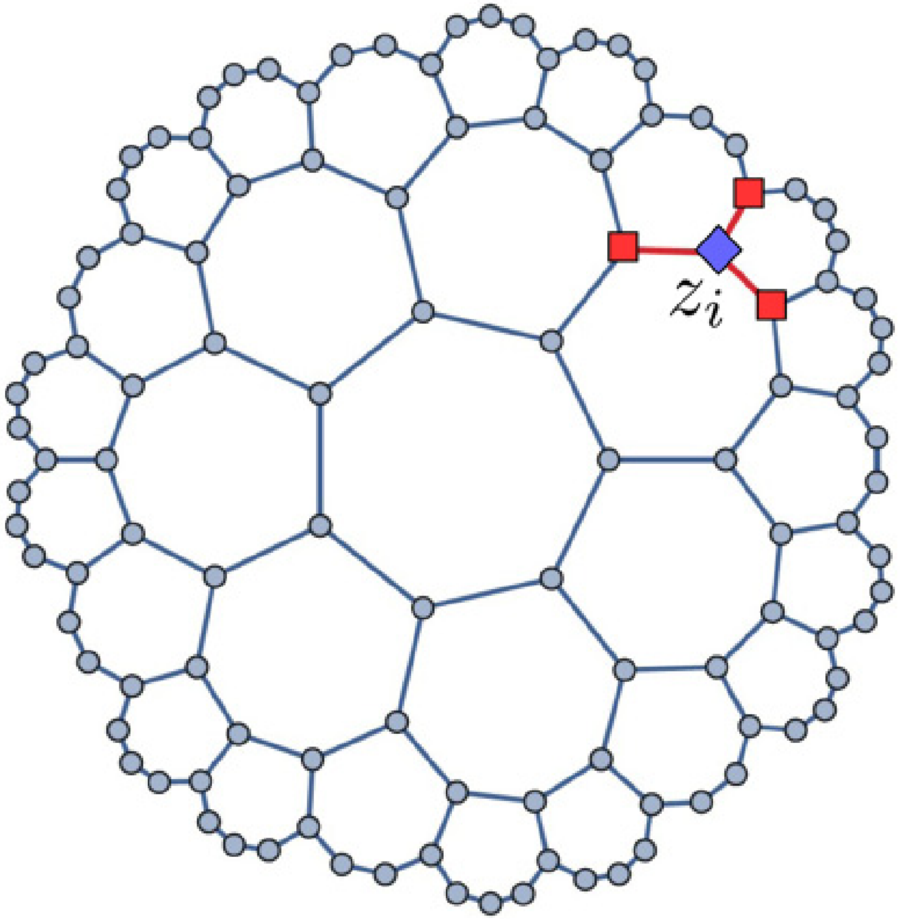
4. **the.**

flat-band degeneracy in hyperbolic kagome

Nearest-neighbor models approximate continuum



NN-hopping Hamiltonian is the ***adjacency matrix*** of the graph

Discrete (lattice) Hamiltonian: $\sum_j A_{ij} f_j = \lambda f_i$

I. Boettcher, P. Bienias, R. Belyansky, A. J. Kollár, A. V. Gorshkov, *Phys. Rev. A* **102**, 032208 (2020)

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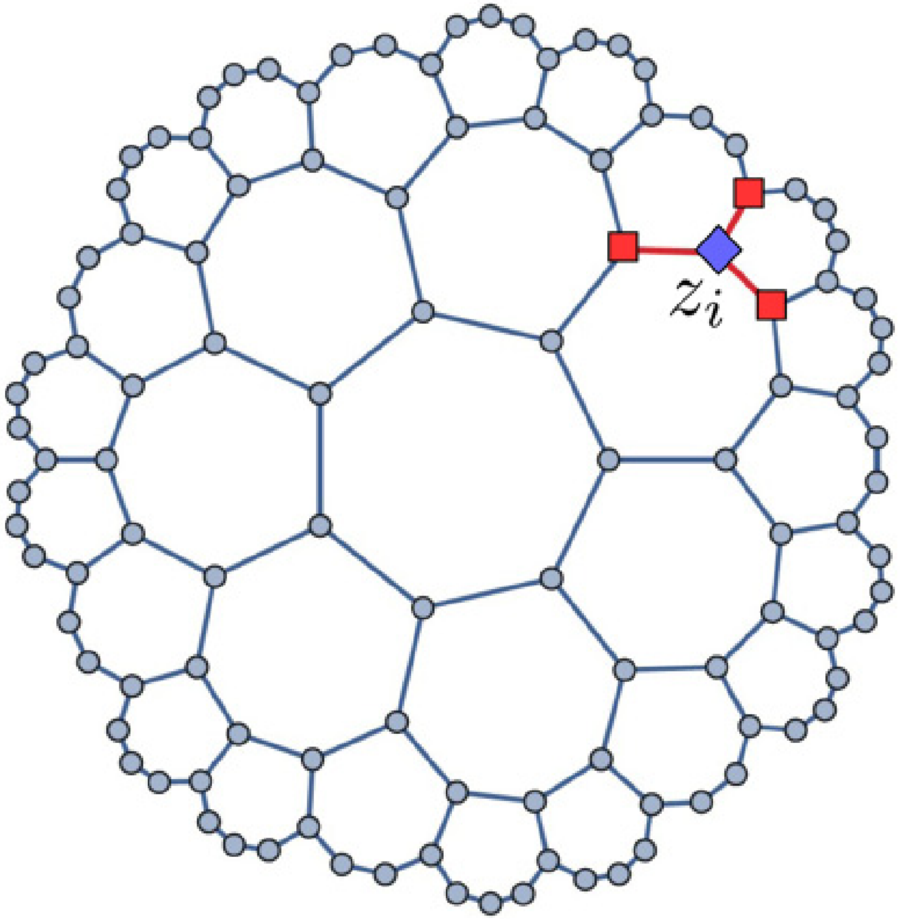

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BUT, assume that $f_j = f(z_j)$ are particular values of a *smooth function on the Poincare disk*. Then:

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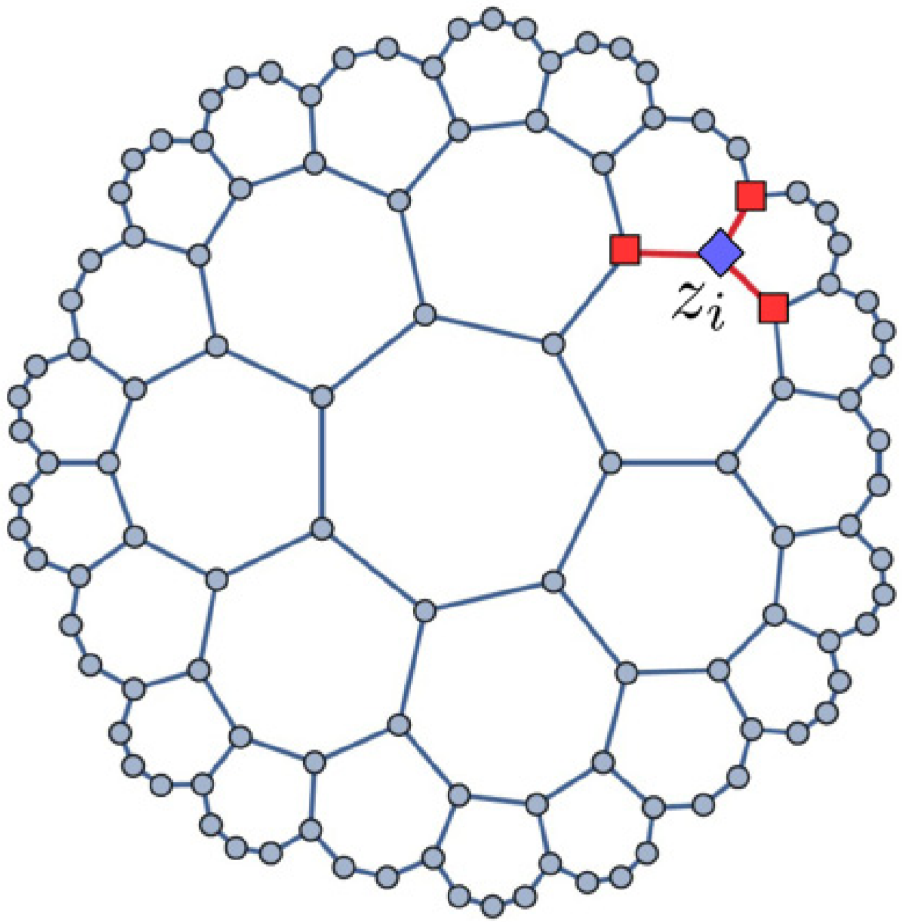

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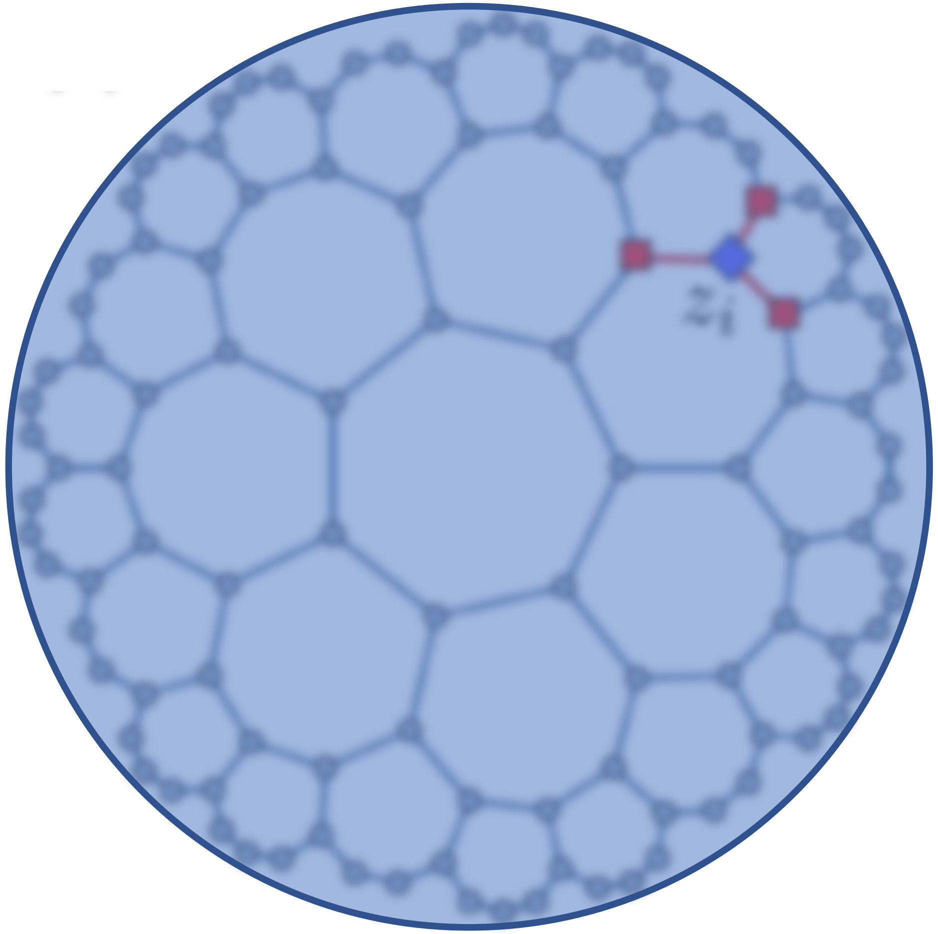
$\approx$ 0.276…

Laplace-Beltrami operator in continuum

I. Boettcher, P. Bienias, R. Belyansky, A. J. Kollár, A. V. Gorshkov, *Phys. Rev. A* **102**, 032208 (2020)

Goal: study standing waves of a “hyperbolic drum”



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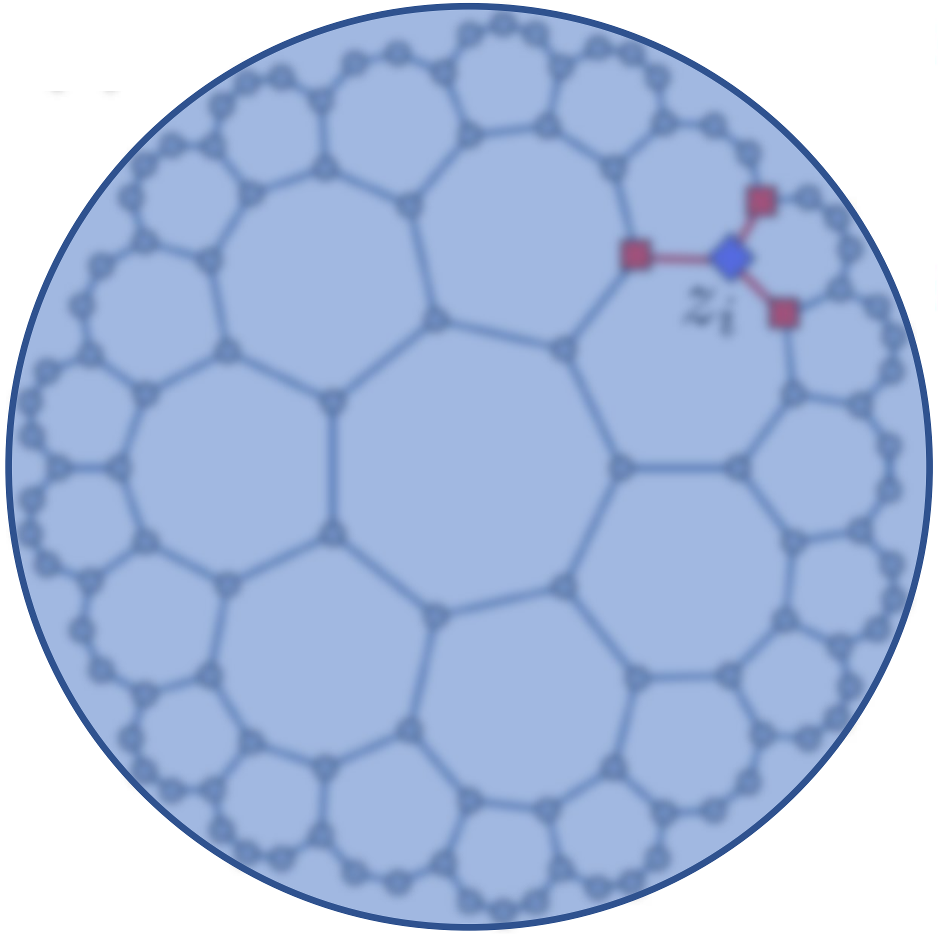

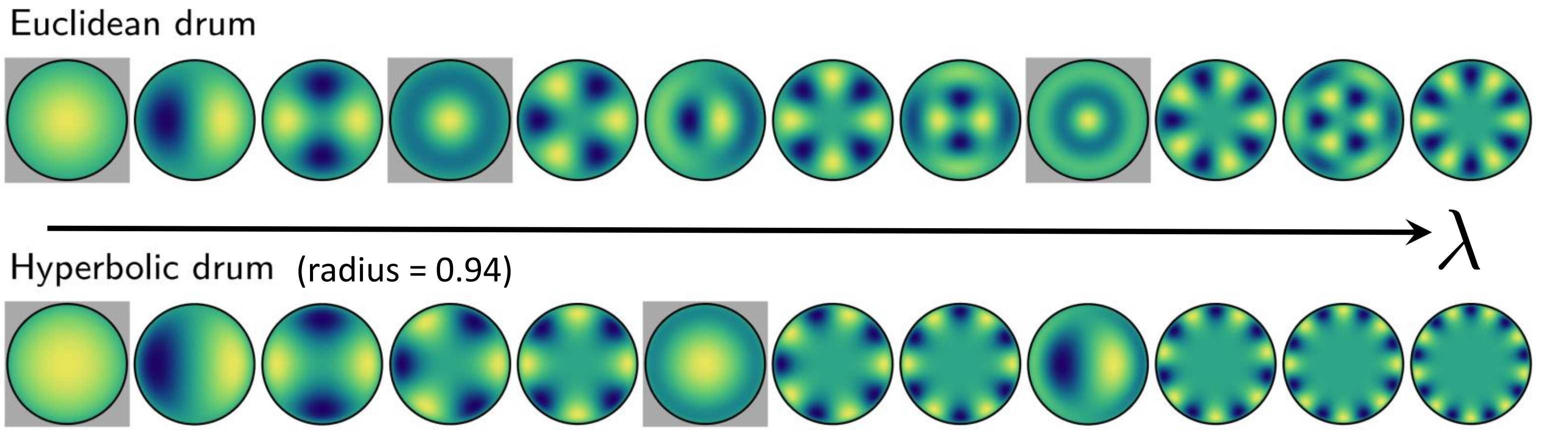

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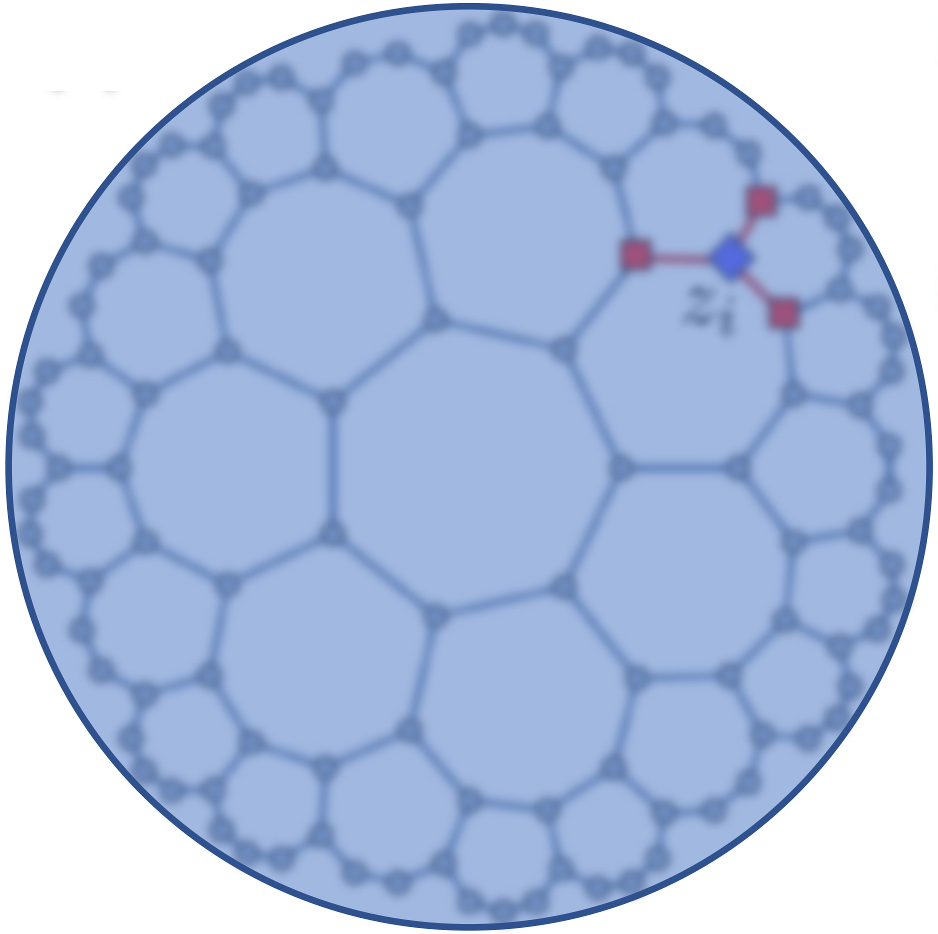

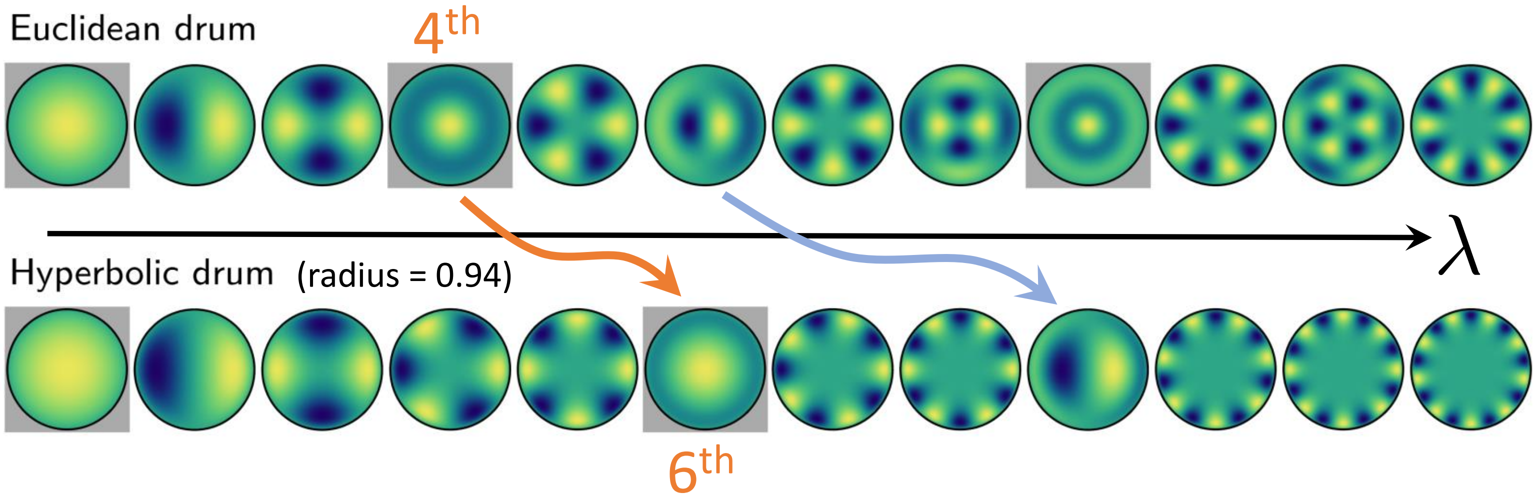

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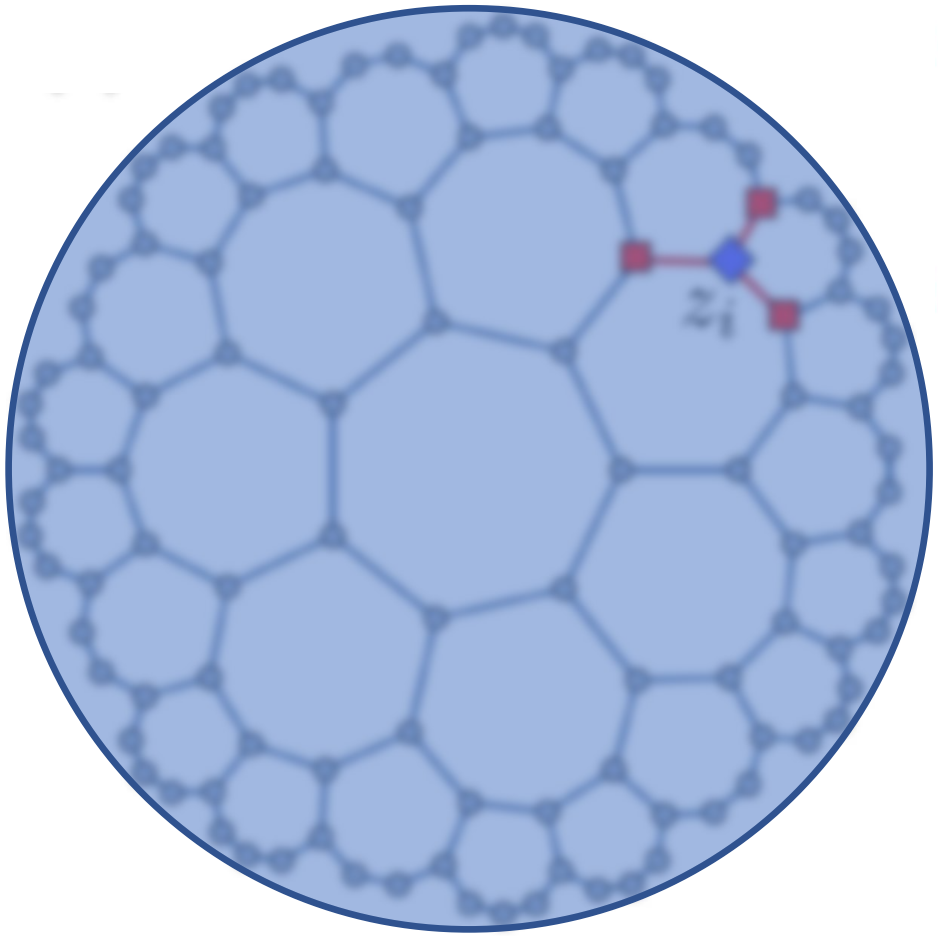

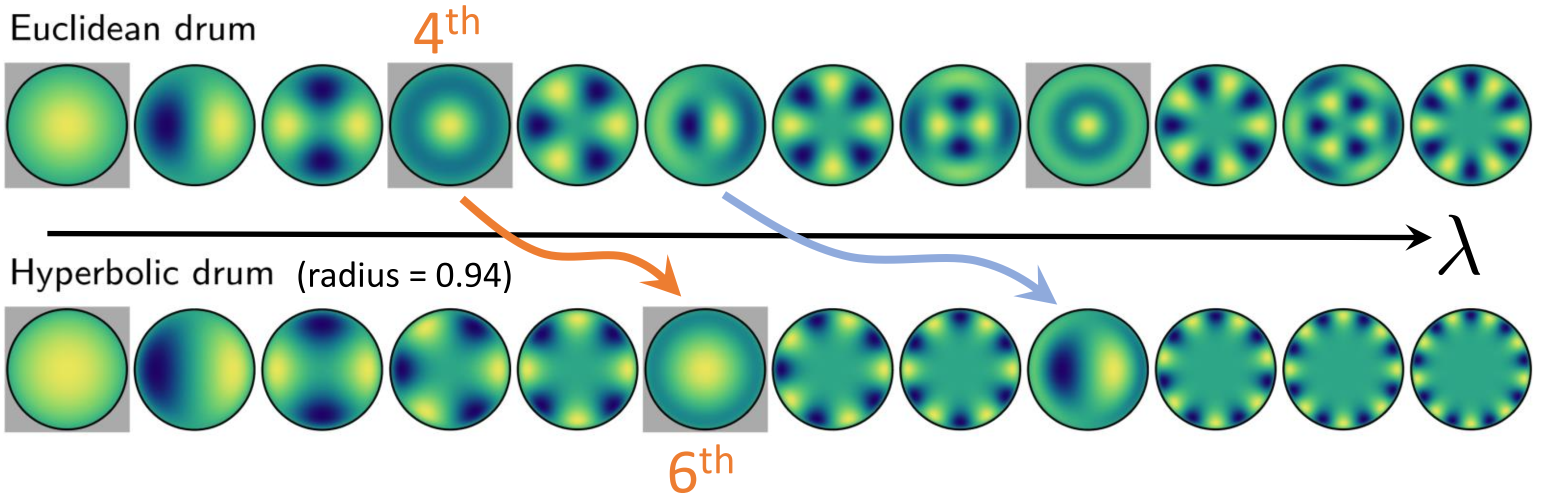

The closer the radius is to r=1, the stronger is the re-ordering!

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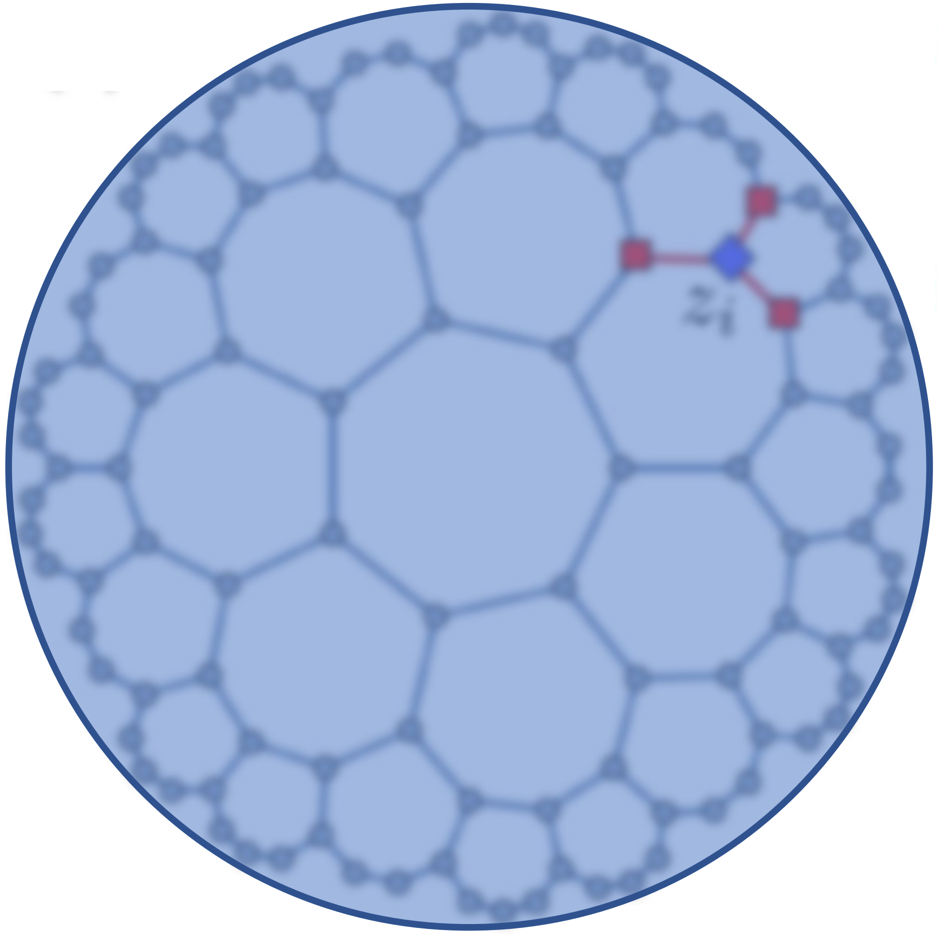

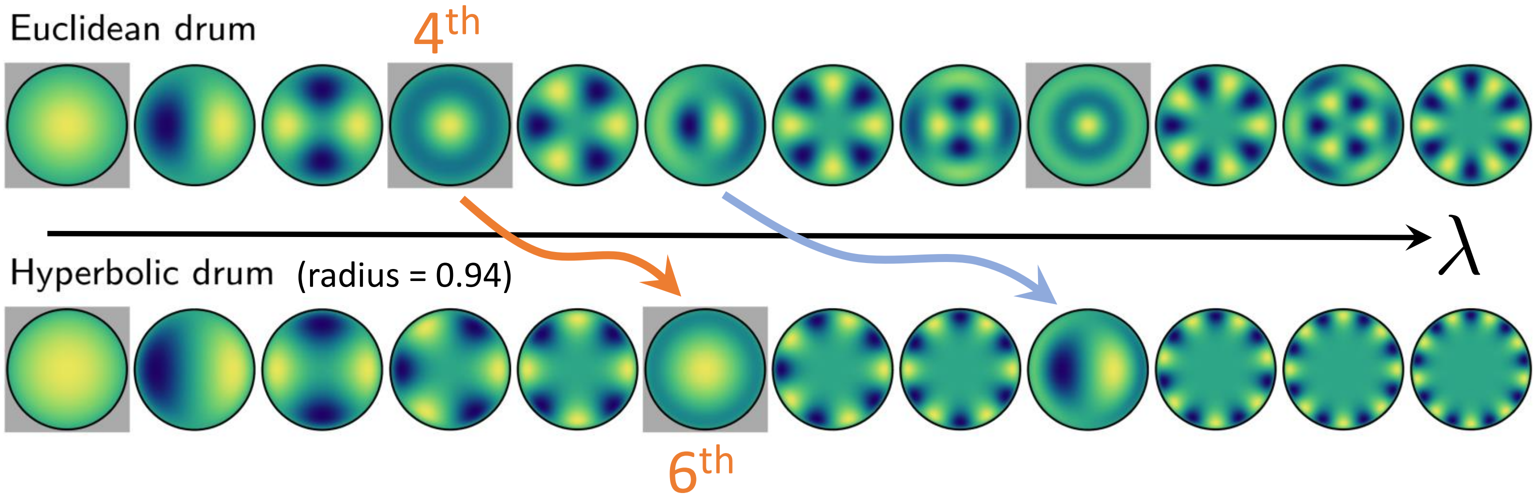

The closer the radius is to r=1, the stronger is the re-ordering!

Can we experimentally reproduce this spectral re-ordering?

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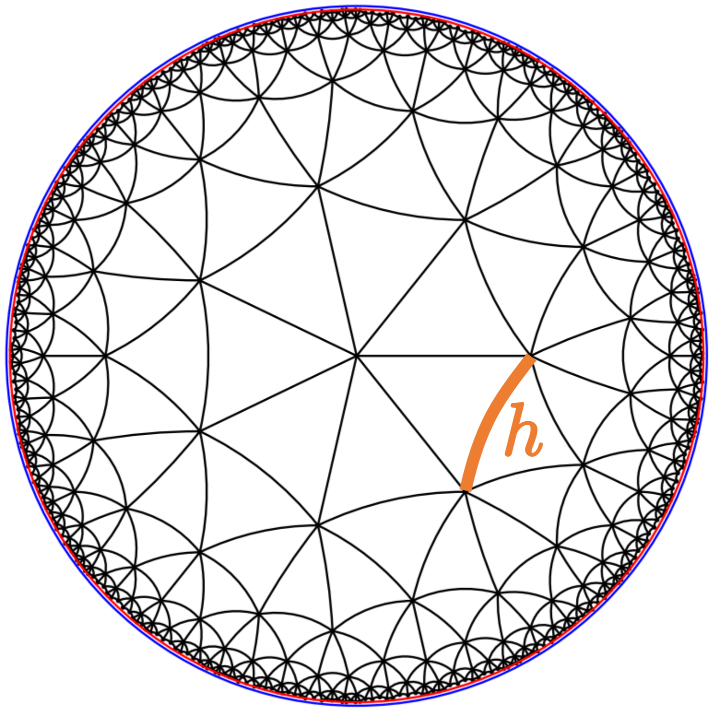
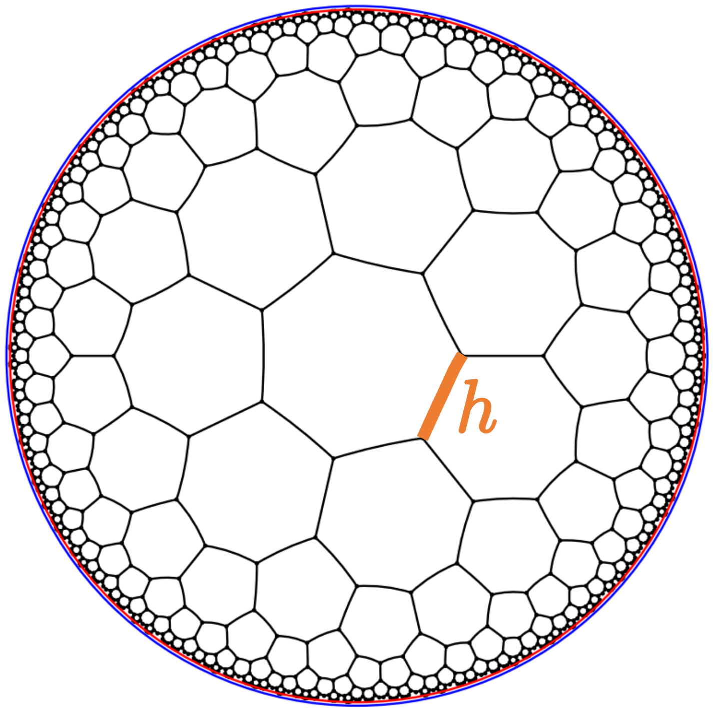
Laplace-Beltrami operator in continuum

Modelling of a “hyperbolic drum” ($R_0 = 0.99$)



{3,7}



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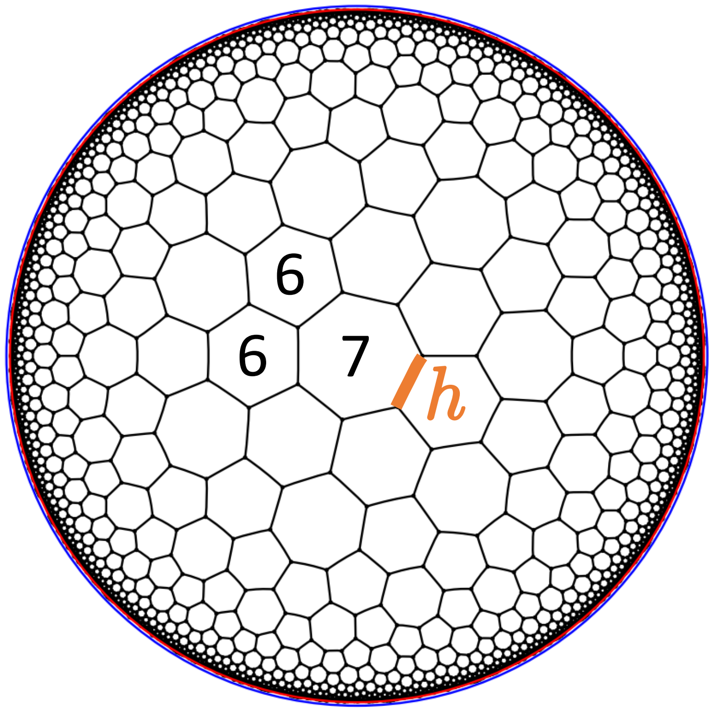

“hyperbolic soccerball” t{3,7}

P. M. Lenggenhager, A. Stegmaier, *Nat. Commun.* **13**, 4373(2022)

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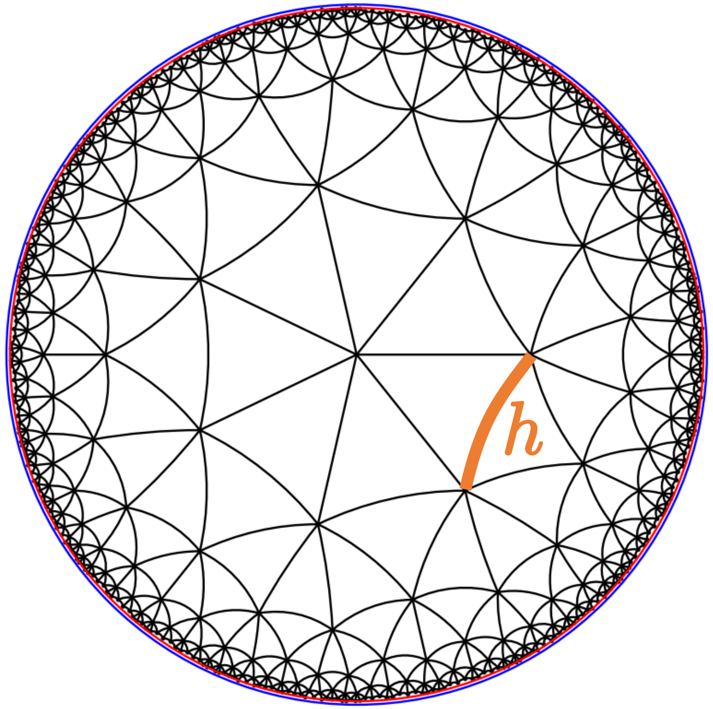

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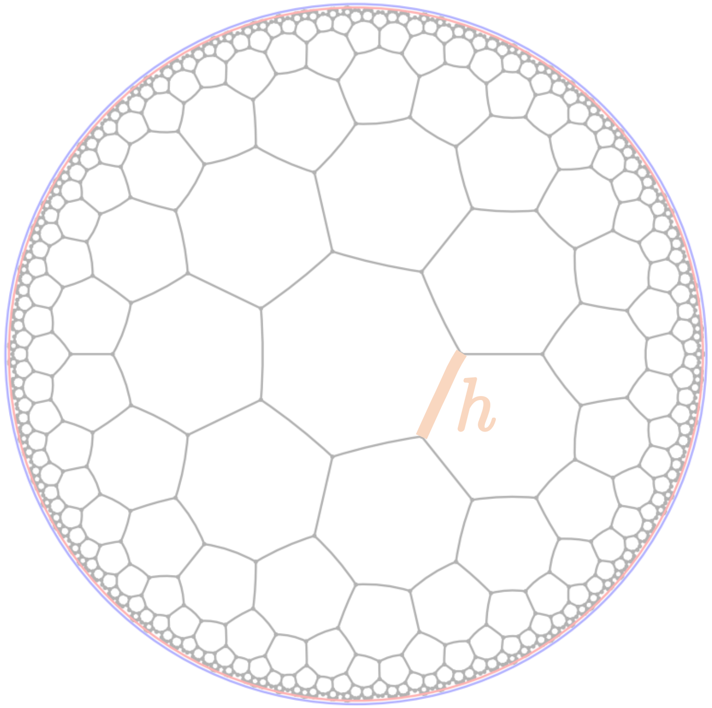

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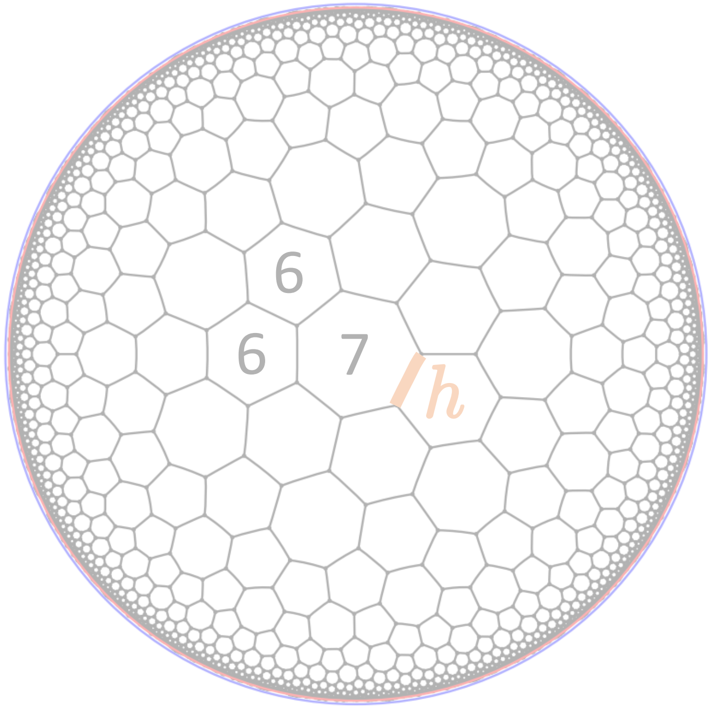

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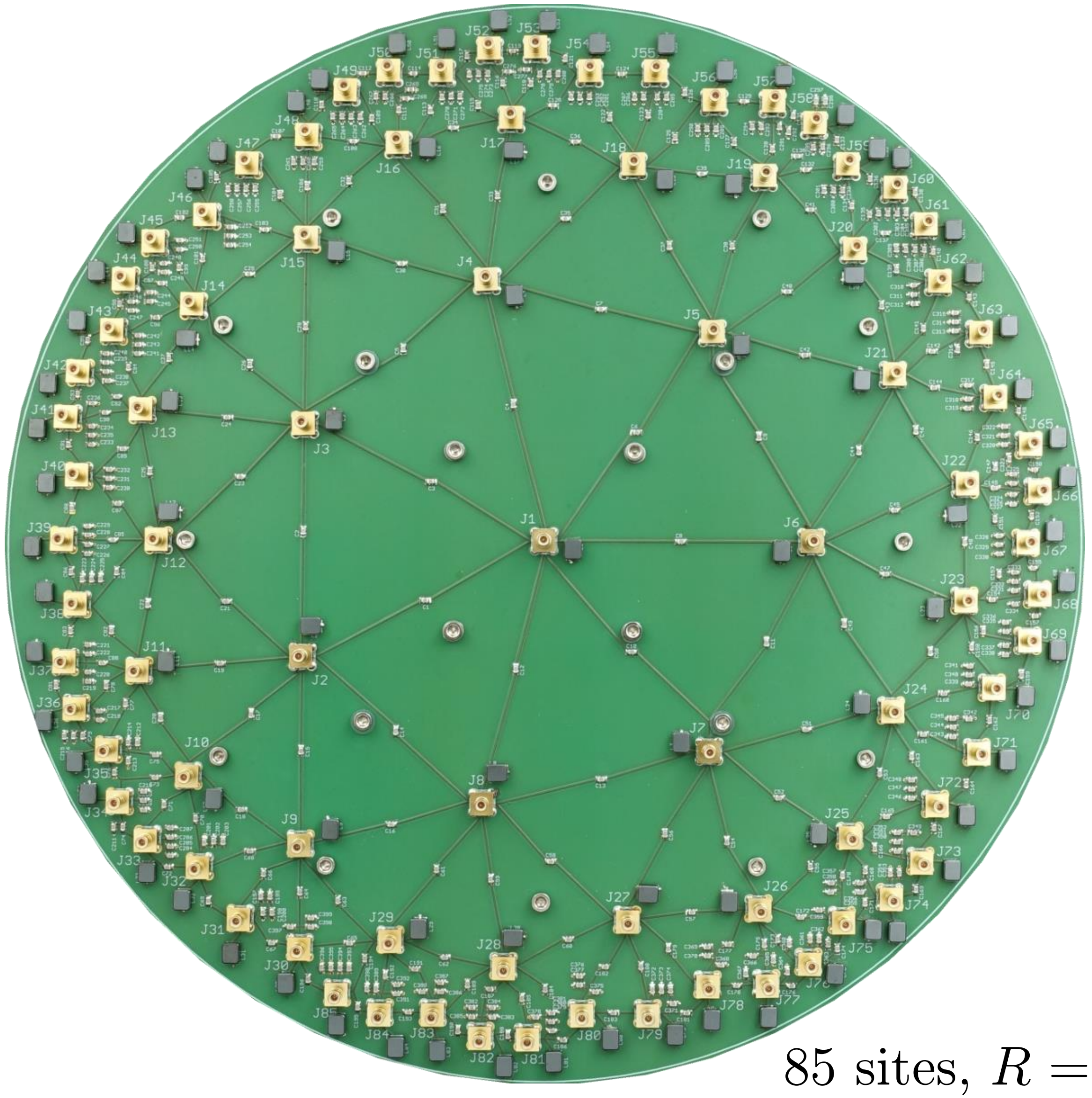
Experimental realization



$85 \text{ sites}, R = 0.94$

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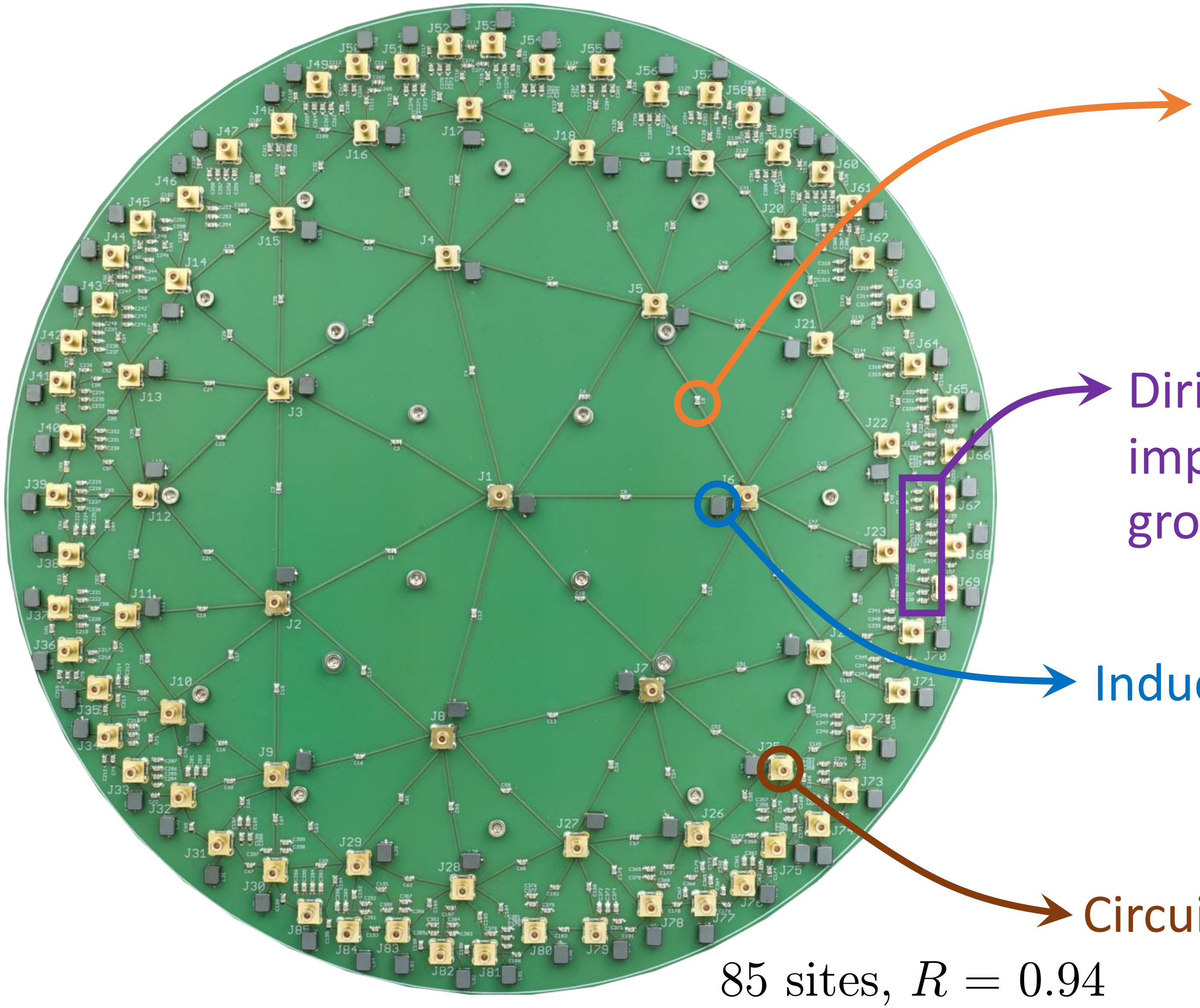

Capacitive coupling of sites (C = 1nF)

Dirichlet boundary condition implemented by additional capacitive grounding of boundary nodes C’ = nC.

