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Physics in a Strong Magnetic Field

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1 Introduction

A glance at the behavior of resistance of a two dimensional electron system as a function of the perpendicular magnetic field (Fig.1), reveals immediately why the quantum Hall effect has attracted so much attention in the past years. One usually plots the resistivities along the direction of the current (ρ_{xx}) and in the direction perpendicular to it (ρ_{xy}) as a function of the field B. Very schematically, for certain range of the field ρ_{xx} is nearly equal to zero, and for other ranges it develops a bump. On the average ρ_{xy} grows linearly with the field, but in the regions where ρ_{xx} is equal to zero, ρ_{xy} presents a flat plateau which is a fraction times h/e^2 to an extraordinary accuracy. This is the quantized Hall effect which has led to two Nobel prizes, one in 1985 to Von Klitzing for the discovery of the integer Hall effect, and the other in 1992 to Laughlin, Störmer and Tsui for the fractional Hall effect.

The basic experimental observation is best recast using the conductivities σ_{xx} and σ_{xy} which give the components of the inverse of the resistivity tensor [1]. The quantized Hall regime corresponds to a nearly vanishing dissipation:

$$\sigma_{xx} \to 0 \tag{1}$$

accompanied by the quantization of the Hall conductance:

$$\sigma_{xy} = \nu \frac{e^2}{h} \tag{2}$$

In the integer Hall effect case, ν is an integer with a precision of about 10^{-10}. In the fractional case, ν is a fraction which reveals the bizarre properties of many electron physics. The fractions are universal and independent of the type of semiconductor material, the purity of the sample and so forth. The effect occurs when the electrons are at a particular density encoded in the fraction ν as if the electrons locked their separation at particular values. Changing the electron density by a small amount does not destroy the effect but changing it by a larger amount does, this is the origin of the plateaus.

In this introductory seminar we shall present the basic tools needed to understand these phenomena. We first review briefly the classical motion of an electron in a magnetic field and the

[1] $\sigma_{xx} = \frac{\rho_{xx}}{\sqrt{\rho_{xx}^2+\rho_{xy}^2}}$, $\sigma_{xy} = \frac{\rho_{xy}}{\sqrt{\rho_{xx}^2+\rho_{xy}^2}}$

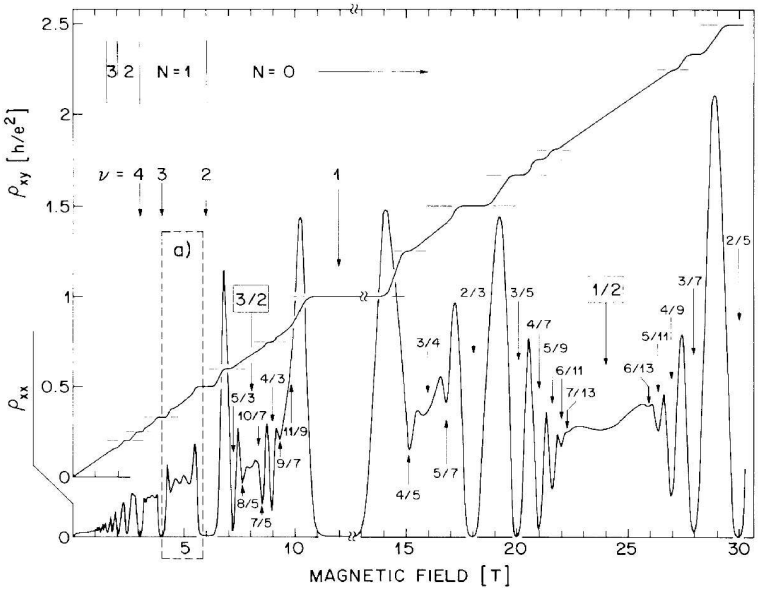

Figure 1: Overview of the diagonal resistivity ρ_{xx} and Hall resistance ρ_{xy} After ref. [4].

classical Hall effect. We then move on to the quantum mechanical description. We introduce the Landau levels and show their relevance to understand the integer Hall effect. Various theoretical ideas to account for the robustness of the plateau values of the Hall conductance are then discussed. Finally, we give some hints of how taking into account electron-electron interactions can explain the occurrence of the fractional Hall effect.

Some classical review papers and references on the quantum Hall effect can be found in ref. [1],[2],[3].