Inductive coupling to ground (L = 10μH)

Circuit nodes & connectors for oscilloscopes

$85 \text{ sites}, \ R = 0.94$

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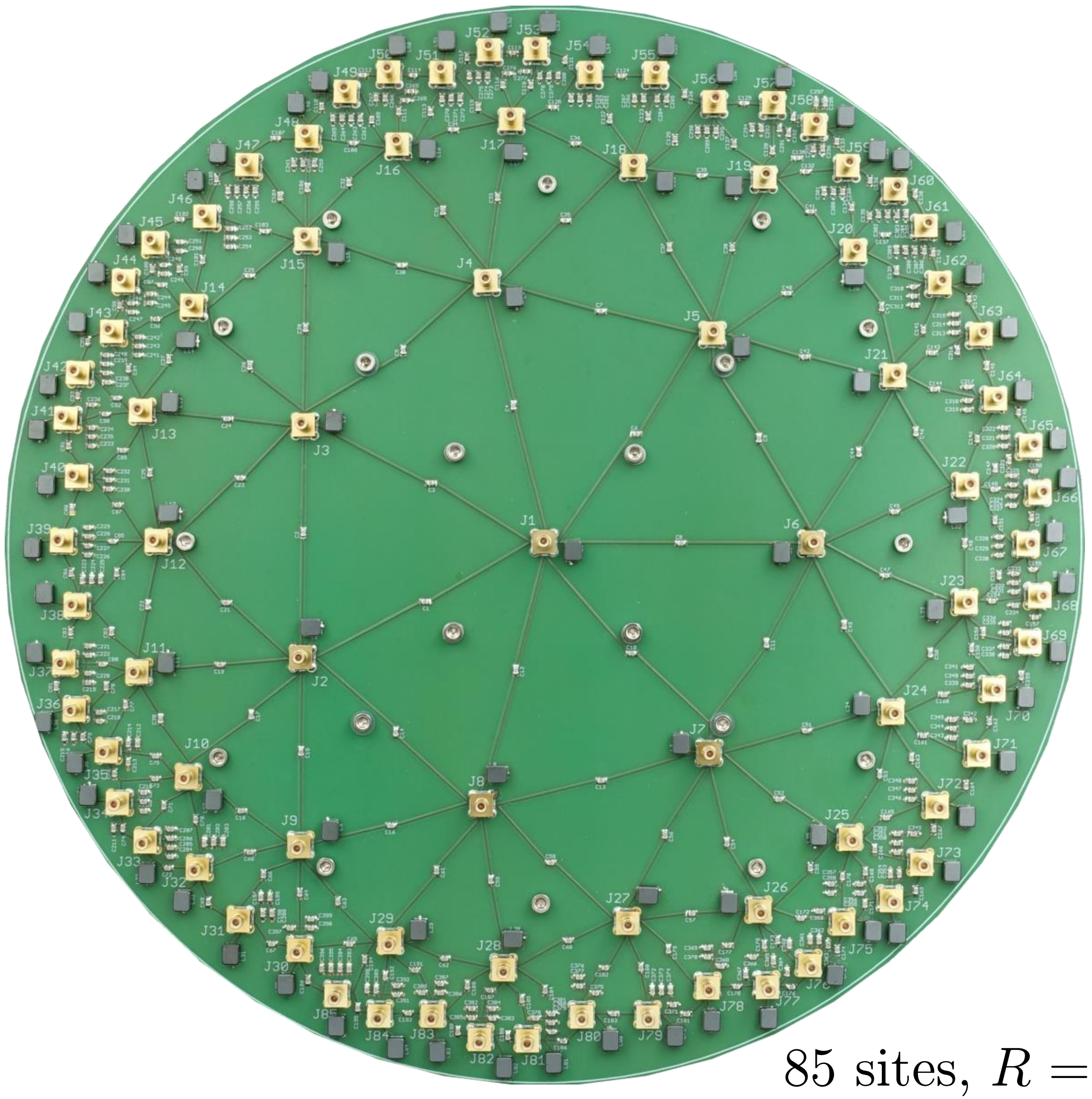

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Spectral reversal:

$$\lambda^{\beta} = \frac{1}{7\pi^2 h^2 LC} \frac{1}{(f^{\beta})^2}$$

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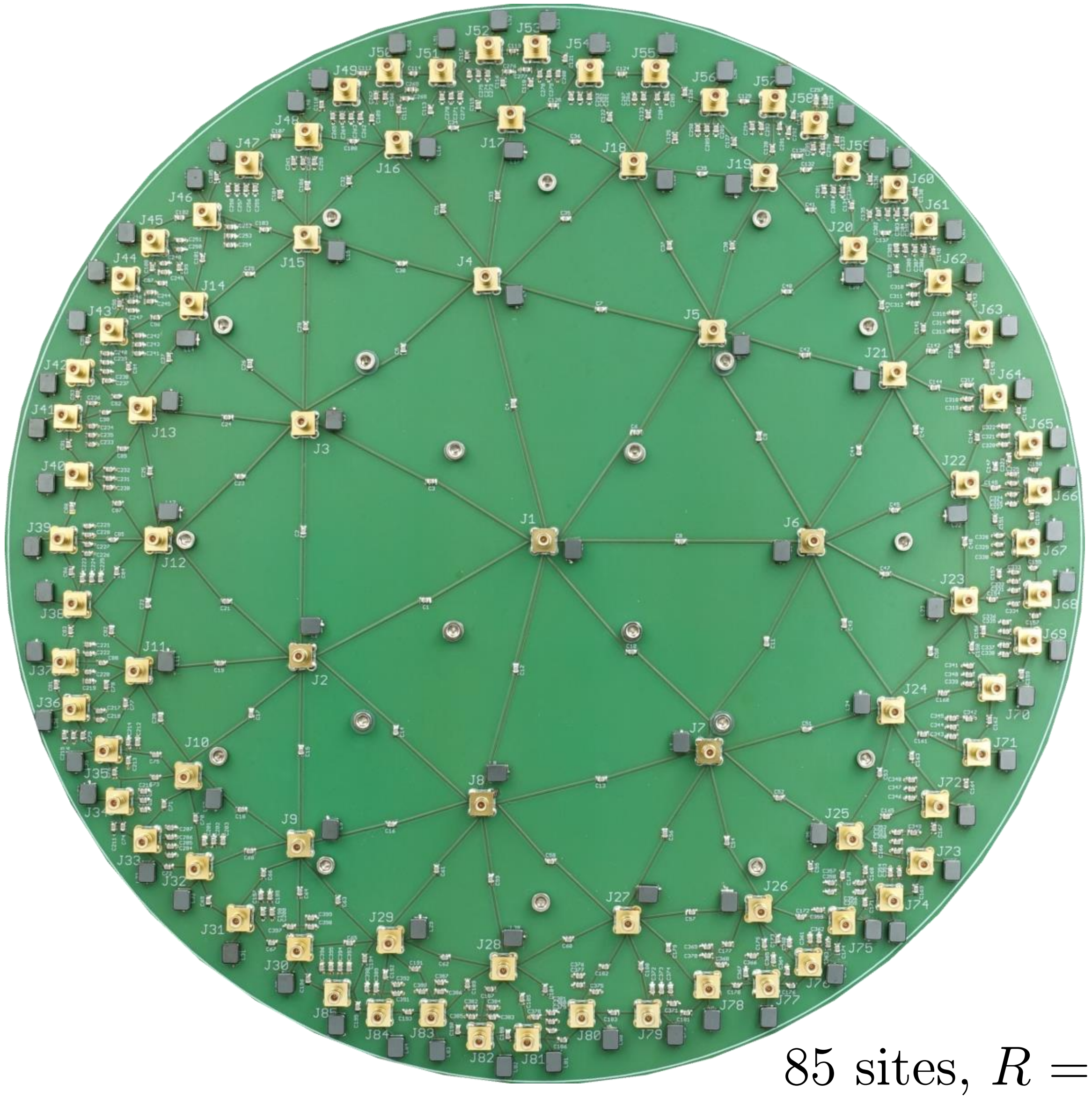

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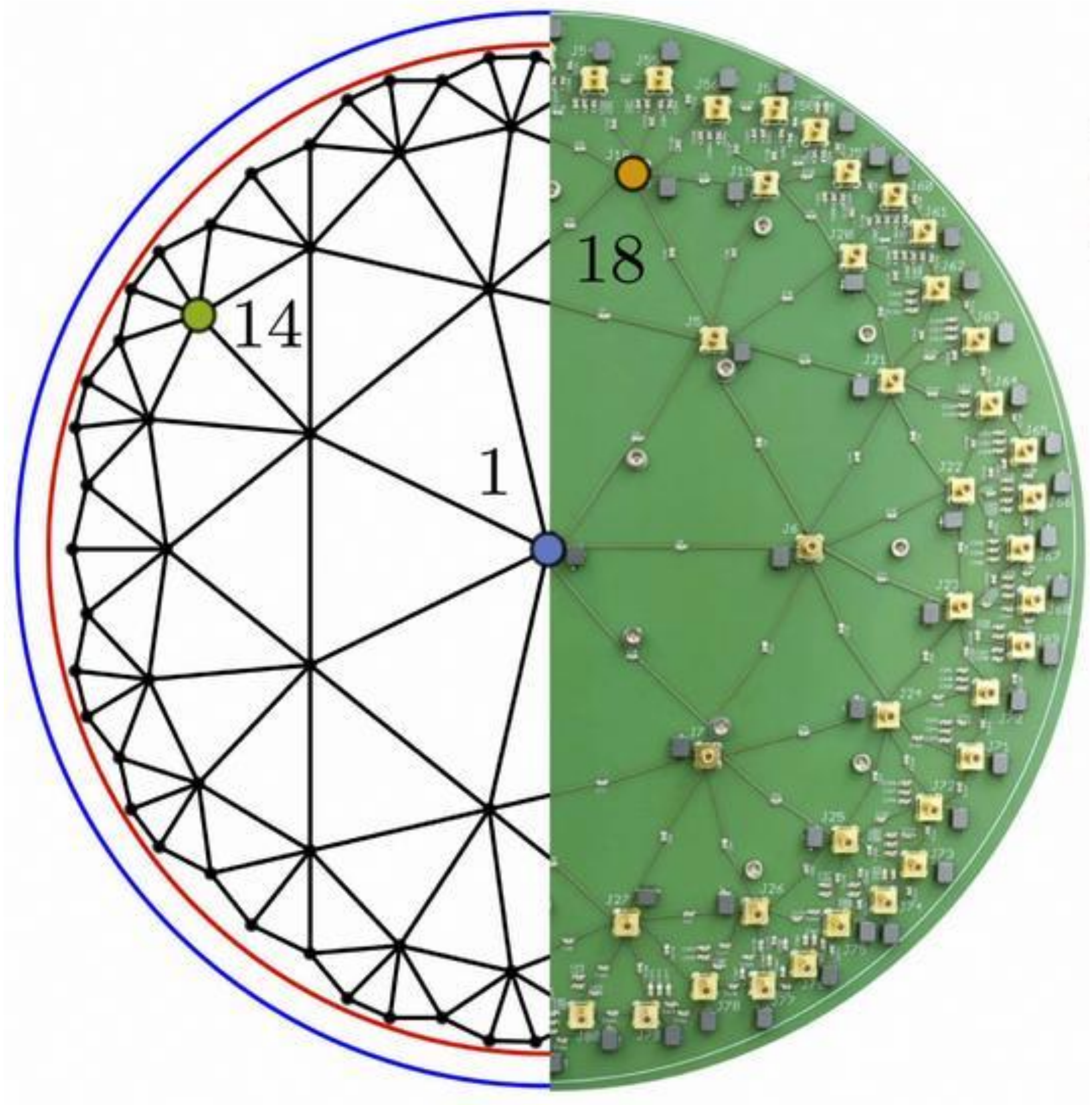
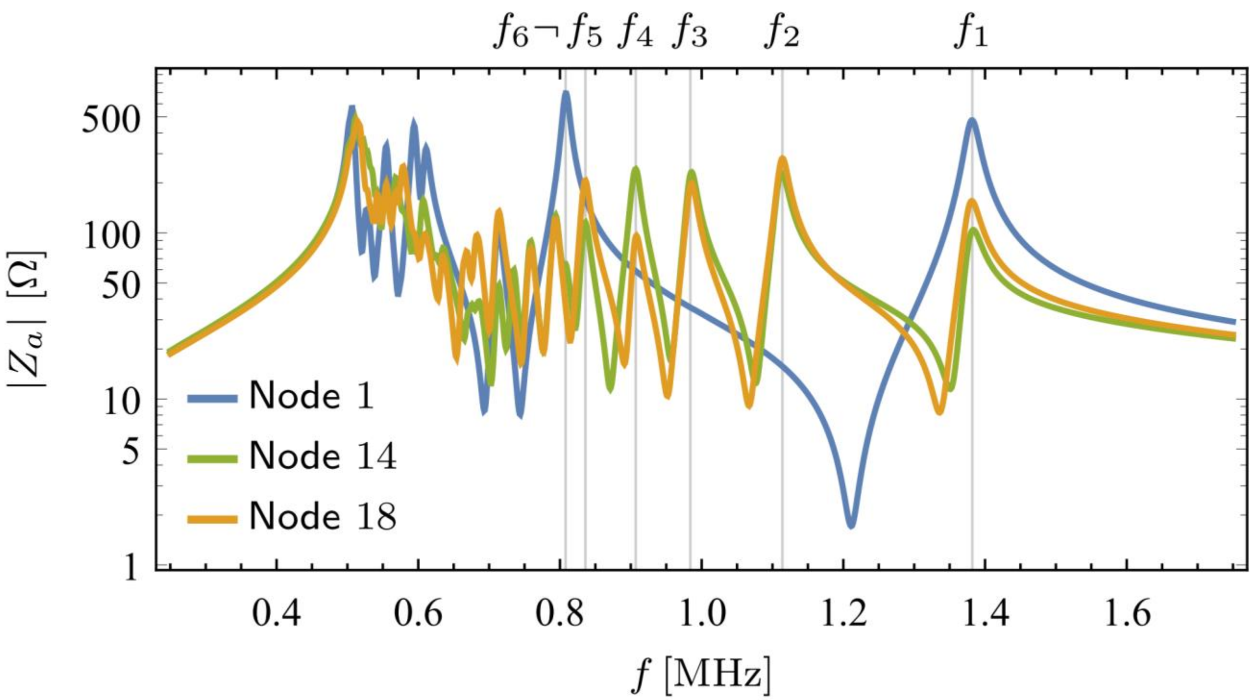
Eigenmodes of Laplace-Beltrami

Circuit resonance frequencies

P. M. Lenggenhager, A. Stegmaier, *Nat. Commun.* **13**, 4373(2022)

Experiment #1 – impedance measurements





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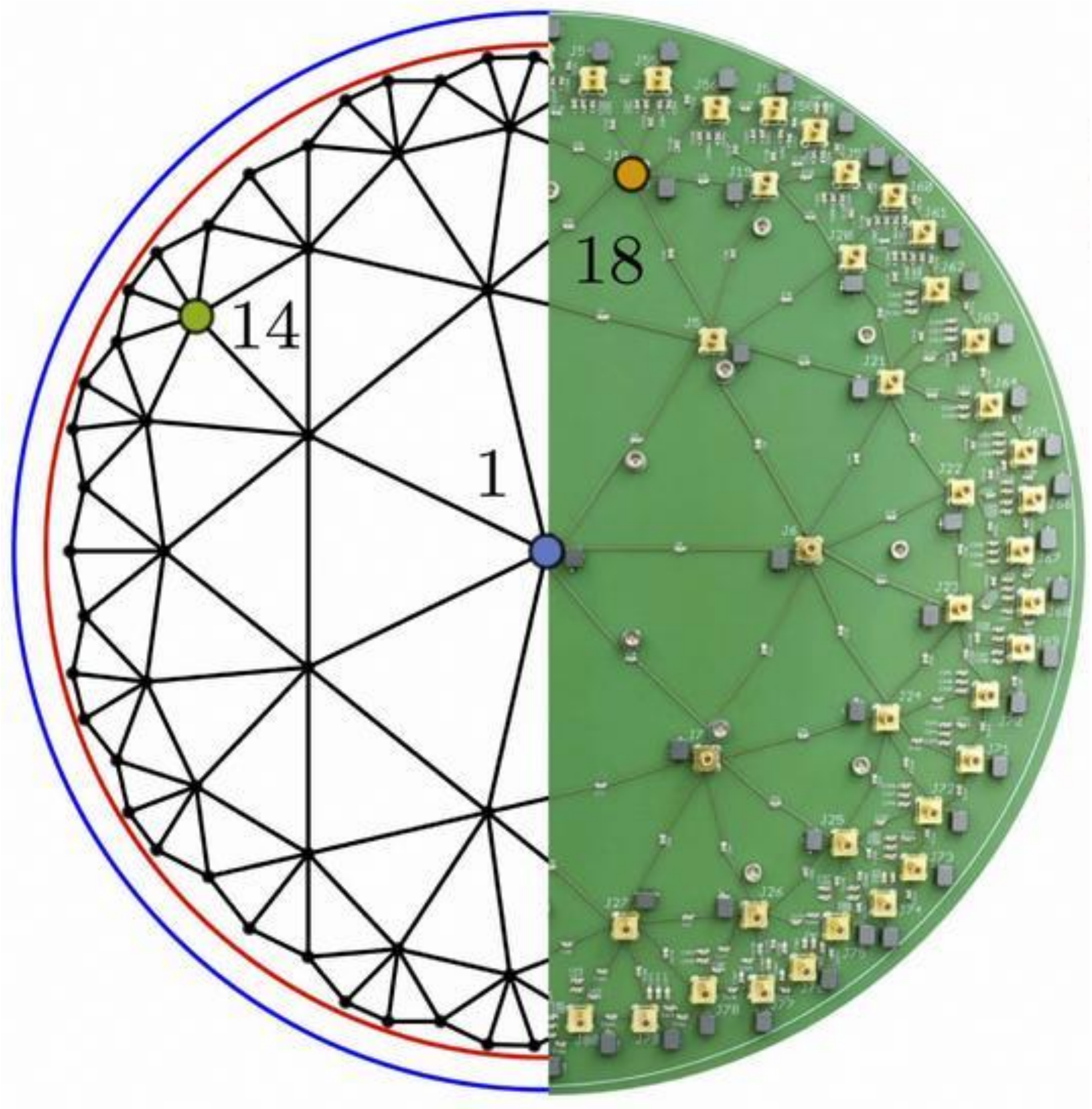
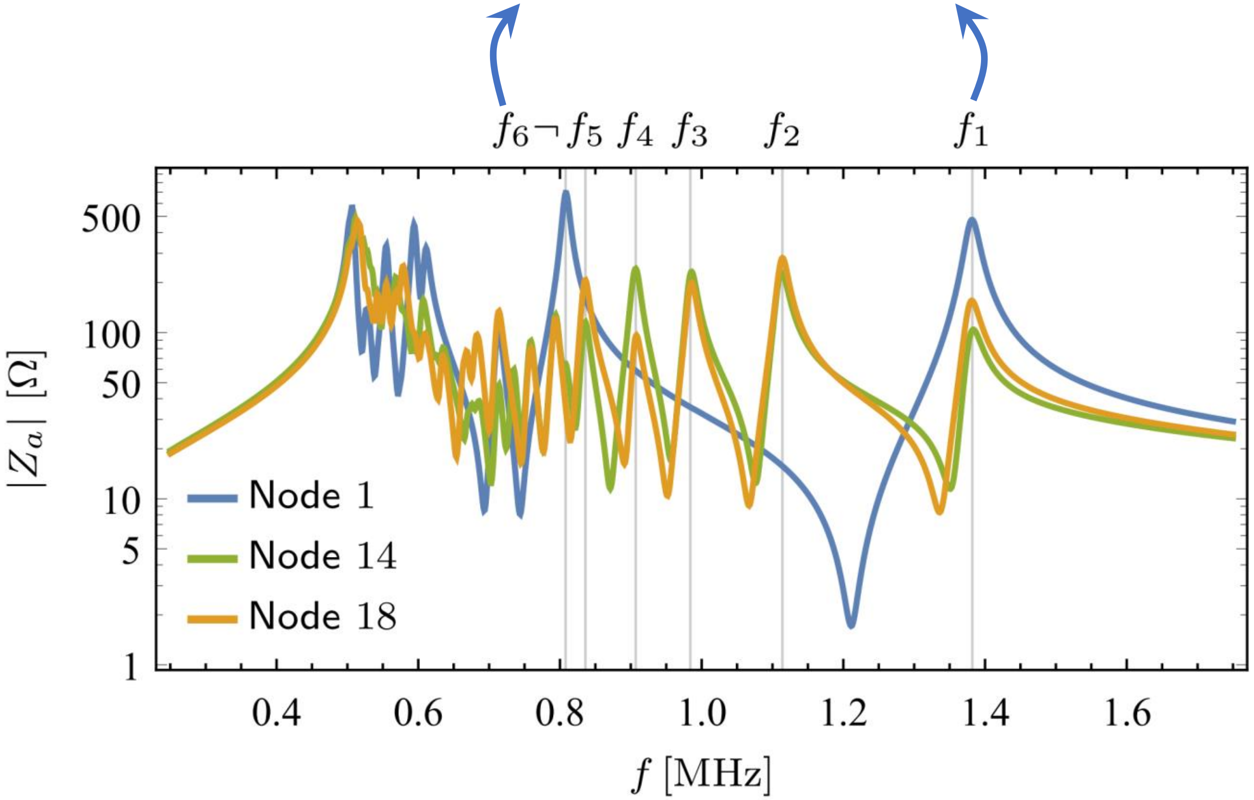


Only s-modes visible at site “1”.



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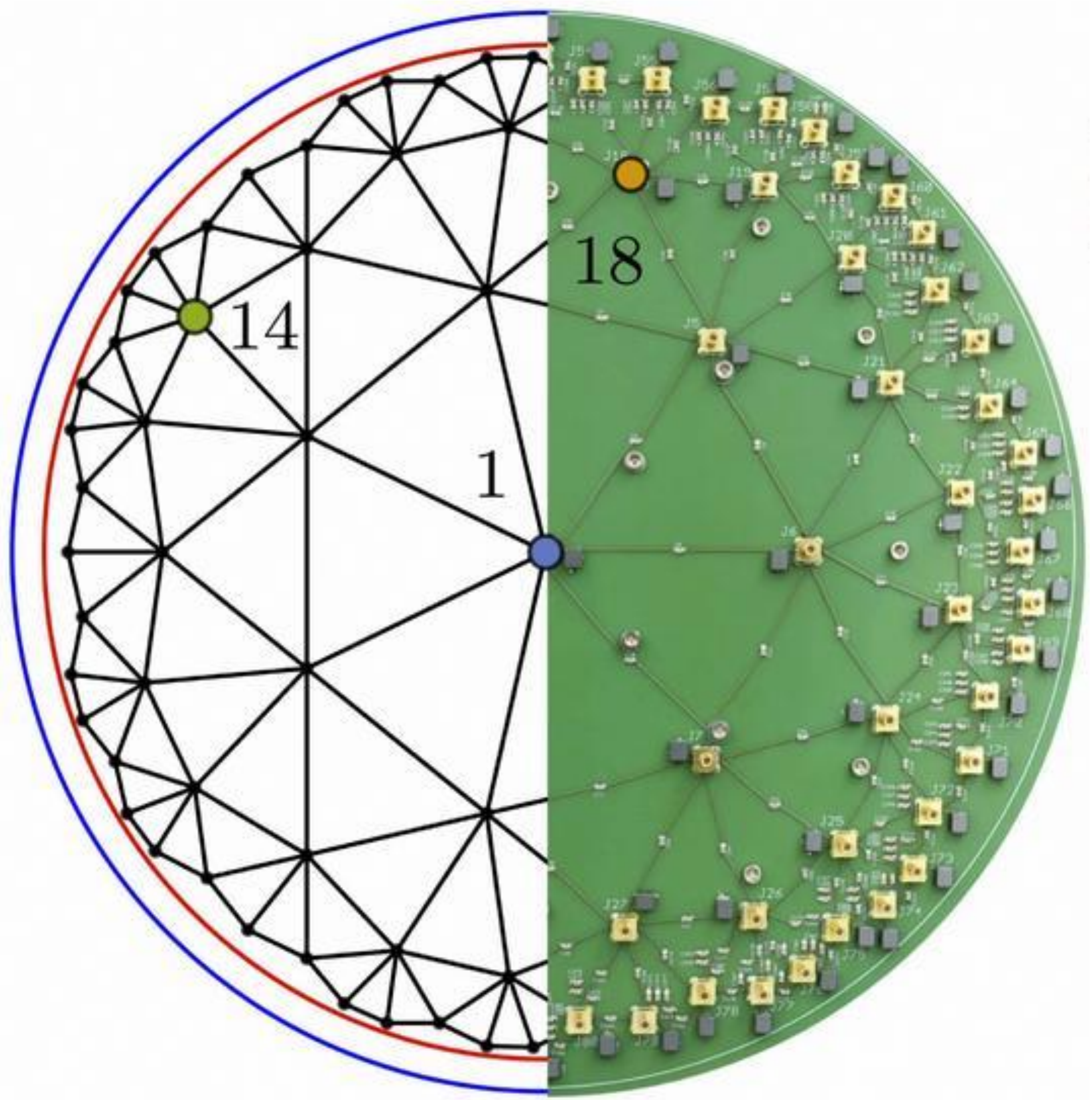

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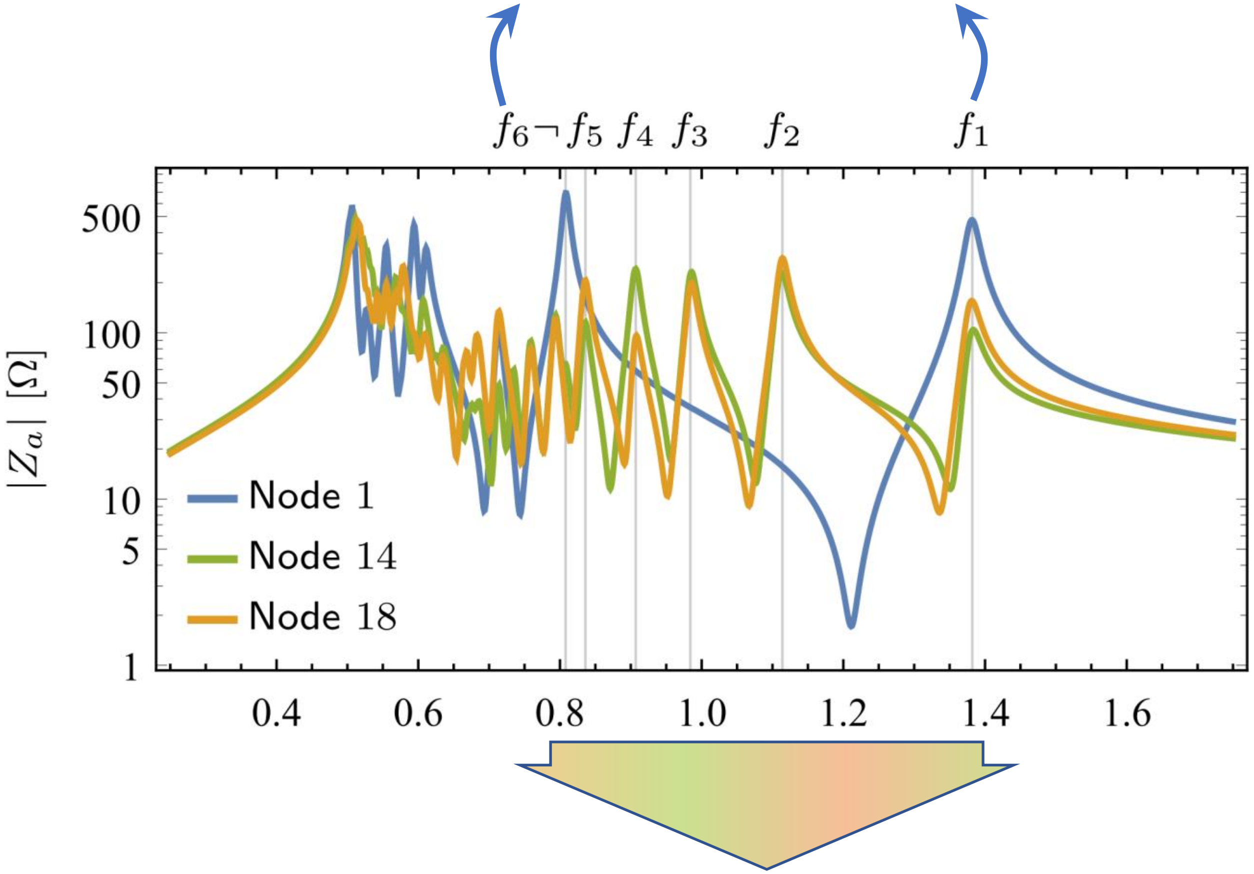

All modes visible at sites “14” and “18”.

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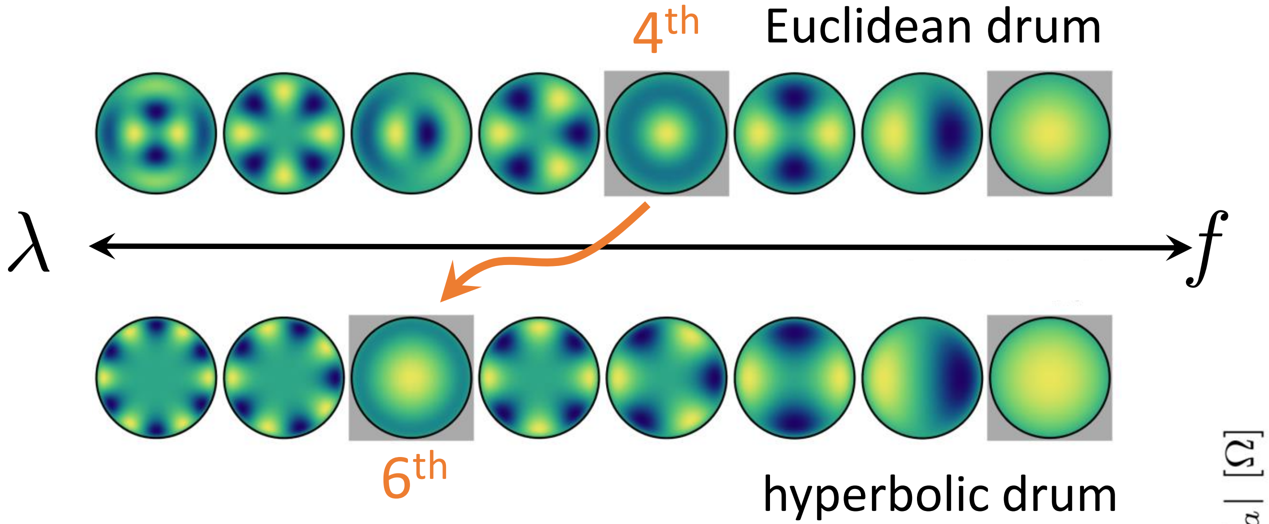

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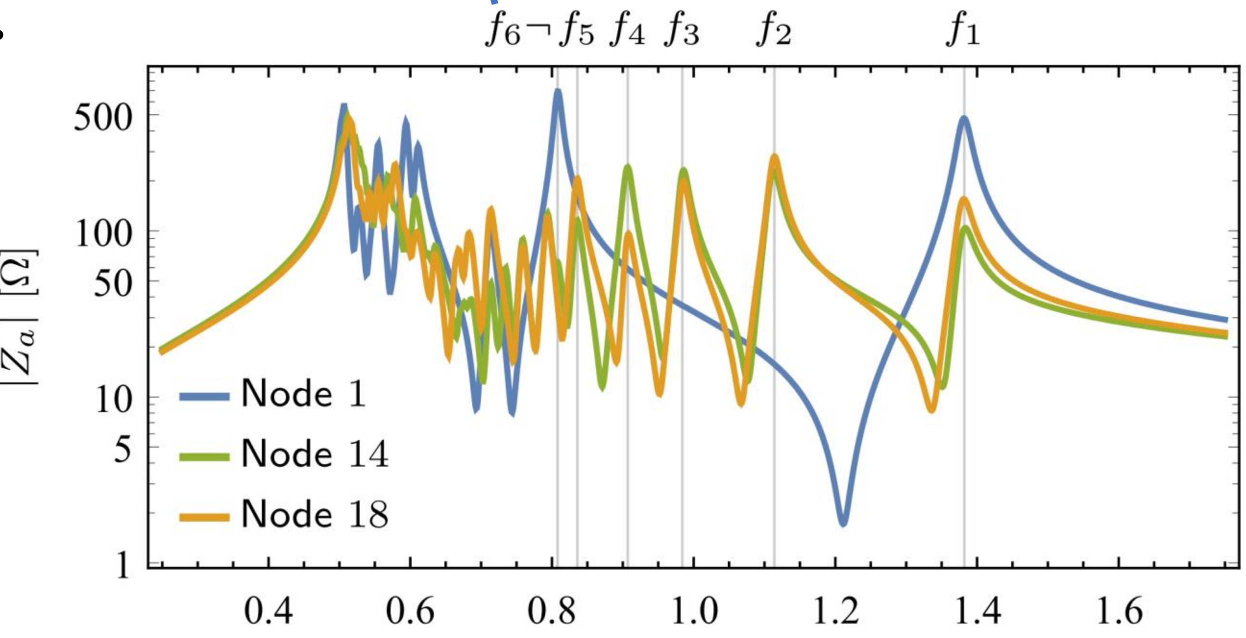

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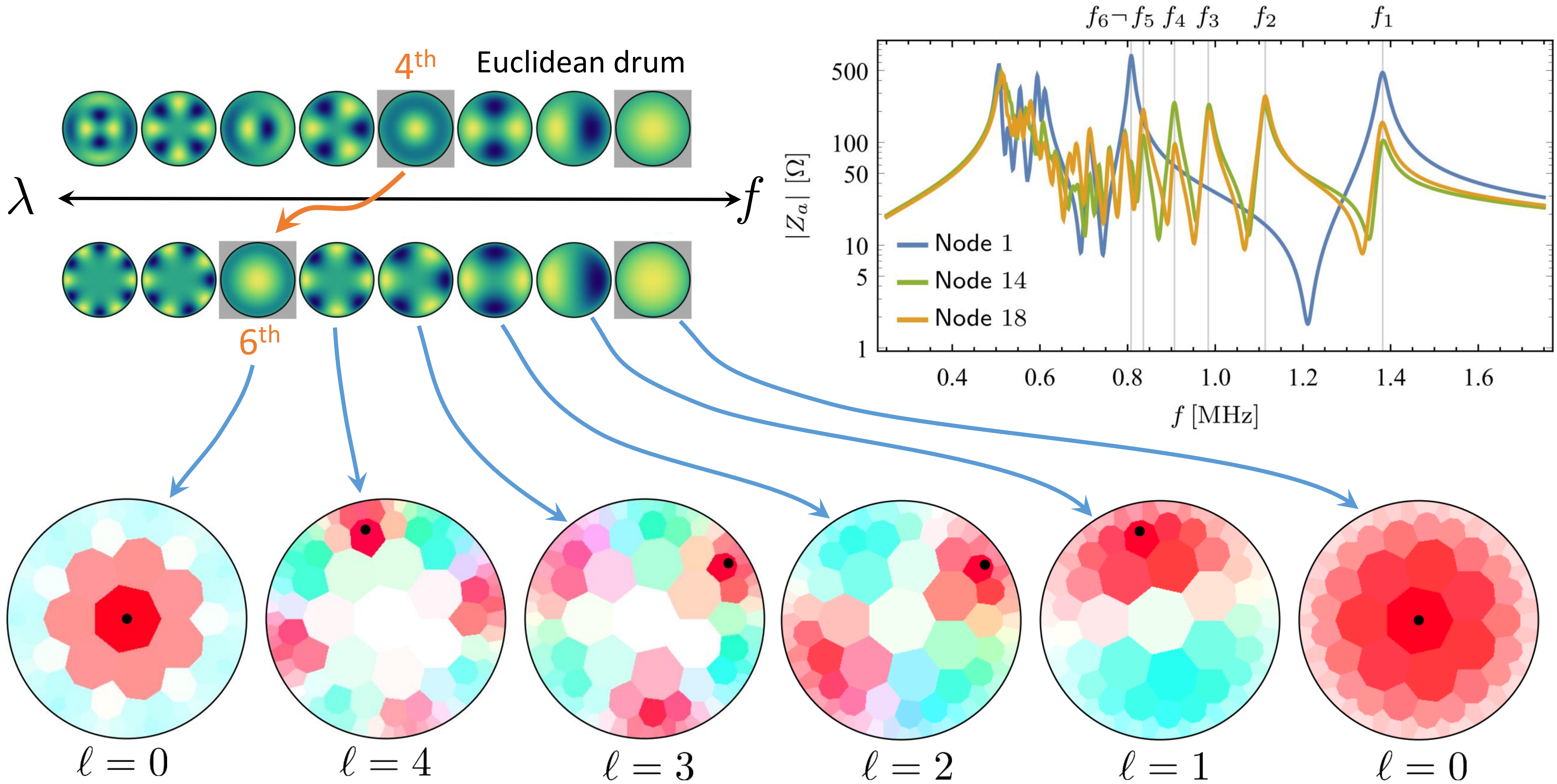
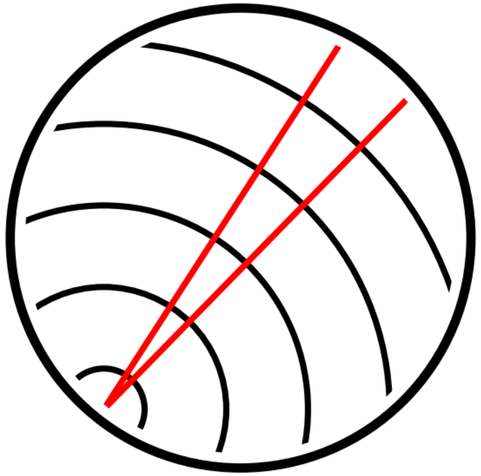
Experiment #2 – Measuring eigenmode profiles

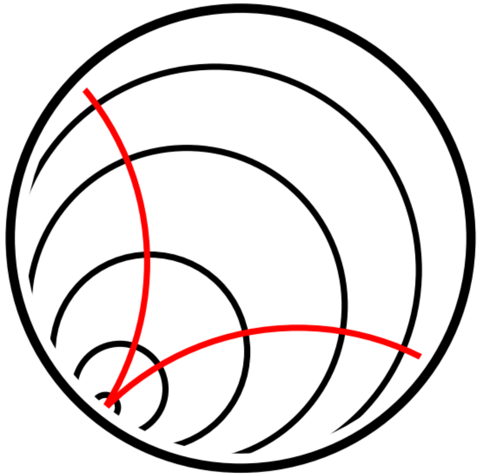


Experiment #3 – Pulse propagation

Euclidean drum



Hyperbolic drum



geodesics, wave fronts

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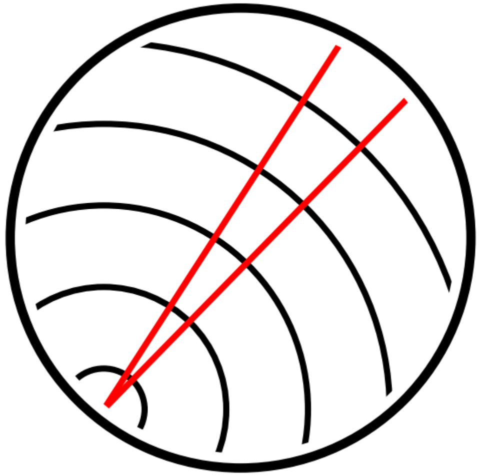
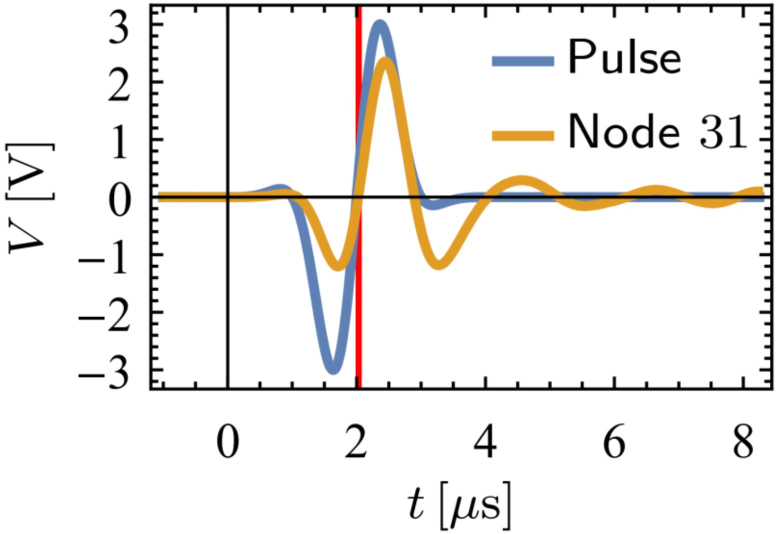


current pulse & induced voltage $V(t)$

(at the same node)



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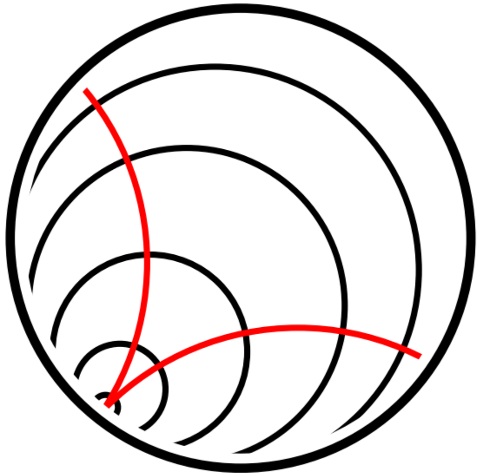

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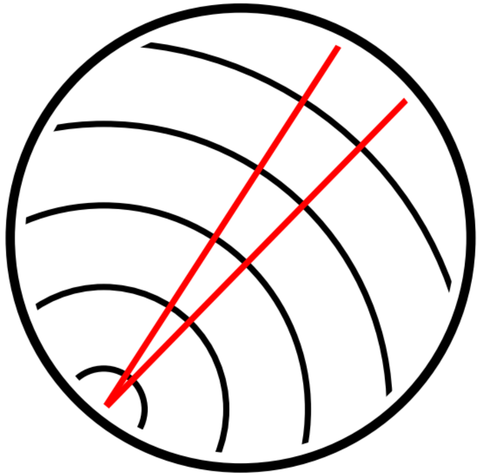

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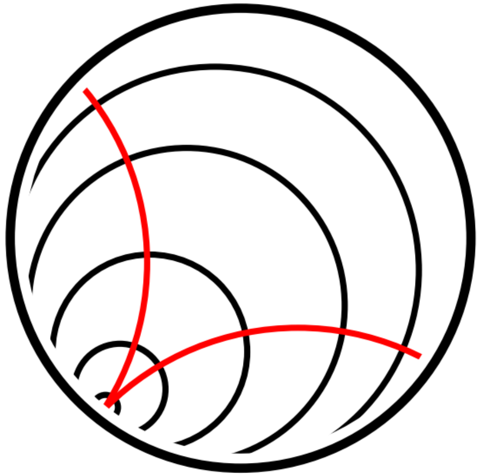

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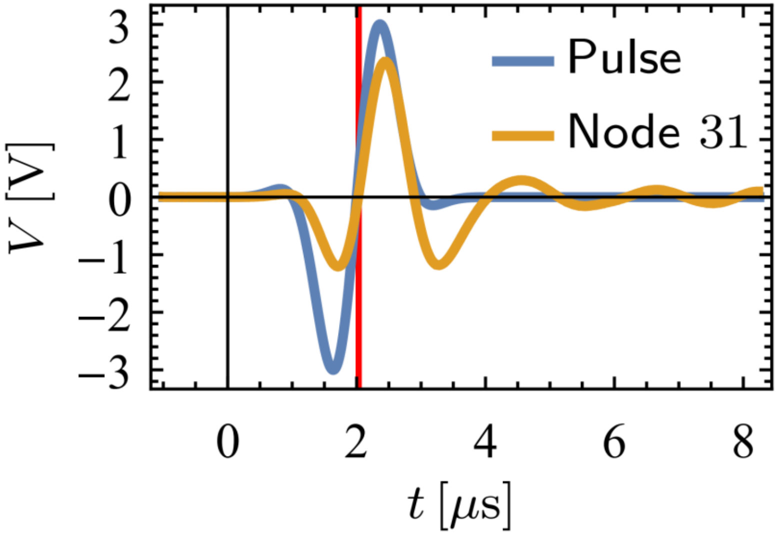
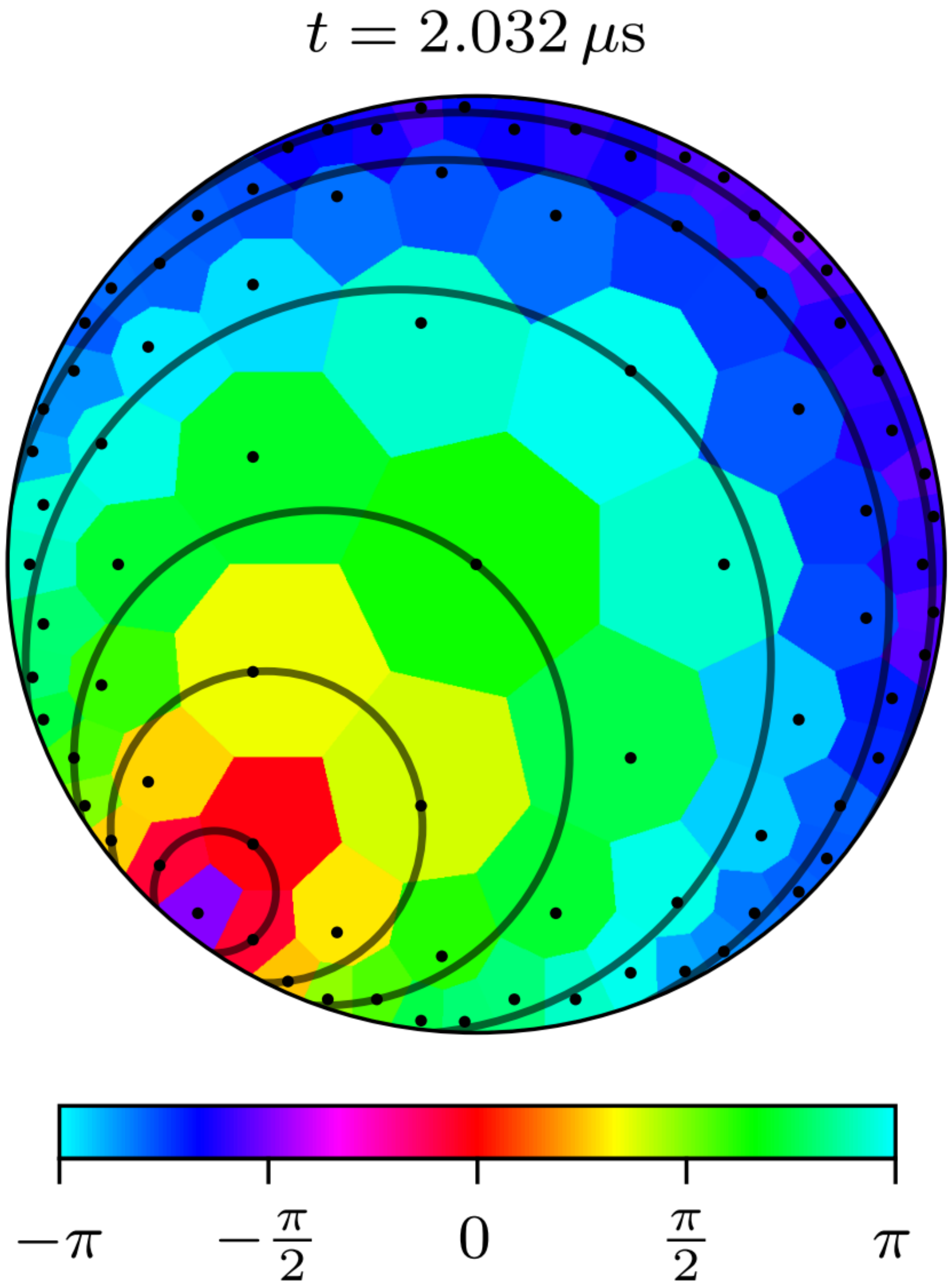


Complexified data obtained from **Hilbert's transform**:

$$v(t) = V(t) + \frac{\mathrm{i}}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}\tau \frac{V(\tau)}{t-\tau}$$



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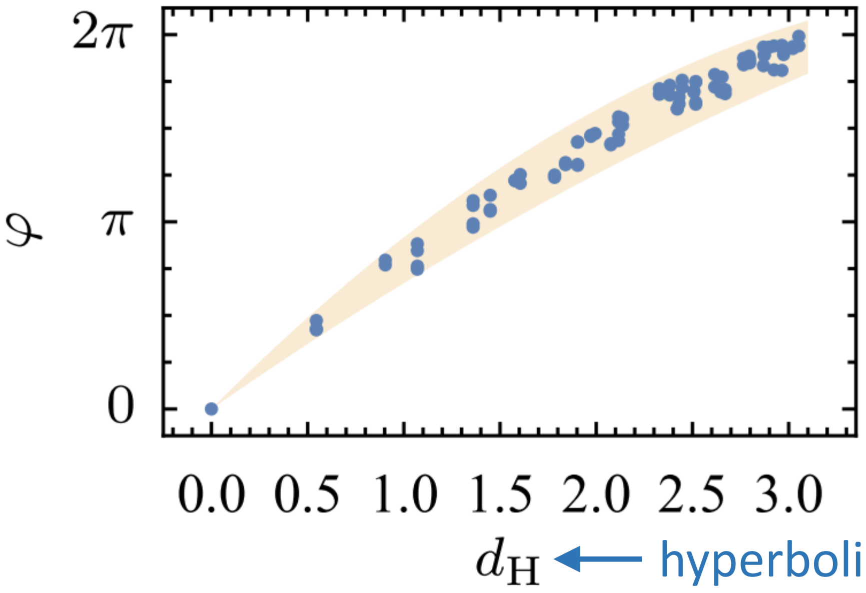
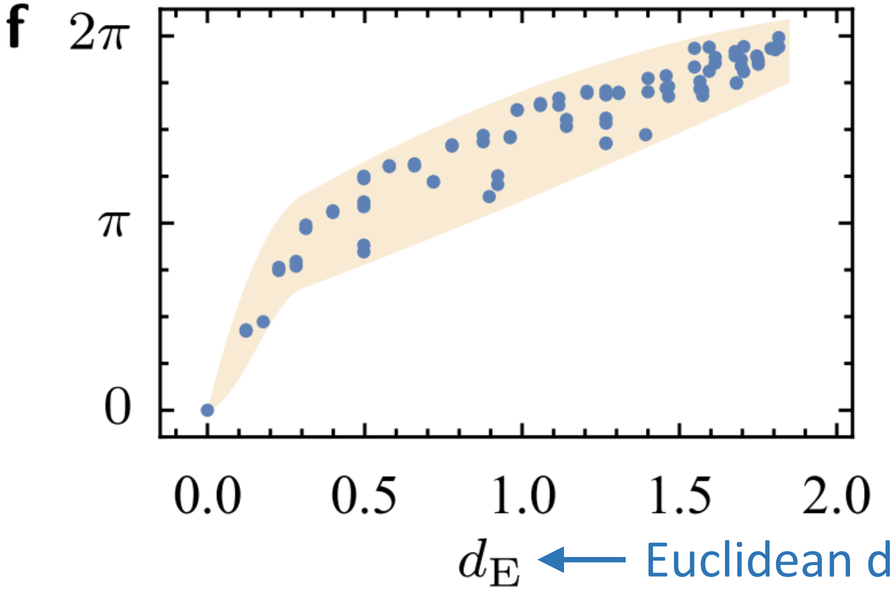


← hyperbolic distance



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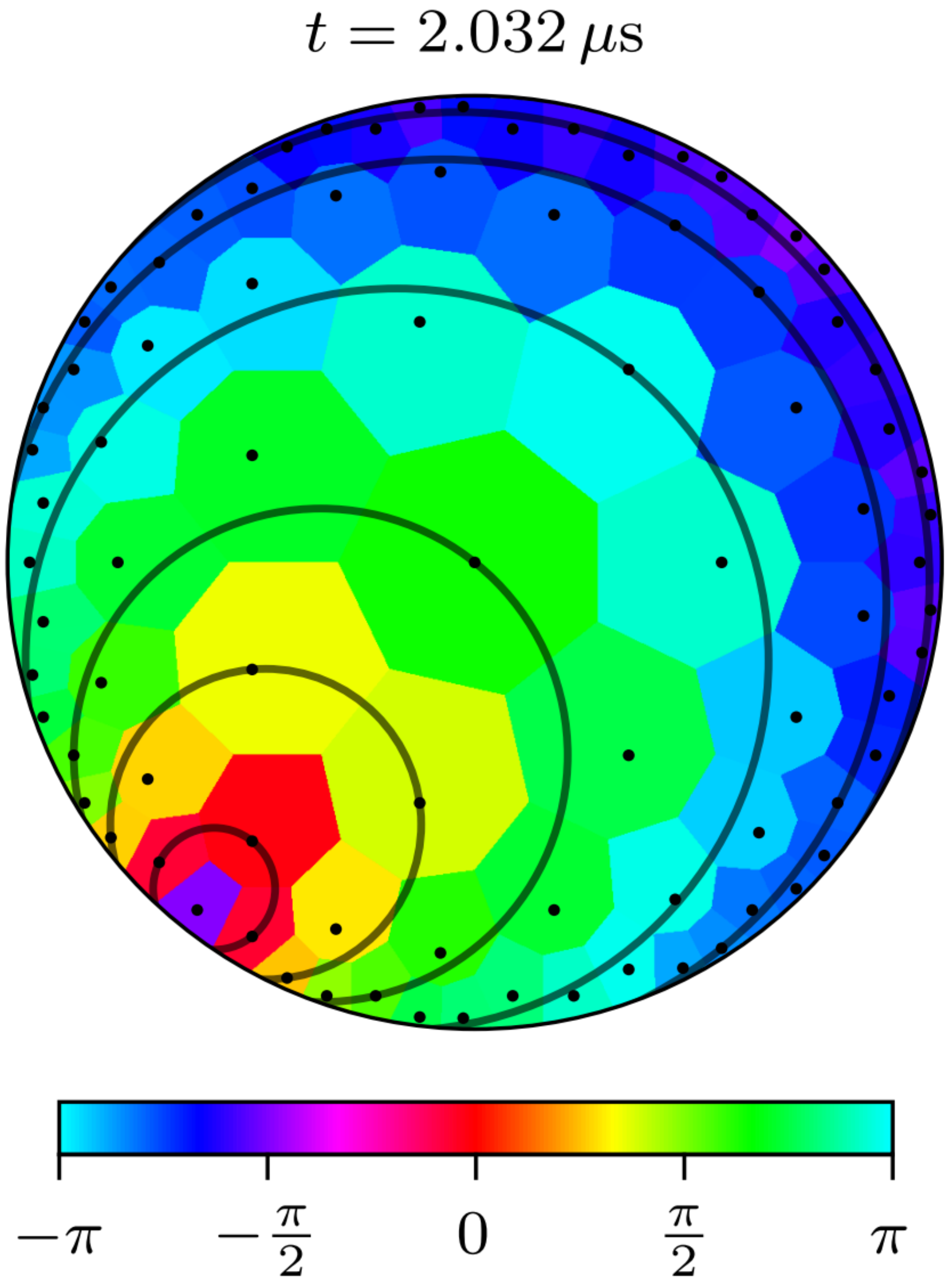

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Isometries of Euclidean plane $SE(2)$

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Isometries of Euclidean plane $SE(2)$

Isometries of hyperbolic plane $SO(2,1) \cong PSL(2,\mathbb{R}) \cong PSU(1,1)$

How to define momentum space?

Isometries of sphere	$SO(3) \cong PSU(2)$
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Discrete subgroups	*2D space groups (wallpaper groups)*
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Discrete subgroups	*point groups*

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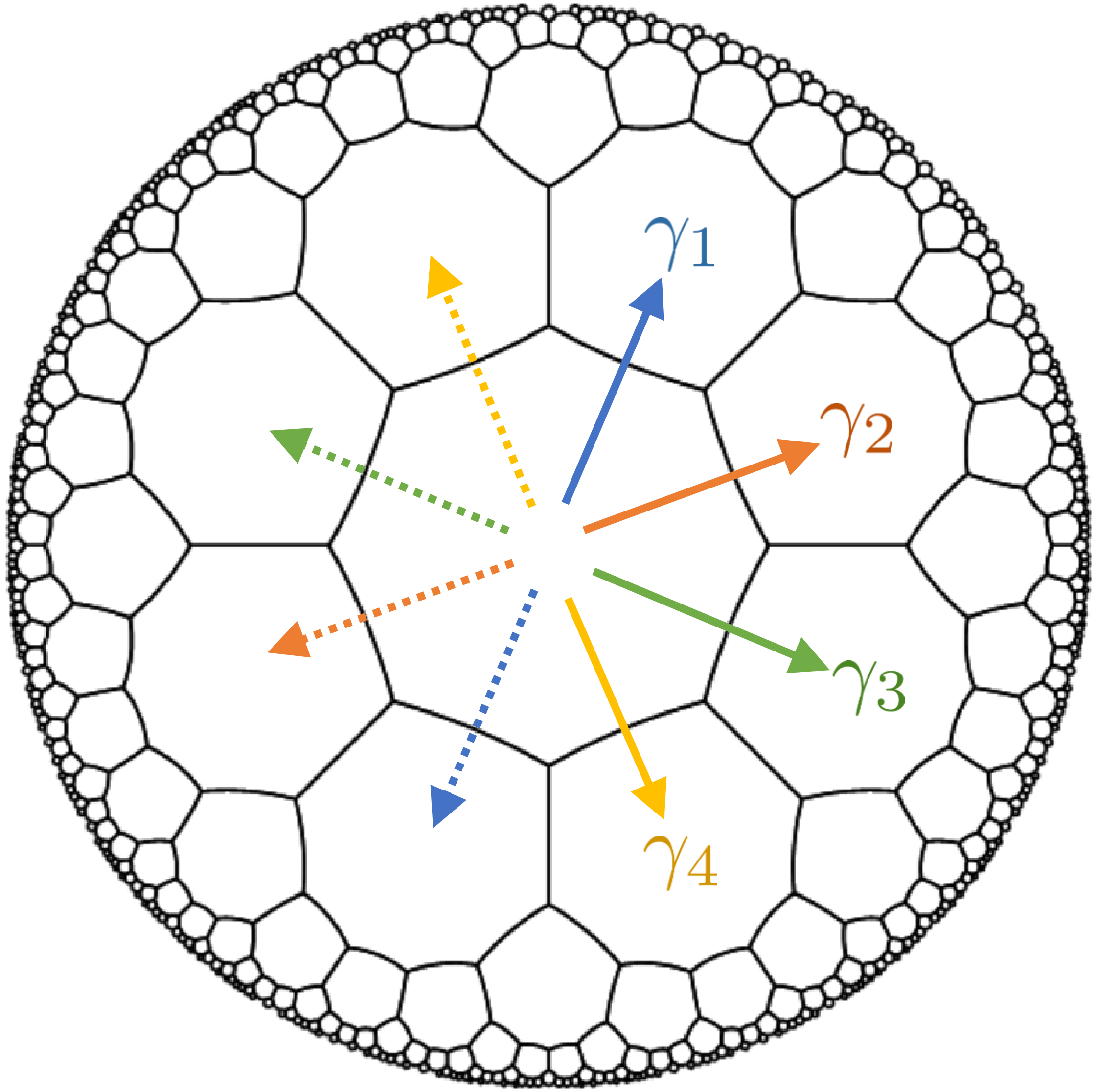
Discrete subgroups *point groups*

Isometries of Euclidean plane $SE(2)$
Discrete subgroups *2D space groups (wallpaper groups)*

Isometries of hyperbolic plane $SO(2,1) \cong PSL(2,\mathbb{R}) \cong PSU(1,1)$
Discrete subgroups *Fuchsian groups*

From Fuchsian groups to hyperbolic translation groups

Four translation generators on {8,3} lattice:



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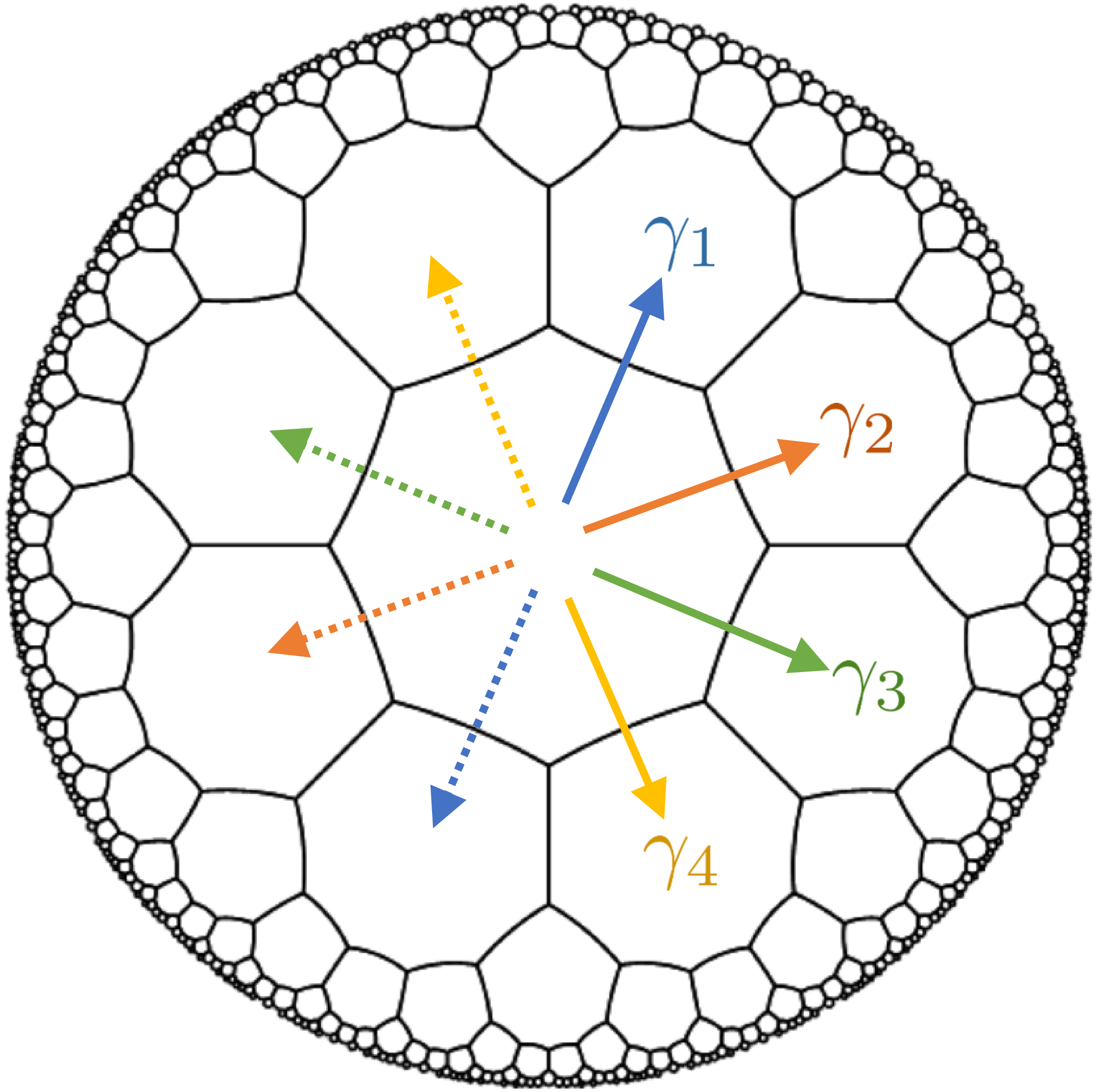
$$\gamma_1 \gamma_2^{-1} \gamma_3 = \text{rotation by } \frac{2\pi}{8}$$



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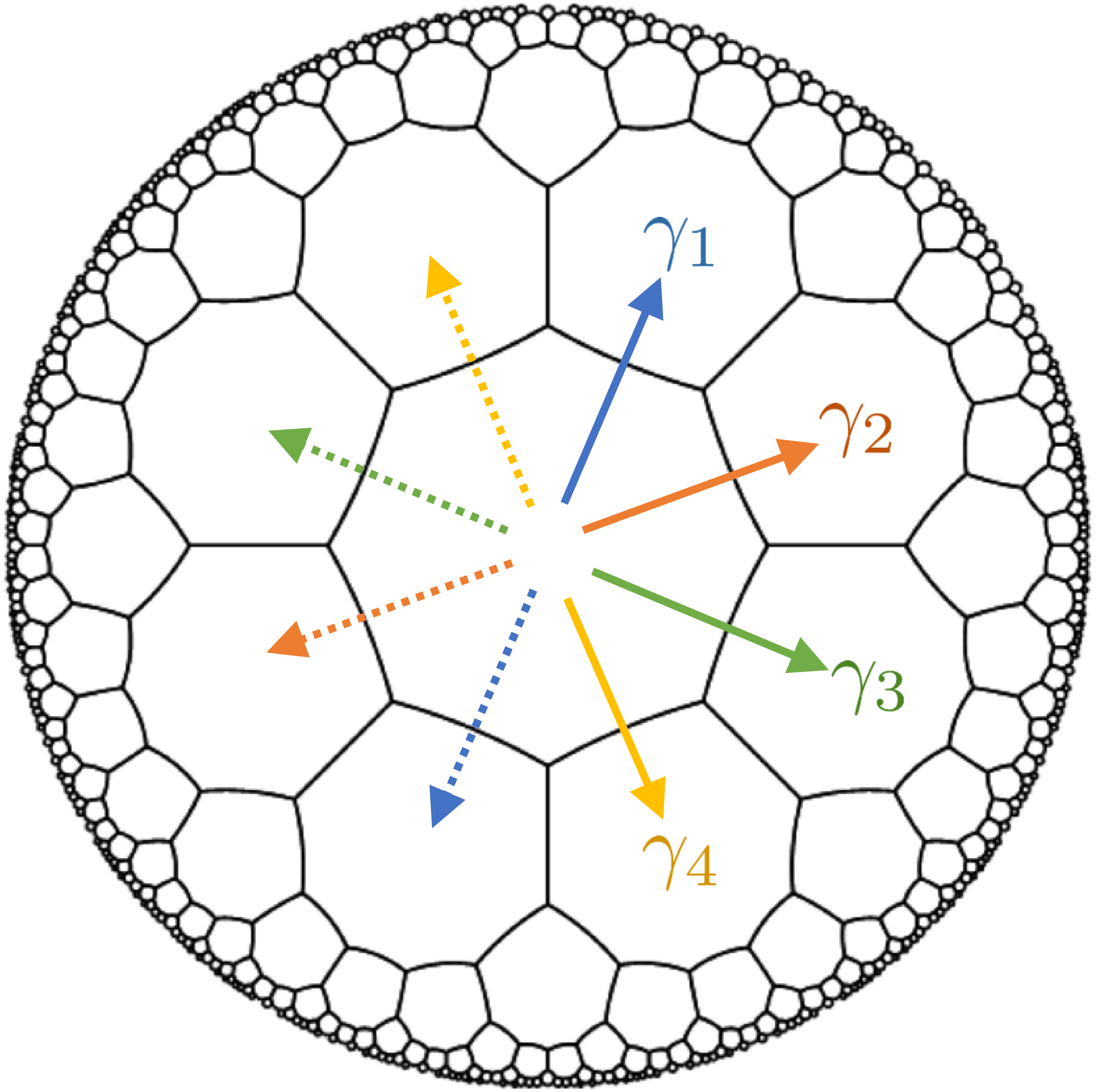
“Abstract presentation” of the {8,3} Fuchsian:

$$\Gamma = \left\langle \underbrace{\gamma_1, \gamma_2, \gamma_3, \gamma_4}_{\text{generators}} : \underbrace{\left(\gamma_1\gamma_2^{-1}\gamma_3\right)^8 = 1, \ldots}_{\text{constraints}} \right\rangle$$



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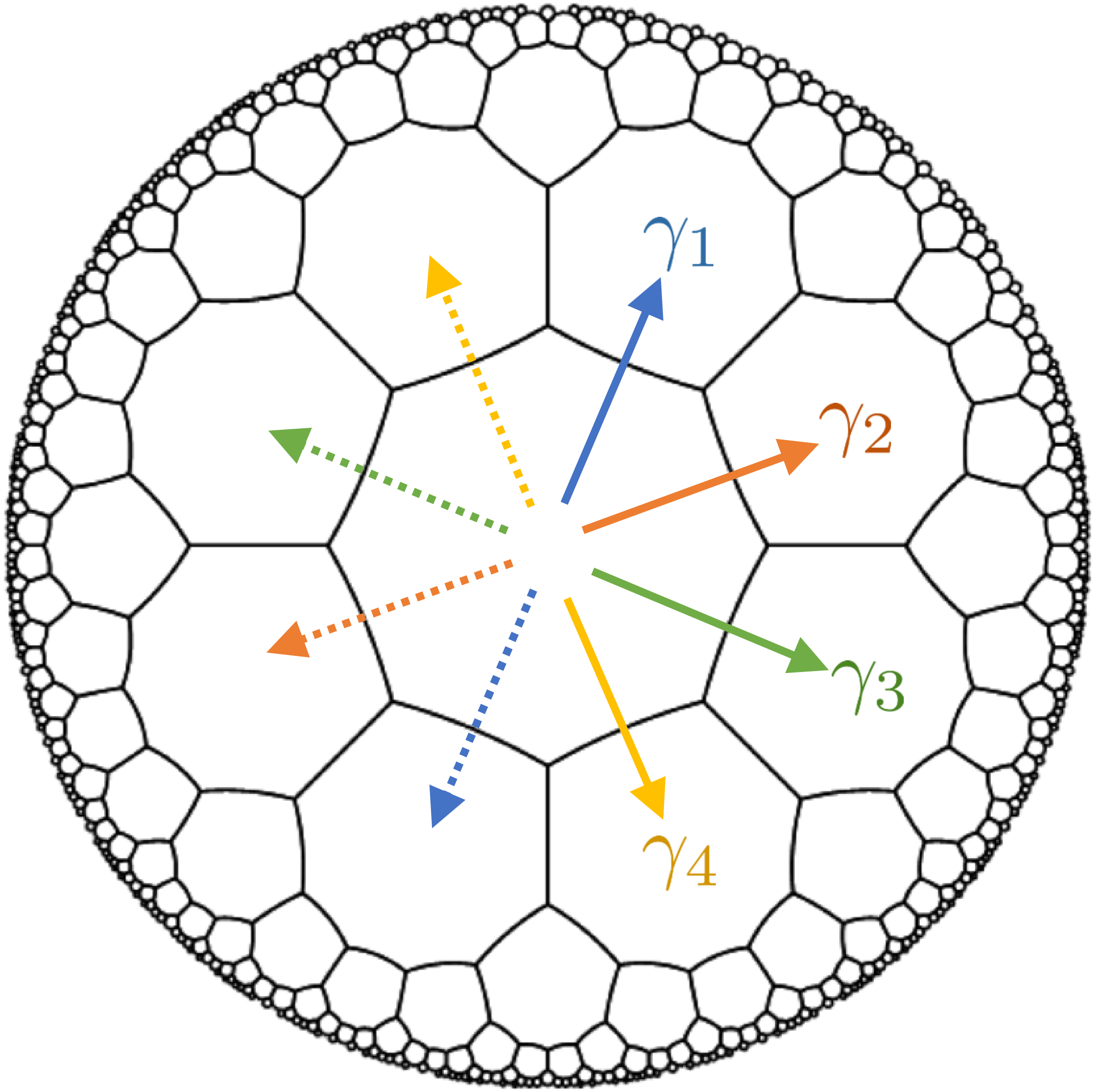
Torsion-free Fuchsian group:

→ no element g of finite order, i.e.,

$$g^n = 1 \;\Leftrightarrow\; n = 0 \text{ or } g = 1$$



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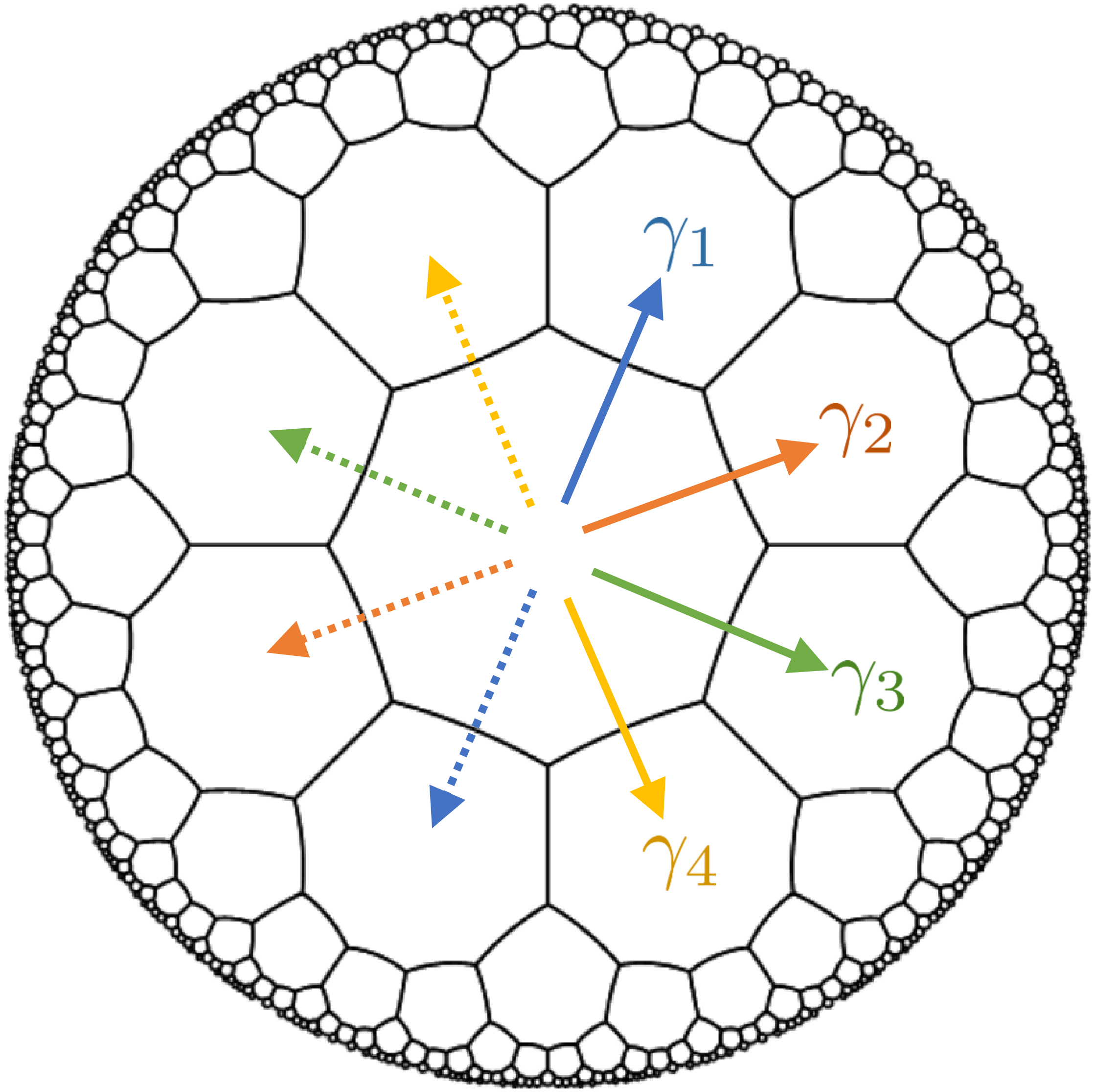
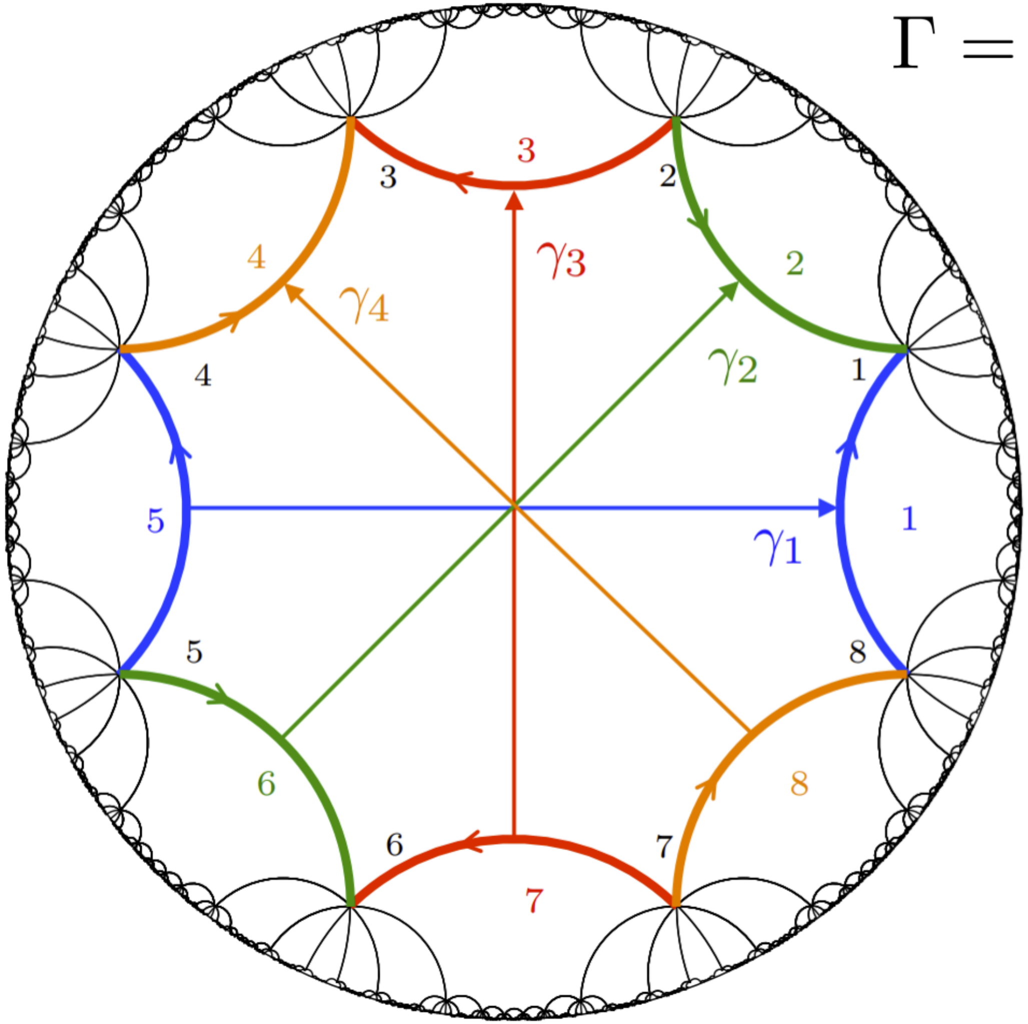


Define hyperbolic translation group: ***maximal torsion-free*** *(normal)* ***subgroup***.

From Fuchsian groups to hyperbolic translation groups

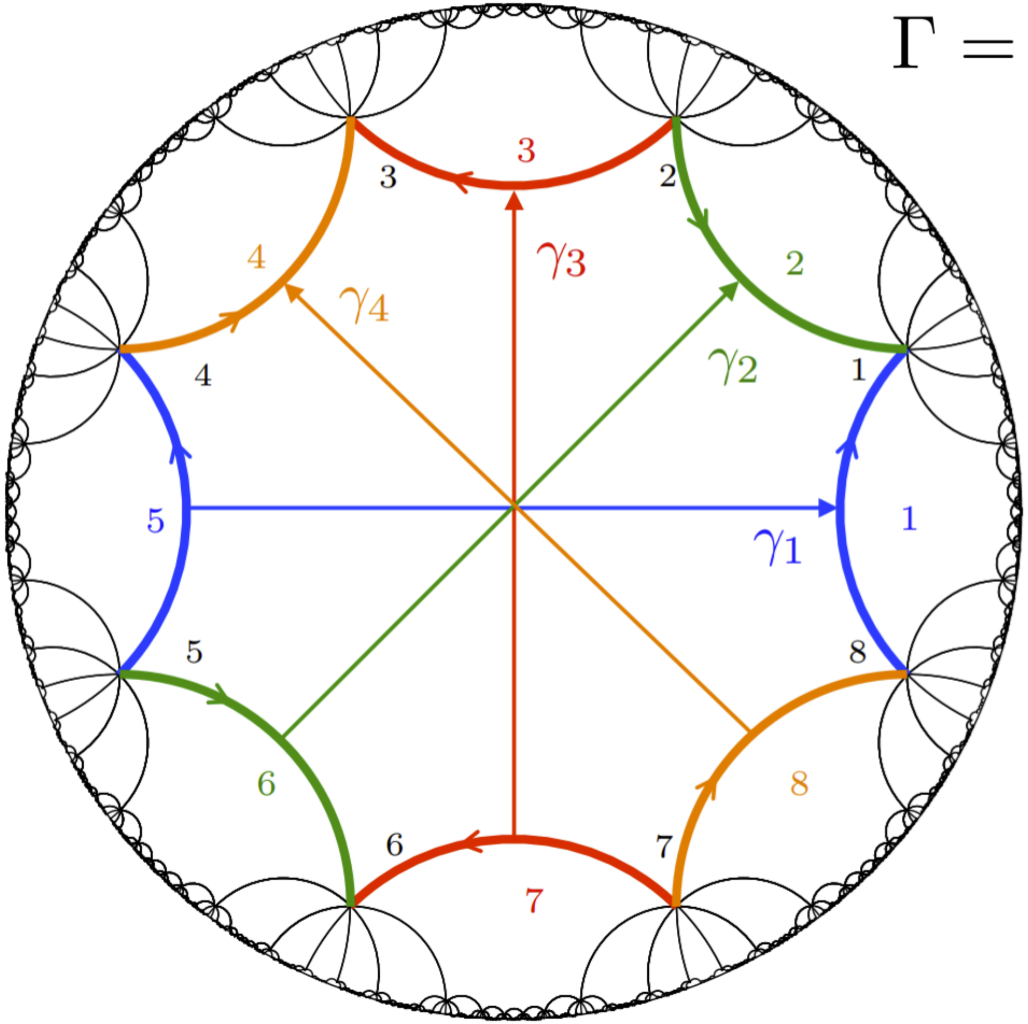


{8,8} “Bolza” tessellation

$$\Gamma = \langle \gamma_1, \gamma_2, \gamma_3, \gamma_4 : \gamma_1 \gamma_2^{-1} \gamma_3 \gamma_4^{-1} \gamma_1^{-1} \gamma_2 \gamma_3^{-1} \gamma_4 = 1 \rangle$$

J. Maciejko and S. Rayan, *Hyperbolic band theory*, Sci. Adv. **7**, abe9170 (2021)

Hyperbolic band theory on {8,8} lattice



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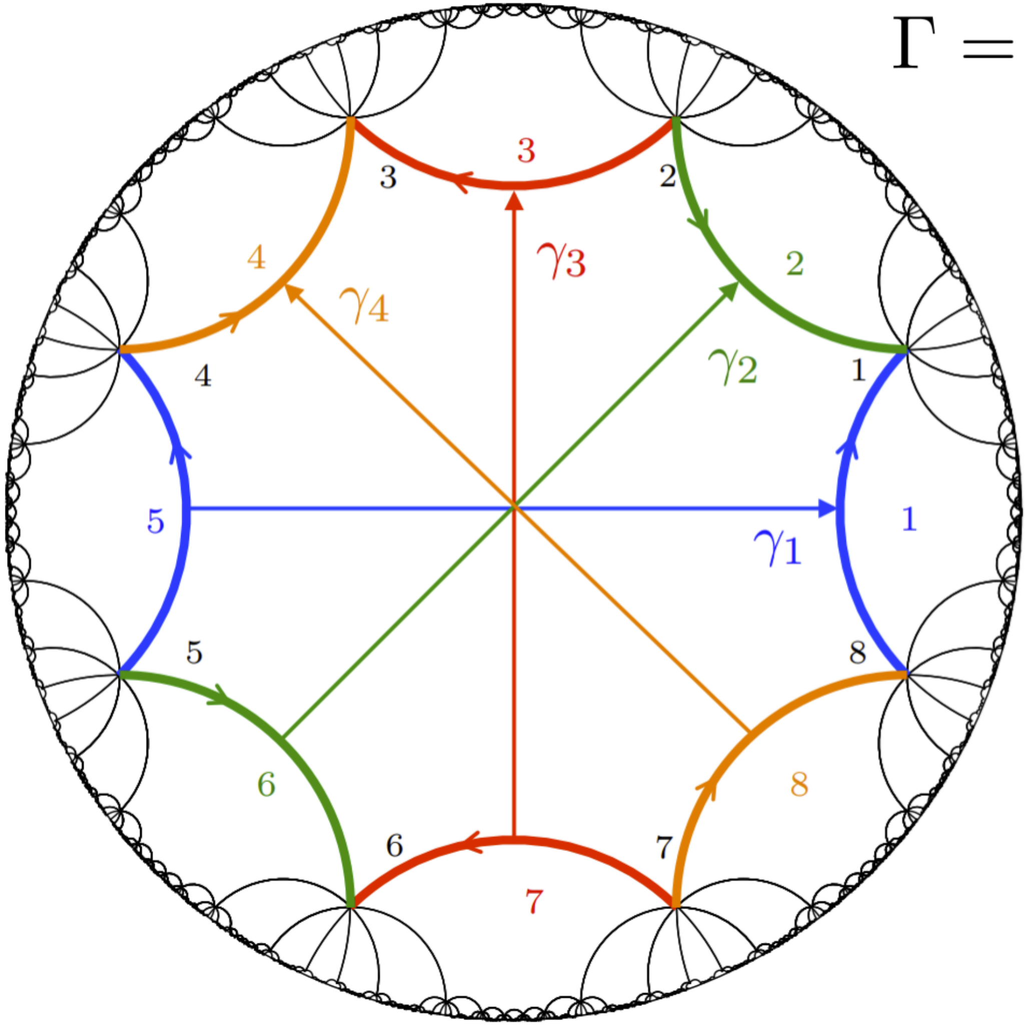

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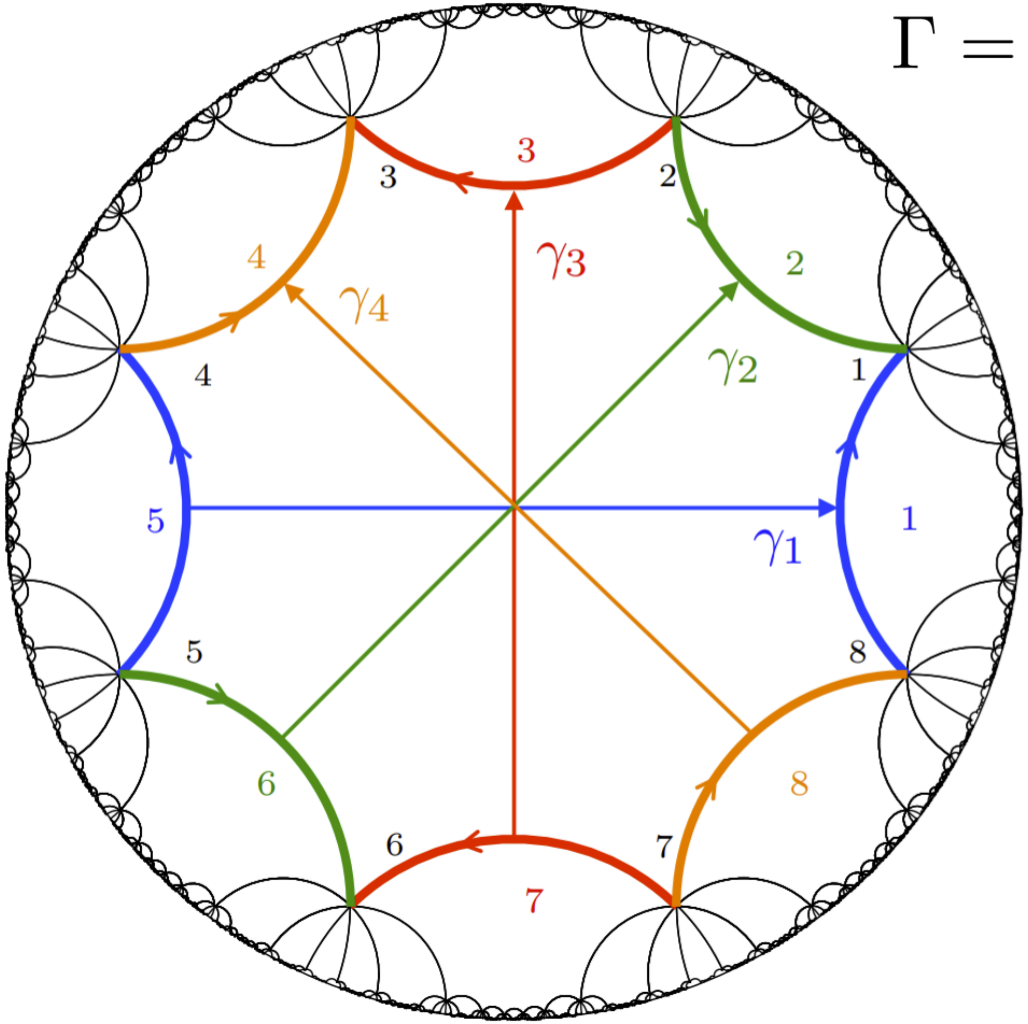

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$$\left.\begin{aligned} \gamma_1 &\mapsto e^{ik_1} \\ \gamma_2 &\mapsto e^{ik_2} \\ \gamma_3 &\mapsto e^{ik_3} \\ \gamma_4 &\mapsto e^{ik_4} \end{aligned}\right\} \text{Four-dimensional Brillouin zone}$$

J. Maciejko and S. Rayan, *Hyperbolic band theory*, Sci. Adv. **7**, abe9170 (2021)

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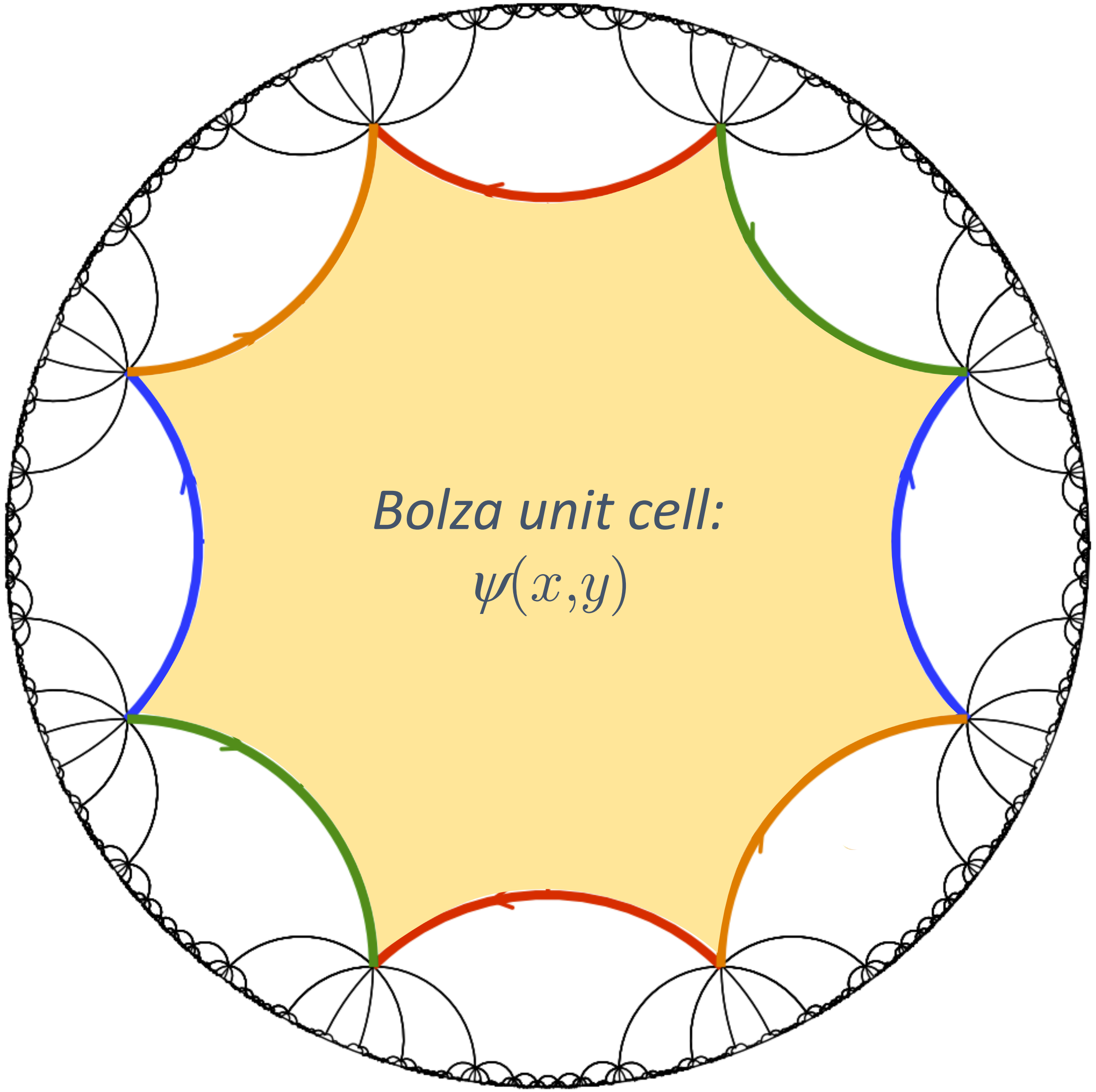

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Four-dimensional Brillouin zone

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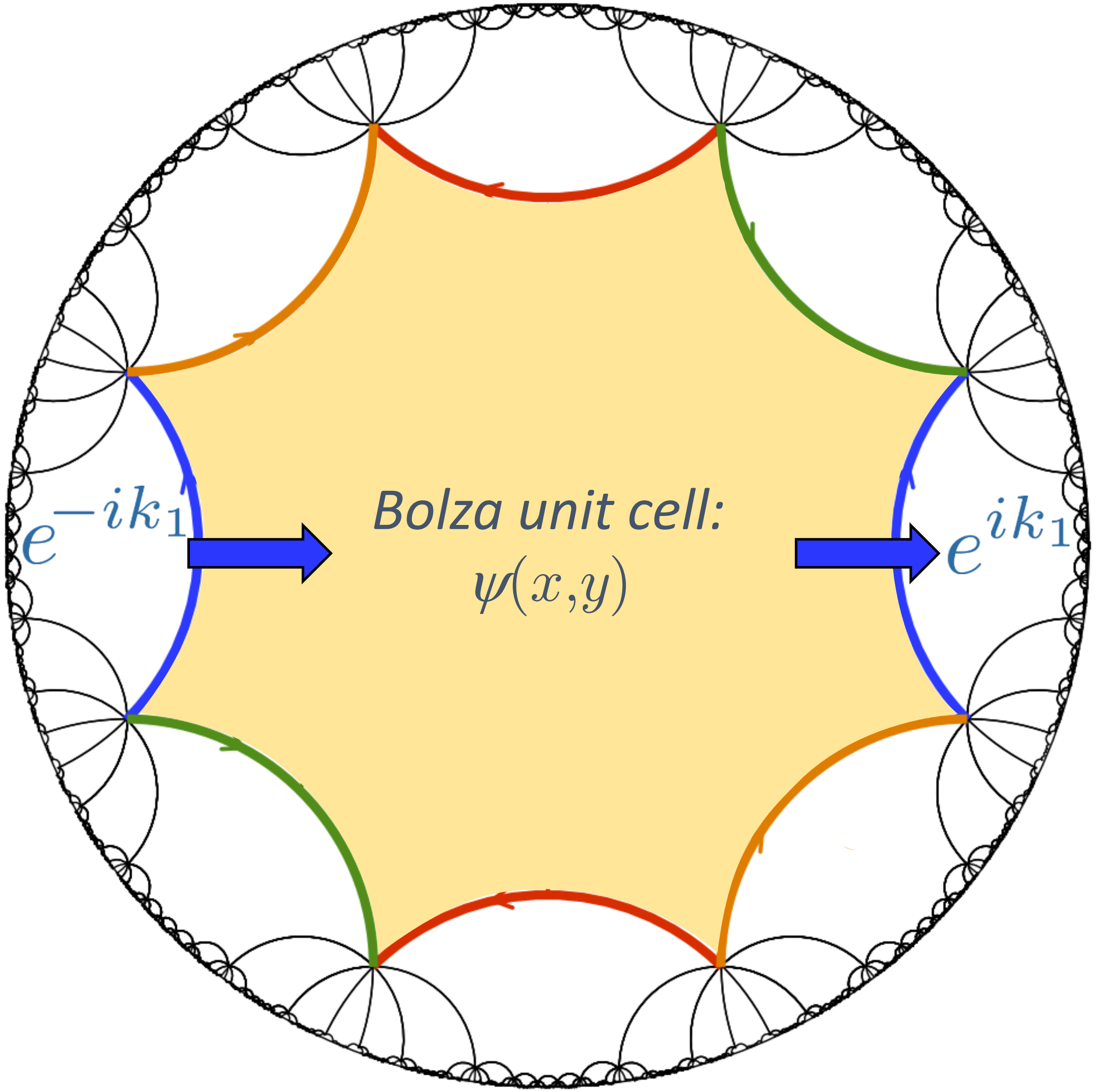

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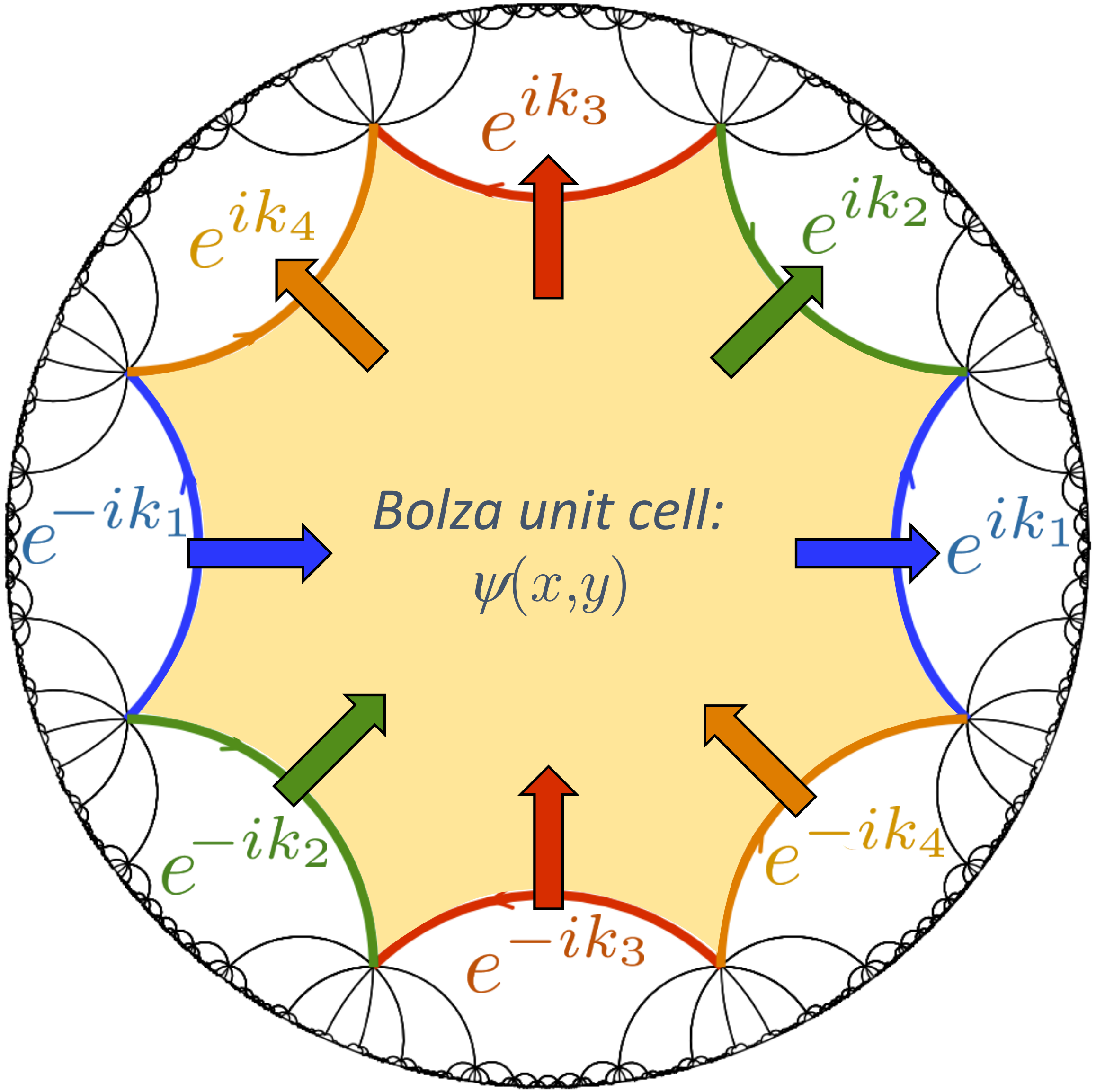