2 Single particle in a magnetic field

2.1 Classical motion in a magnetic field

As a first step we must understand the classical motion of an electron of charge $-e$ confined in a two-dimensional plane (x, y), and subject to a constant magnetic field $B\hat{z}$ perpendicular to this plane. The Newtonian equations of motion due to the Lorentz force are given by:

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \frac{eB}{m} \begin{pmatrix} -\dot{y} \\ \dot{x} \end{pmatrix}, \tag{3}$$

m is the mass of the particle. In complex notations $z = x + iy$, with ω defined as:

$$\omega = eB/m \tag{4}$$

(3) rewrites:

$$\ddot{z} = i\omega\dot{z} \tag{5}$$

The solution is given by:

$$z(t) = z_0 + de^{i\omega t} \tag{6}$$

The trajectory is a circle of radius $|d|$ run at a constant angular velocity. The frequency ω is independent of the initial conditions and fixed by the magnetic field, the charge and mass of the

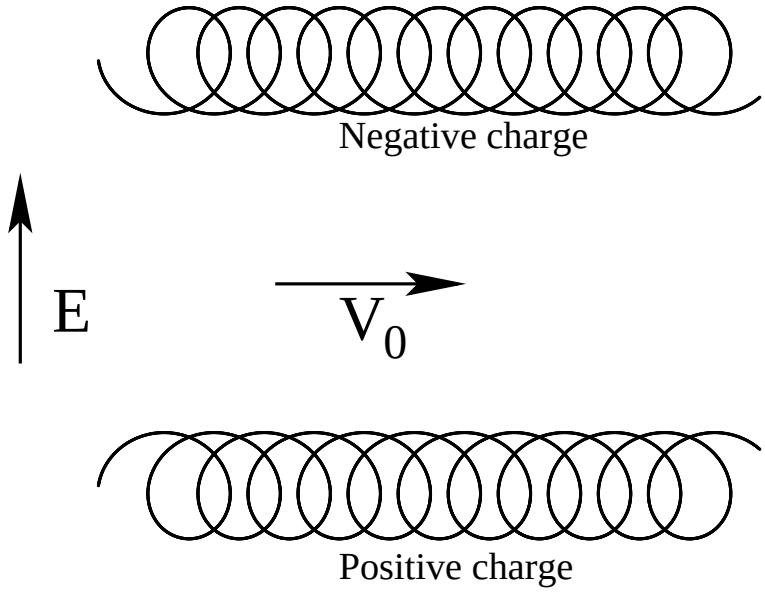

Figure 2: Illustration of cyclotron motion for a classical charged particle in the presence of uniform perpendicular magnetic and electric fields. The magnetic field is perpendicular to the plane of the figure, pointing upwards. The electric field **E** lies in the plane as shown. The drift velocity $\mathbf{v}_0 = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2}$ is also represented. We have drawn trajectories for both possible signs of the particle electric charge.

particle. It is called the cyclotron frequency. The average position of the particle over the time, $z_0 = x_0 + iy_0$, is arbitrary, and is called the guiding center. The radius $|d|$ of the trajectory is proportional to the speed of the particle times its mass. In a Fermi liquid, the speed of the electrons times their mass is frozen and equal to the Fermi momentum. The measurement of the cyclotron radius can thus be used to determine the Fermi momentum [2].

Let us add to the magnetic field an electric field $E\hat{\boldsymbol{y}}$ in the y direction. The equation of motion now becomes,

$$\ddot{z} = i\omega\dot{z} - ieE/m. \tag{7}$$

This equation can be put into the form (5) if we use the variable $z' = z - Et/B$. It results from the fact that the electric field can be eliminated through a Galilean transformation to the frame moving at the speed E/B in the x direction perpendicular to E with respect to the laboratory frame. As a result, in presence of an electric field **E**, the guiding center moves perpendicularly to the electric field at a speed:

$$\mathbf{v}_0 = \frac{\mathbf{E}\wedge\mathbf{B}}{B^2}. \tag{8}$$

This classical motion is illustrated on Fig. 2.