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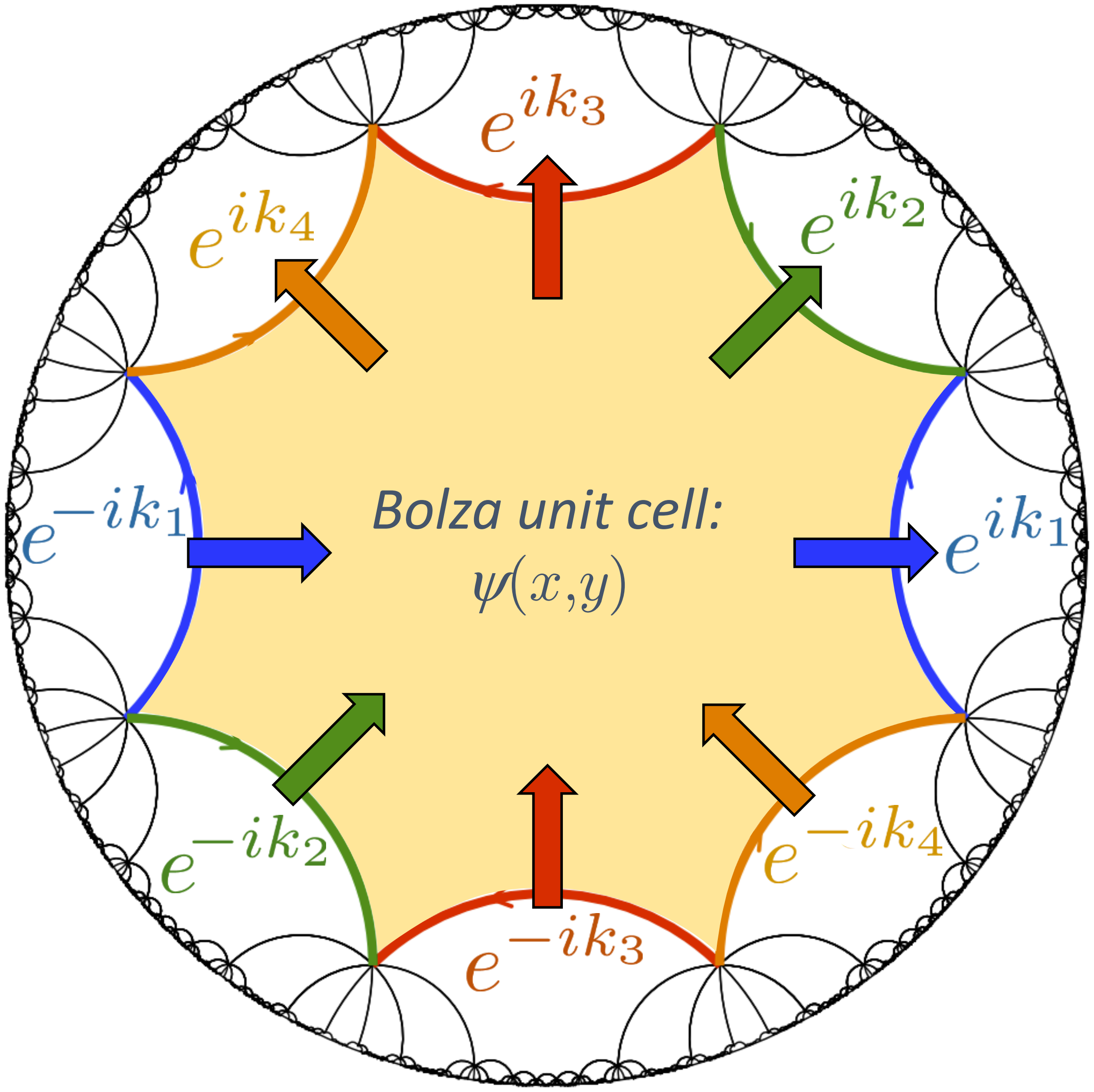
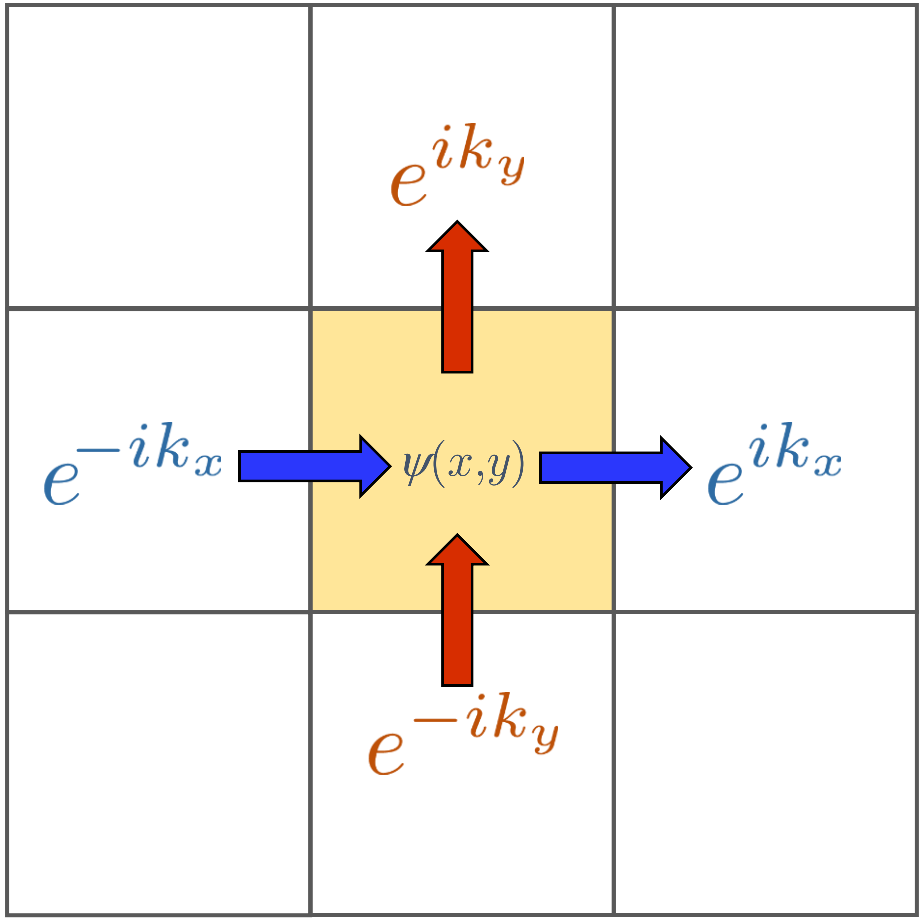
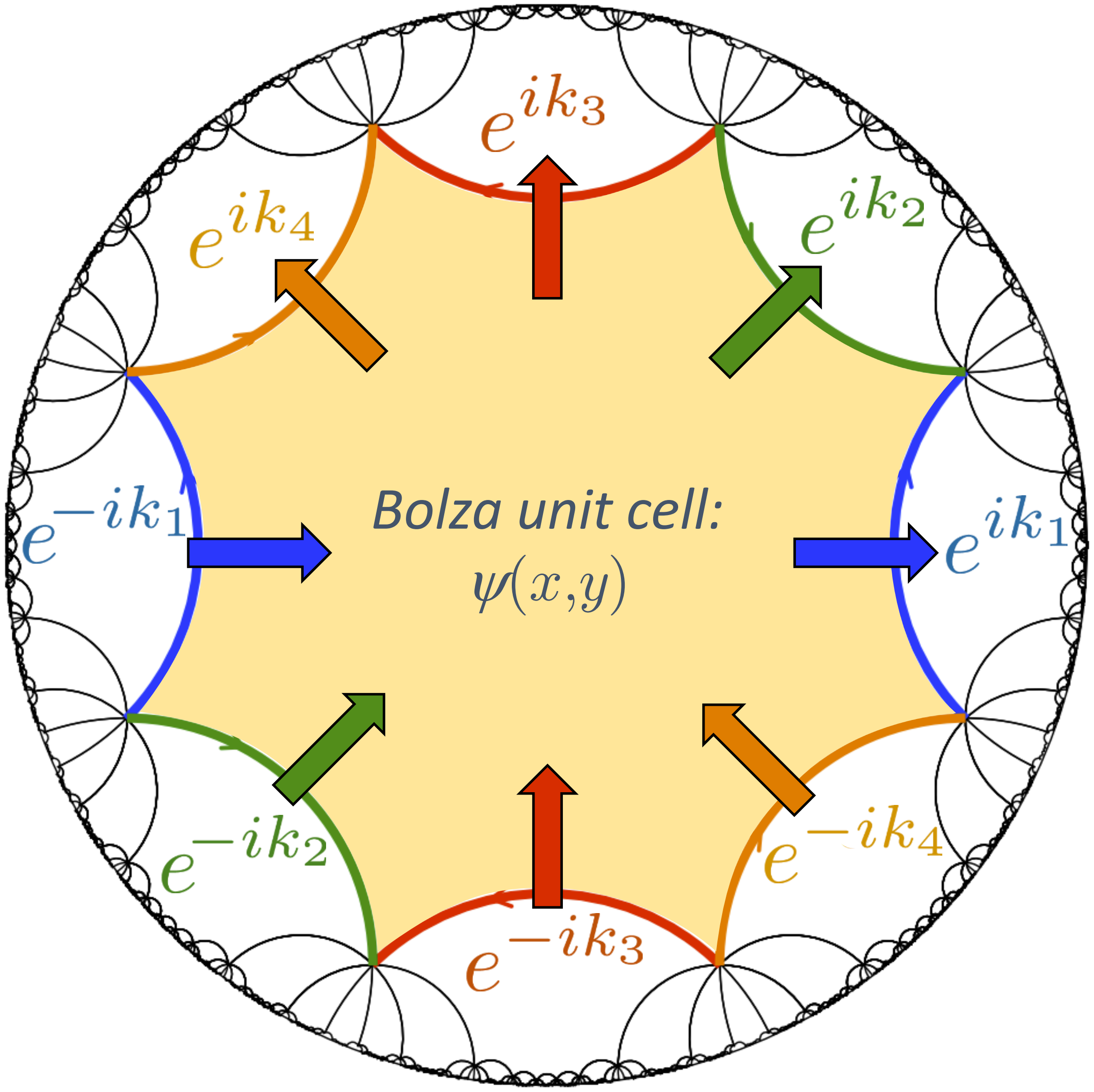


4D Brillouin zone

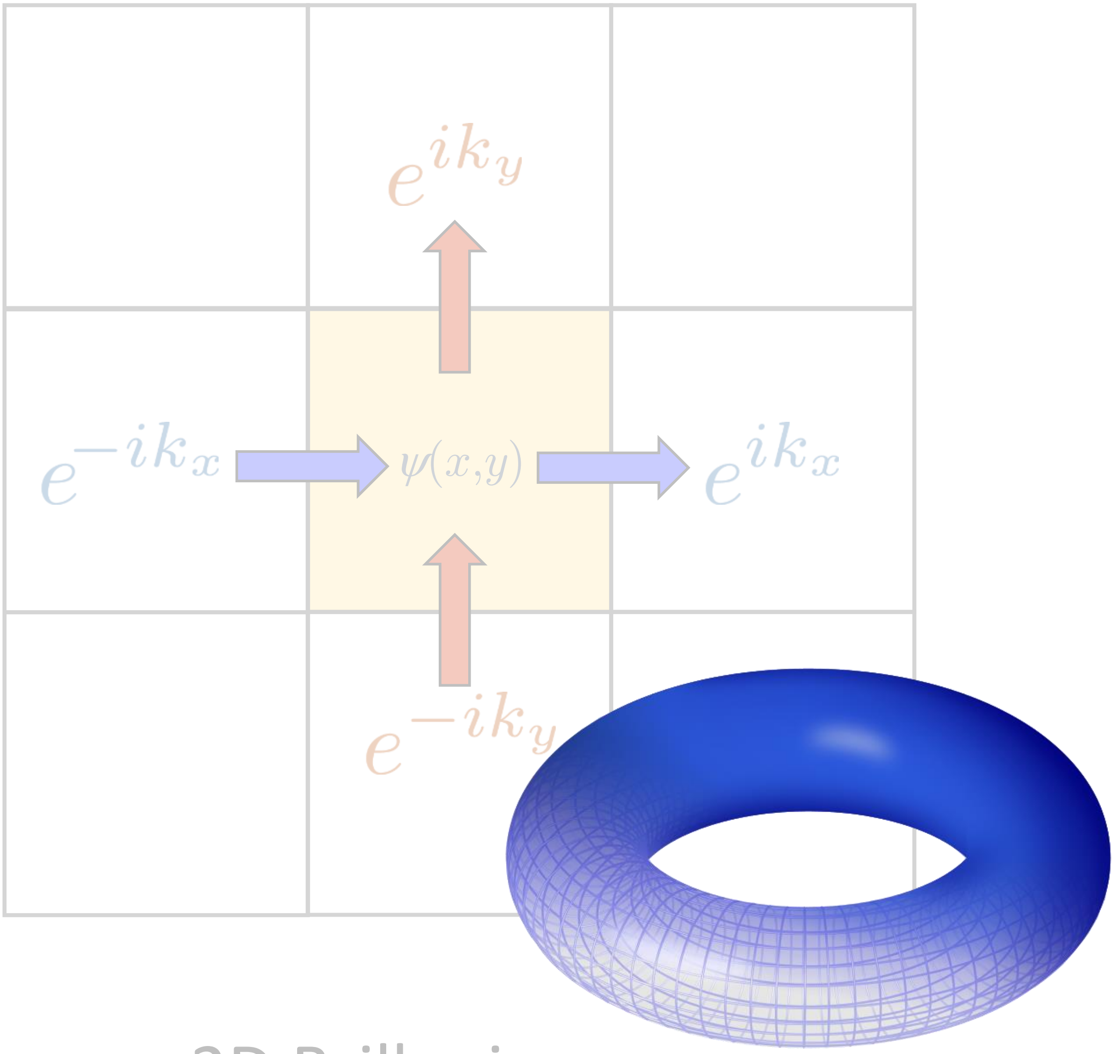


2D Brillouin zone

Hyperbolic band theory on {8,8} lattice



4D Brillouin zone



2D Brillouin zone

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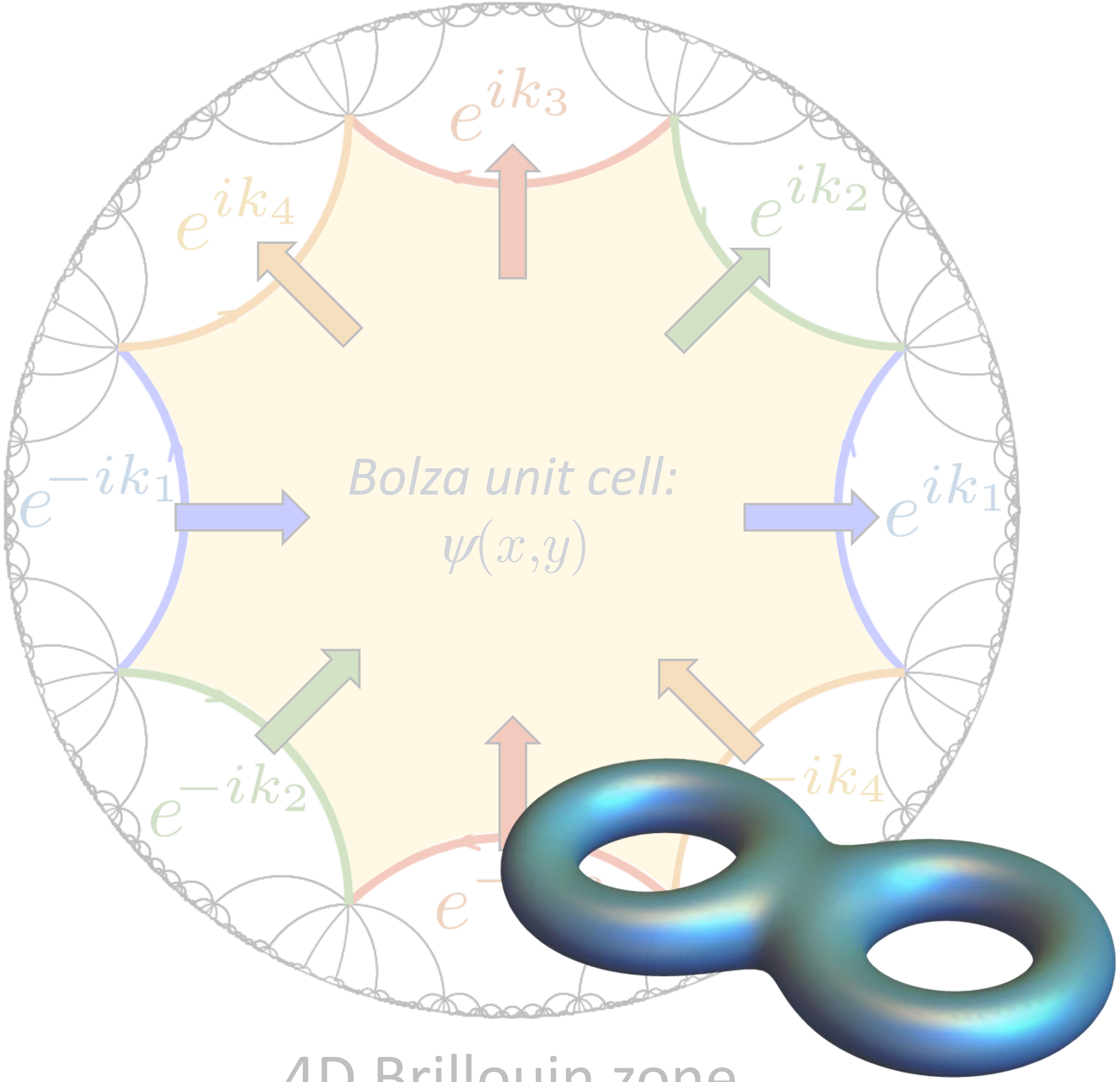
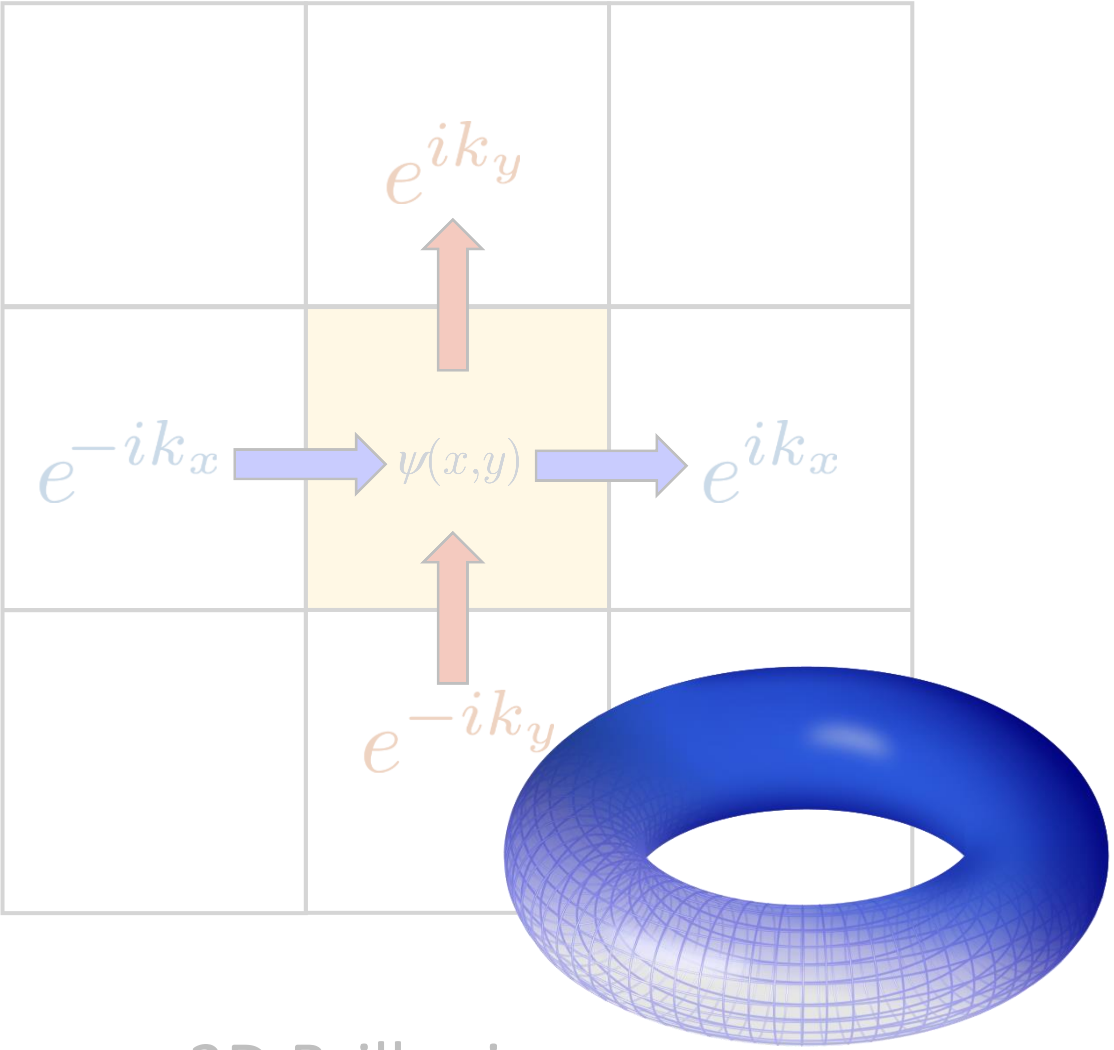


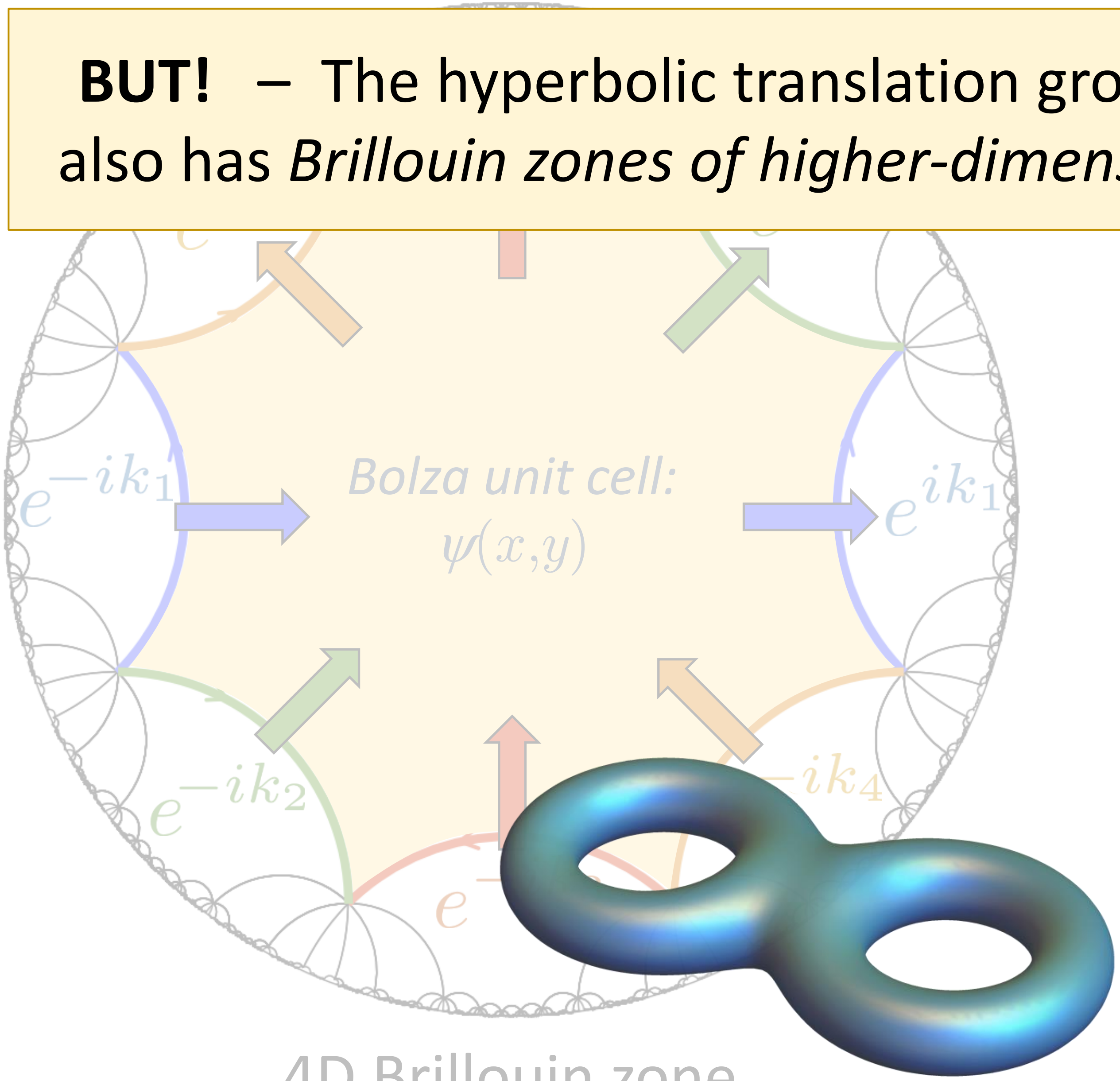
4D Brillouin zone



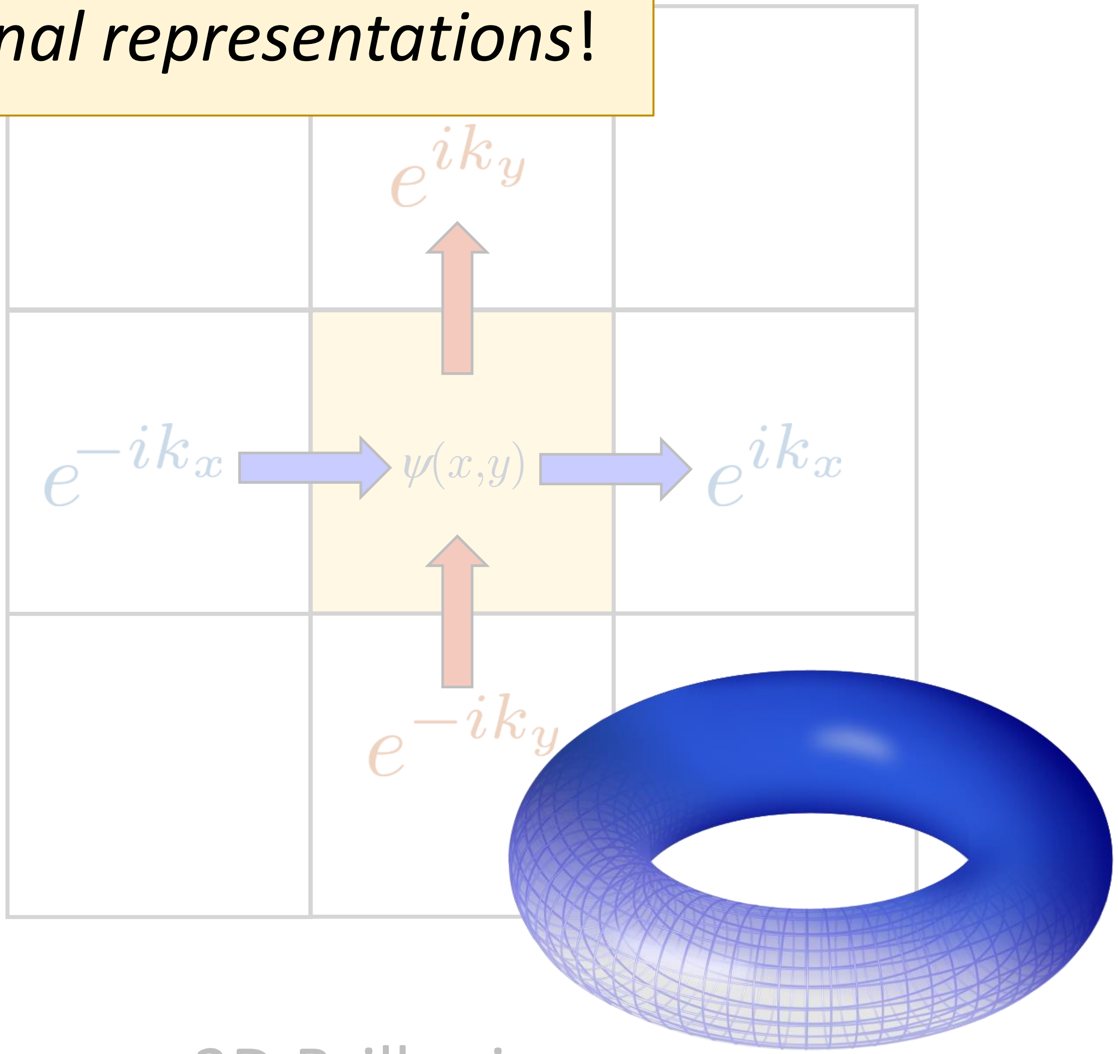
2D Brillouin zone

Hyperbolic band theory on {8,8} lattice

BUT! – The hyperbolic translation group is non-Abelian and also has *Brillouin zones of higher-dimensional representations*!



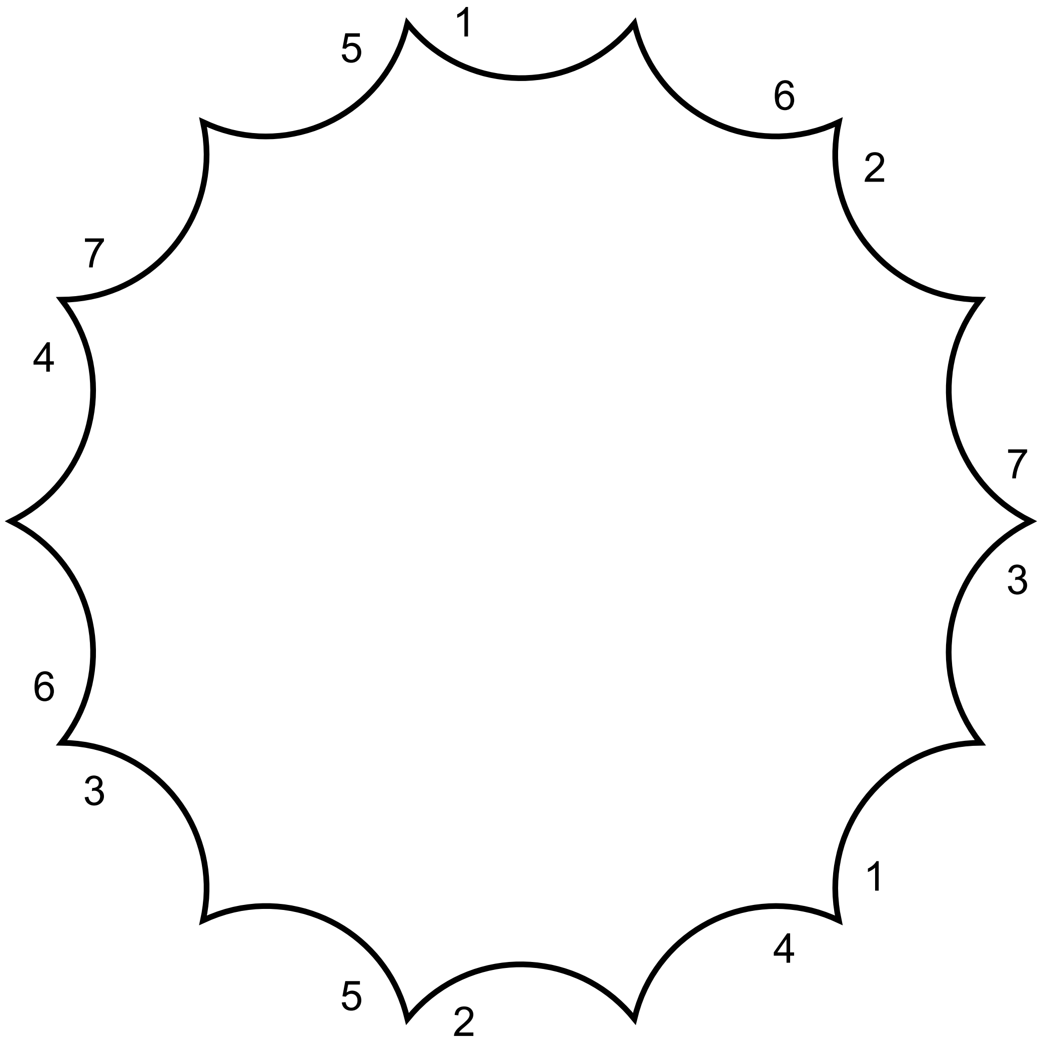
4D Brillouin zone



2D Brillouin zone

Hyperbolic band theory on {14,7} lattice

$$\Gamma_{\{14,7\}} = \langle \gamma_1, \ldots, \gamma_7 \colon \gamma_2\gamma_4\gamma_6\gamma_1\gamma_3\gamma_5\gamma_7 = \mathbb{1},$$
$$\gamma_3\gamma_6\gamma_2\gamma_5\gamma_1\gamma_4\gamma_7 = \mathbb{1}\rangle$$



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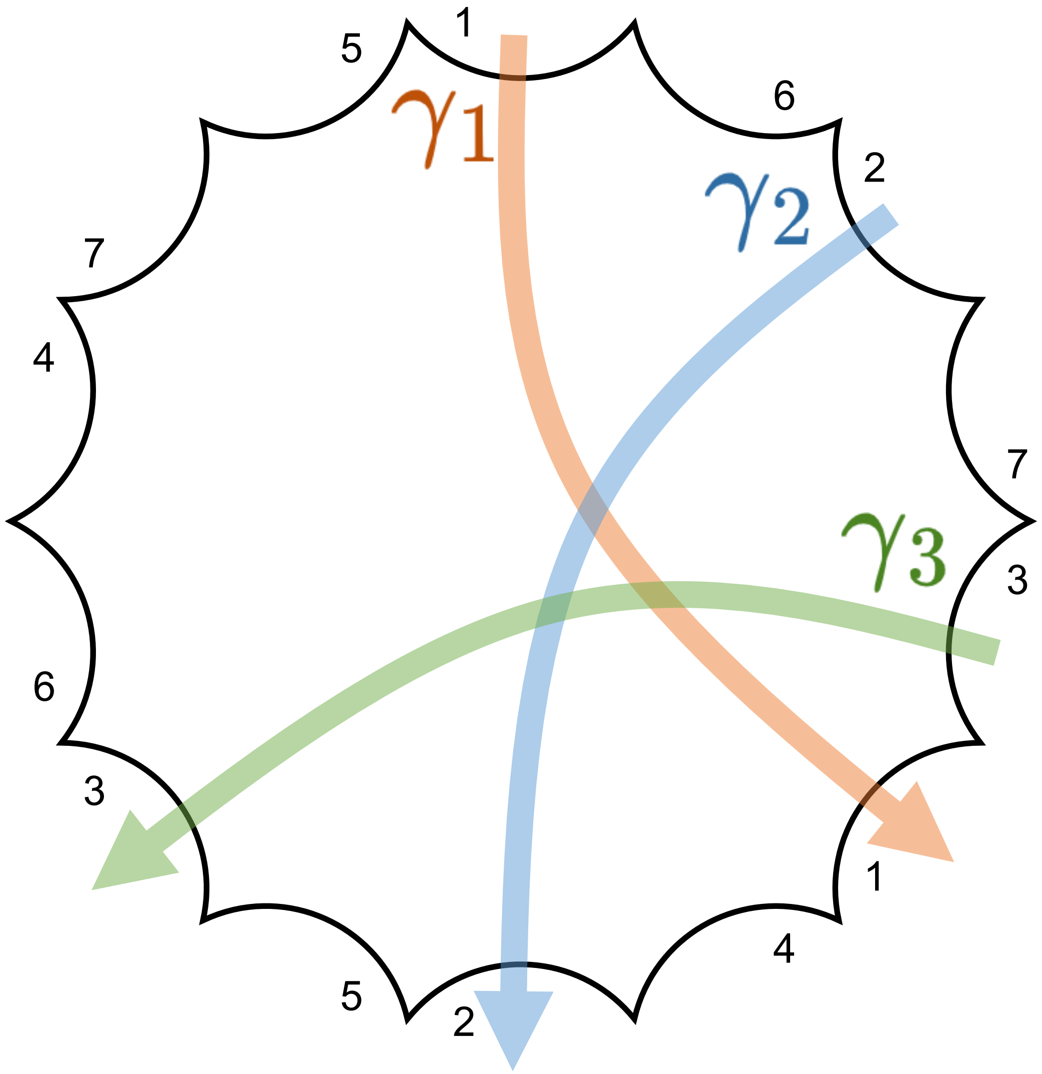

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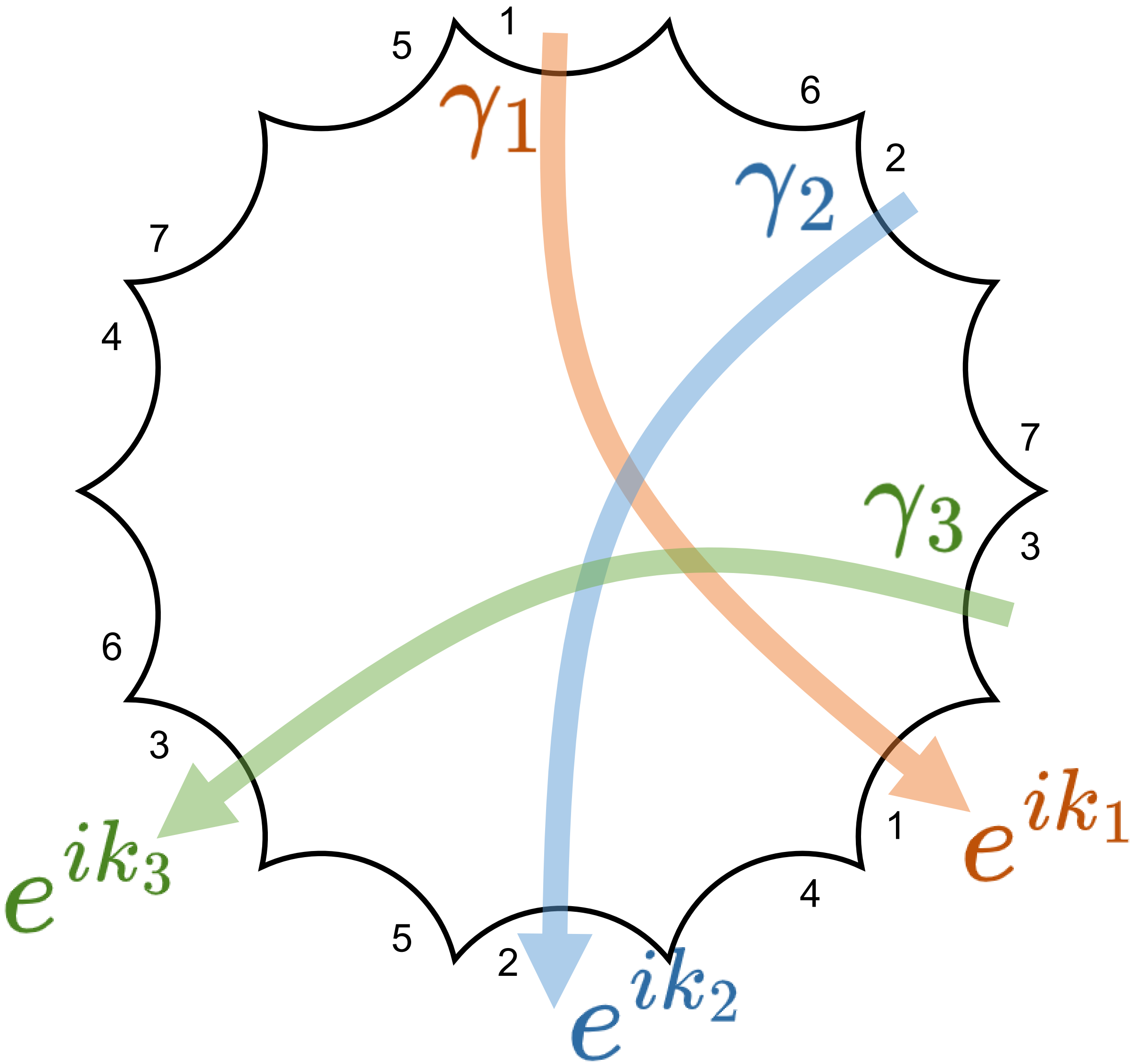

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$$\gamma_5 \mapsto e^{ik_5}$$

$$\gamma_6 \mapsto e^{ik_6}$$

$$\gamma_7 \mapsto e^{-i(k_1+k_2+k_3+k_4+k_5+k_6)}$$

Hyperbolic band theory on {14,7} lattice

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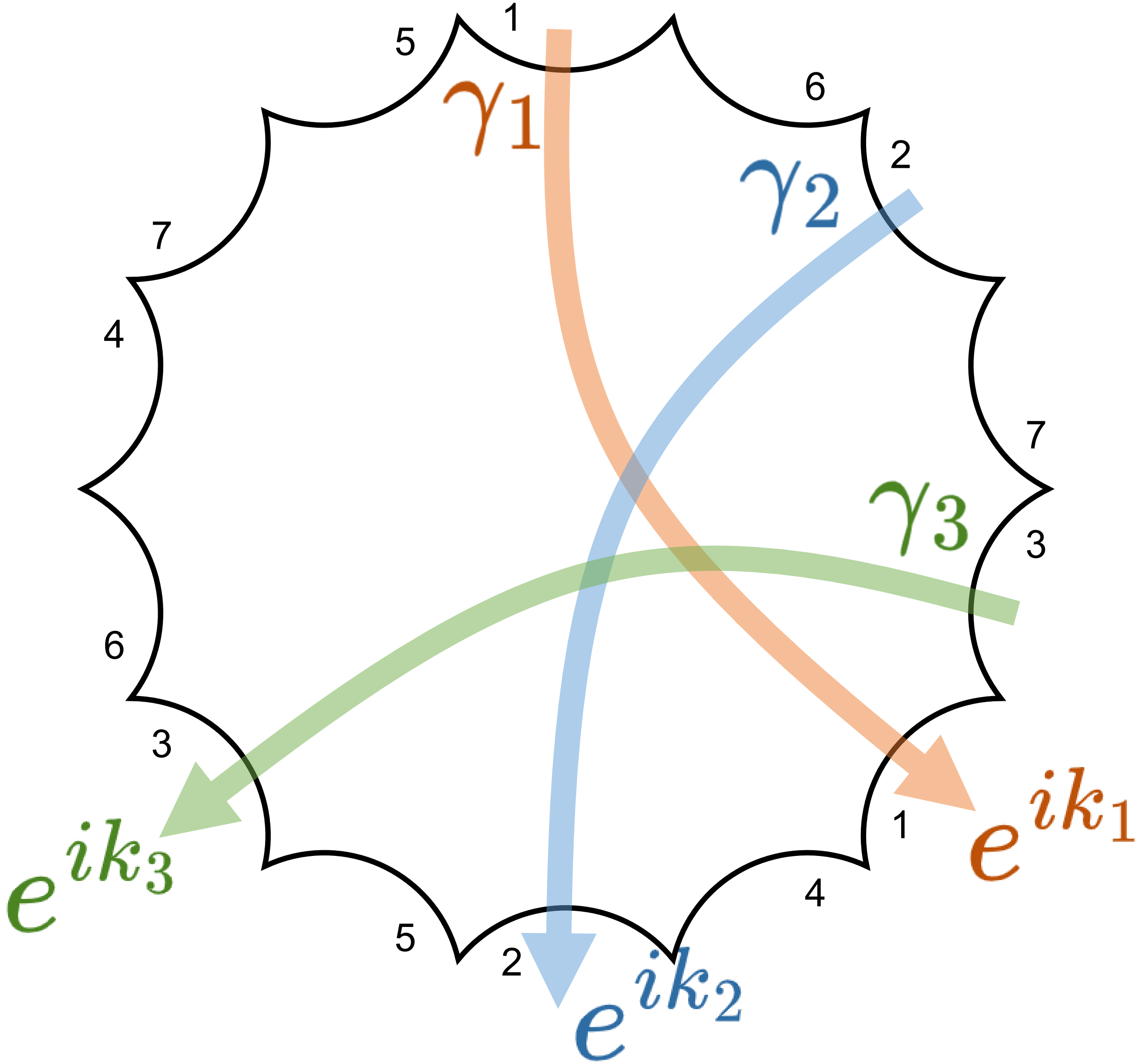

$$\left.\begin{aligned} \gamma_1 &\mapsto e^{ik_1} \\ \gamma_2 &\mapsto e^{ik_2} \\ \gamma_3 &\mapsto e^{ik_3} \\ \gamma_4 &\mapsto e^{ik_4} \\ \gamma_5 &\mapsto e^{ik_5} \\ \gamma_6 &\mapsto e^{ik_6} \end{aligned}\right\} \text{Six-dimensional Brillouin zone}$$

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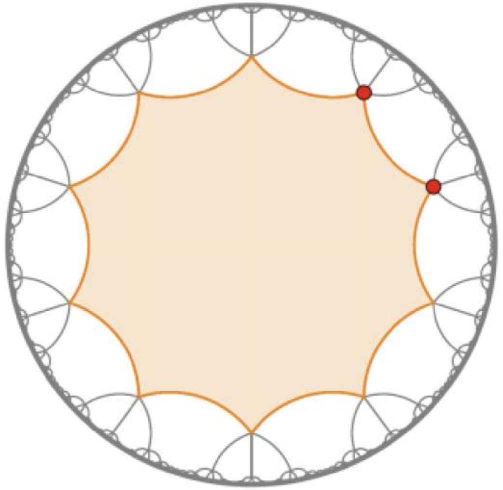
Crystallography of hyperbolic lattices



{10,5}

4D BZ

I. Boettcher *et al.*, *Crystallography of Hyperbolic Lattices*, Phys. Rev. B **105**, 125118 (2022)

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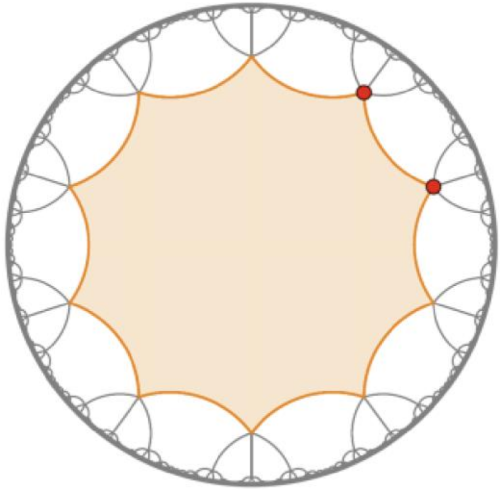

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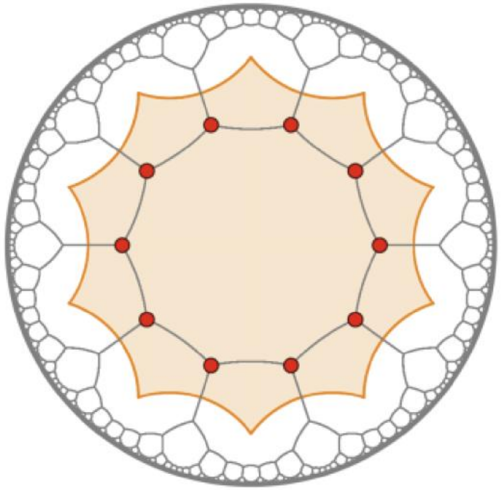
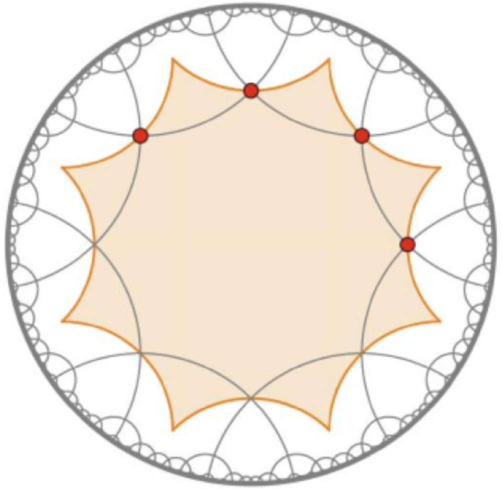


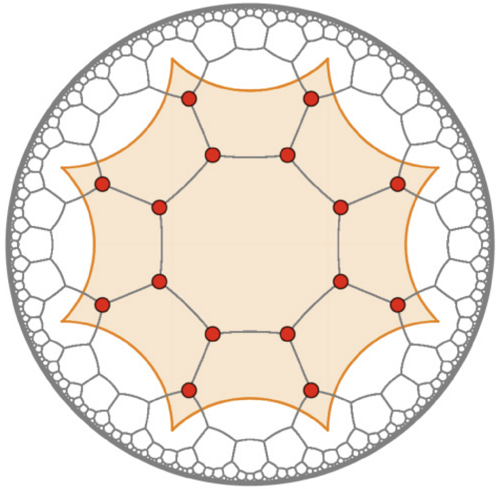
{10,3}
fits onto {10,5}

4D BZ



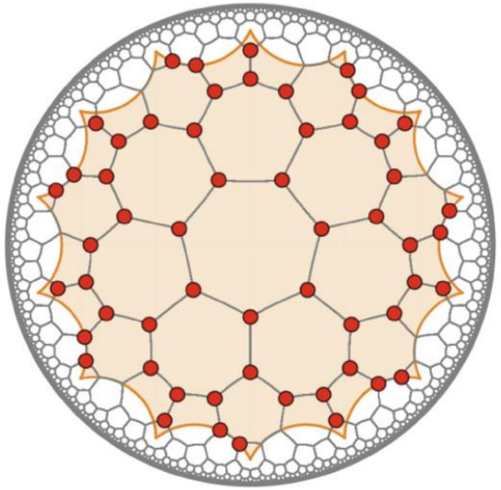
{8,4}
fits onto {8,8}

4D BZ



{8,3}
fits onto {8,8}

4D BZ



{7,3}
fits onto {14,7}

6D BZ

I. Boettcher *et al.*, *Crystallography of Hyperbolic Lattices*, Phys. Rev. B **105**, 125118 (2022)

From Haldene to hyperbolic Haldane model

Discussed here:

D. M. Urwyler, P. M. Lenggenhager, I. Boettcher, R. Thomale, T. Neupert, TB, "*Hyperbolic topological band insulators*", arXiv:2203.07292 (2022)

David M. Urwyler, "*Hyperbolic topological insulators*", Master's Thesis (2021), http://dx.doi.org/10.13140/RG.2.2.34715.34081

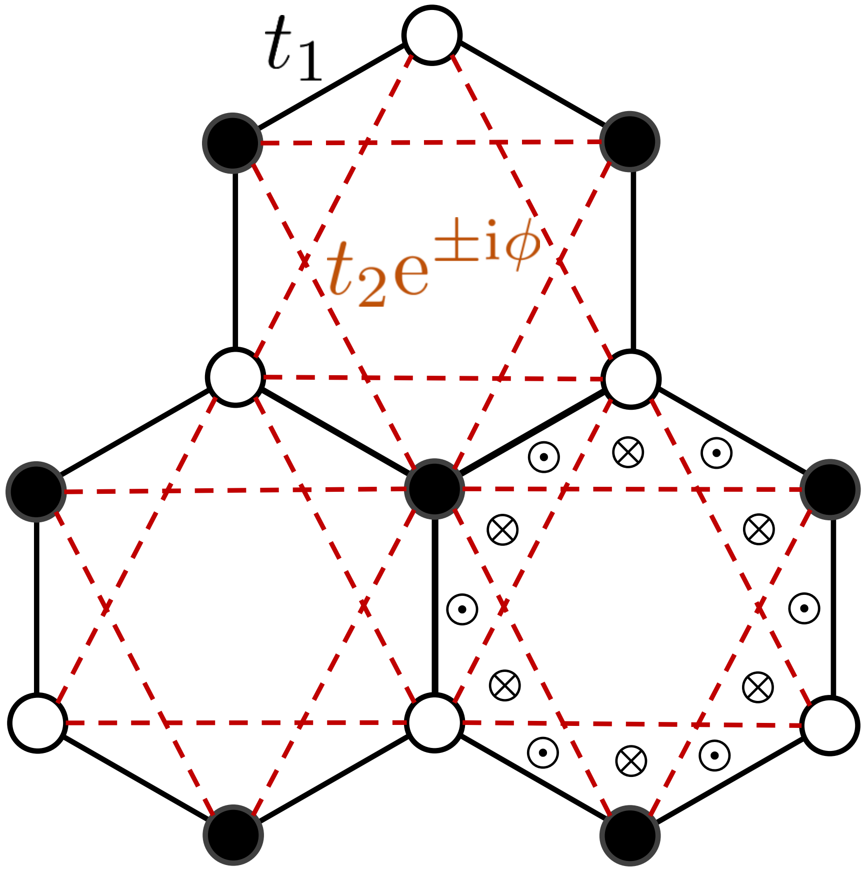
See also related works:

W. Zhang, H. Yuan, N. Sun, H. Sun, X. Zhang, "*Observation of novel topological states in hyperbolic lattices*", Nat. Commun. **13**, 2937 (2022) (arXiv:2203.03214)

Z.-R. Liu, C.-B. Hua, T. Peng, B. Zhou, "*Chern insulator in a hyperbolic lattice*", Phys. Rev. B **105**, 245301 (2022) (arXiv:2203.02101)

From Haldene to hyperbolic Haldane model

Replace hexagons of the honeycomb lattice by <u>octagons</u> -- this produces {8,3} lattice.



F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988)

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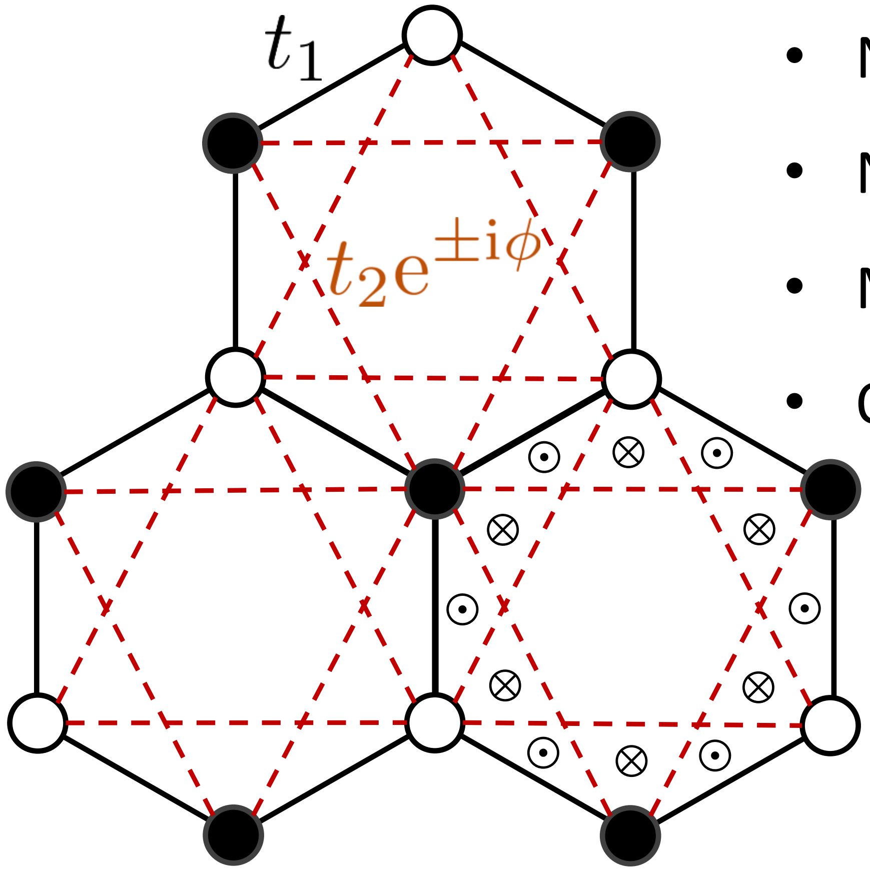

4 ingredients:

- NN hopping ($t_1 = 1$)
- NNN hopping (t_2)
- Magnetic flux (ϕ)
- On-site potential (±M)

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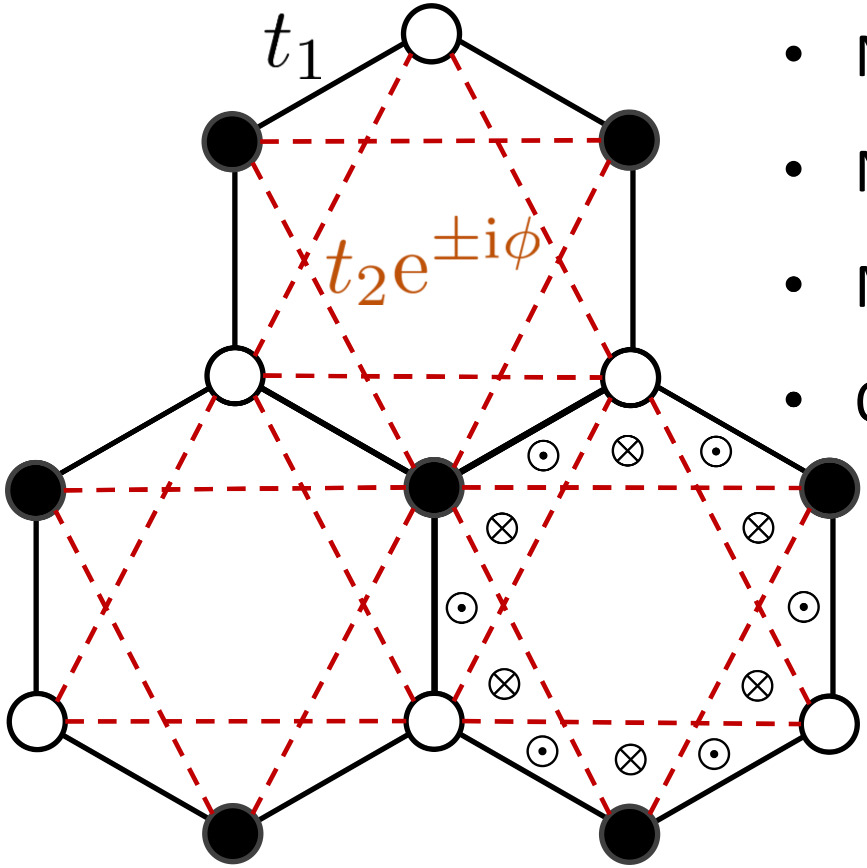

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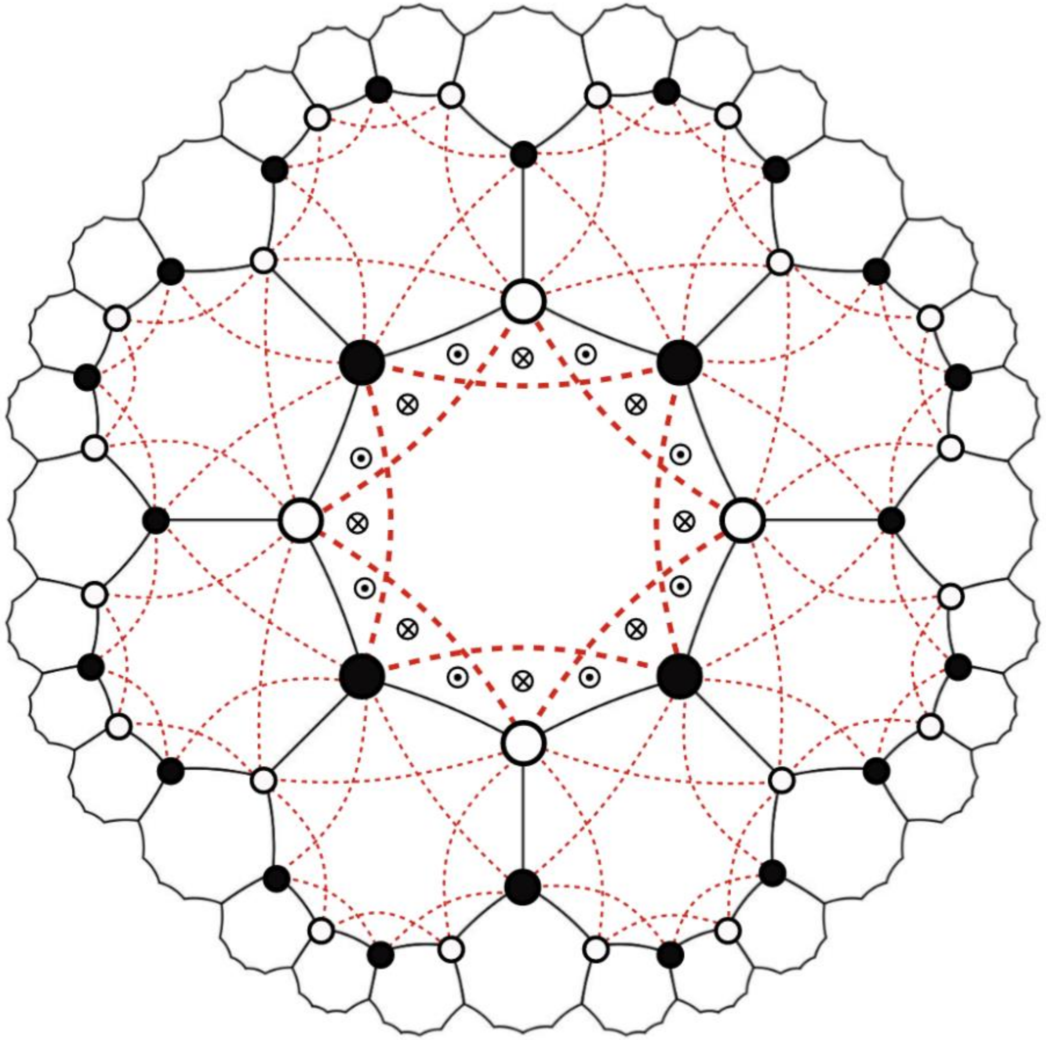

D. M. Urwyler, Master's thesis, University of Zürich (2021)
D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

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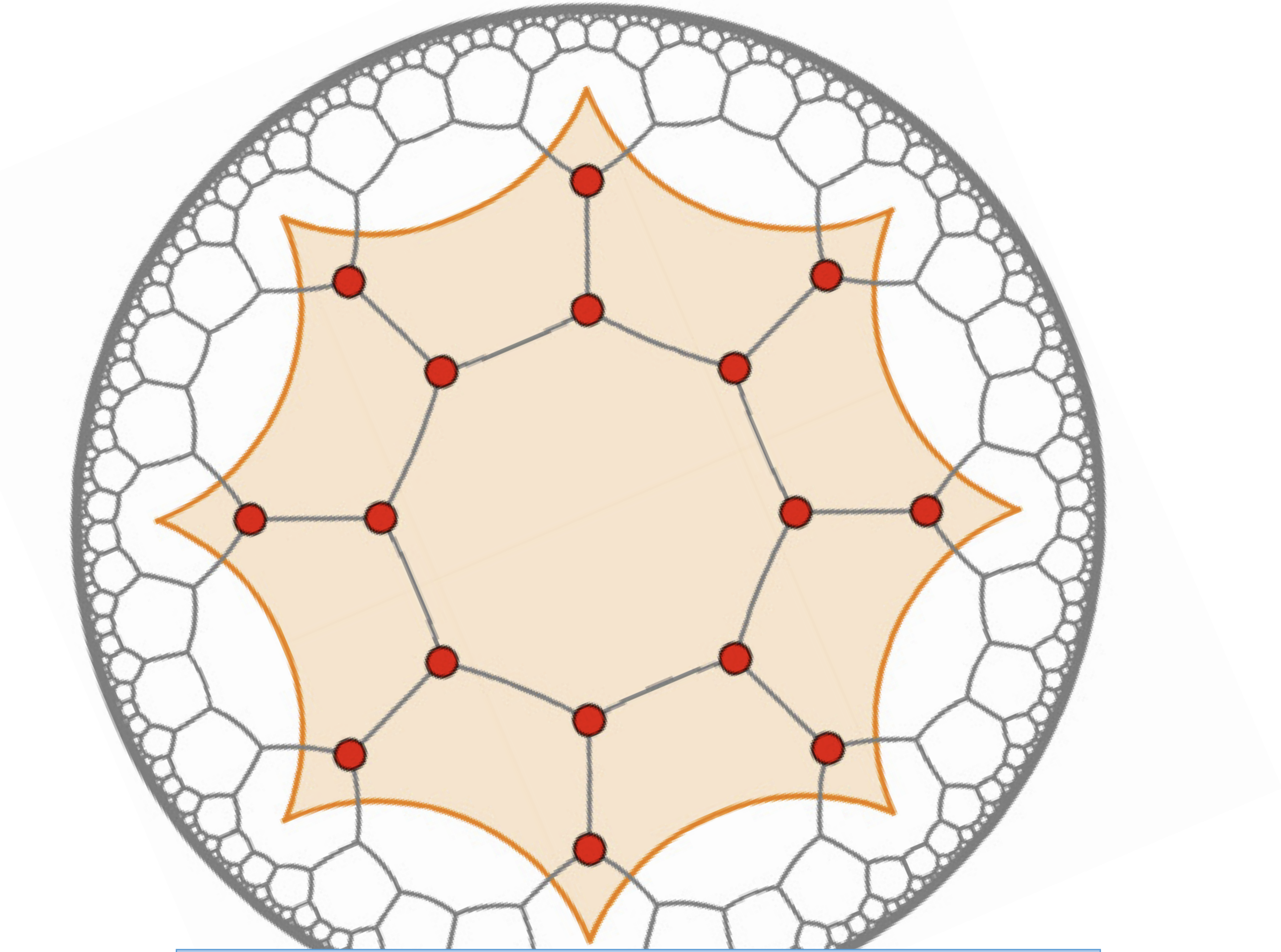
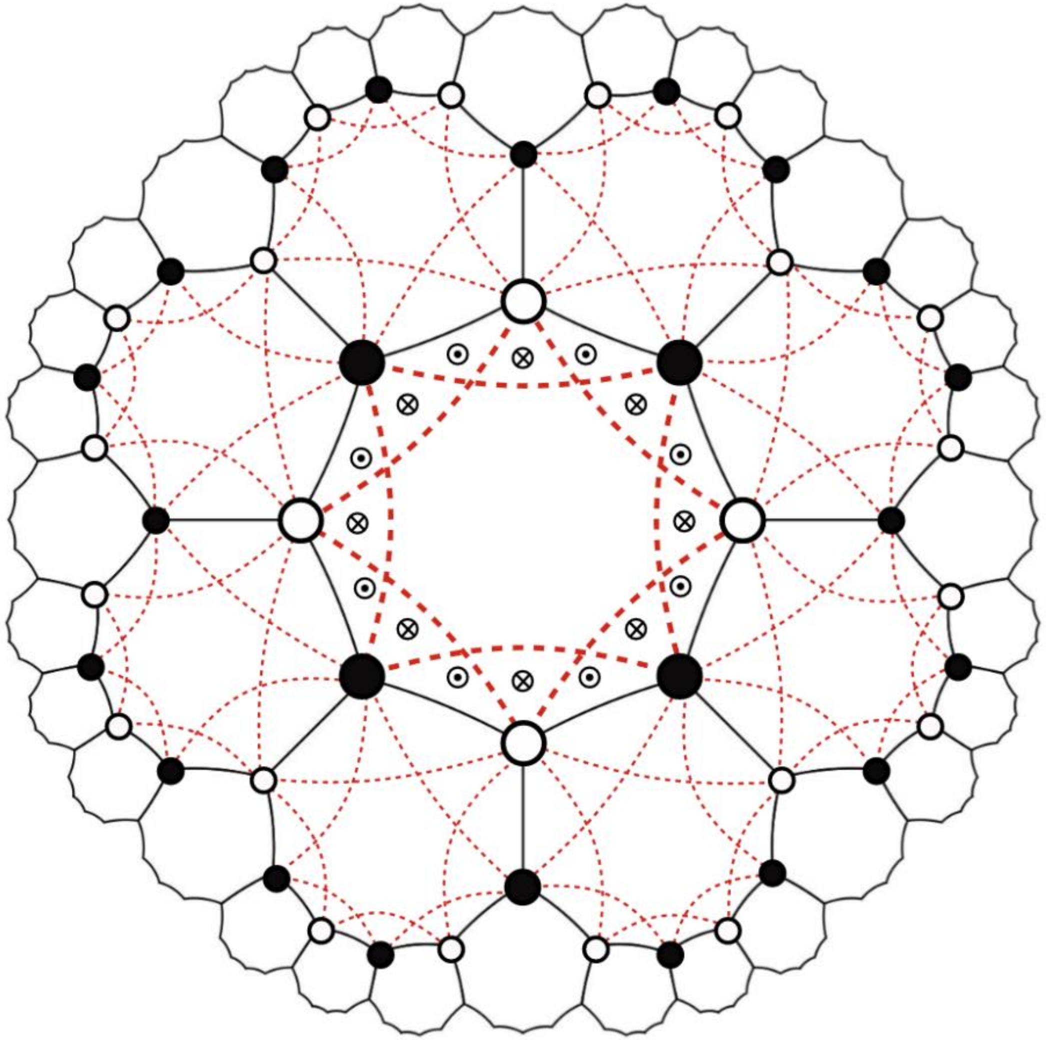


16 sites per cell

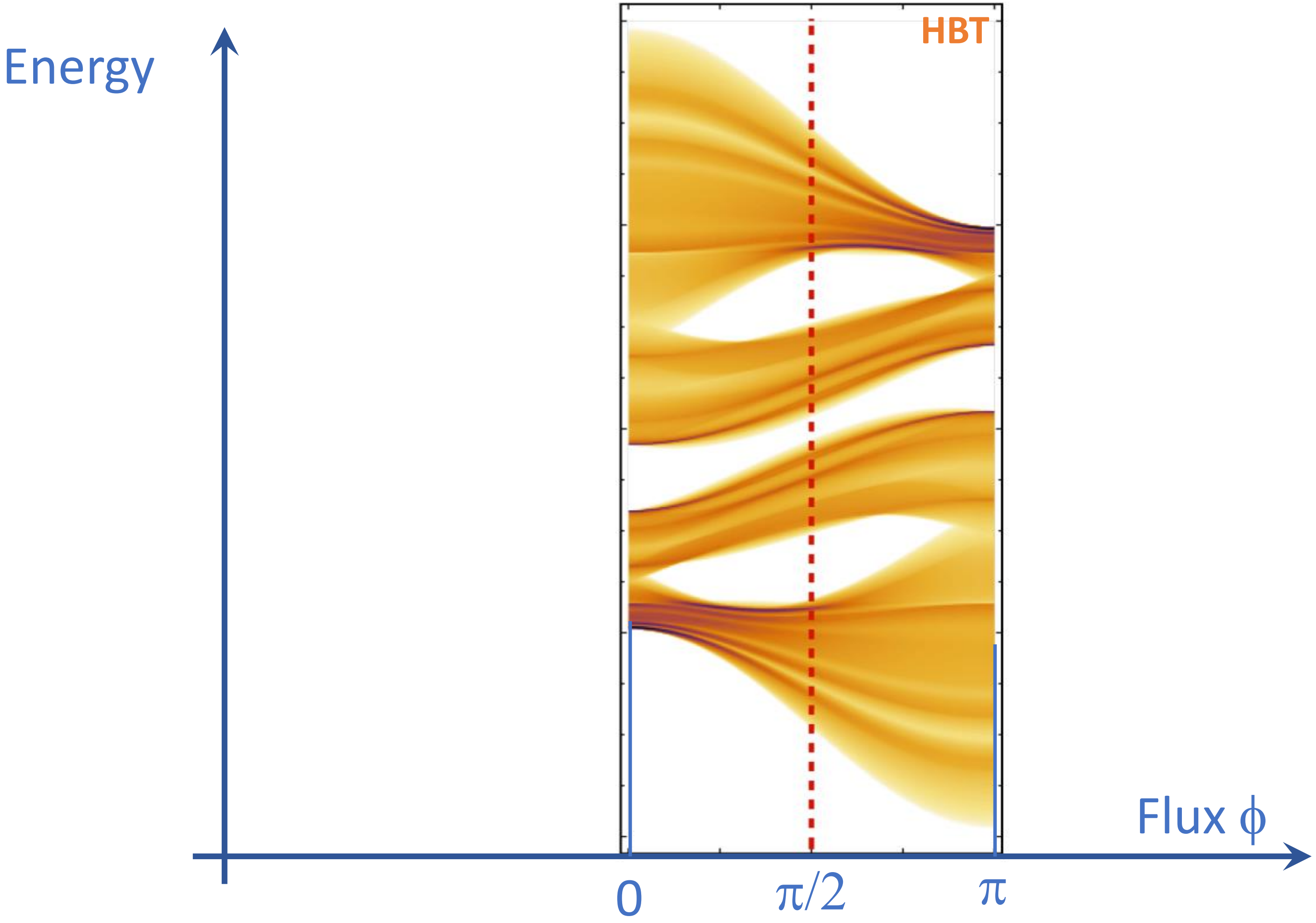
→ **16 energy bands** in the 4D $\boldsymbol{k}$-space



D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

Hyperbolic Haldane model

Model parameters: $t_1 = 1,\ t_2 = \frac{1}{6},\ M = \frac{1}{3},\ \phi = \frac{\pi}{2}$



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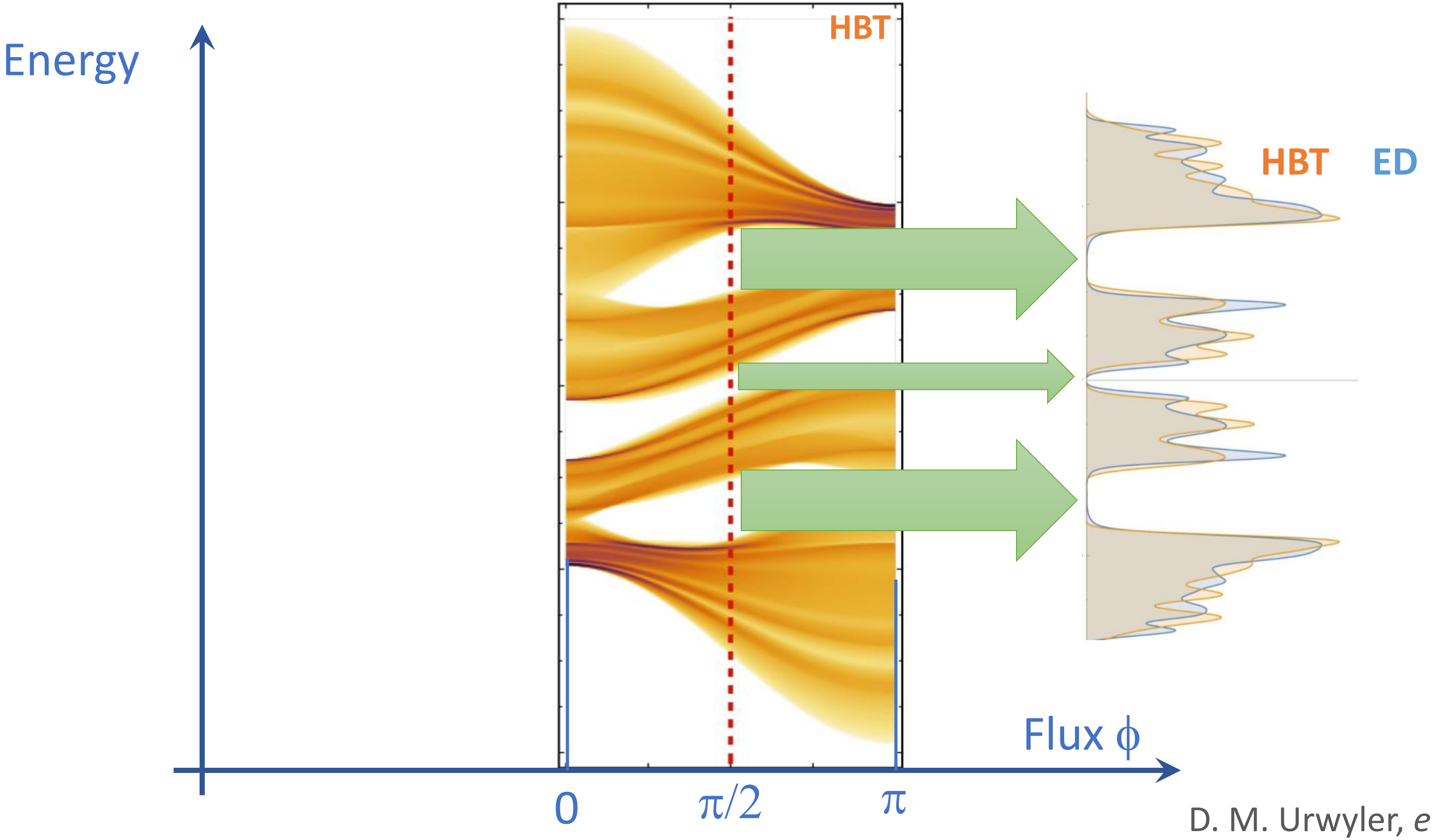

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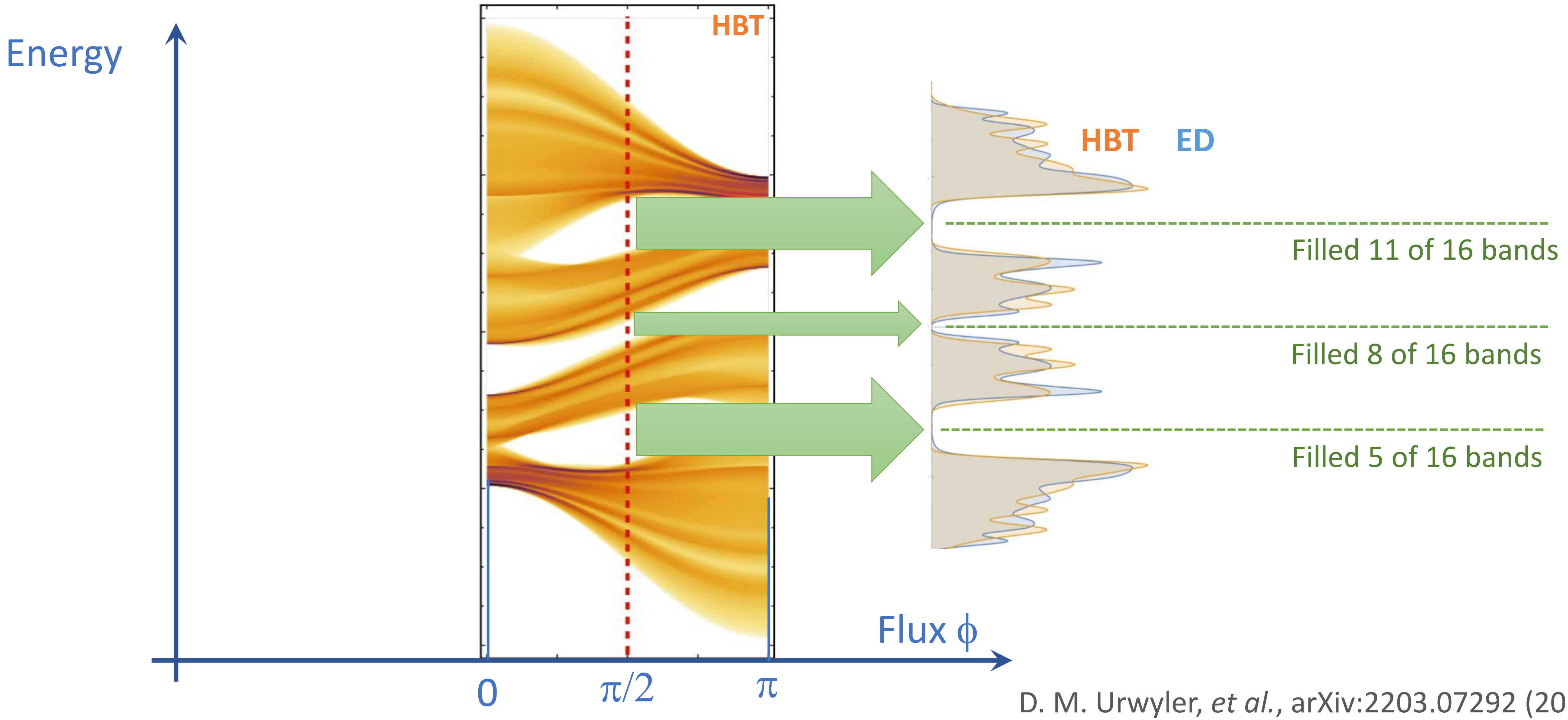

D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

Chern numbers of the hyperbolic Haldane model

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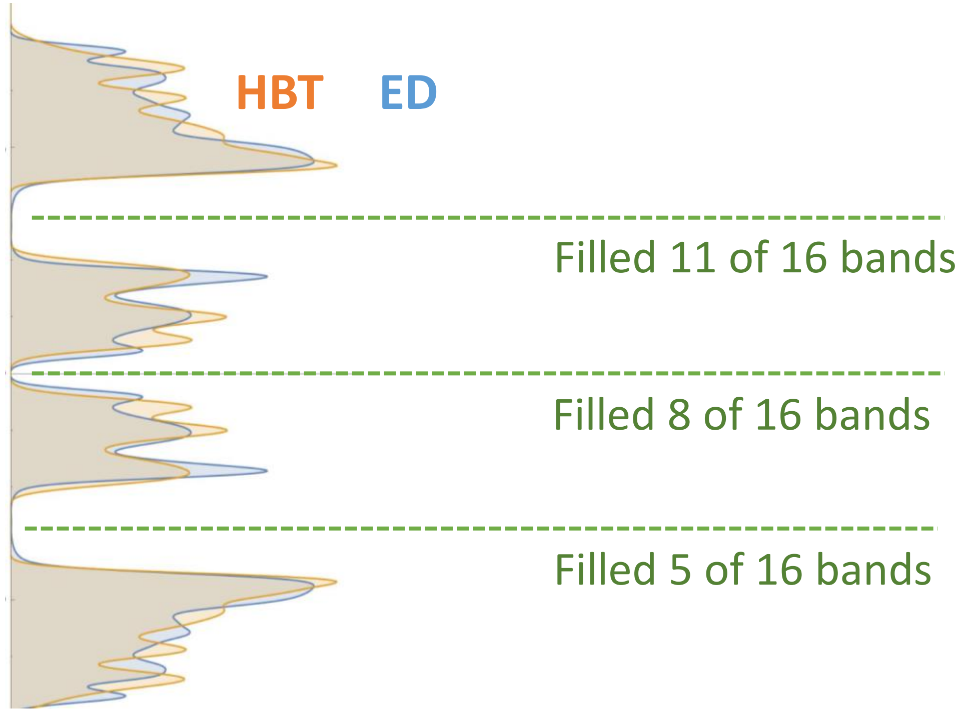
(Computed from the U(1) HBT states:)

$C_{k_i,k_j}/\ N_{occ}$	$N_{occ} = 5$	$N_{occ} = 8$	$N_{occ} = 11$
C_{k_x,k_y}	-1	0	-1
C_{k_x,k_z}	1	0	1
C_{k_x,k_w}	-1	0	-1
C_{k_y,k_z}	-1	0	-1
C_{k_y,k_w}	1	0	1
C_{k_z,k_w}	-1	0	-1



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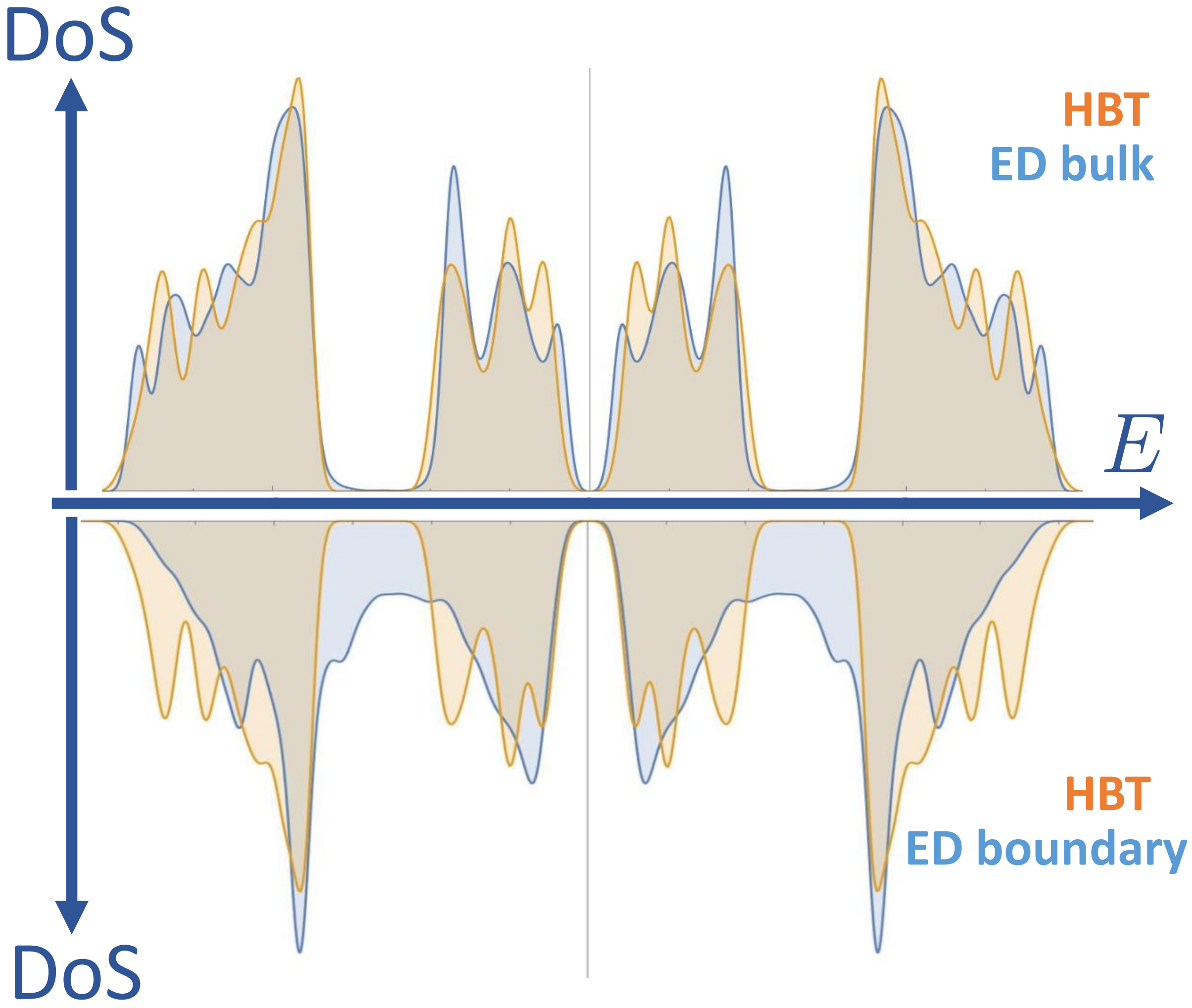
(Computed from the U(1) HBT states:)

$C_{ki,k_J} /\ N_{occ}$	$N_{occ} = 5$	$N_{occ} = 8$	$N_{occ} = 11$
C_{k_x,k_y}	-1	0	-1
C_{k_x,k_z}	1	0	1
C_{k_x,k_w}	-1	0	-1
C_{k_y,k_z}	-1	0	-1
C_{k_y,k_w}	1	0	1
C_{k_z,k_w}	-1	0	-1



Chern numbers of the hyperbolic Haldane model

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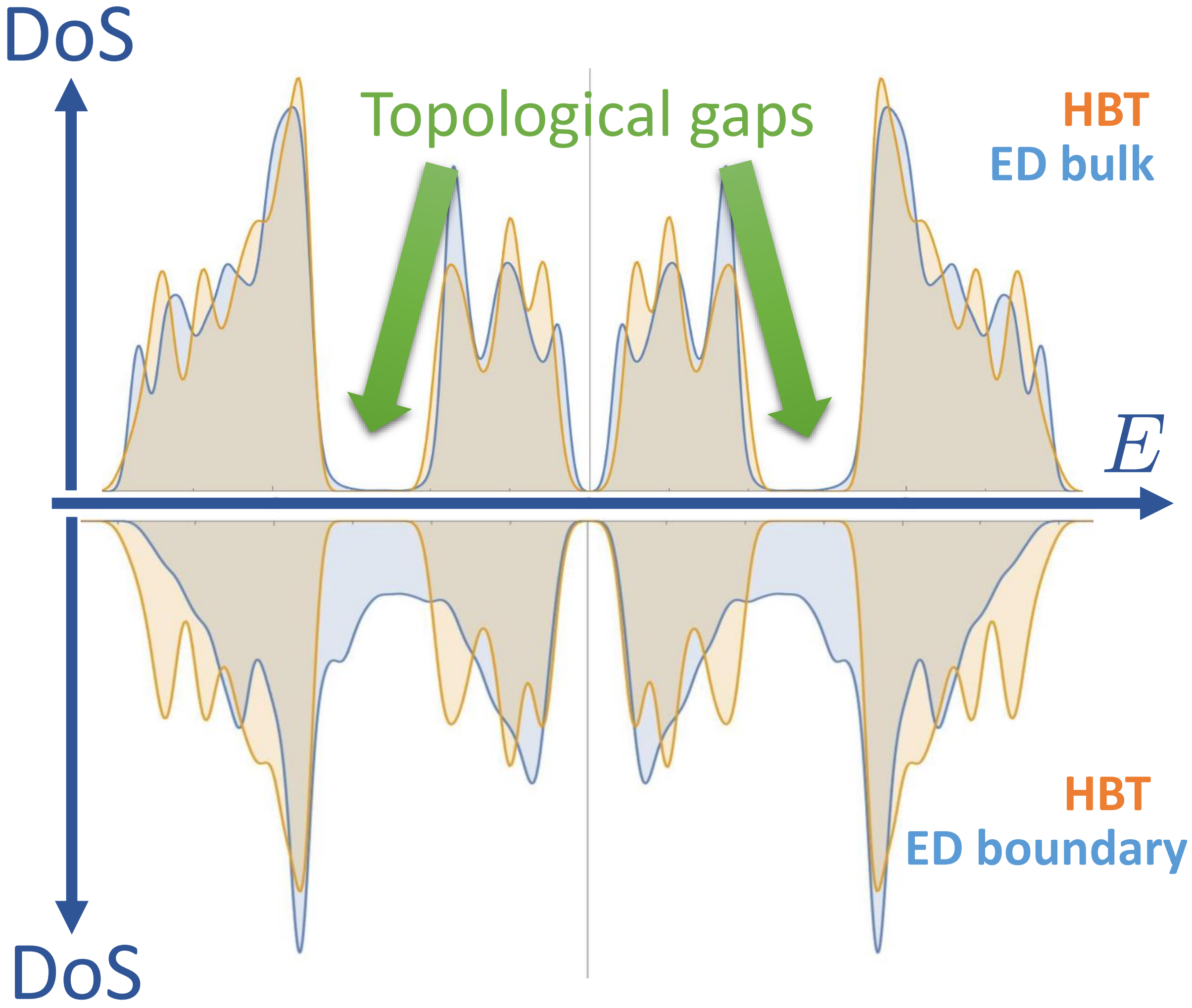

Topological gaps

Chern numbers of the hyperbolic Haldane model

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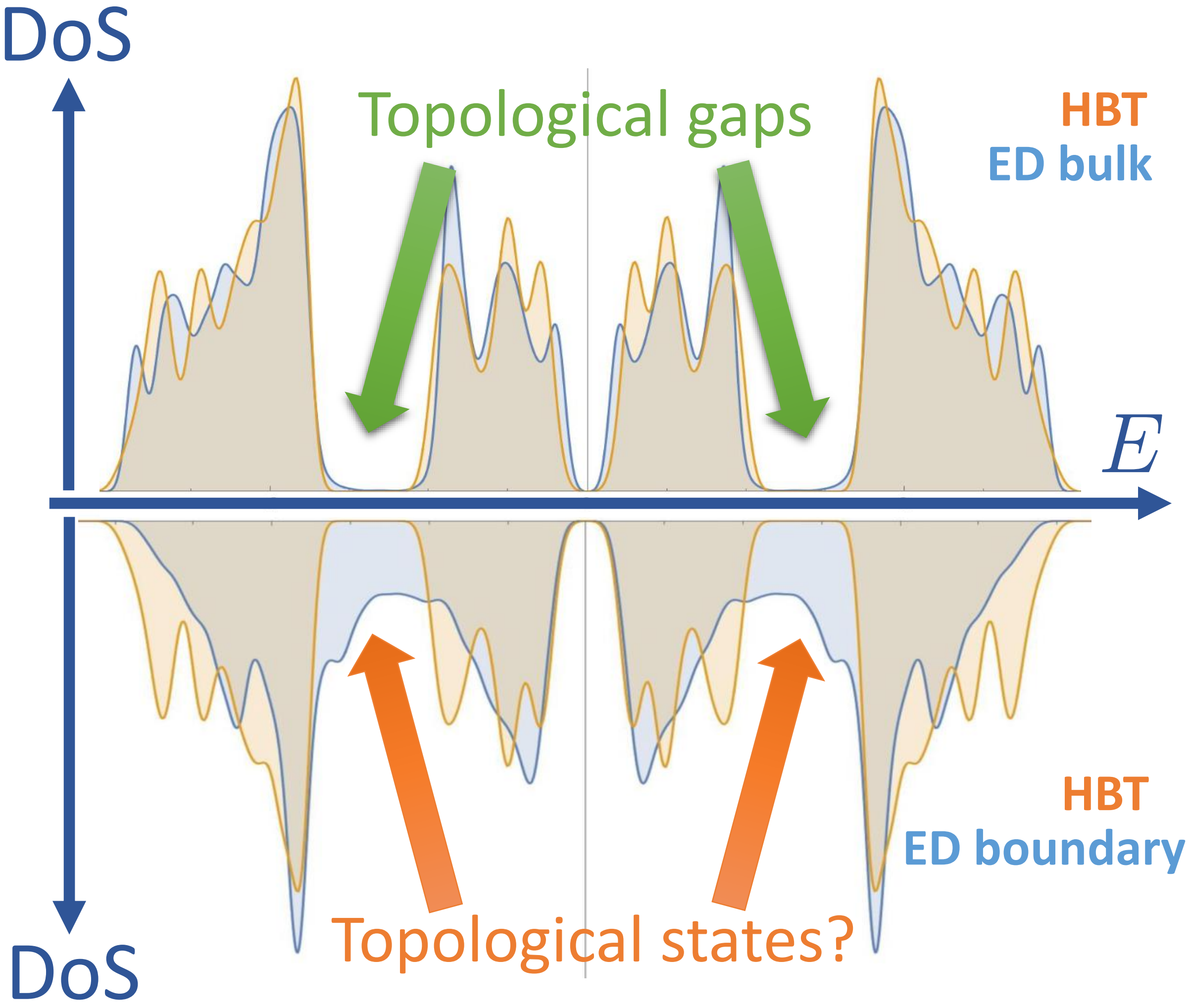

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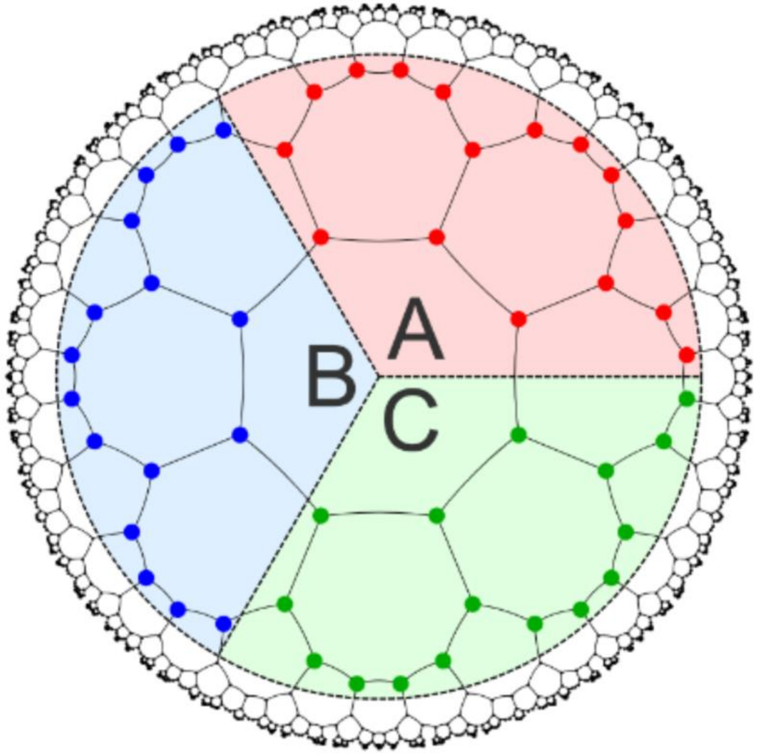
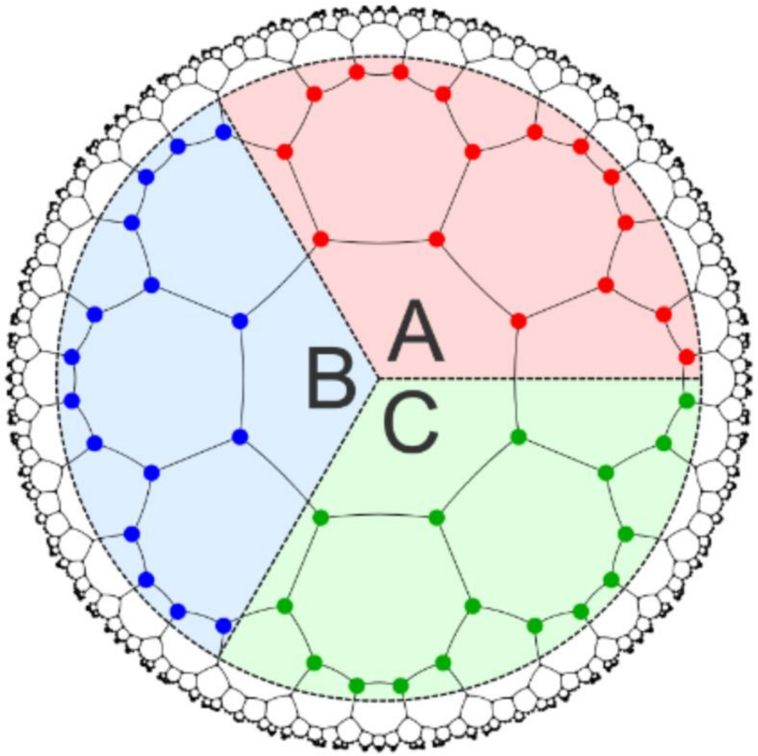


$$C_{RS} = 12\pi i \sum_{j\in A}\sum_{k\in B}\sum_{l\in C}\left(\mathbf{P}_{jk}\mathbf{P}_{kl}\mathbf{P}_{jl} - \mathbf{P}_{jl}\mathbf{P}_{lk}\mathbf{P}_{kj}\right)$$

A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Phys. **321**, 2—111 (2006)

Chern numbers of the hyperbolic Haldane model



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Result:

		Haldane		
f	μ	C_{xy}	C_{xz}	C_{RS}
5/16	-1.3	-1	1	-0.986
8/16	0	0	0	0
11/16	+1.3	-1	1	-0.986

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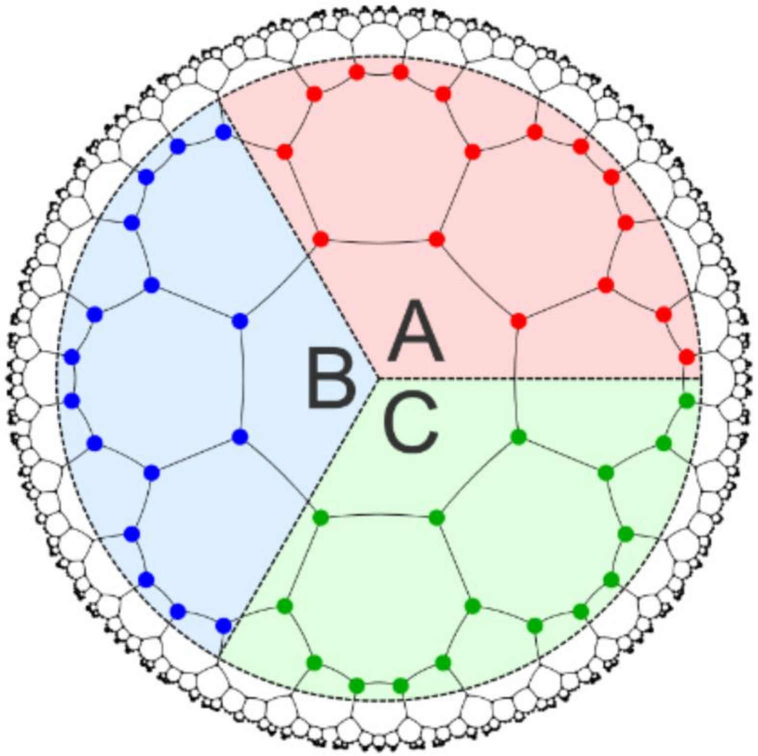

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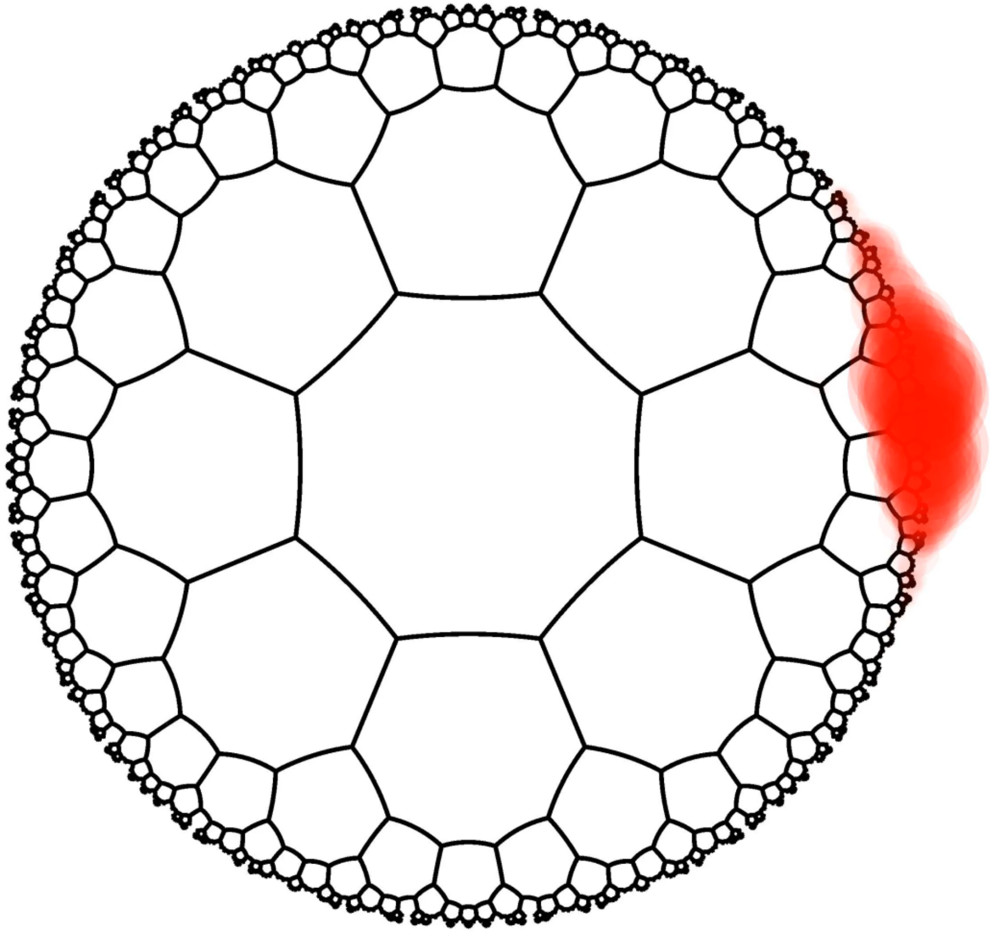
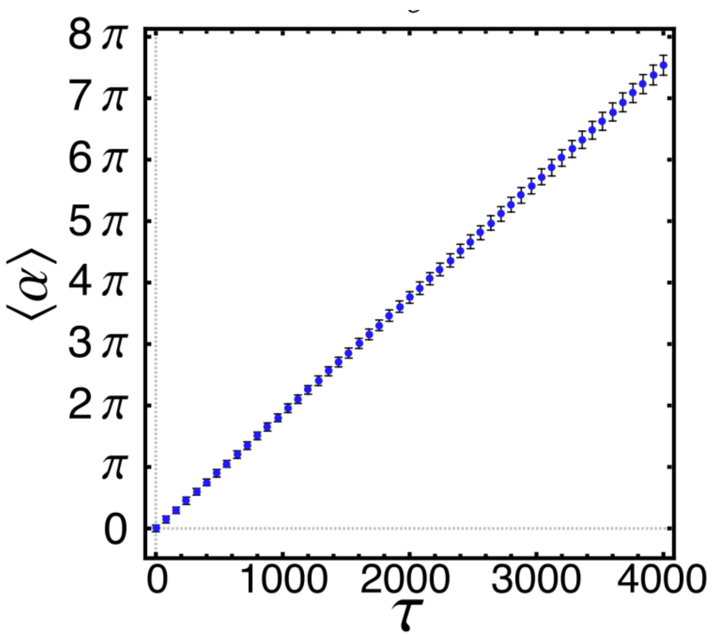
Is there a universal relation between Chern numbers in real space vs. in momentum space?

A. Kitaev, *Anyons in an exactly solved model and beyond*, Ann. Phys. **321**, 2—111 (2006)

Chiral edge states on the hyperbolic boundary





Wave packet motion along the boundary.