2.2 Classical Hall effect

Let us study the simple consequences of the classical equation of motion for the resistivity tensor. In a simple model, an electron travels with the Fermi velocity v_f uniformly distributed over all possible directions on a distance given by its mean free path $l_0 = v_f\tau_0$. Here the scattering time τ_0 is the average time between two collisions. This electron is then scattered with the velocity v_f over all possible directions. In the presence of an electric field **E** it is uniformly accelerated with the

[2] Recently, the same experiment has been performed to determine the charge of quasiparticles when the Fermi momentum was known. [32]

acceleration $-e\mathbf{E}/m$ in the direction of the electric field in between two collisions. It thus acquires a mean velocity $\mathbf{v} = -e\mathbf{E}/m\tau_0$ directed parallel to $\mathbf{E}$. In the presence of a uniform magnetic field B, the particles also acquire a uniform speed perpendicular to $\mathbf{E}$. Adding up the contributions of independent electrons with a two dimensional electron number density n, we deduce the resistivity tensor $\boldsymbol{\rho}$ which expresses the linear relation between the current density $\mathbf{j}$ and the electric field $\mathbf{E}$:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ -\rho_{xy} & \rho_{xx} \end{pmatrix} \begin{pmatrix} j_x \\ j_y \end{pmatrix}, \tag{9}$$

where the longitudinal resistivity is:

$$\rho_{xx} = \frac{m}{ne^2\tau_0}, \tag{10}$$

and the transverse (or Hall) resistivity (ρ_{xy}), which relates the current density $j_\perp$ perpendicular to the electric field E to the field itself, has the expression:

$$\rho_{xy} = \frac{B}{ne} \tag{11}$$

This result relies on Galilean invariance only and is not modified by interactions. The simplest way to derive this expression is to assume that the current is known. To this current, we associate an average electronic velocity $\bar{\boldsymbol{v}}$ so that $\mathbf{j} = -ne\bar{\boldsymbol{v}}$. This velocity generates a Lorentz force $\mathbf{f}_{\mathrm{L}} = -e\bar{\boldsymbol{v}} \wedge \mathbf{B}$, which has to be balanced by a transverse electric field $\mathbf{E}_\perp$ ($\mathbf{E}_\perp.\mathbf{j} = \mathbf{0}$) so that $\mathbf{f}_{\mathrm{L}} - e\mathbf{E}_\perp = \mathbf{0}$, thus giving $\mathbf{E}_\perp = \frac{\mathbf{j}\wedge\mathbf{B}}{ne}$, in agreement with (9) and (11). An alternative viewpoint is that a Galilean transformation to a frame with relative velocity $\bar{\boldsymbol{v}}$ suppresses the transverse electric field $\mathbf{E}_\perp$. If we compare these predictions with the experimental situation mentioned in the introduction, we see that classically the transport is dissipative with a constant longitudinal resistivity, and therefore, the observed vanishing of ρ_{xx} is not predicted by this simple model. However, the average slope of the transverse resistivity with respect to the magnetic field is accurately predicted. At the values for which $\rho_{xx} = 0$, the transverse conductivity is given by the inverse transverse resistivity (11). Comparing the prediction with the experimental result (2) we deduce that the Hall effect occurs when the electron density is close to the value:

$$n = \nu\frac{eB}{h} \tag{12}$$

The quantity h/eB has the dimension of an area and we shall have more to say about it. The fact that Planck's constant appears explicitly in this expression suggests that quantum mechanics plays a crucial role in the formation of those plateaus in ρ_{xy}. But before analyzing the new features induced by quantum mechanics, it is useful to first describe in more detail the Hamiltonian approach to classical motion.

2.3 Hamiltonian formalism

Let us introduce a vector potential $\mathbf{A}(\mathbf{r})$ for the magnetic field:

$$B = \partial_x A_y - \partial_y A_x \tag{13}$$

The vector potential $\mathbf{A}(\mathbf{r})$ is defined up to a gauge transformation $\mathbf{A}(\mathbf{r}) \to \mathbf{A}(\mathbf{r}) + c\boldsymbol{\nabla}\chi(\mathbf{r})$. The action from which the equations of motion of a mass m and charge $-e$ particle (confined to the plane) in presence of the magnetic field $B\hat{\boldsymbol{z}}$ derive, is given by:

$$S = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \left(\frac{m}{2}\dot{\mathbf{r}}^2 - e\mathbf{A}.\dot{\mathbf{r}}\right) dt \tag{14}$$

Note that the action is not gauge invariant and under a gauge transformation $S \to S - e(\chi(\mathbf{r}_2) - \chi(\mathbf{r}_1))$. Of course, since this change only involves the end points of the electron path, the classical equations of motion are not affected by such a gauge transformation. In presence of a uniform

magnetic field, if the particle makes a closed path and returns back to its position, the action accumulated by the potential on the trajectory is eB times the area surrounded by the trajectory.

Using the canonical rules, we obtain a Hamiltonian:

$$H_0 = \frac{1}{2m}\left(\mathbf{p} + e\mathbf{A}\right)^2 = \frac{\boldsymbol{\pi}^2}{2m}, \tag{15}$$

where $\mathbf{p} = m\dot{\mathbf{r}} - e\mathbf{A}$ is the momentum conjugated to $\mathbf{r}$, the quantities $\mathbf{p}$ and $\mathbf{r}$ obey the Poisson brackets:

$$\{p_i, p_j\} = 0, \quad \{r_i, r_j\} = 0, \quad \{p_i, r_j\} = \delta_{ij} \tag{16}$$

and the so-called dynamical momenta:

$$\boldsymbol{\pi} = m\dot{\mathbf{r}} = \mathbf{p} + e\mathbf{A}, \tag{17}$$

obey the Poisson brackets:

$$\{\pi_i, \pi_j\} = \epsilon_{ij} eB, \quad \{r_i, r_j\} = 0, \quad \{\pi_i, r_j\} = \delta_{ij}, \tag{18}$$

where ϵ_{ij} is the antisymmetric tensor $\epsilon_{xy} = -\epsilon_{yx} = 1$.

We can also define new coordinates R_x, R_y which have zero Poisson brackets with the dynamical momenta:

$$R_x = x - \frac{1}{eB}\pi_y, \quad R_y = y + \frac{1}{eB}\pi_x, \tag{19}$$

with the Poisson brackets given by:

$$\{R_i, R_j\} = -\epsilon_{ij}\frac{1}{eB}, \quad \{\pi_i, R_j\} = 0. \tag{20}$$

One can verify that the coordinates so defined coincide with the guiding center defined in (6): $R_x + iR_y = z_0$.

To understand the physical meaning of the guiding center it is instructive to consider the motion of a charged particle in presence of an external potential $V(\mathbf{r})$, which is supposed to vary slowly ($|\partial_i \partial_j V| << m\omega^2$ for all i, j):

$$H = H_0 - eV(\mathbf{r}). \tag{21}$$

The case of the electric field we looked at in the last section, corresponds to $V(\mathbf{r}) = -Ey$. We are interested in the motion of the guiding center in presence of $V(\mathbf{r})$. If the radius of the cyclotron orbital is sufficiently small and the speed of rotation sufficiently fast so that the potential seen during a rotation is approximatively constant, we can average over time; the dynamical momenta acquire a zero expectation value and we can replace the position $\mathbf{r}$ by the guiding center $\mathbf{R}$. In this approximation, the guiding center motion is given by:

$$\dot{\mathbf{R}} = \frac{\mathbf{B} \wedge \boldsymbol{\nabla} V}{B^2} \tag{22}$$

The motion decomposes into a fast rotation around the cyclotron orbit and a slow motion of the guiding center along the the equipotential lines of $V(\mathbf{r})$.

If the potential is smoothly varying, we can divide the equipotential lines into two kinds: Those located near the maxima of V which are closed, and those located near its mean value which can wind a long way through the saddle points of V. We can qualitatively understand why the preceding picture of the transport can be dramatically affected if we take into account the influence of an external potential. In presence of an electric field the potential seen by the electrons becomes $V(\mathbf{R}) - e\mathbf{E}.\mathbf{R}$, and according to whether the equipotential line we consider is closed or extended, the electron traveling along it is localized or not. We shall return to this point later.

A very important consequence of this effective dynamics for guiding centers in the high magnetic field limit is that area is preserved under the time-evolution. This is a special case of Liouville's theorem on the conservation of phase-space volumes for Hamiltonian systems. But here, very remarkably, phase-space has to be identified with the physical plane, since the two coordinates of the guiding center are canonically conjugated according to (20). Physically, this means that if the initial condition is such that the electronic density is constant inside a domain Ω_0, and zero outside, after the system has evolved according to the dynamics (22), it is still constant inside a deformed domain Ω_t of the *same* area as Ω_0 and zero outside. In other words, the electronic fluid is incompressible. As we shall see in section 3.2, this property plays a crucial role in understanding the quantization of σ_{xy}. This can be formalized further if consider the set of Poisson brackets between plane waves $e^{i\mathbf{k}.\mathbf{R}}$ given by:

$$\{e^{i\mathbf{k}.\mathbf{R}}, e^{i\mathbf{k}'.\mathbf{R}}\} = \frac{\mathbf{k}\wedge\mathbf{k}'}{eB} e^{i(\mathbf{k}+\mathbf{k}').\mathbf{R}}. \tag{23}$$

This algebra is known as the algebra of diffeomorphisms which preserve the area.