D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

Flat bands in hyperbolic frustrated-hopping models

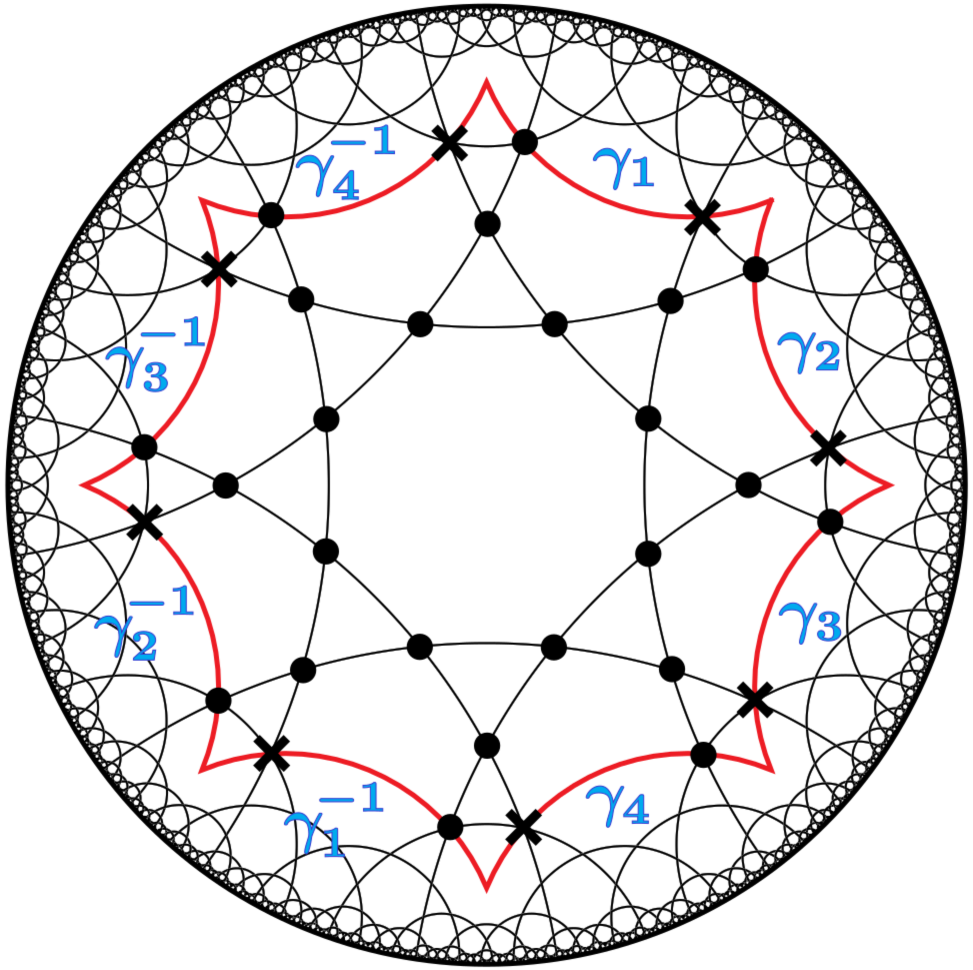
Discussed here:

TB and Joseph Maciejko, "*Flat bands and band touching from real-space topology in hyperbolic lattices*", arXiv:2205.11571 (2022)

See also related work:

R. Mosseri, R. Vogeler, J. Vidal, "*Aharonov-Bohm cages, flat bands, and gap labeling in hyperbolic tilings*", Phys. Rev. B **106**, 155120 (2022) (arXiv:2206.04543)

Flat bands on octagon kagome lattice (with PBC)



TB and Joseph Maciejko, arXiv:2205.11571 (2022)

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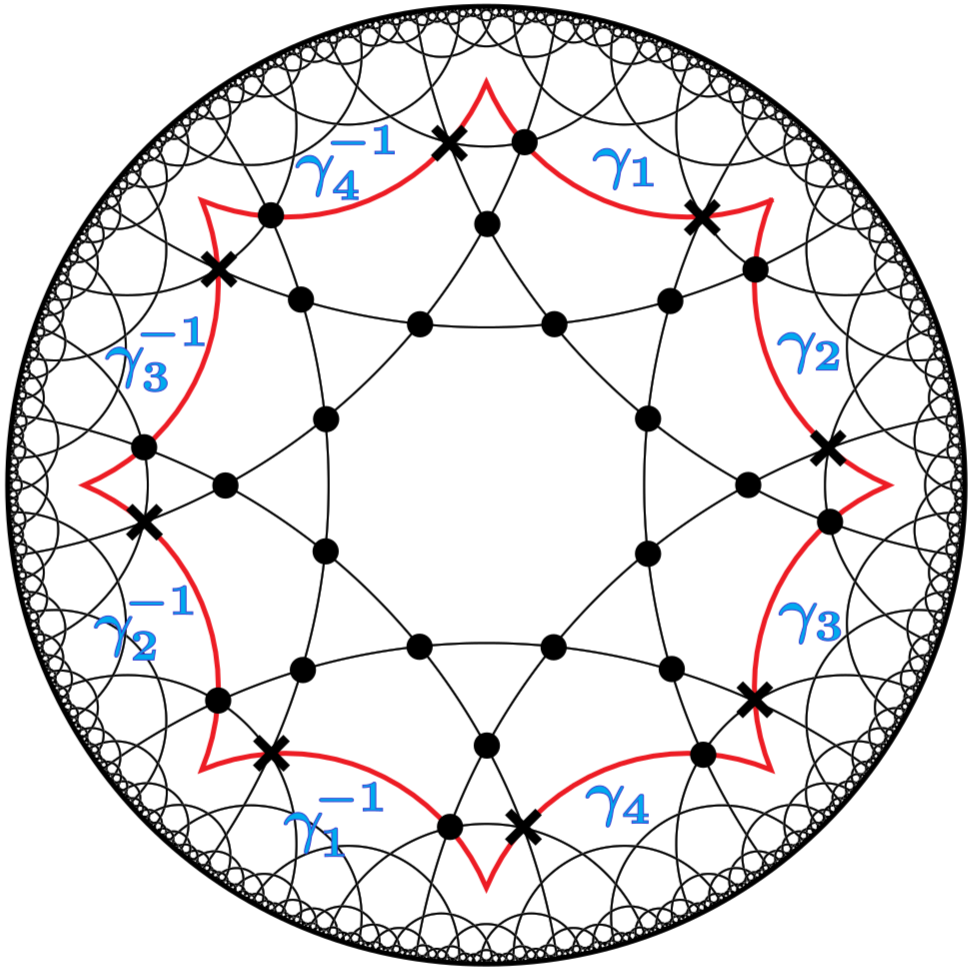

24 sites (orbitals) per Bolza cell
→ 24*N* states per *N* cells

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

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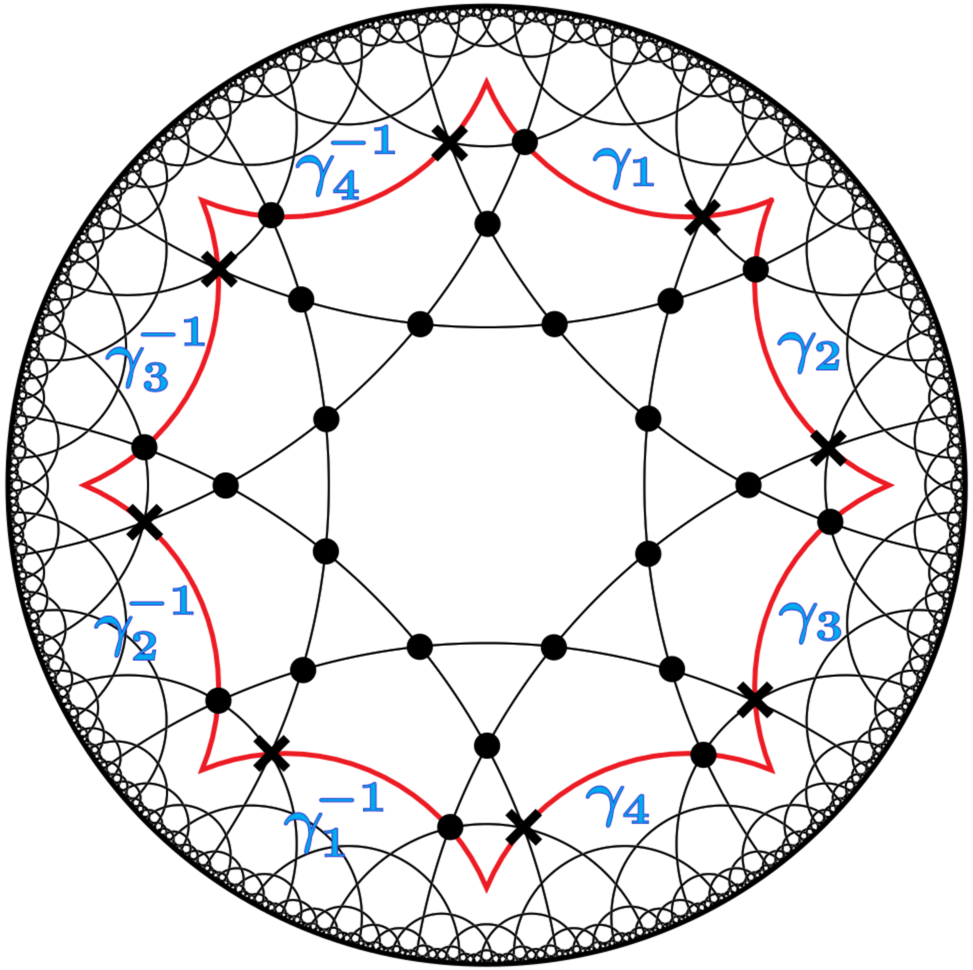

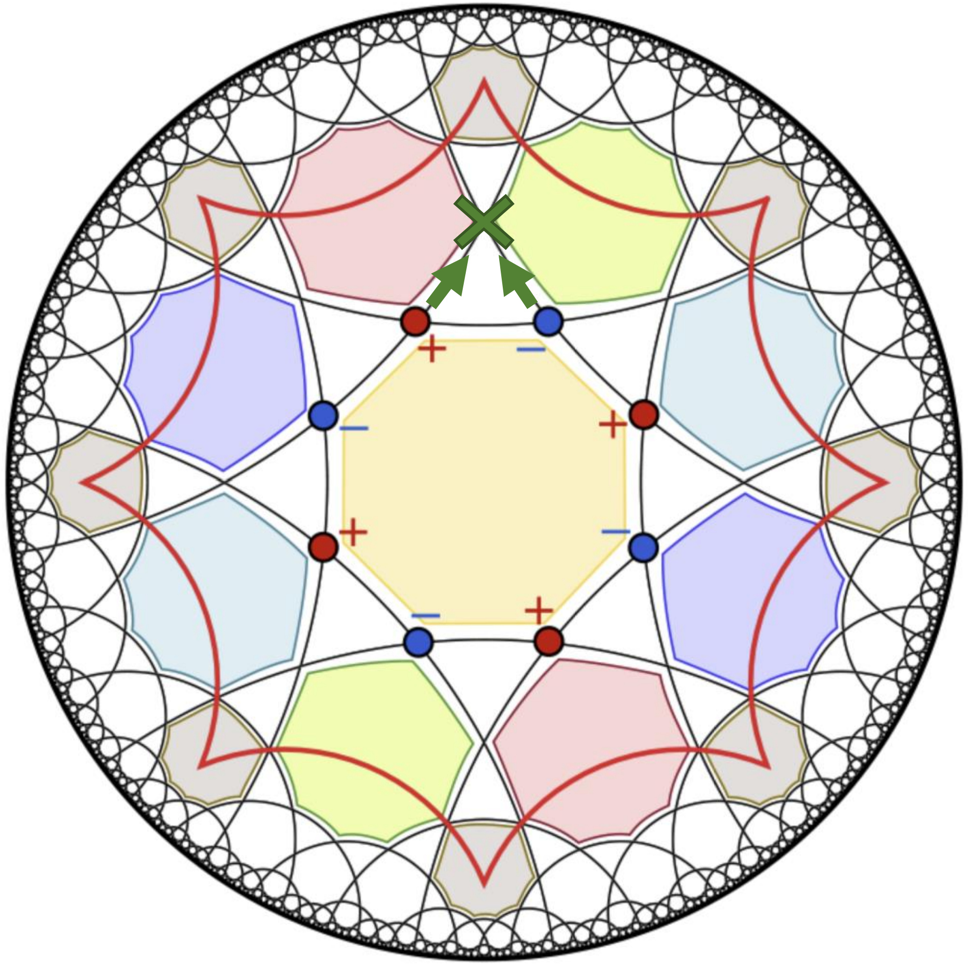

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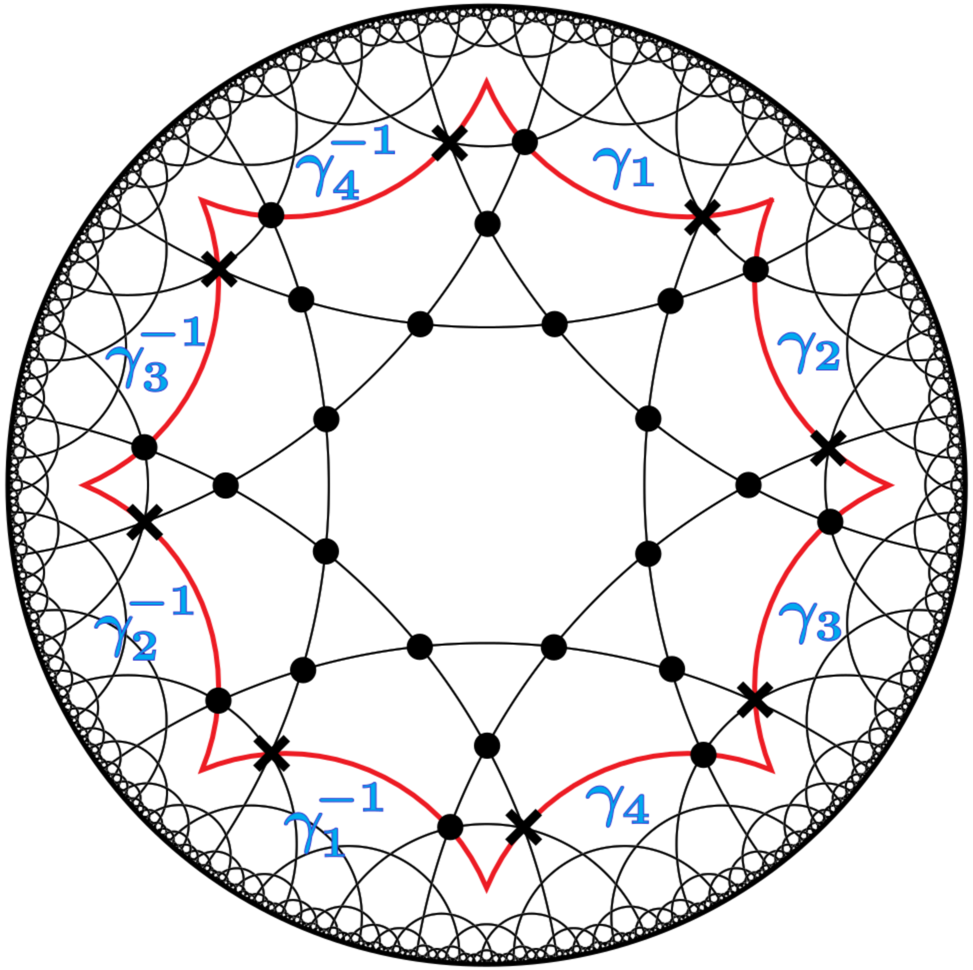

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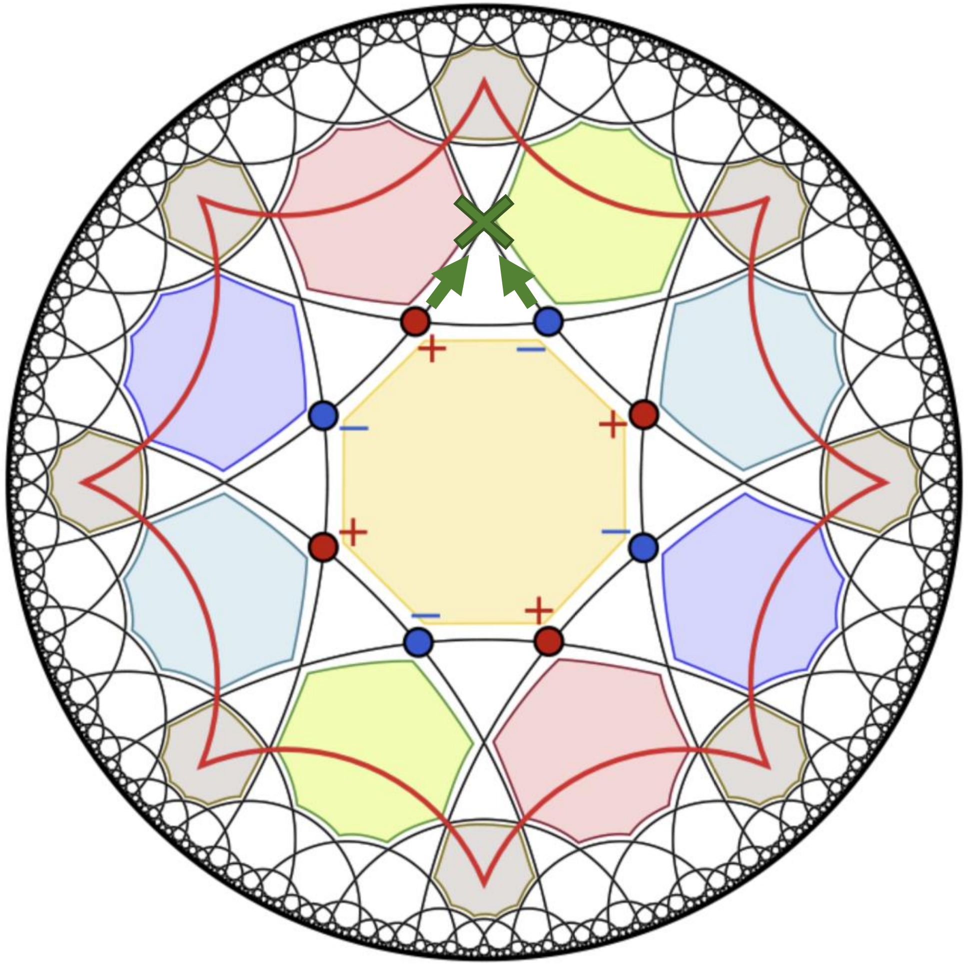

6 flat-band states per Bolza cell
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TB and Joseph Maciejko, arXiv:2205.11571 (2022)

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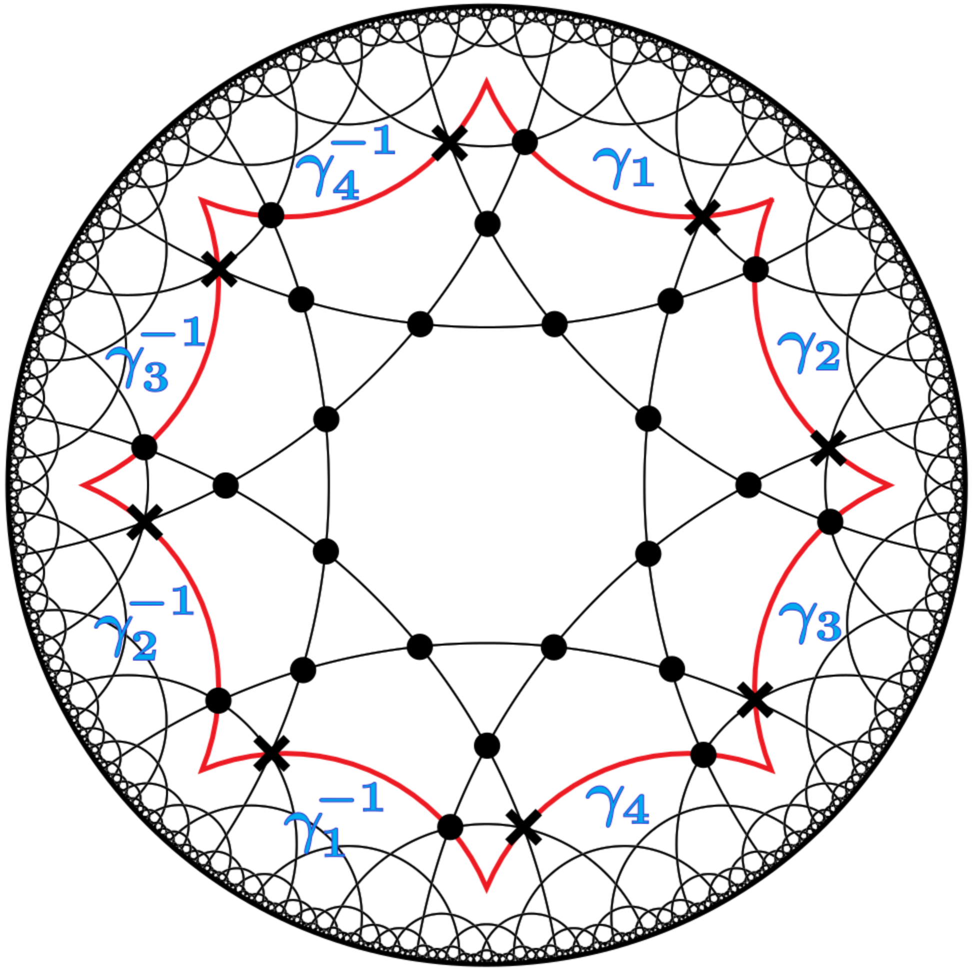

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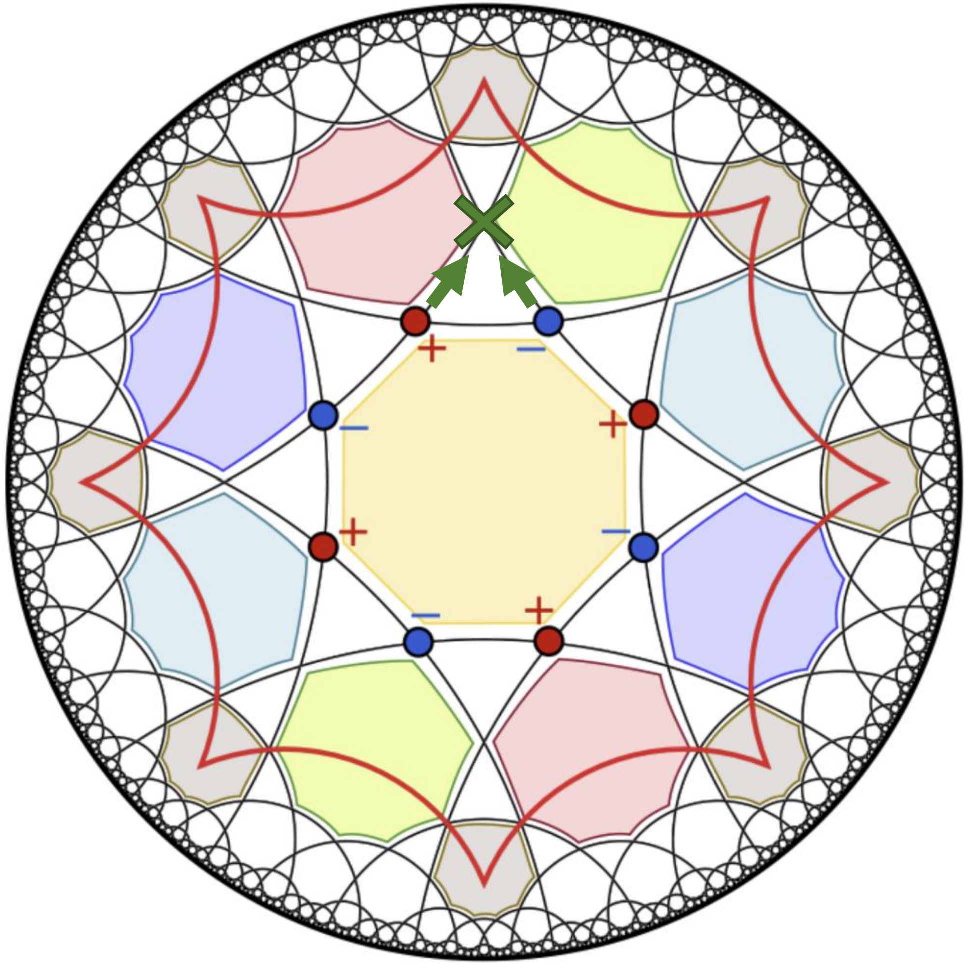

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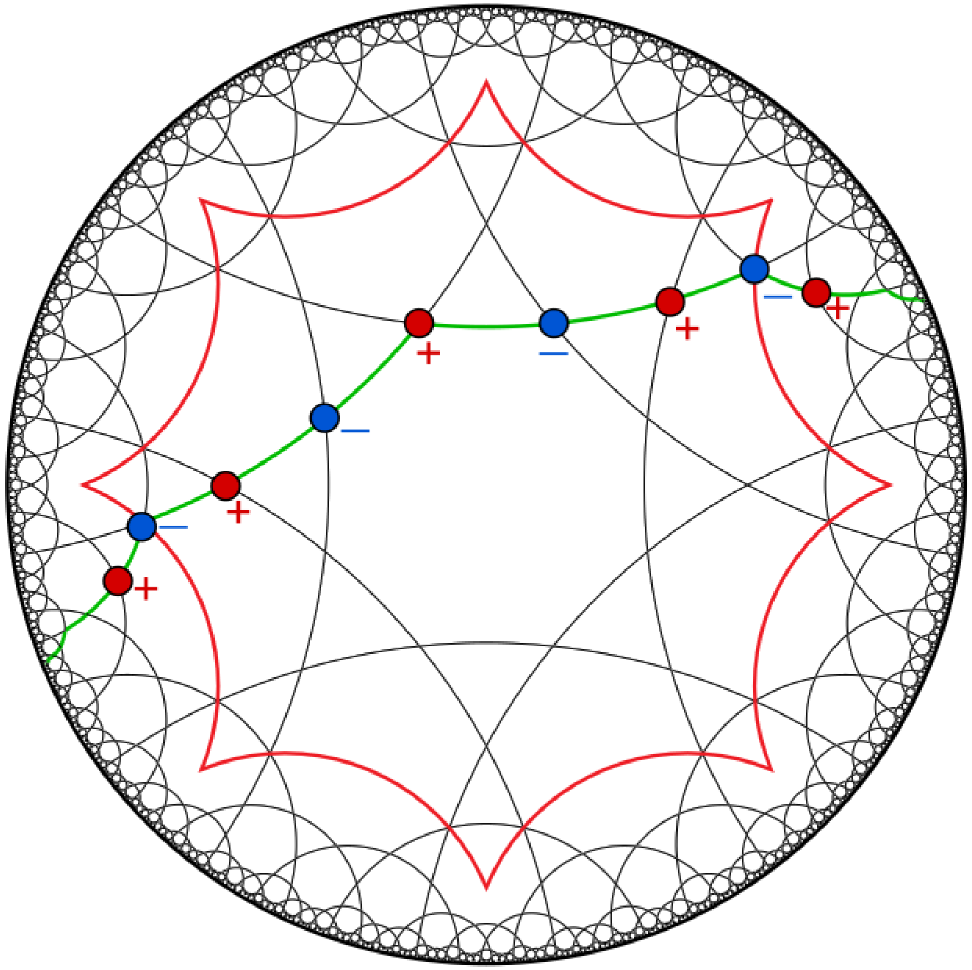
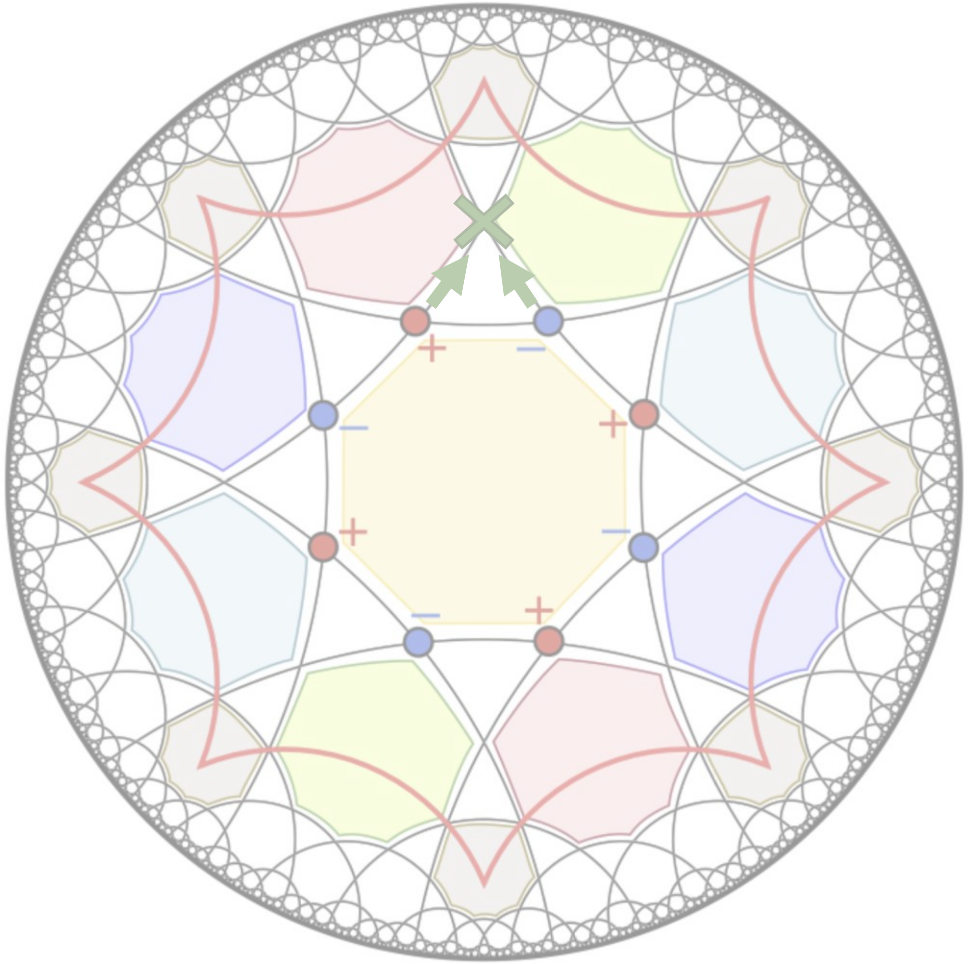


“string states” due to non-trivial homology of the compactified system?



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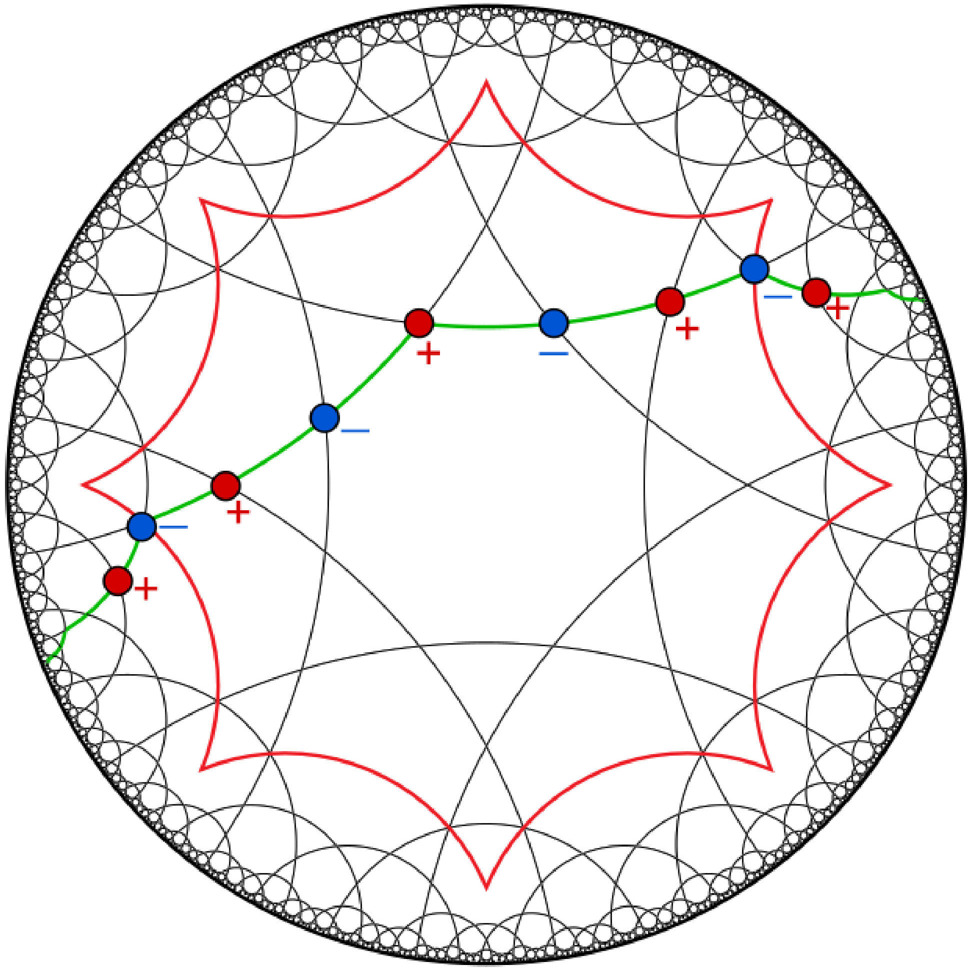

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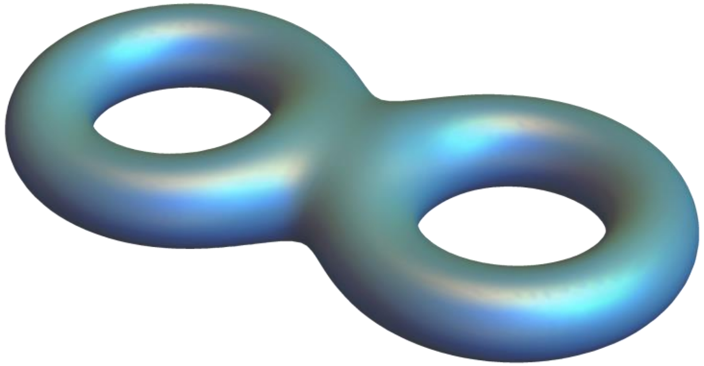
Single unit cell is (g = 2)-hole torus, which supports 4 non-trivial cycles.



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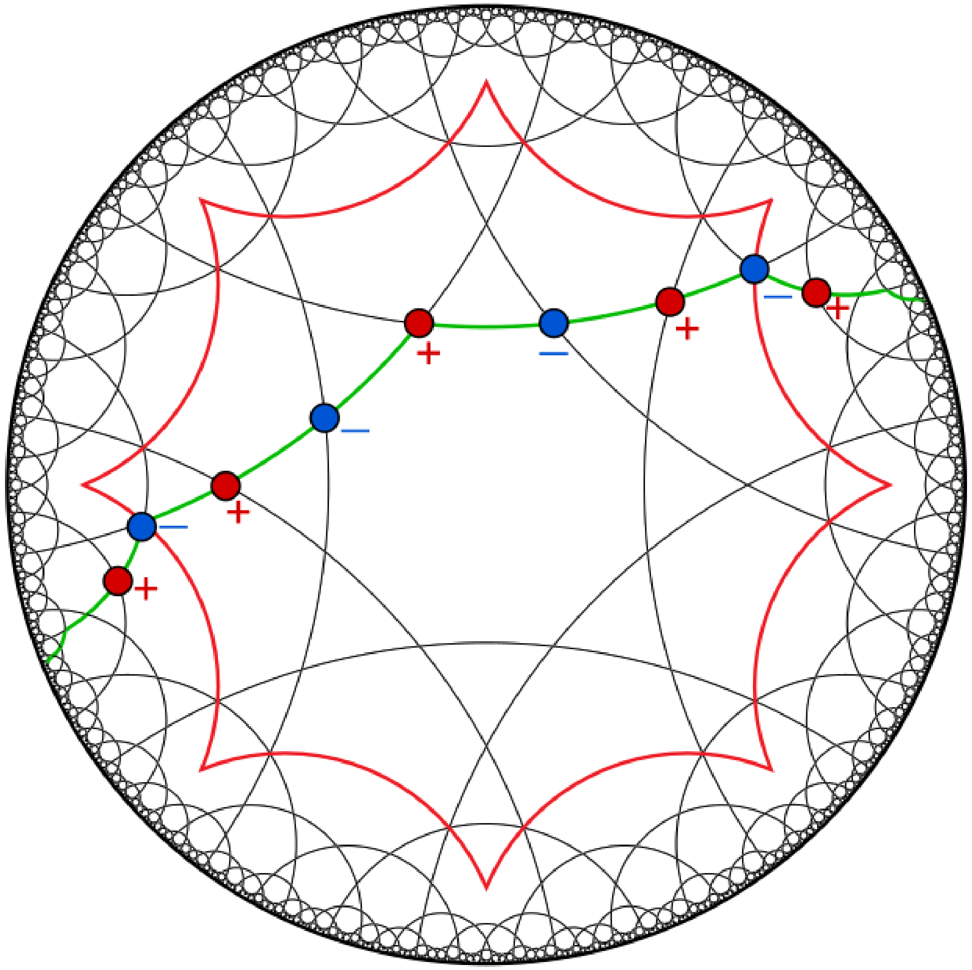

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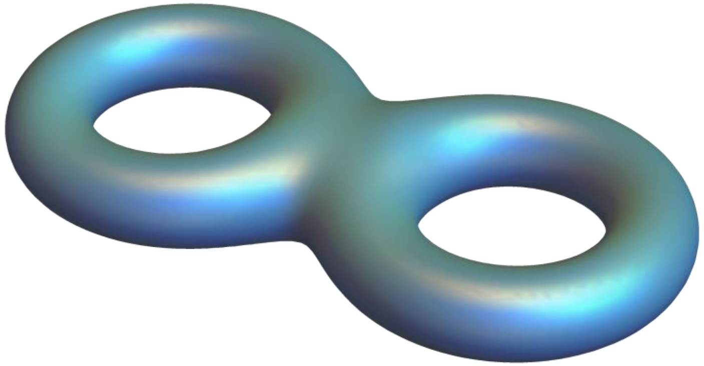

N-cell cluster (with compactified boundary) has genus h given by Riemann-Hurwitz theorem:

$$h = N(g-1) + 1$$

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

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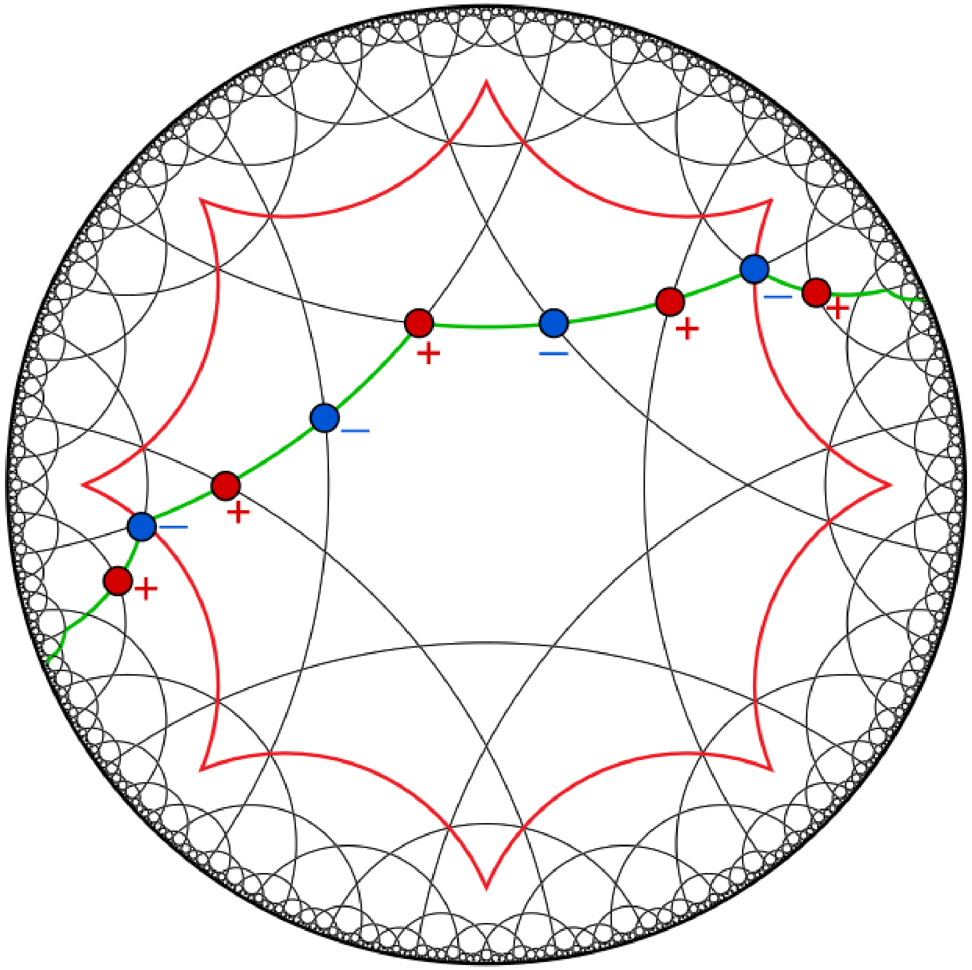

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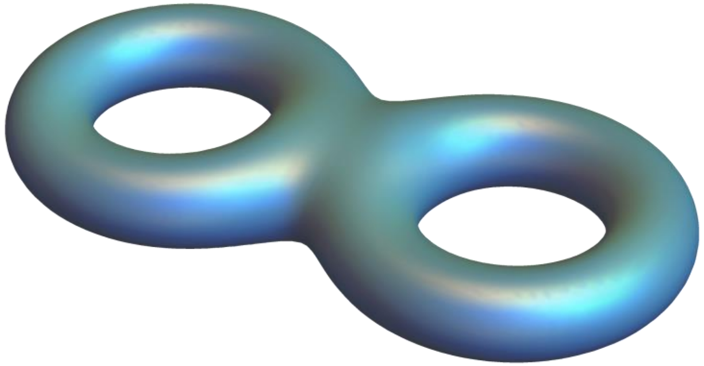

N-cell cluster (with compactified boundary) has genus *h* given by Riemann-Hurwitz theorem:

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h-hole torus has $2h = 2N + 2$ non-trivial cycles, i.e., that many additional “string states”.

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

Flat bands on octagon kagome lattice (with PBC)

Of the 24*N* states, the number of linearly-independent states in the flat band is:

$$(6N - 1) + (2N + 2) = 8N + 1$$

single-octagon states

string states

total flat-band states

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

Abelian vs. non-Abelian flat-band states

The real-space argument captures *the whole spectrum*, i.e., Abelian <u>and</u> non-Abelian irreps.

$\mathrm{frac}_{\mathrm{all}} = 1/3$

<u>TB</u> and Joseph Maciejko, arXiv:2205.11571 (2022)

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Diagonalization of momentum-space Hamiltonian (Abelian irreps) reveals 8 of the 24 bands are flat.

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TB and Joseph Maciejko, arXiv:2205.11571 (2022)

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—

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=

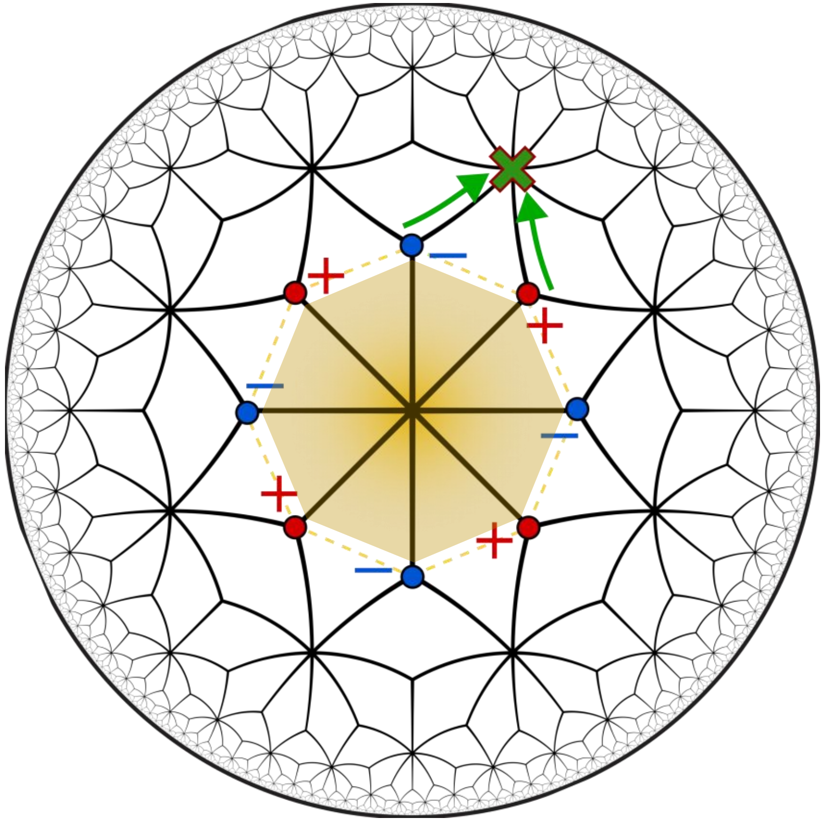
By taking the different, also 1/3 of the non-Abelian states lie at the flat-band energy.

$\mathrm{frac}_{\text{non-Ab.}}$ = 1/3

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

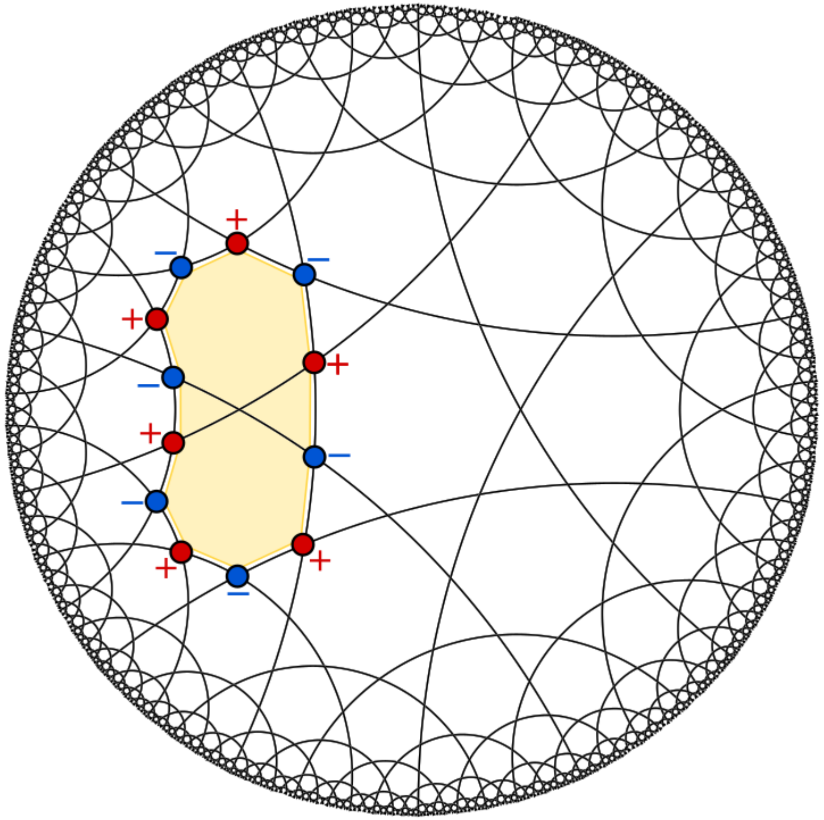
Other hyperbolic frustrated-hopping models

octagon-dice



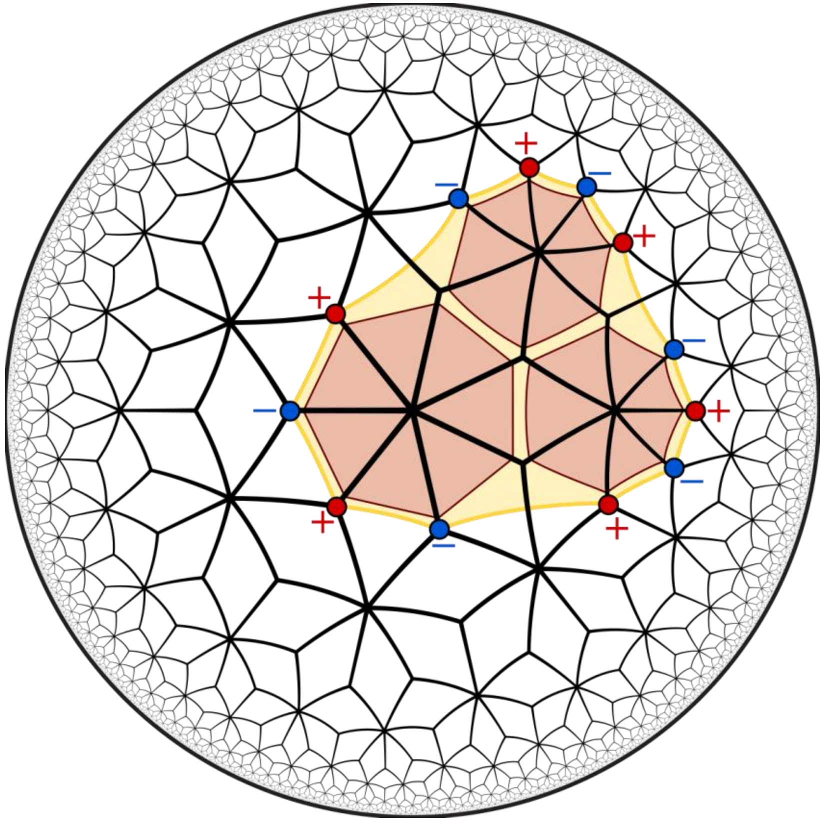
frac = 5/11
touching = 2

heptagon-kagome



frac = 1/3
touching = 0

heptagon-dice



frac = 2/5
touching = 0

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

Other hyperbolic frustrated-hopping models

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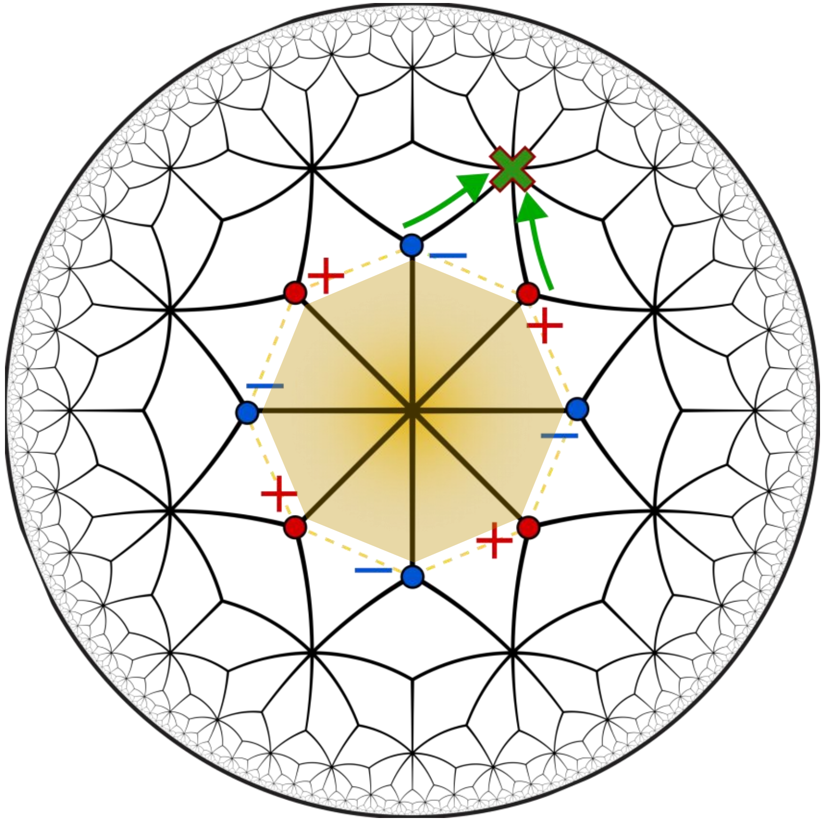

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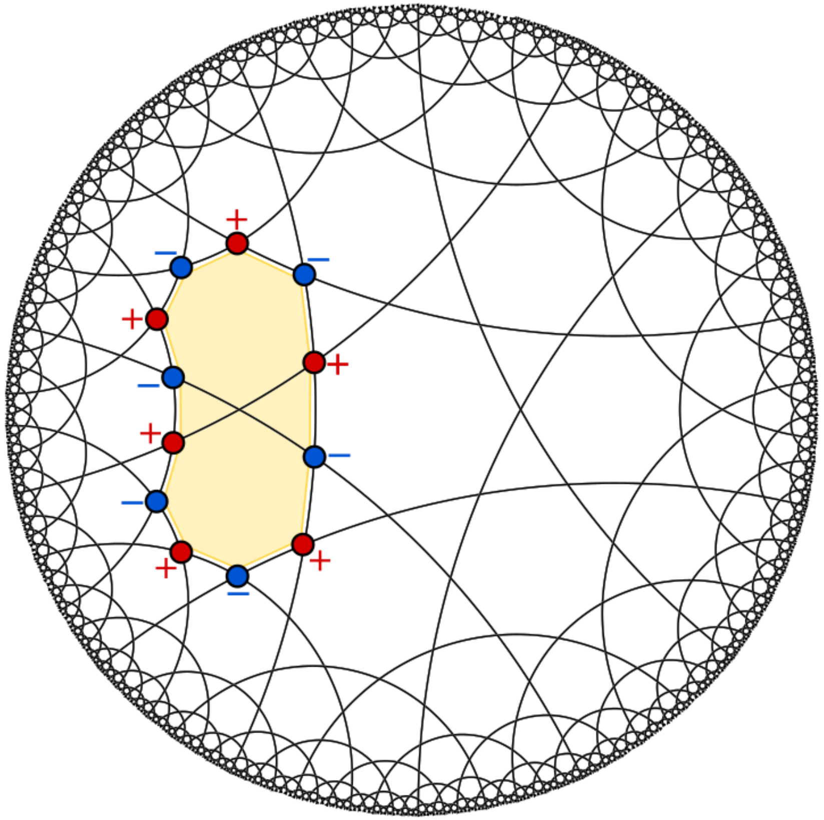