2.4 Quantum-mechanical description

2.4.1 Quantum formalism

In the Hamiltonian formalism, the quantization of a charged particle in a magnetic field is straightforward. The momenta are operators $p_i = \frac{\hbar}{i}\partial_{r_i}$, where $\hbar = h/2\pi$. The discussion of the preceding section can be repeated with the Poisson brackets replaced by commutators: $\{X, Y\} \to \frac{\hbar}{i}[X, Y]$.

The dynamical momenta and the guiding center define two sets of operators which obey the commutation relations analogous to (18,20):

$$[\pi_i, \pi_j] = -i\hbar\epsilon_{ij}eB, \quad [R_i, R_j] = i\hbar\epsilon_{ij}\frac{1}{eB}, \quad [\pi_i, R_j] = 0. \tag{24}$$

Note that the commutation relations (24) for the dynamical momenta π_i involve the magnetic field at the numerator, whereas those involving the guiding centers R_i are inversely proportional to the magnetic field. We may therefore expect to recover two different classical limits, when the magnetic field is weak and when it becomes very strong.

To compute the spectrum of H_0 we can define creation and annihilation operators as linear combinations of the two dynamical momenta :

$$a = \sqrt{\frac{1}{2\hbar eB}}(\pi_x - i\pi_y), \quad a^+ = \sqrt{\frac{1}{2\hbar eB}}(\pi_x + i\pi_y), \tag{25}$$

obeying the Heisenberg relations:

$$[a, a^+] = 1. \tag{26}$$

In terms of these oscillators the unperturbed Hamiltonian is:

$$H_0 = \hbar\omega(a^+a + \frac{1}{2}), \tag{27}$$

and its spectrum is that of an oscillator:

$$E_n = \hbar\omega(n + \frac{1}{2}), \tag{28}$$

with $n \geq 0$. Each energy branch is called a Landau level.

The strong magnetic field limit is when the cyclotron radius gets frozen, and the dynamics is fully controlled by the guiding center coordinates. The fact that the guiding center coordinates commute with H_0 implies that its spectrum is extremely degenerate. The two coordinates R_x, R_y do not commute with each other and cannot be fixed simultaneously. There is a quantum uncertainty

$\Delta R_x \Delta R_y = \frac{\hbar}{eB}$ to determine the position of the guiding center. It is customary to define the magnetic length l by:

$$l = \sqrt{\frac{\hbar}{eB}}. \tag{29}$$

Due to the uncertainty principle, the physical plane can be thought of as divided into disjoint cells of area $2\pi l^2$ where the guiding center can be localized. This area coincides precisely with the area threaded by one magnetic flux quantum $\Phi_0 = 2\pi\hbar/e$. The degeneracy per energy level and per unit area is $1/2\pi l^2$ so that in an area Ω, the number of degenerate states is:

$$N_\Omega = \frac{\Omega}{2\pi l^2}, \tag{30}$$

so that electrons behave "as if " they acquire some size under a magnetic field, the area being inversely proportional to B.

Imagine now that we continuously fill a bounded region of the plane with noninteracting electrons. Let n be the electron number density. We introduce the so called filling factor as the number of electrons per cell:

$$\nu = n2\pi l^2 \tag{31}$$

Due to the Pauli principle, a cell can be occupied by one electron only per energy level. Therefore, each time the filling factor reaches an integer, an energy level gets filled and the next electron must be added to the next energy level. Thus, the energy per added electron (chemical potential) jumps by a quantity $\hbar\omega$. This is the integer quantum Hall regime, and indeed, comparing (31) with (12) we can identify the filling factors of the Hall effect with the fraction entering the expression of the transverse conductivity (2). Not surprisingly thus, when the filling factor takes integer values, the (integer) Hall effect is observed. This naive approach however, seems to indicate that the integer Hall effect should be observed only at the specific values of the magnetic field for which ν given by (12) or (31) is an integer, instead of some extended regions of B, as seen experimentally. Also, the explanation for the fractional values of ν is out of reach in this approach.

The energy separation between levels must be compared with the other energy scales introduced by the impurities and the interactions which will split the degeneracy. A necessary condition to observe the Hall effect is that the splitting of the energy levels within each Landau level remains small compared to $\hbar\omega$, so that the Landau levels are well separated in energy. This picture however is too naive to account for the width of the plateaus. We must invoke the existence two kinds of energy levels. Extended levels narrowly dispersed around the Landau energy and localized levels which do not carry current but spread in energy. The presence of these localized states is necessary to enable the chemical potential to vary smoothly between two Landau levels instead of jumping abruptly. This seems to ruin the quantization argument made just before, and we shall have more to say to reconcile the quantized picture with the existence of localized states later.

We have just seen that the $\nu = n$ integer Hall effect occurs precisely when the density is such that an integer number of electrons n occupy a magnetic cell. Conversely, we can expect that the $\nu = 1/3$ Hall effect occurs when one electron occupies three cells by himself! This locking of the separation between electrons cannot be accounted for by the Pauli principle. The alternative explanation is that it is due to the interactions between the electrons. This is the starting of Laughlin's theory for the fractional Hall effect, and this aspect is discussed in S. Girvin's lecture.

2.4.2 Landau gauge

Although the bulk properties of a system of electrons must be independent of the gauge choice, it is instructive to carry out the quantization procedure in different gauges. Different gauges can be better suited to different geometries because the shape of the wave functions depends on the gauge choice.

Let us consider the so-called Landau gauge which is well suited to a cylindrical geometry:

$$A_x = -By, \quad A_y = 0. \tag{32}$$

In this gauge the dynamical momenta are:

$$\pi_x = p_x - eBy,\ \pi_y = p_y, \tag{33}$$

and the guiding center coordinates are:

$$R_x = x - \frac{1}{eB}p_y, \quad R_y = \frac{1}{eB}p_x, \tag{34}$$

We can find the simultaneous spectrum of H_0 and R_y, and thus fix the value of the x-momentum $p_x = \hbar k$. We therefore look for eigenfunctions of H_0 in the form:

$$\Psi_k(\mathbf{r}) = e^{ikx} f_k(y). \tag{35}$$

Each value of $R_y = kl^2$ determine an effective one dimensional Hamiltonian for $f_k(y)$:

$$H_k \quad = \quad \frac{1}{2m}p_y^2 + \frac{1}{2}m\omega^2(y - kl^2)^2, \tag{36}$$

where ω is the cyclotron frequency (4). This is the Hamiltonian of a harmonic oscillator centered at a position $y = kl^2$ determined by the momentum in the x direction. The spectrum is independent of k and given by:

$$\epsilon_n = (n + \frac{1}{2})\hbar\omega. \tag{37}$$

Let us for the moment concentrate on the lowest level $n = 0$. The wave functions f_{k0} are Gaussian centered on kl^2 of width l:

$$f_{k0}(y) = \exp\left(-\frac{(y - kl^2)^2}{2l^2}\right). \tag{38}$$

To recover the degeneracy, imagine we impose periodic boundary conditions in the x direction ($x + L_x \equiv x$). This imposes a quantization condition on k which must take the values $k_m = 2\pi m/L_x$ for the wave function (35) to be periodic. For each value of m the Gaussian wave packet $f_{m0}(y)$ is centered on $y_m = 2\pi m l^2/L_x$. The number of allowed values of m in an interval of length L_y is thus $\frac{L_x L_y}{2\pi l^2}$ and we recover the degeneracy (30).

The n^{th} Landau Level wave functions are obtained by acting with the creation operator $(a^+)^n$ on the ground state wave functions $\Psi_k(\mathbf{x})$. One can verify that the wave functions are expressed in terms of Hermite polynomials H_n as:

$$f_{kn}(y) = H_n(\frac{y - kl^2}{l}) \exp\left(-\frac{(y - kl^2)^2}{2l^2}\right). \tag{39}$$

2.4.3 Symmetric gauge

Another useful gauge well suited to study the system on a disc is the so-called symmetric gauge defined by:

$$A_x = -\frac{By}{2}, \quad A_y = \frac{Bx}{2}. \tag{40}$$

In this gauge the guiding center coordinates are:

$$R_x = \frac{x}{2} - \frac{1}{eB}p_y, \quad R_y = \frac{y}{2} + \frac{1}{eB}p_x. \tag{41}$$

We combine them into two oscillators:

$$b = \frac{1}{\sqrt{2}l}(R_x + iR_y), \quad b^+ = \frac{1}{\sqrt{2}l}(R_x - iR_y). \tag{42}$$