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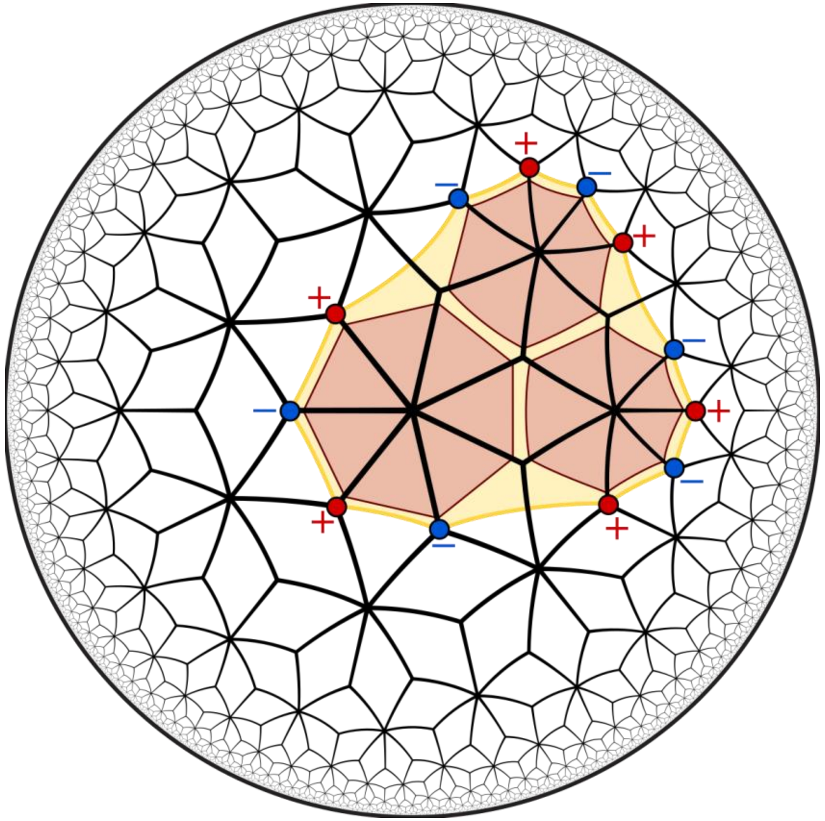

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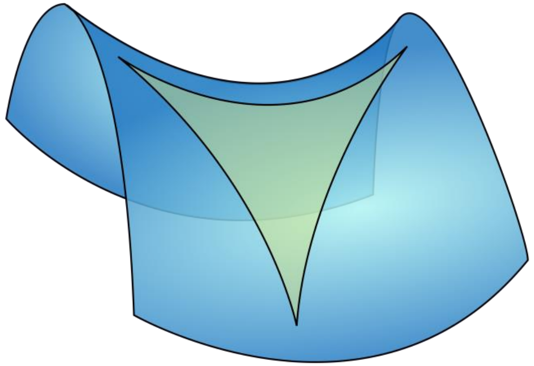
touching = 0

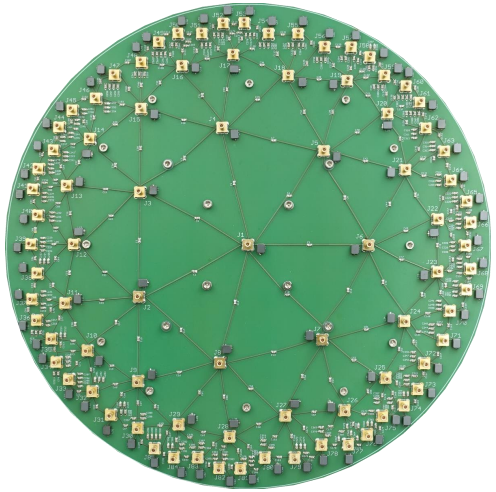
i.e. the flat band in these is *gapped!*

TB and Joseph Maciejko, arXiv:2205.11571 (2022)

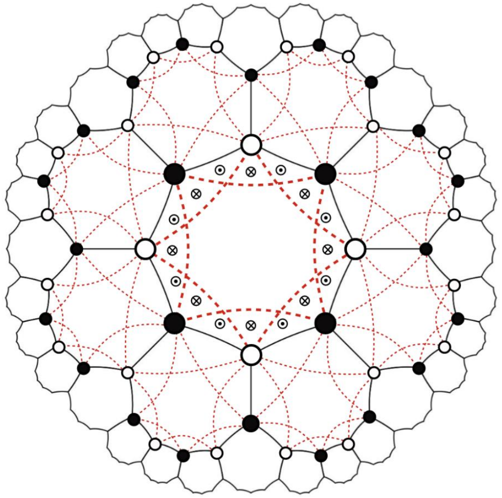
Summary



Negative curvature

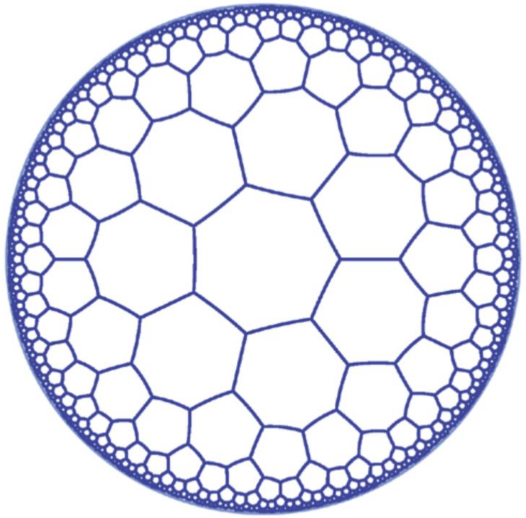


Hyperbolic continuum

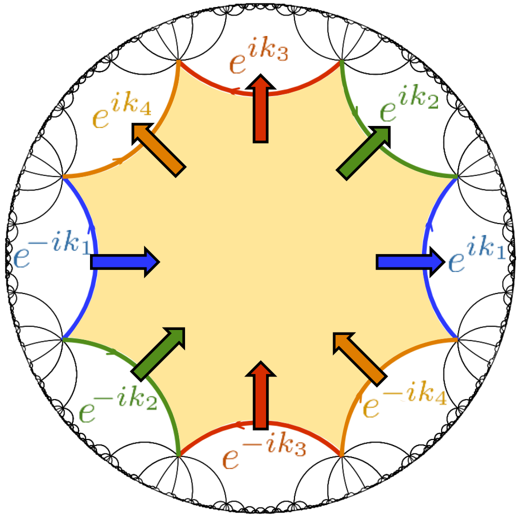


Hyperbolic Haldane model

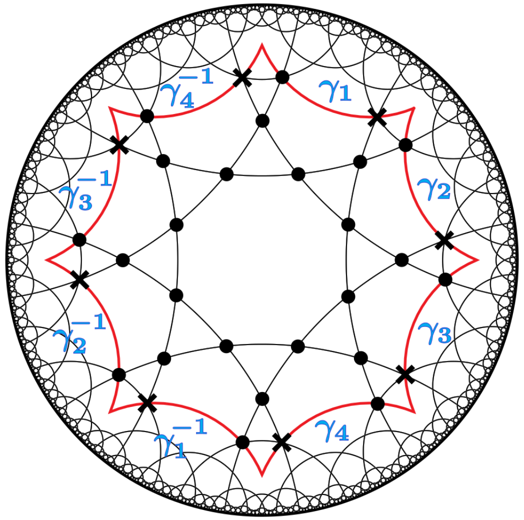
Hyperbolic {*p*,*q*} lattices



Hyperbolic band theory



Hyperbolic flat bands



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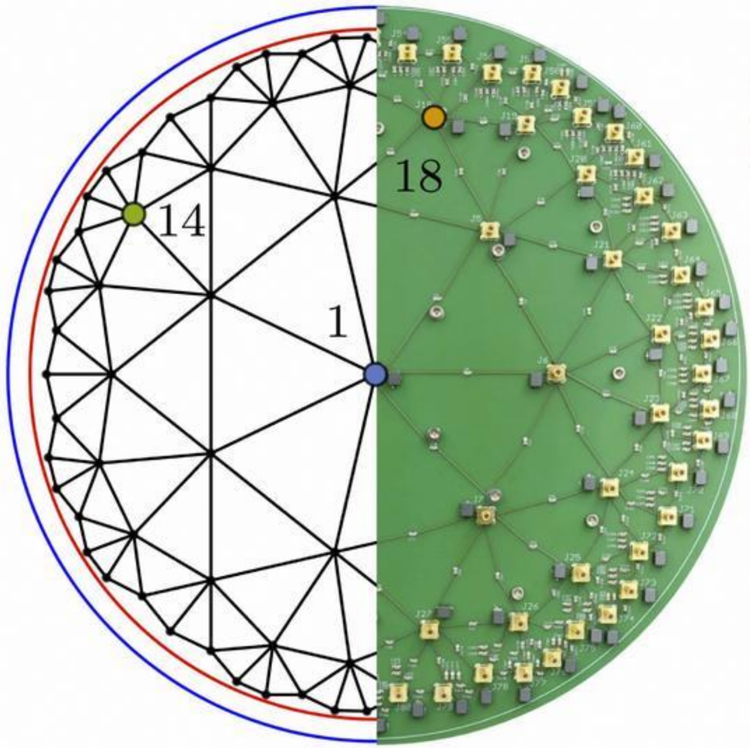

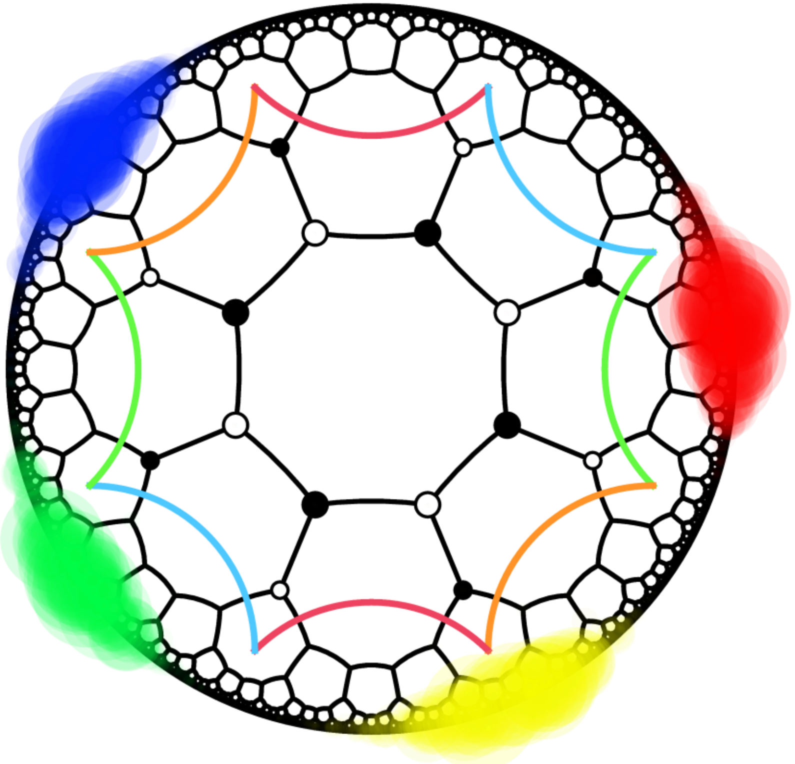

Tomáš Bzdušek: ***From hyperbolic drum towards hyperbolic topological matter***

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20. October, 2022:

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Hyperbolic drum: Nat. Commun. 13, 4373 (2022)

Mapping 4D *k*-space: arXiv:2205.05106 (2022)

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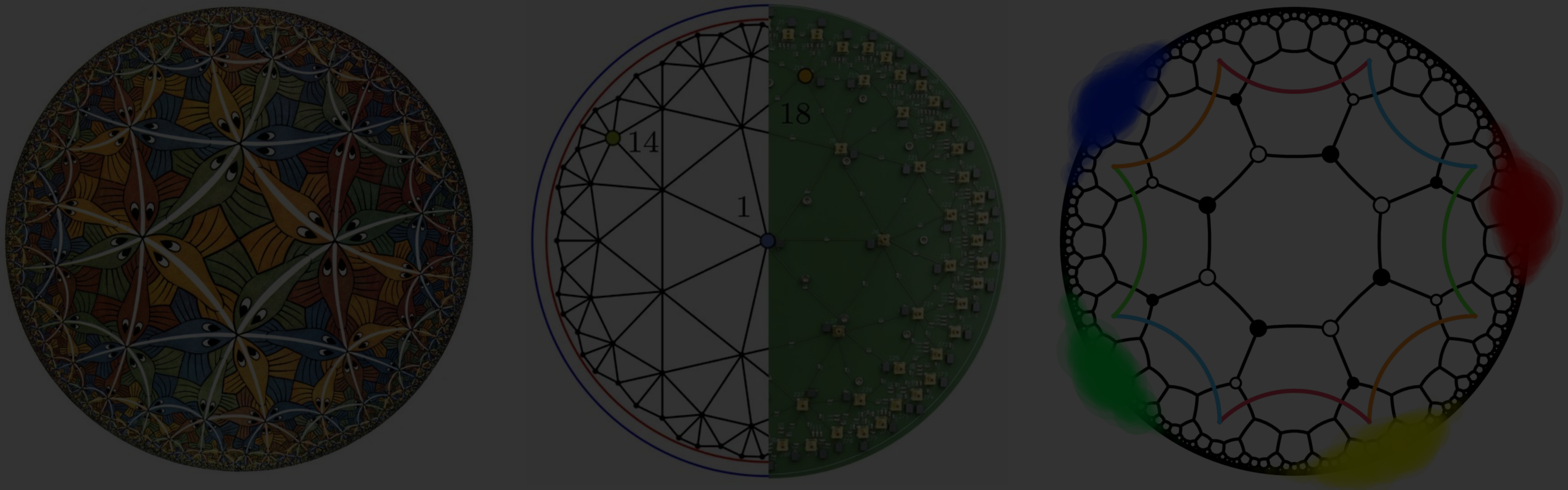

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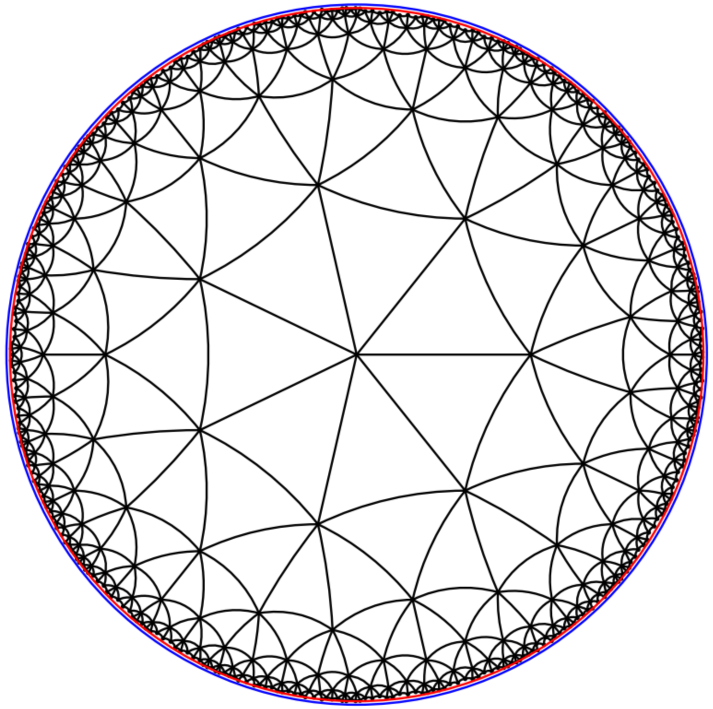
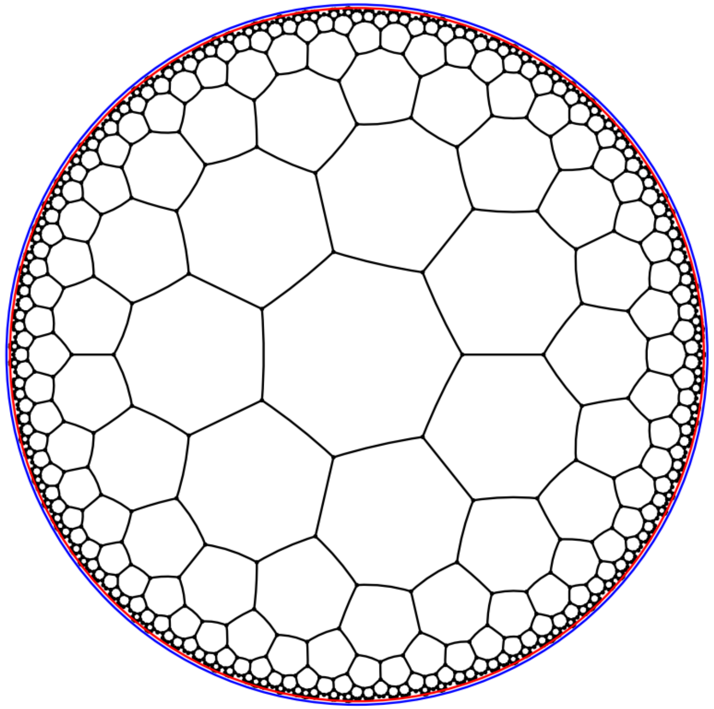
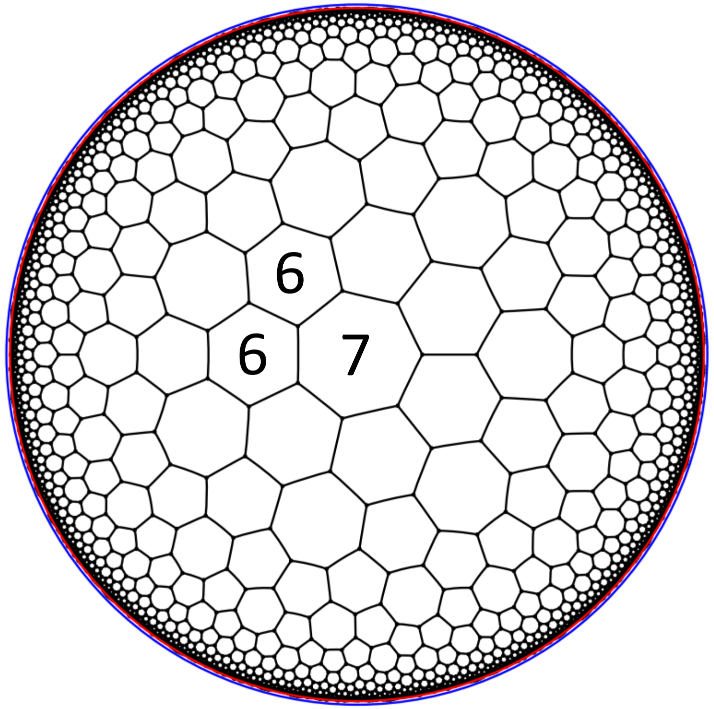
Modelling of a “hyperbolic drum” ($R_0 = 0.99$)



{3,7}



{7,3}



“hyperbolic soccerball” t{3,7}

P. M. Lenggenhager, A. Stegmaier, *et al.*, arXiv:2109.01148 (2021)

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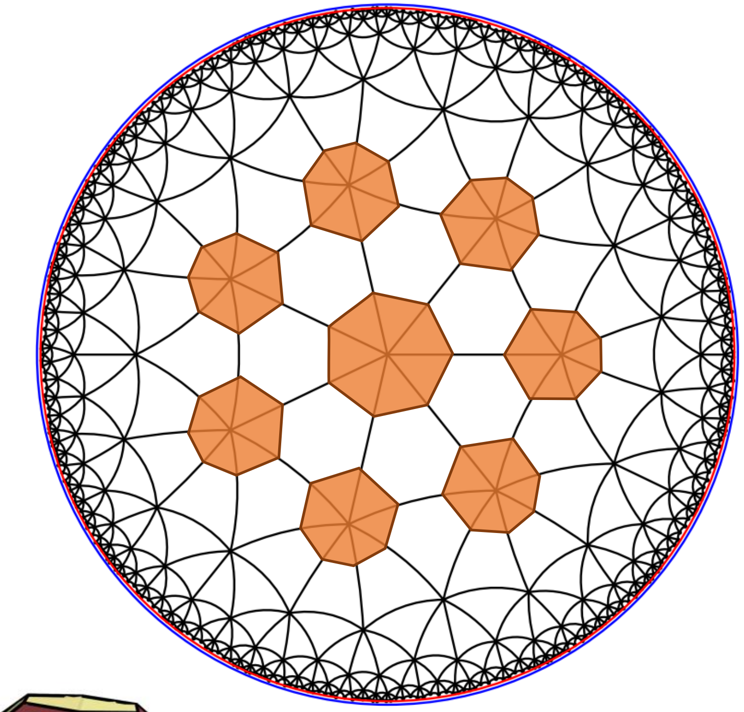

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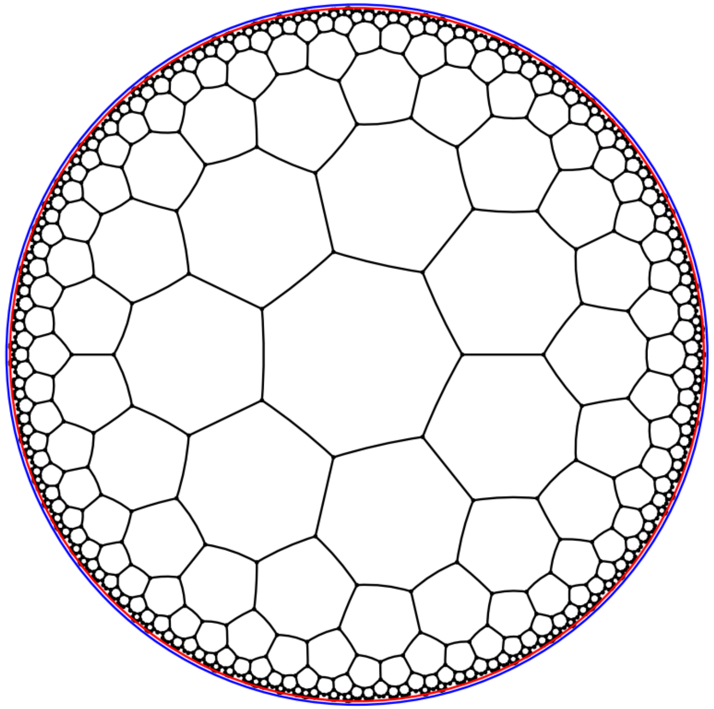

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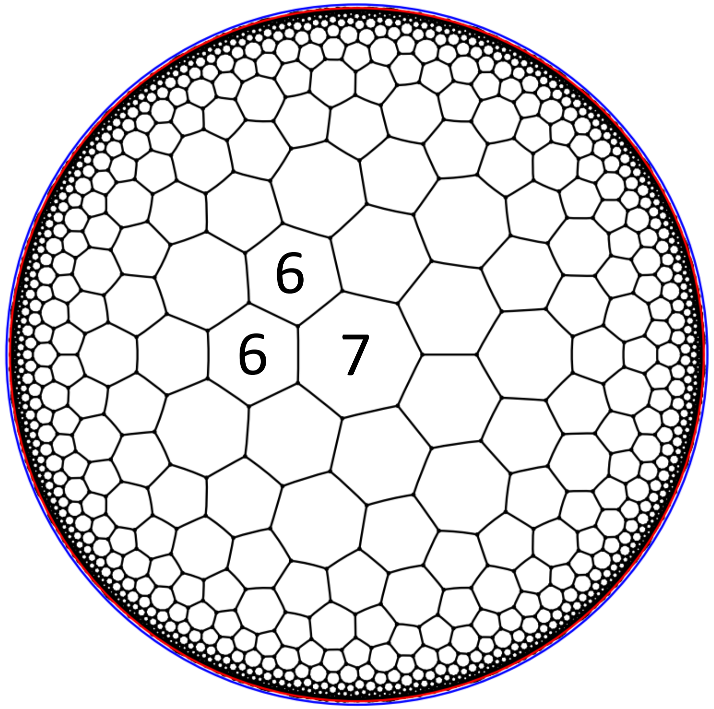
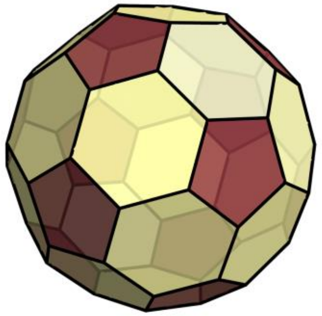
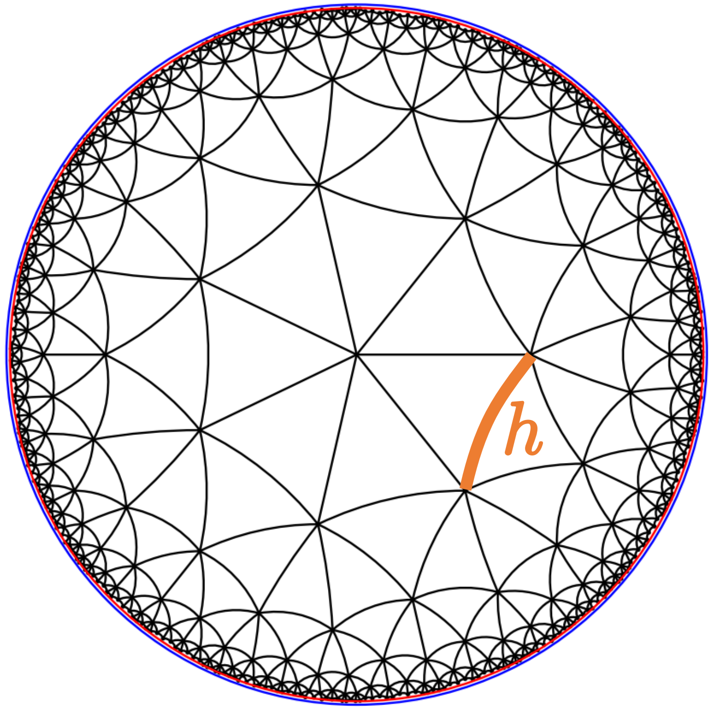


“hyperbolic soccerball” t{3,7}

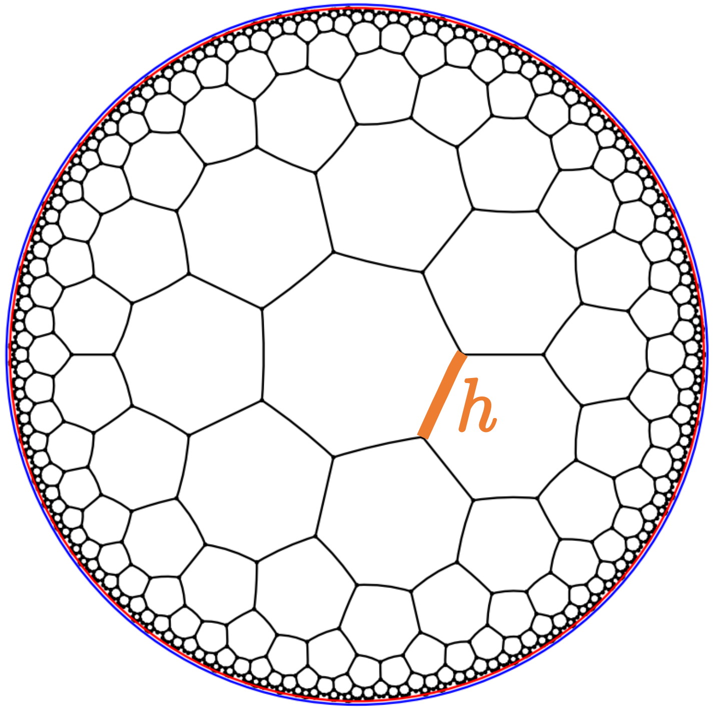


P. M. Lenggenhager, A. Stegmaier, *et al.*, arXiv:2109.01148 (2021)

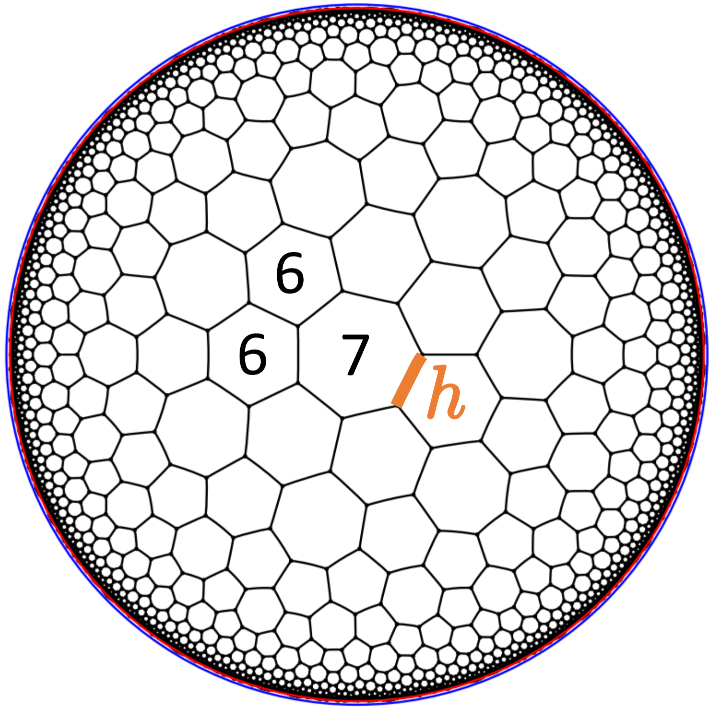
Modelling of a “hyperbolic drum” ($R_0 = 0.99$)



{3,7}



{7,3}



“hyperbolic soccerball” t{3,7}

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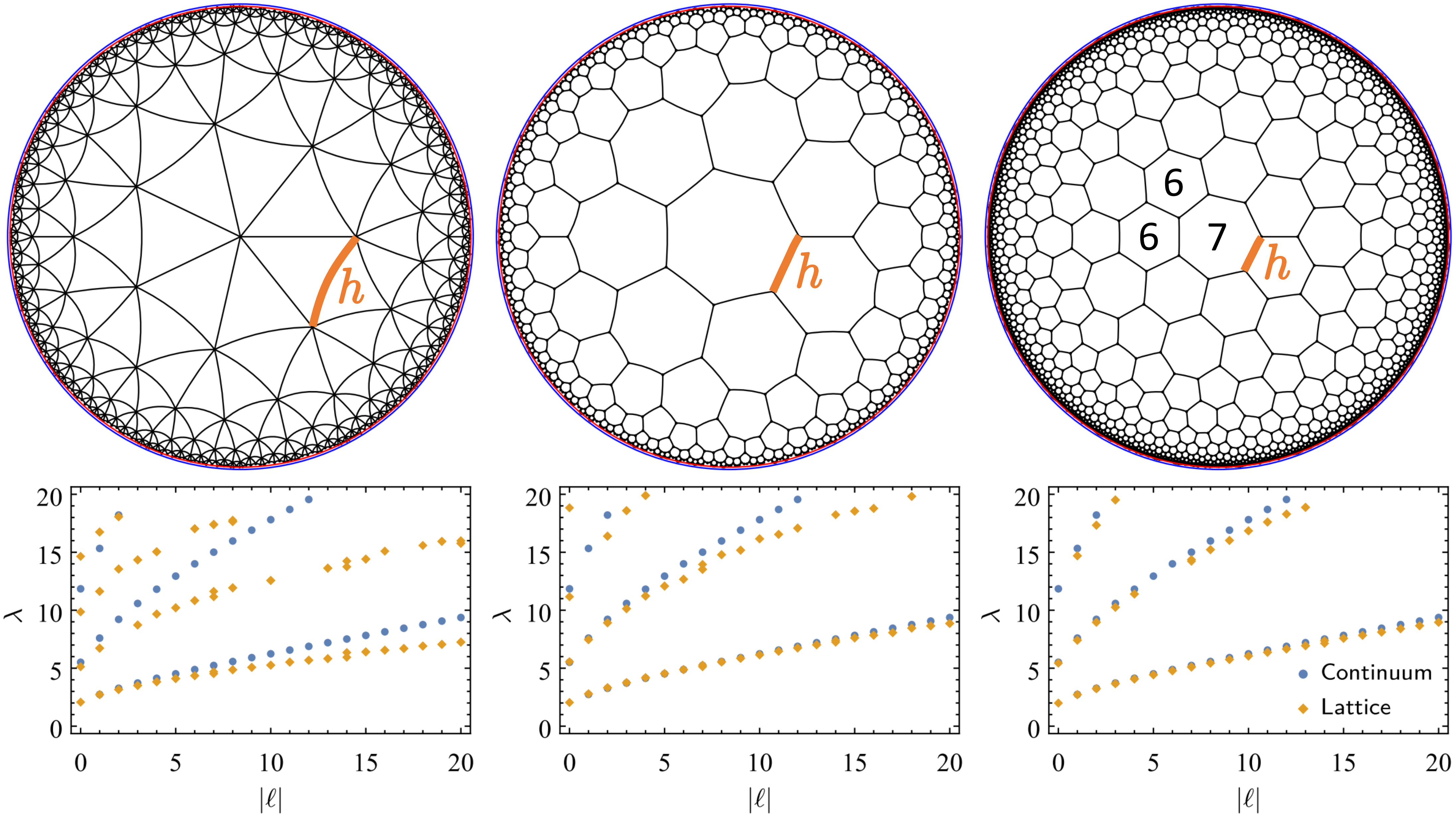

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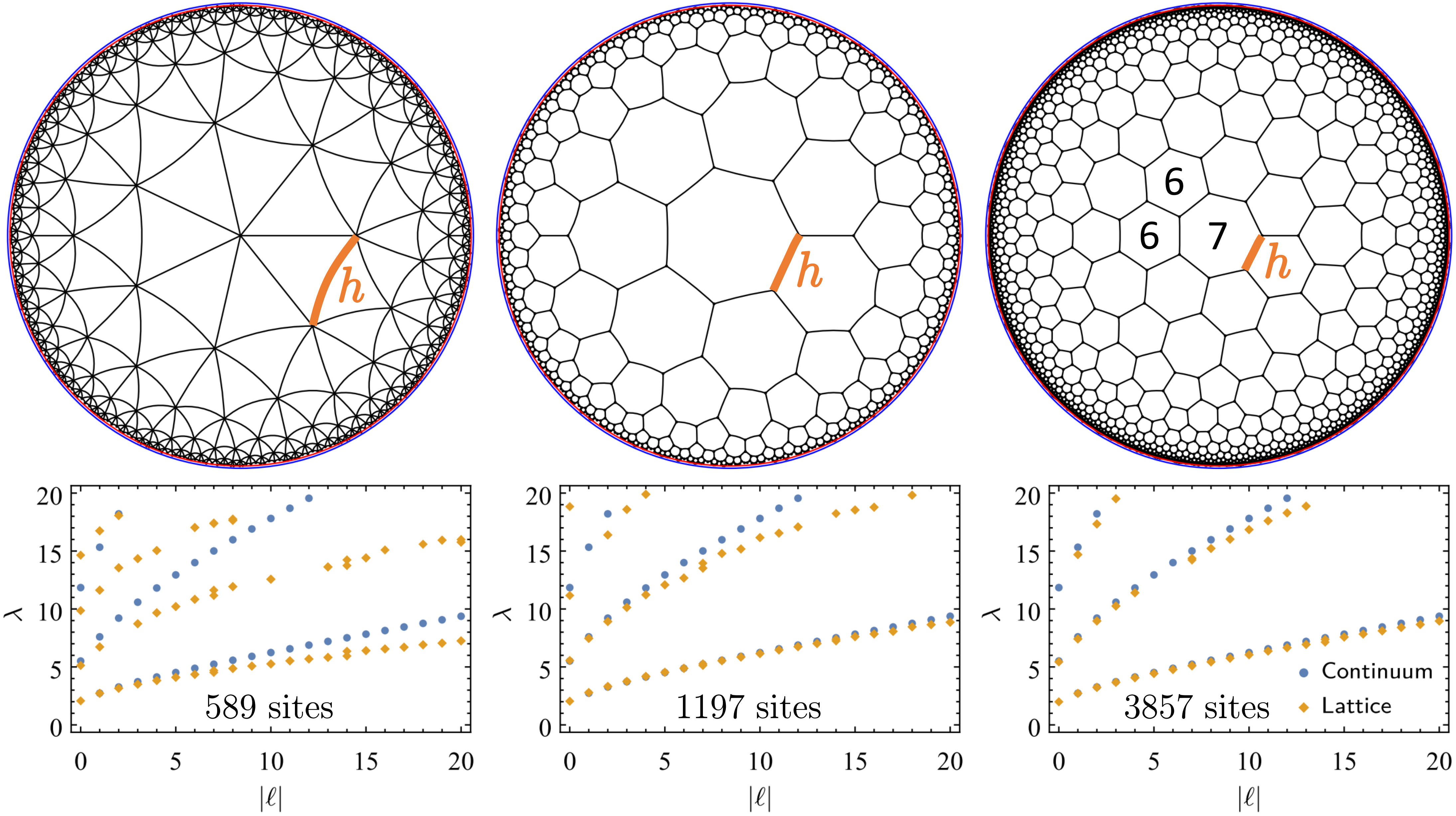
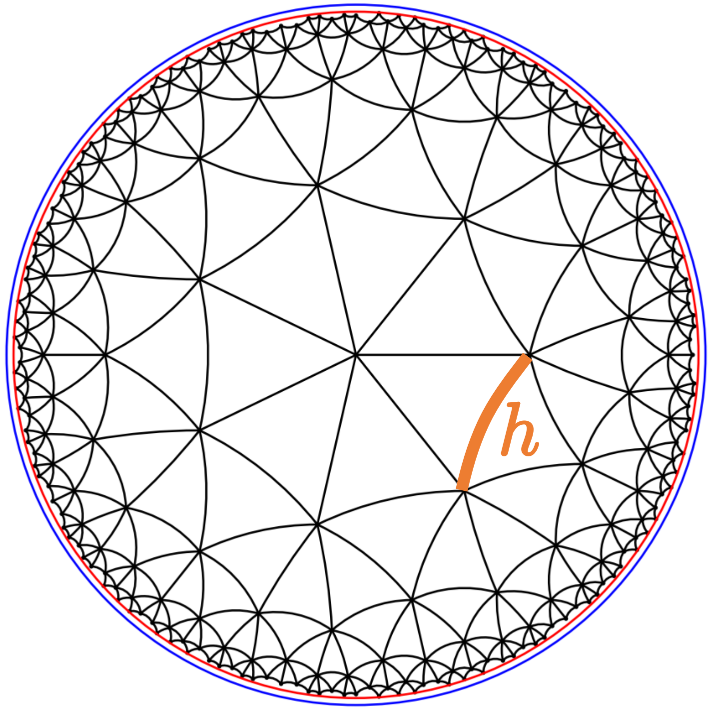
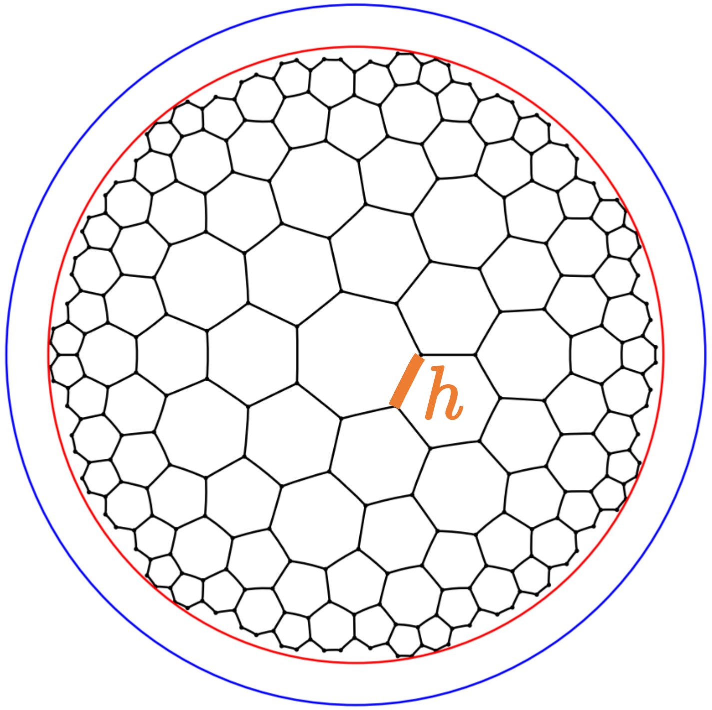


P. M. Lenggenhager, A. Stegmaier, *et al.*, arXiv:2109.01148 (2021)

… with a fixed number of 275 sites





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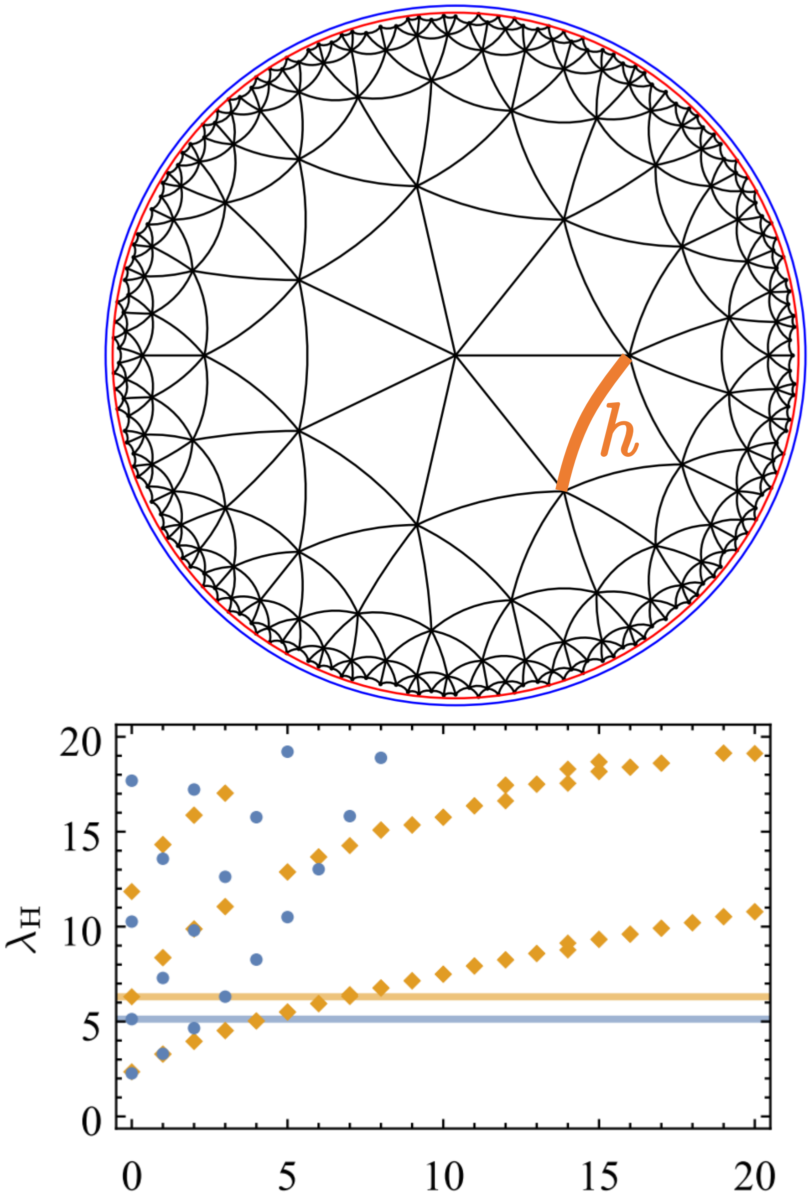

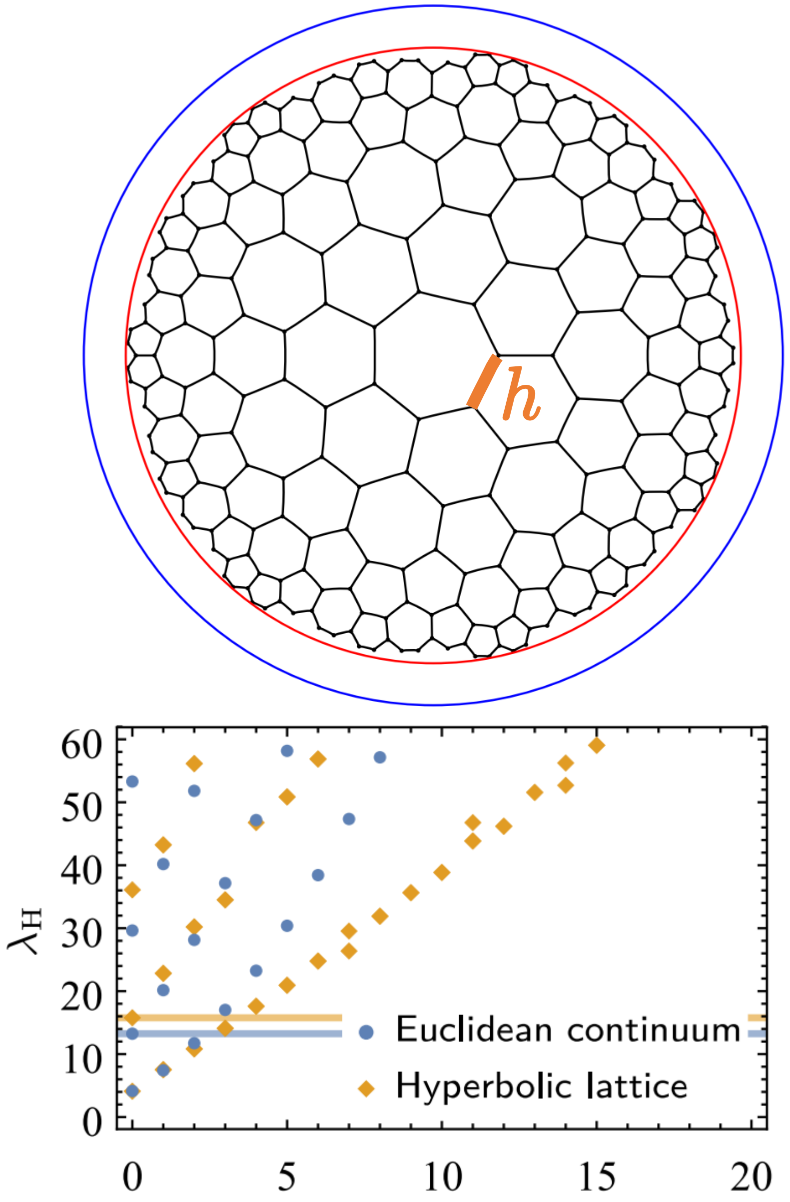

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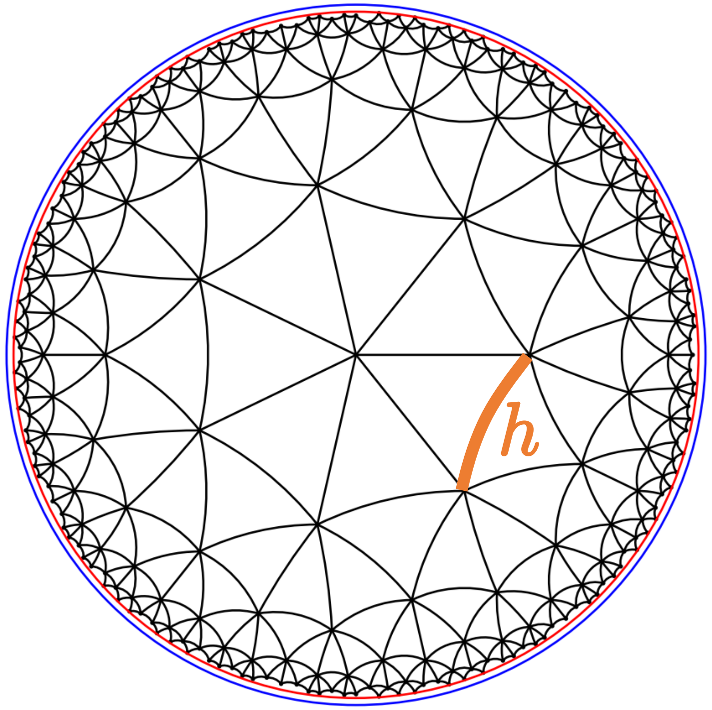
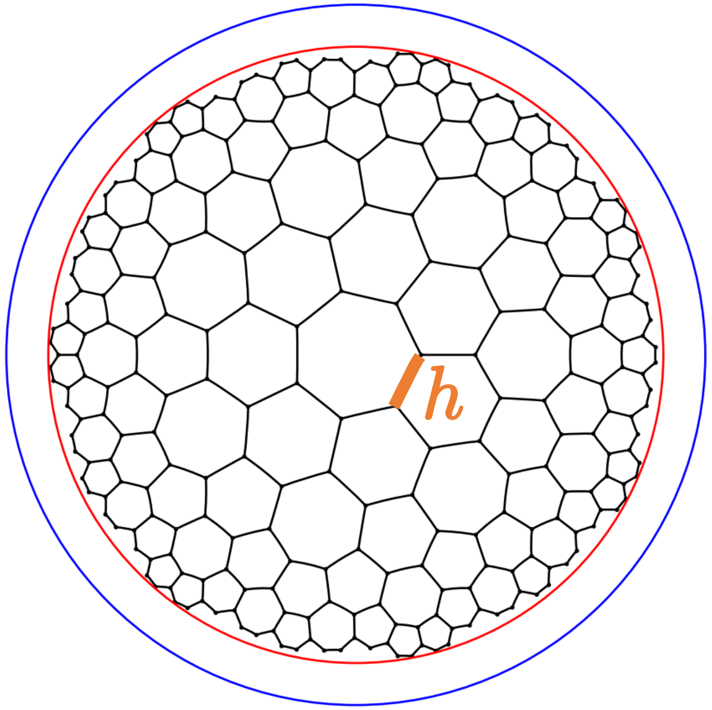
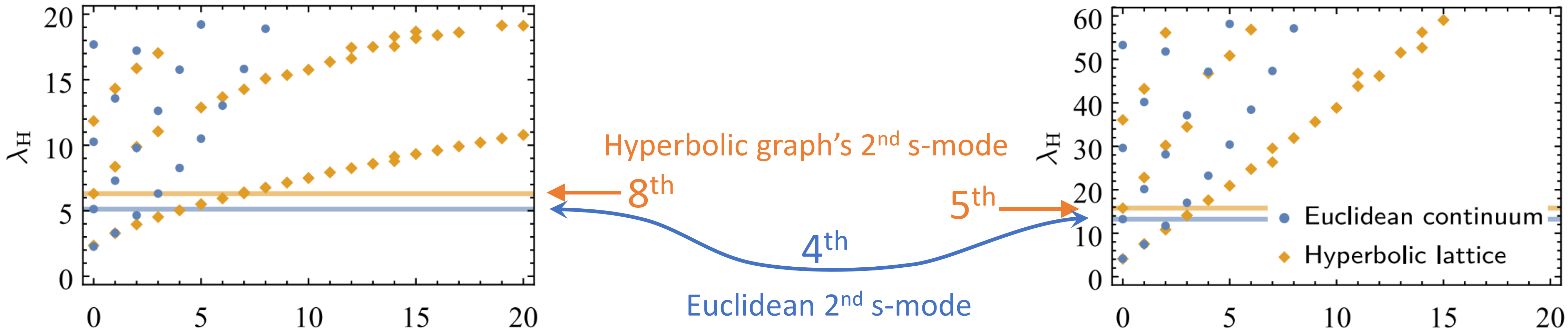