The lowest Landau level wave functions are obtained upon acting onto the ground state of (27) with $(b^+)^m$. In this gauge, the angular momentum L is a good quantum number and they carry an angular momentum $L = -m$. Their expression is proportional to [3]:

$$\Psi_{m0}(\bar{z}) = (\bar{z}/l)^m \exp\left(-\frac{z\bar{z}}{4l^2}\right), \tag{43}$$

and they can be visualized as thin circular shells of radius $\sqrt{2m}l$ around the origin. Thus if we quantize the system in a a disk of finite radius R, we recover the expected degeneracy (30) by keeping only the wave functions confined into the disk $m \leq m_0 = R^2/2l^2$. By taking linear combinations of wave functions (43) we see that the general wave functions are proportional to polynomials of fixed degree m_0 in z. A useful way to characterize them is through the location of their zeros $\bar{Z}_i$:

$$\Psi_0(\bar{z}) = \prod_{i=1}^{m_0}(\bar{z} - \bar{Z}_i) \exp\left(-\frac{z\bar{z}}{4l^2}\right). \tag{44}$$

3 Hall conductance Quantization: the Integer Effect

3.1 Galilean invariant systems

The first important thing to emphasize is that for a two-dimensional Galilean invariant system, (in the absence of impurities or boundaries) a full quantum-mechanical treatment yields the *same* resistivity tensor as for the pure classical system, namely $\rho_{xx} = 0$ and $\rho_{xy} = B/(ne)$, or equivalently, $\sigma_{xx} = 0$ and $\sigma_{xy} = ne/B$.

To check this, let us consider the following Hamiltonian, for a system of N interacting electrons in the presence of uniform time-independent perpendicular magnetic ($\mathbf{B}$) and electric fields ($\mathbf{E}$):

$$H = \frac{1}{2m}\sum_{j=1}^{N}\left((\mathbf{P}_j + e\mathbf{A}(\mathbf{r}_j))^2 - eV(\mathbf{r}_j)\right) + \frac{1}{2}\sum_{i\neq j} U(\mathbf{r}_i - \mathbf{r}_j) \tag{45}$$

where as usual, $\mathbf{B} = \nabla \wedge \mathbf{A}$, $\mathbf{E} = -\nabla V$, and U is the pair interaction potential. Inspired by the discussion of the classical case, let us now introduce the following transformation on the N-particle wave-function $\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N, t)$:

$$\Psi(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_N, t) = \exp\left(\frac{i}{\hbar}\sum_{j=1}^{N}\theta(\mathbf{r}_j, t)\right)\tilde{\Psi}(\mathbf{r}'_1, \mathbf{r}'_2, ..., \mathbf{r}'_N, t) \tag{46}$$

where $\mathbf{r}_j$ denotes the position of particle j in the laboratory frame and $\mathbf{r}'_j$ its position in the moving frame, with constant velocity $\mathbf{v}_0$ given by:

$$\mathbf{v}_0 = \frac{\mathbf{E} \wedge \mathbf{B}}{B^2} \tag{47}$$

Therefore, we have the relation: $\mathbf{r}'_j = \mathbf{r}_j - \mathbf{v}_0 t$. It is possible to choose the phase $\theta(\mathbf{r}, t)$ in such a way that $\tilde{\Psi}$ satisfies the time-dependent Schrödinger equation associated to the simplified Hamiltonian $\tilde{H}$ deduced from H by removing the potential term $-e\sum_j V(\mathbf{r}_j)$. So in the inertial frame moving with constant velocity $\mathbf{v}_0$, there is no electrical field, and only the original magnetic field remains. Note that the expression of $\theta(\mathbf{r}, t)$ does depend on the choice of gauge. For instance, in the radial gauge $\mathbf{A} = \frac{1}{2}\mathbf{B} \wedge \mathbf{r}$, we have:

$$\theta(\mathbf{r}, t) = m\mathbf{v}_0.\mathbf{r} - \frac{1}{2}(mv_0^2 + e\mathbf{E}.\mathbf{r})t$$

[3] This convention is not usual, most people prefer to use conventions for which $\bar{z} \to z$.

The phase-factor does not alter the classical composition rule for currents, and we get:

$$\langle\Psi|\mathbf{J}(\mathbf{r})|\Psi\rangle = -ne\mathbf{v}_0 + \langle\tilde{\Psi}|\mathbf{J}(\mathbf{r}')|\tilde{\Psi}\rangle \tag{48}$$

Now since there is no driving electric field in the moving frame, $\langle\tilde{\Psi}|\mathbf{J}(\mathbf{r}')|\tilde{\Psi}\rangle = \mathbf{0}$, so $\langle\Psi|\mathbf{J}(\mathbf{r})|\Psi\rangle = -ne\mathbf{v}_0$ which is exactly the classical result.