Hyperbolic graph's 2nd s-mode

8th

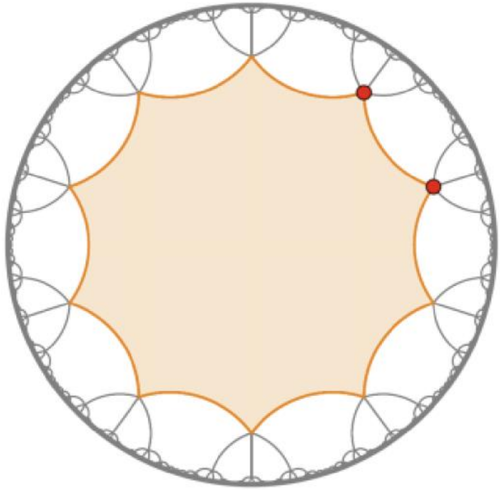
5th

4th

Euclidean 2nd s-mode

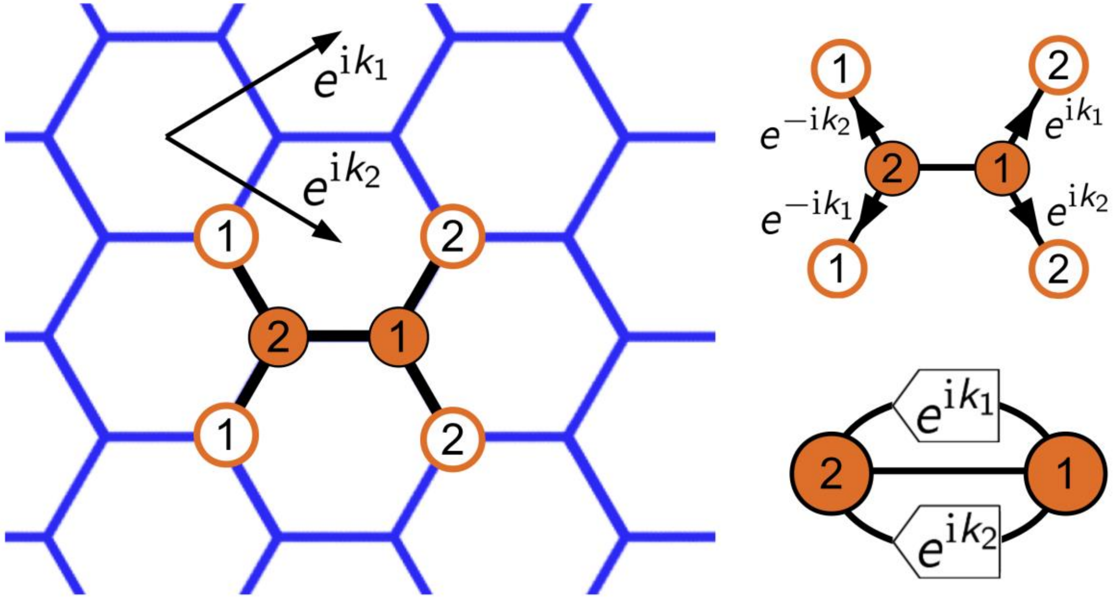
P. M. Lenggenhager, A. Stegmaier, *et al.*, arXiv:2109.01148 (2021)

Mapping out the spectrum in 4D momentum space



{10,5}

4D BZ



Graphene
on {6,3} lattice

A. Chen, TB, *et al.*, *Hyperbolic Matter in Electrical Circuits with Tunable Complex Phases*, arXiv:2205.05106 (2022)

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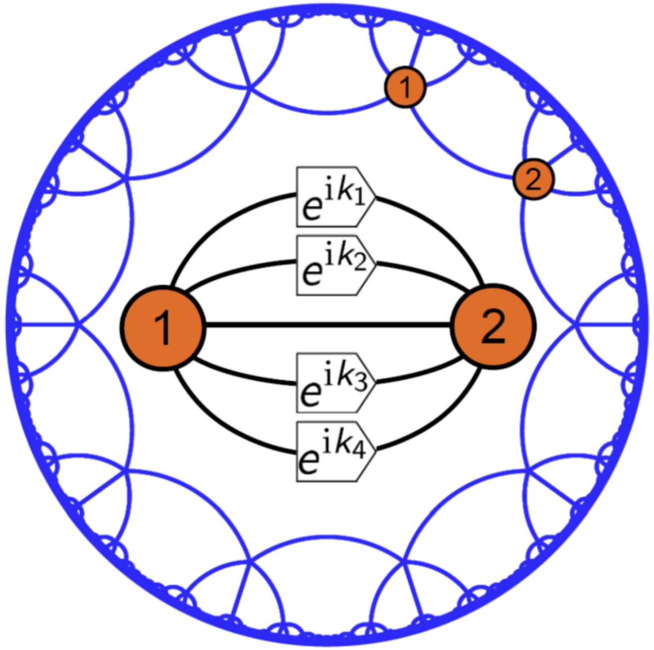
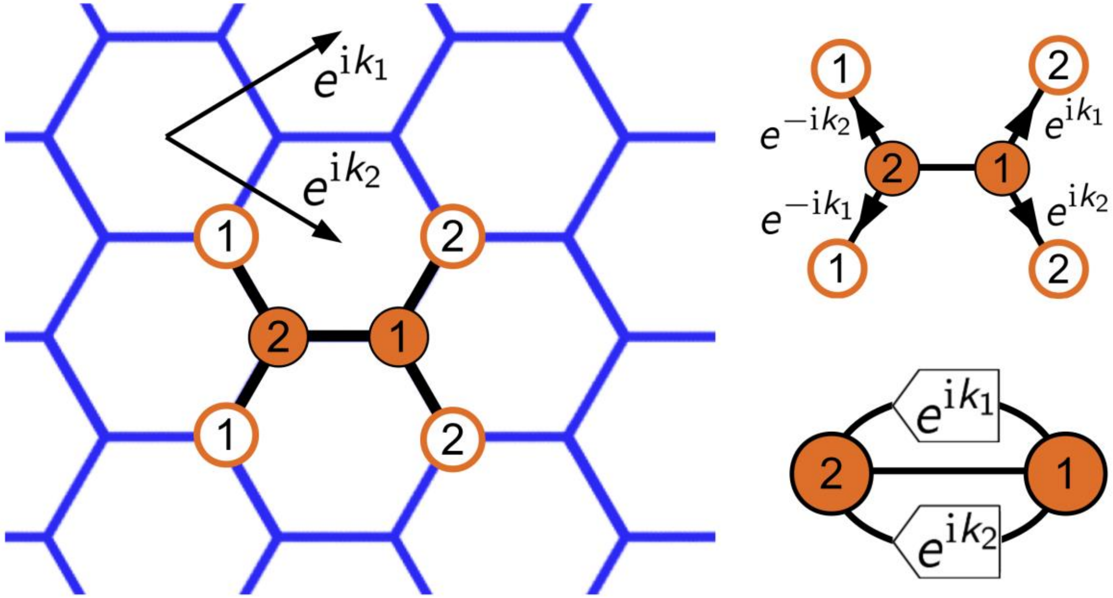


“Hyperbolic graphene”
on {10,5} lattice



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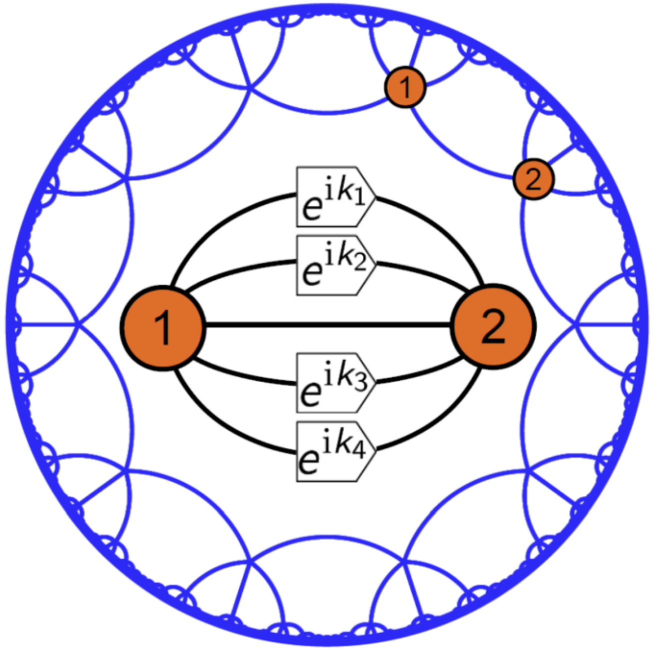

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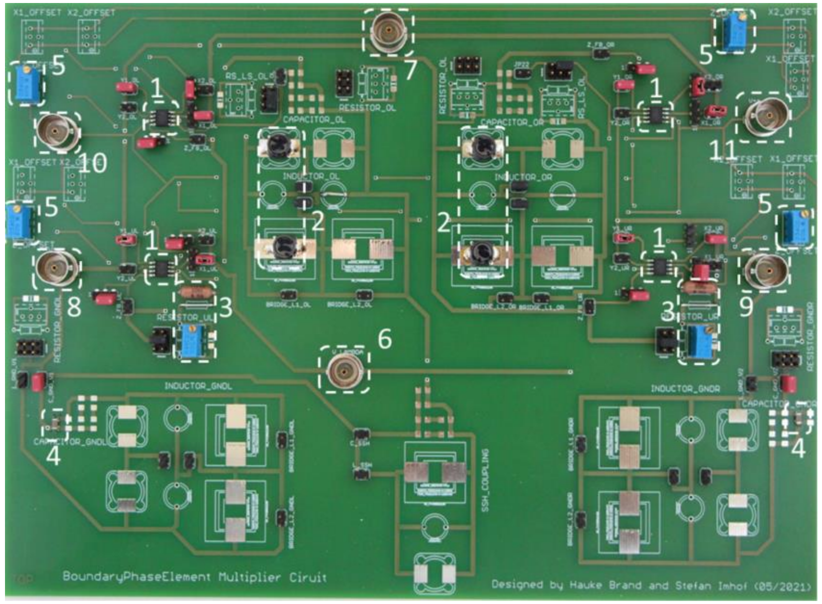

The model as a circuit
with tunable k_1, k_2, k_3, k_4.

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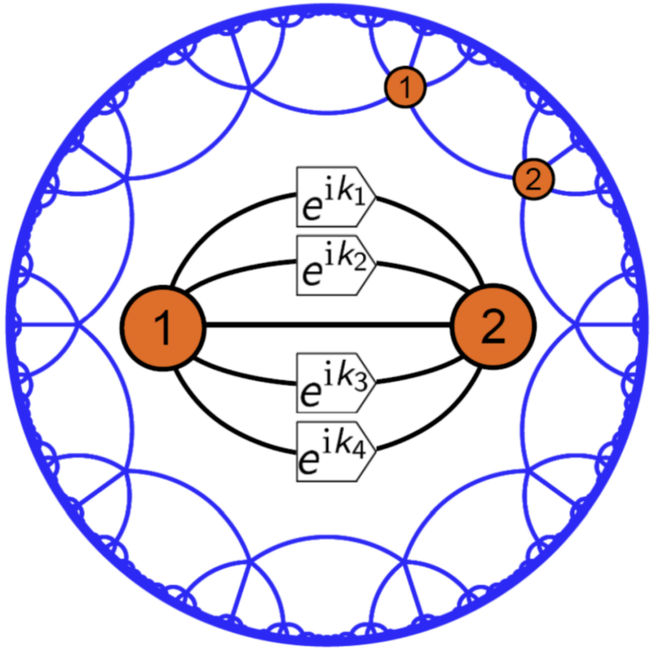

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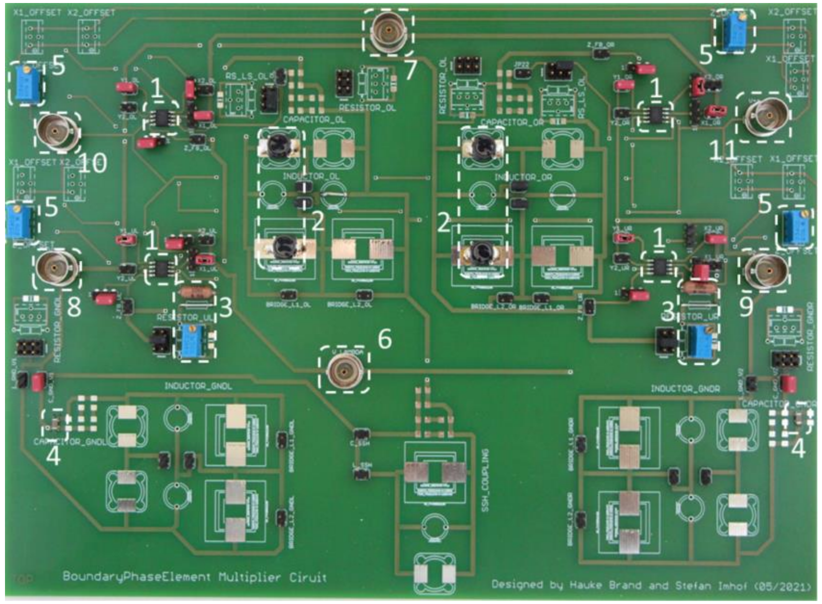

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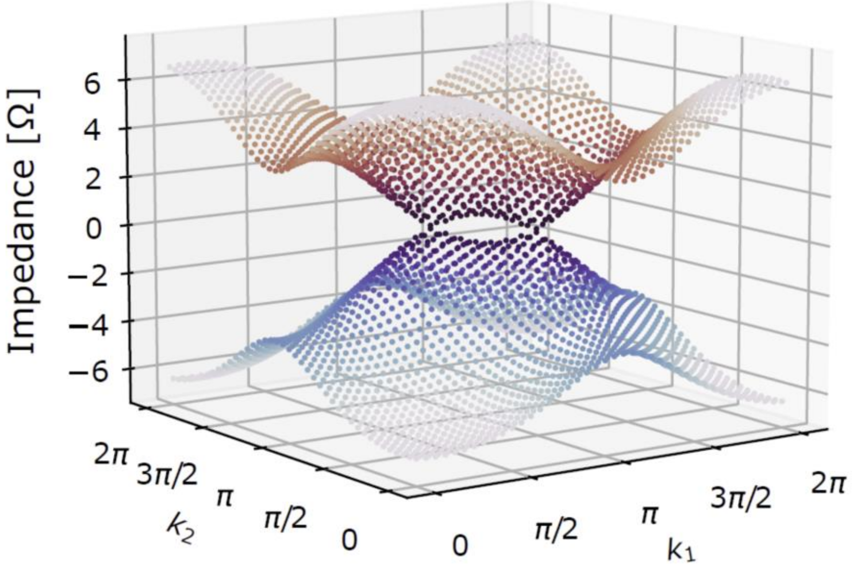
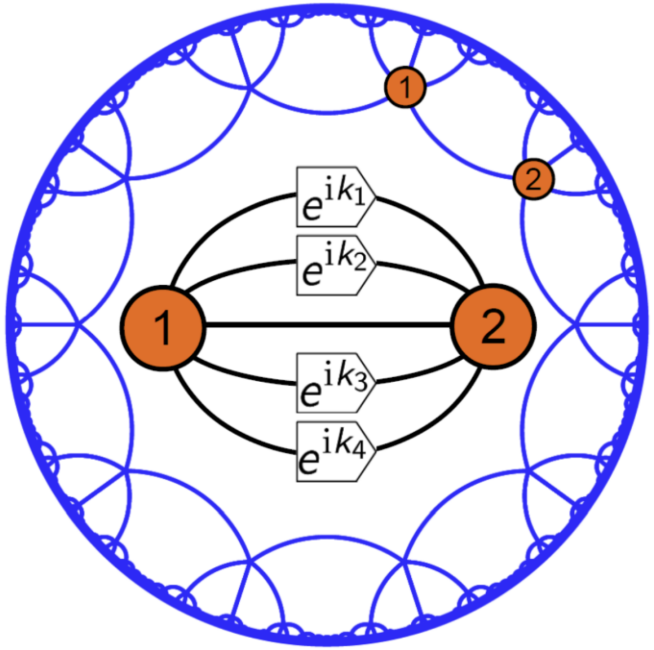


Measured spectrum in momentum space

A. Chen, TB, *et al.*, *Hyperbolic Matter in Electrical Circuits with Tunable Complex Phases*, arXiv:2205.05106 (2022)

Mapping out the spectrum in 4D momentum space



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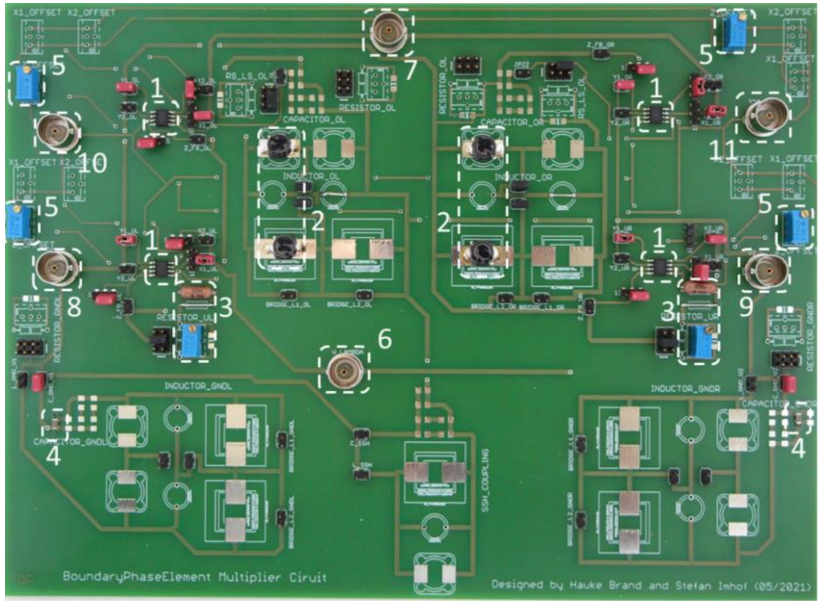

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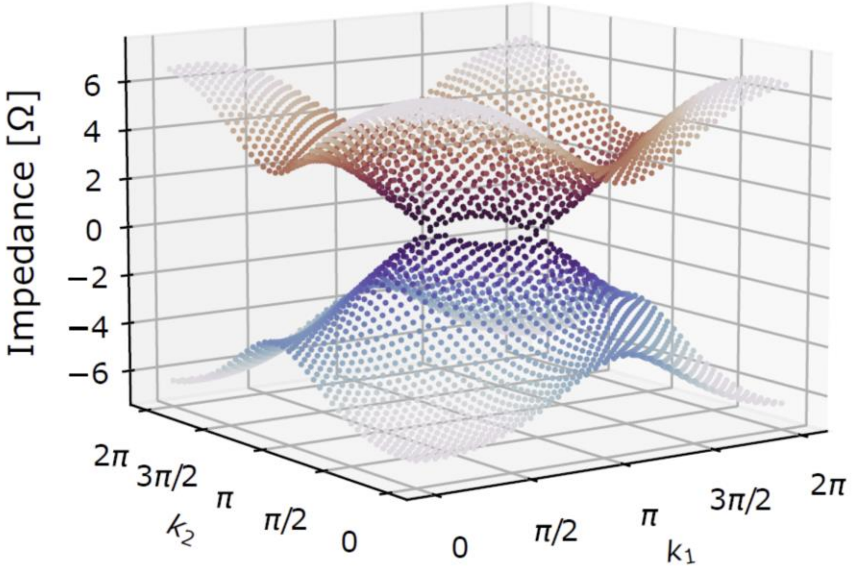
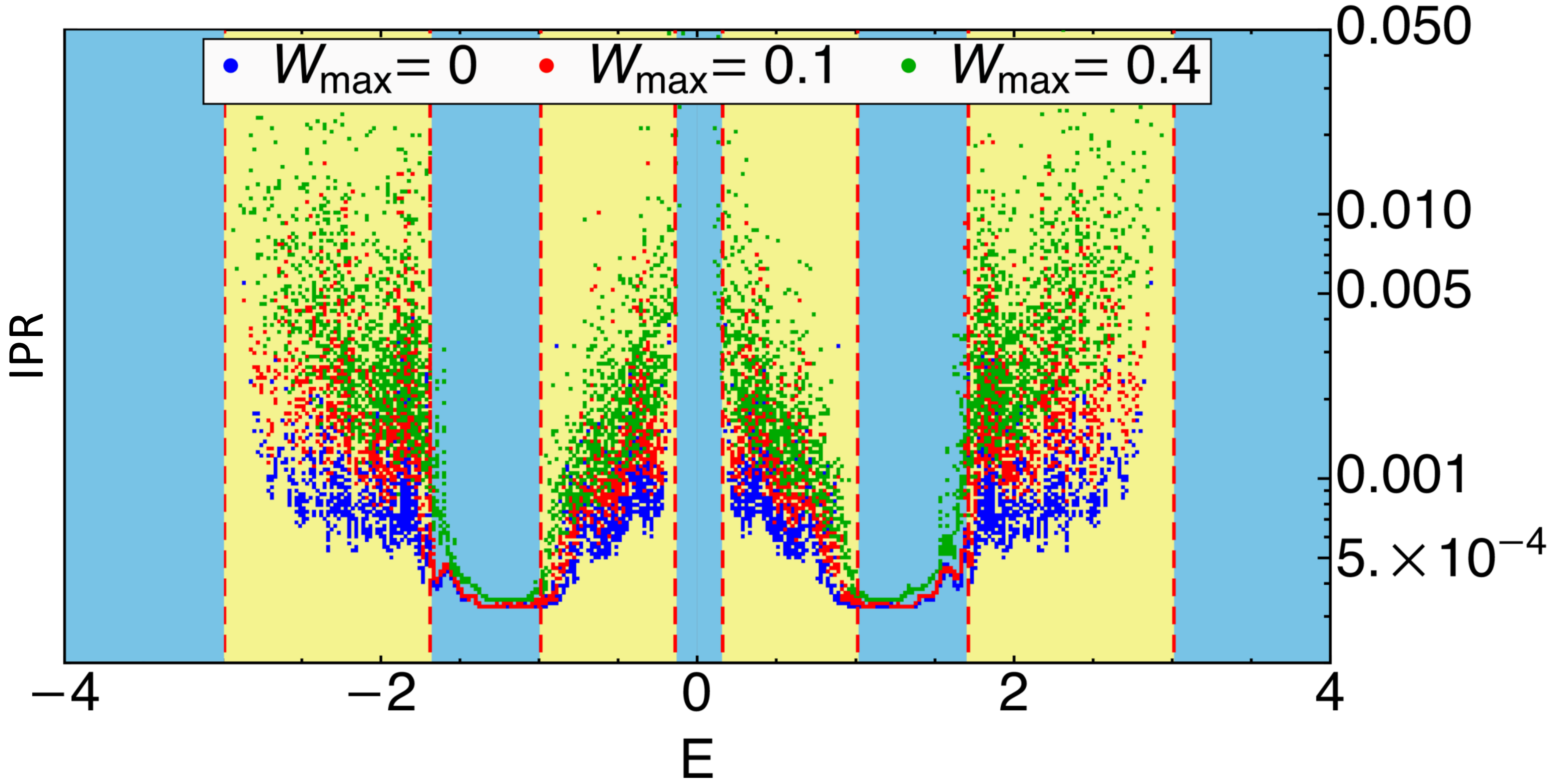


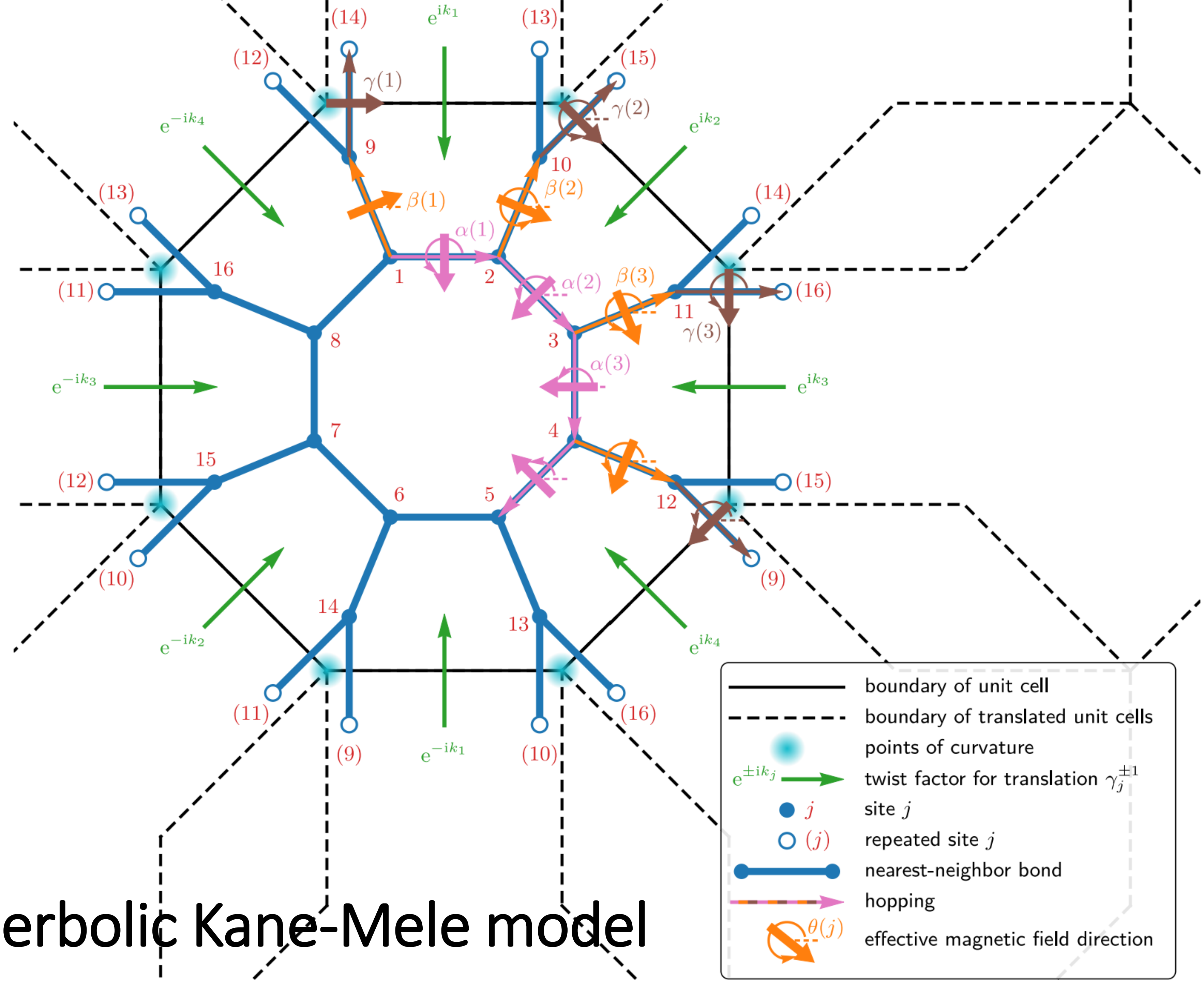
Measured spectrum in
momentum space

BUT! – The hyperbolic translation group is non-Abelian and also has *Brillouin zones of higher-dimensional representations*!

A. Chen, TB, *et al.*, *Hyperbolic Matter in Electrical Circuits with Tunable Complex Phases*, arXiv:2205.05106 (2022)

Effect of random on-site potential on HH model





The “reduced” hyperbolic Kane-Mele model

Hyperbolic Kane-Mele model

Study Z_2 topology protected by time-reversal symmetry.

D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

Hyperbolic Kane-Mele model

Study Z_2 topology protected by time-reversal symmetry.

We fix $t_1 = 1$, $t_2 = 1/6$, $M = 1/3$, $\phi = \pi/2$ and Rashba term $\lambda_\mathrm{R} =$ -$1/6$.

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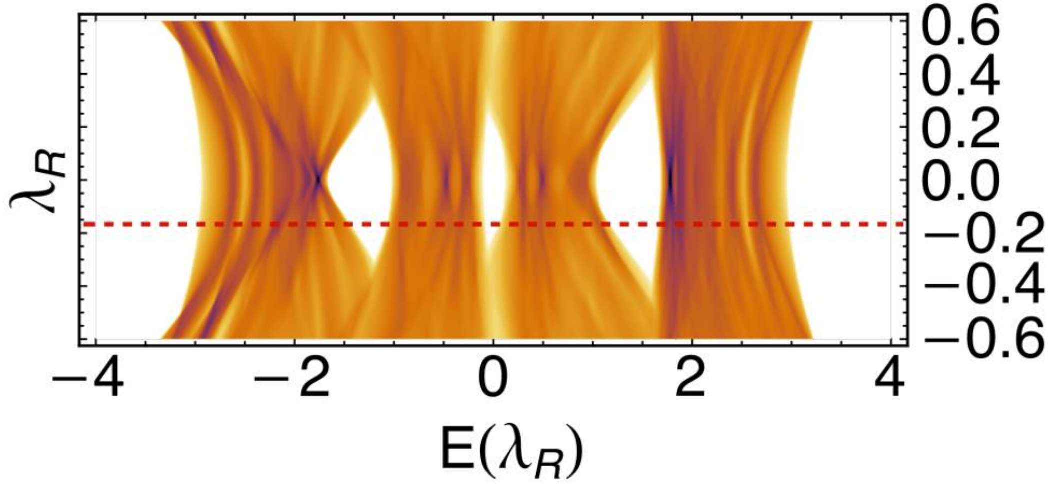

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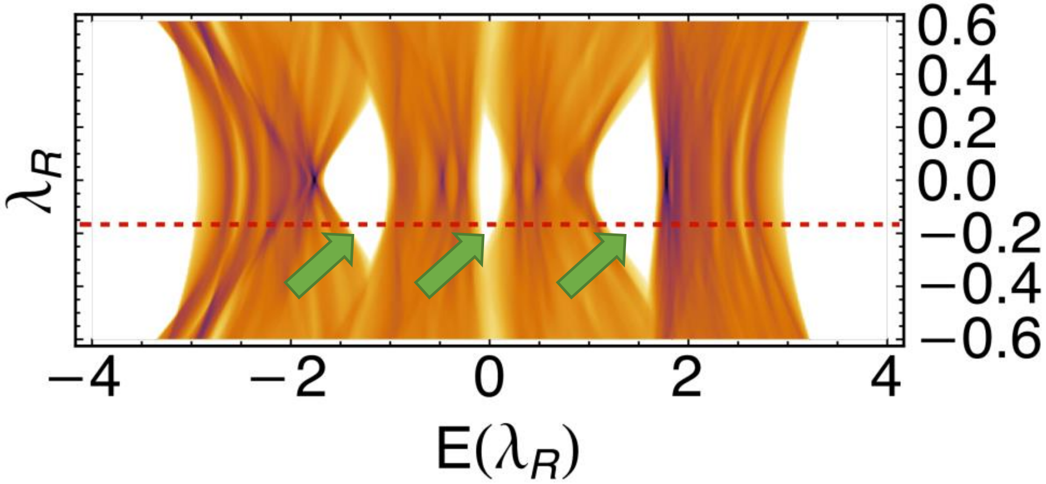

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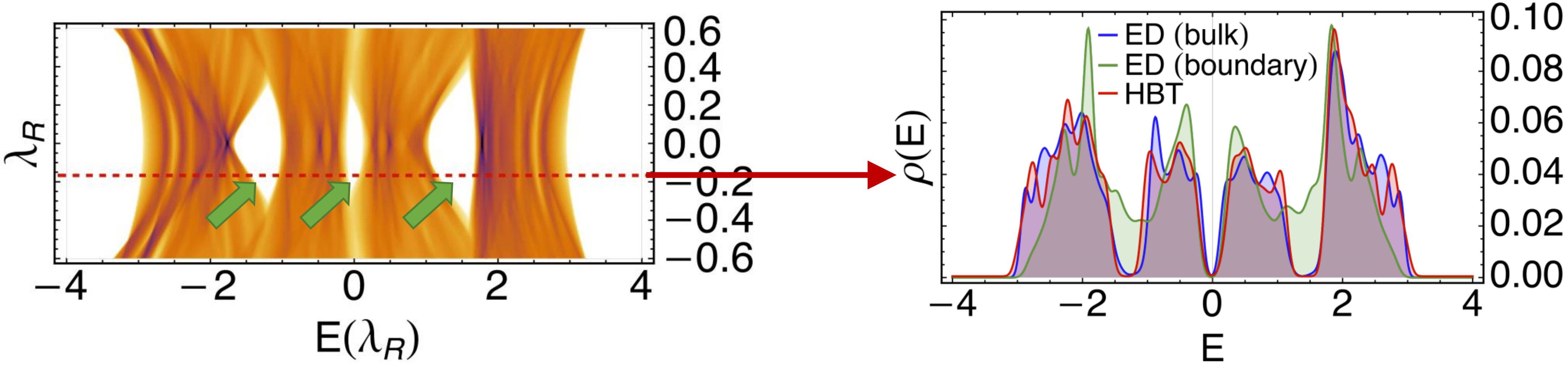

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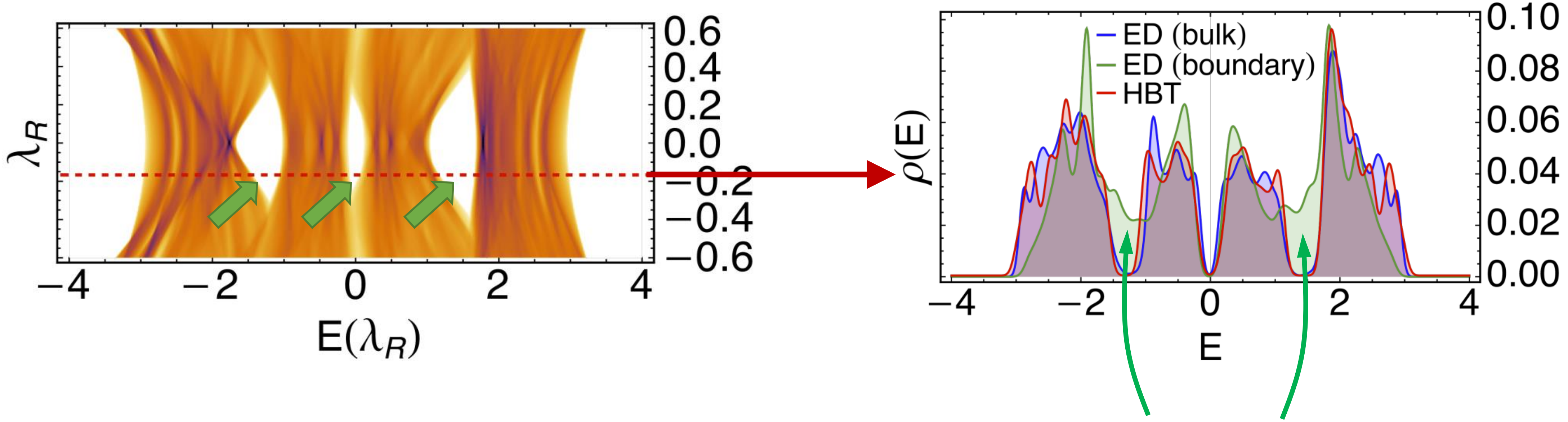
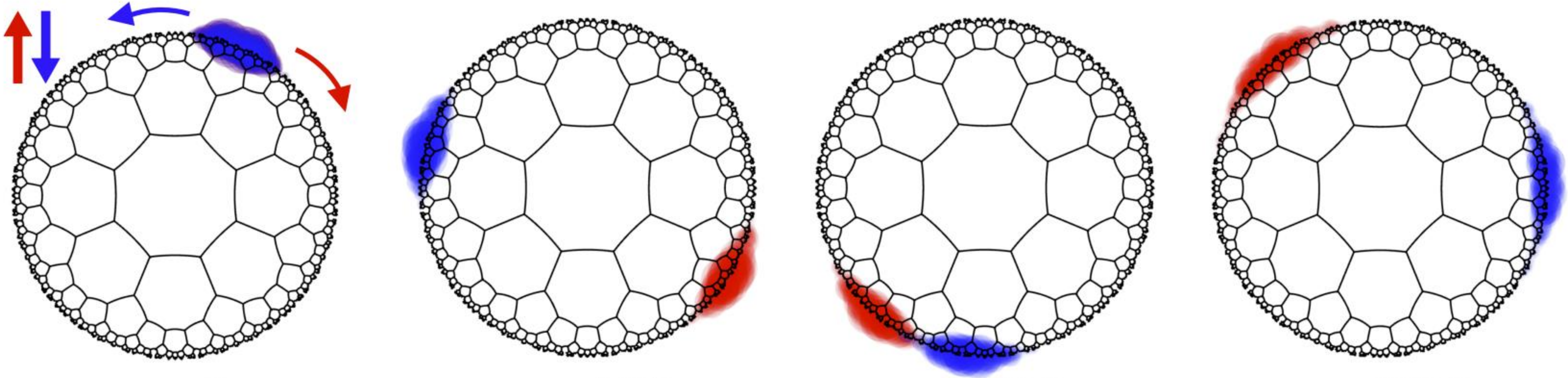


Non-trivial Kane-Mele (Z_2) invariant:

- In all six 2D planes of the 4D $\boldsymbol{k}$-space.
- According to real-space topological marker.

D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

Helical edge states on the hyperbolic boundary



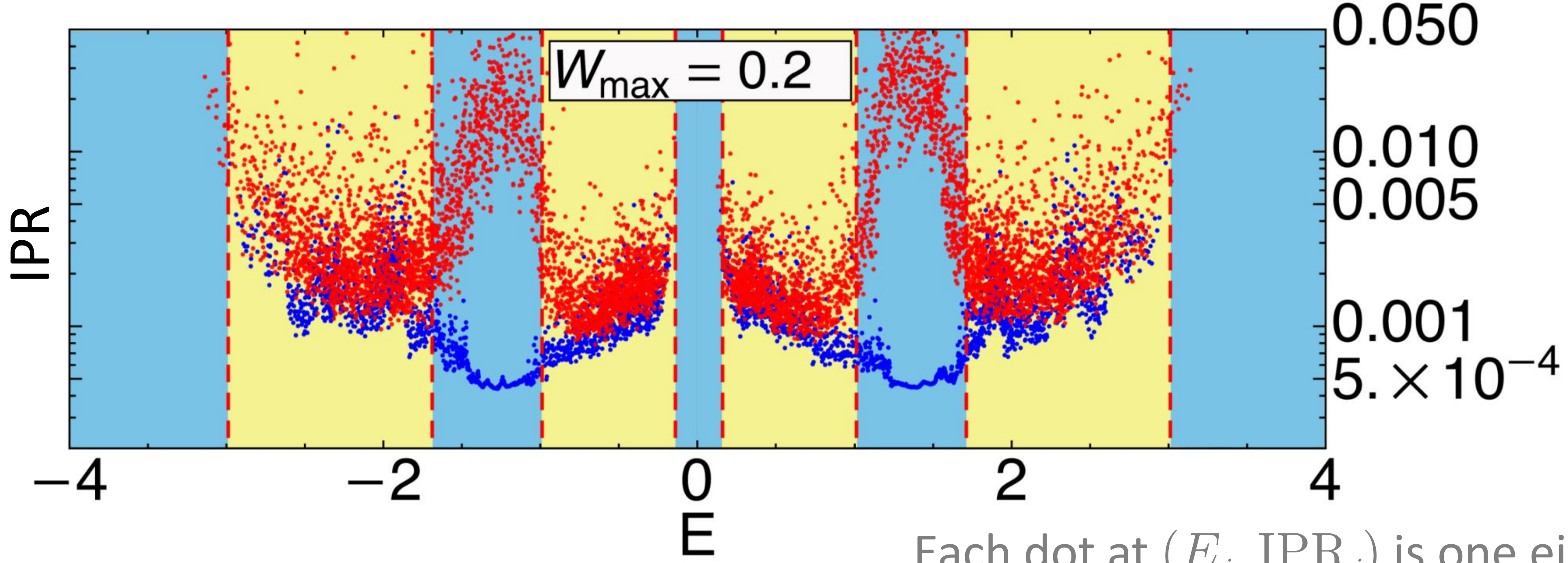
$\tau = 0$ $\tau = 400$ $\tau = 800$ $\tau = 1200$

D. M. Urwyler, *et al.*, arXiv:2203.07292 (2022)

Robustness of edge states against spin disorder

We assume random spin-coupling terms on NN & NNN bonds

(localization quantified by “IPR” = inverse participation ratio:
low IPR = delocalized & high IPR = localized)



Each dot at (E_j, IPR_j) is one eigenstate.
Model: box distribution with $\lambda_{\mathrm{R,max}}$ = 0.2.

Robustness of edge states against spin disorder

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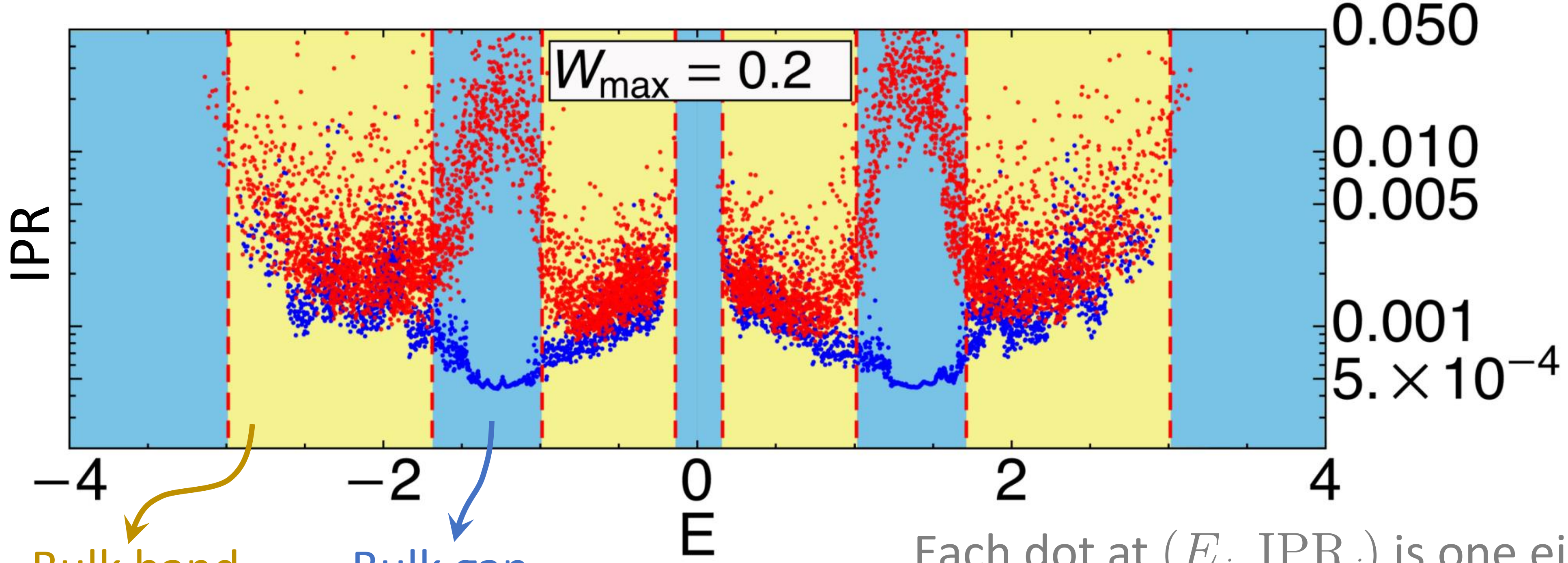
Blue data: TRS-preserving disorder.

Red data: TRS-breaking disorder.



Bulk band

Bulk gap

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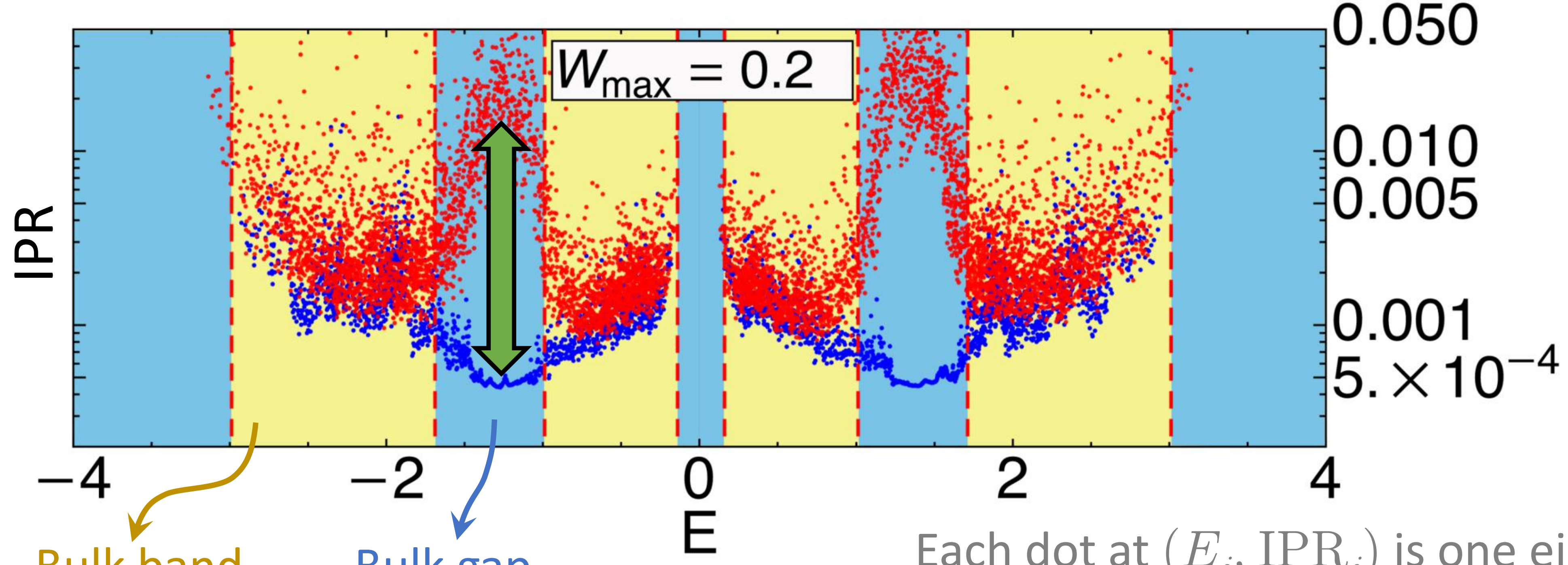

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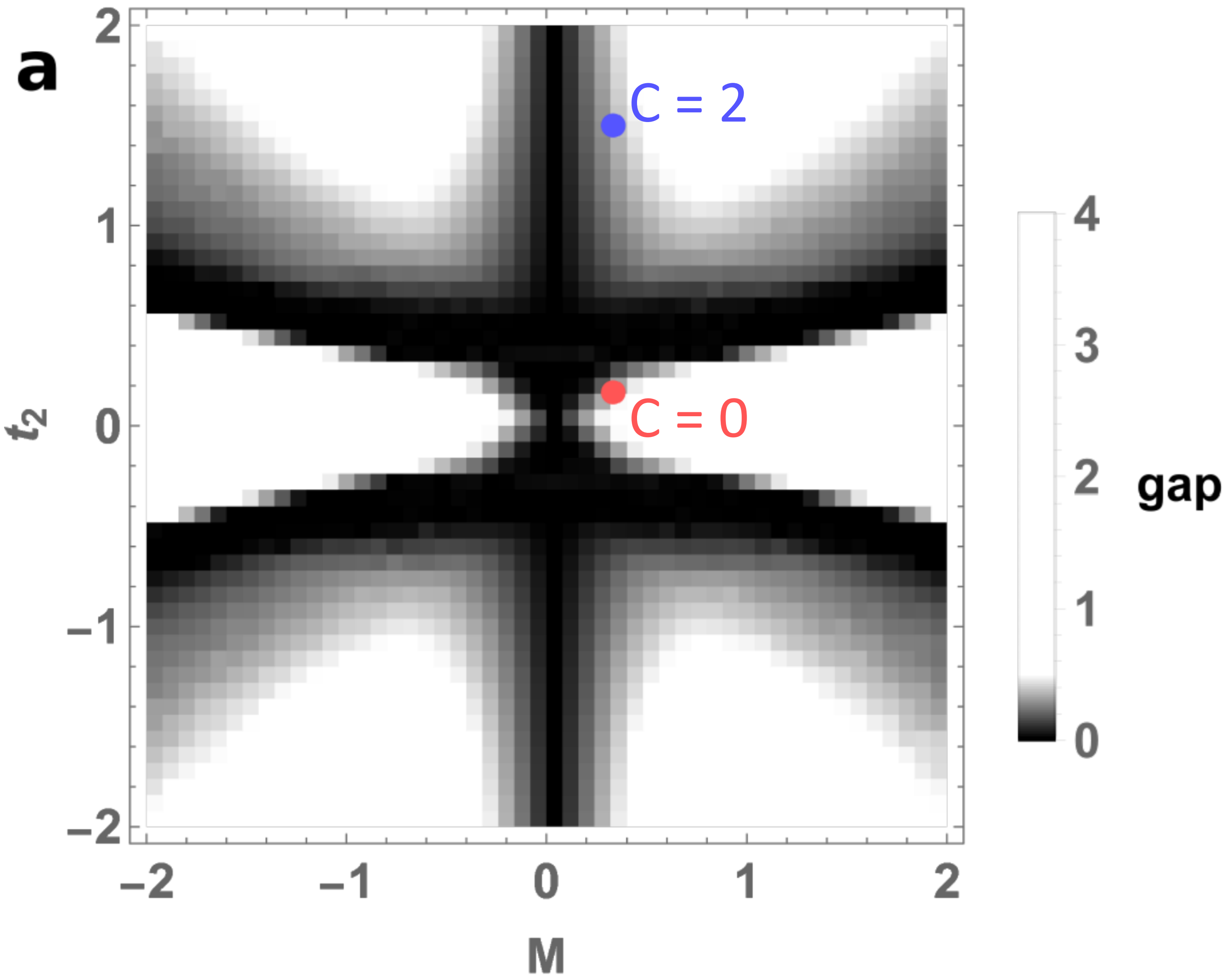
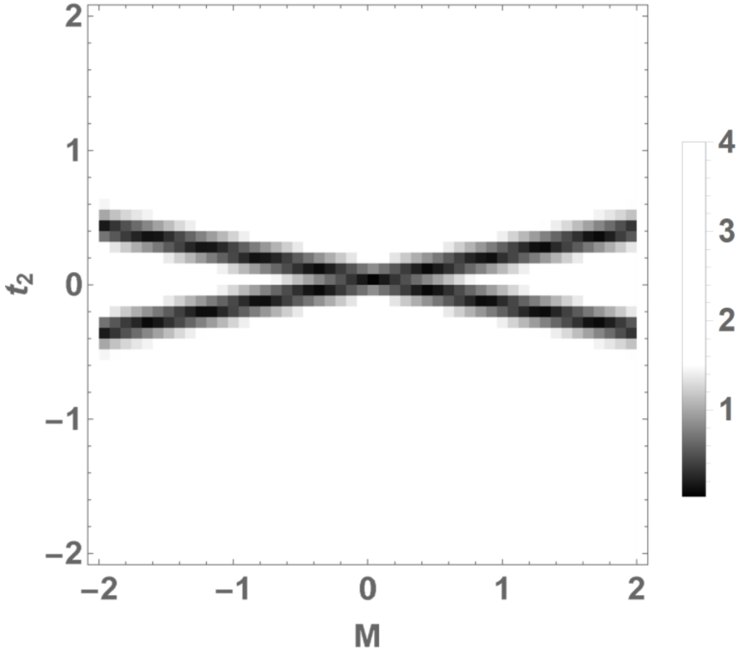
Phase diagram of HH mode at half-filling & t_1=1, $\phi = \pi/2$





Euclidean Haldane
model for comparison