As we have seen, the natural way to measure the electronic density for a two-dimensional quantum system in a magnetic field is the filling factor $\nu = \frac{nh}{eB}$. So we end up with:

$$\sigma_{xy} = \nu\frac{e^2}{h} \tag{49}$$

Although this expression does involve Planck's constant, it is important to note once again that it is *identical* to the the classical prediction for a uniform fluid of electrons of areal density n. As illustrated on Fig. 1, this prediction for a Galilean invariant system coincides with the experimental result for the Hall conductance when the filling factor ν is an integer. The existence of quantized plateaus of the form $\sigma_H = n\frac{e^2}{h}$, with n integer clearly indicates the breakdown of Galilean invariance in real samples, since the two expressions differ when ν is not an integer. This fact is not too surprising, since there is always a random electrostatic potential induced by the impurities which are required to generate charge carriers at the interface between two semi-conductors. The most surprising fact is that despite this random potential (without which there would be no observable Hall quantization!) the measured plateau values are universal with a very high accuracy. Of course, a lot of theoretical work has been dedicated to explain this remarkable phenomenon. To give a simple outline, we may classify most of the existing approaches in the following way:
-The Laughlin argument [5]
-Expressing the Hall conductance as a topological invariant [6]
-The Edge-State picture [7]
We shall try here to give a flavor of these important contributions, but let us first begin to present a rather simple and helpful semi-classical analysis [8].

3.2 An intuitive picture

It is indeed very illuminating to consider the limit of an extremely strong magnetic field, so that the magnetic length $l = (\frac{\hbar}{eB})^{1/2}$ is much smaller than the typical length-scales associated to the spacial variations of the impurity potential $U_{\text{imp}}(\mathbf{r})$. Classically, we have seen that the guiding center $\mathbf{R}$ of classical orbits for a single electron obeys the following equations of motion:

$$\dot{\mathbf{R}} = \frac{\mathbf{B}\wedge\boldsymbol{\nabla}}{B^2}(V - \frac{U_{\text{imp}}}{e})(\mathbf{R}) \tag{50}$$

In particular, this implies that $W(\mathbf{R}) = (V - \frac{U_{\text{imp}}}{e})(\mathbf{R})$ is conserved, so the classical trajectories of guiding centers in the infinite B limit coincide with equipotential curves of the function $W(\mathbf{R})$. Using the intuition gained in section **1**, we expect that after quantization, single particle eigenstates are located along narrow strips of width l centered on these equipotential lines. As usual in semi-classical quantization, only a discrete set of classical orbits are allowed. An extension of the Bohr-Sommerfeld principle indicates that for closed classical orbits, only those which enclose an integer number of flux quanta give rise to quantum eigenstates. Let us denote by W_i the potential energies associated to these selected orbits. We get then the following semi-classical spectrum:

$$E_{i,n} = -eW_i + \hbar\omega(n + \frac{1}{2}) \tag{51}$$

where n is any non-negative integer corresponding to quantizing the fast cyclotron motion around the slow moving guiding center. For a fixed value of n, we may then speak of a generalized n^{th} Landau level, although the degeneracy of this level is lifted by the joint effect of the driving electric field and the impurity potential. When such a level is completely filled, it induces a spacial density of electrons (after coarse-graining on a length-scale of the order of the corresponding cyclotron radius $n^{1/2}l$) equal to $\frac{eB}{h}$, mostly insensitive to the form of the effective potential $W(\mathbf{R})$.

Let us consider now our system to be a horizontal strip defined by $0 \leq y \leq L_y$. We apply an external field along the y direction, in such a way that the edges of the sample $y = 0$ and $y = L_y$ are equipotential lines for V. On average, we expect a global Hall current j_x in the horizontal direction. Each generalized Landau level produces a local current:

$$j_{x,n} = -e\left(\frac{eB}{h}\right)\theta(\mu + eW(\mathbf{R}) - \hbar\omega(n + \frac{1}{2}))v_x(\mathbf{R}) \tag{52}$$

where $v_x(\mathbf{R}) = -\frac{1}{B}\frac{\partial W}{\partial y}(\mathbf{R})$, and μ is the chemical potential of the electronic system. The Heaviside step function $\theta(\mu + eW(\mathbf{R}) - \hbar\omega(n + \frac{1}{2}))$ is the limiting form of the Fermi-Dirac distribution for the semi-classical spectrum (51) at zero temperature. Let us now integrate this local current along a vertical section of the sample, at fixed $x = x_0$. This yields:

$$I_{x,n}(x_0) = \frac{e^2}{h}\int_0^{L_y} dy\, \theta(\mu + eW(x_0, y) - \hbar\omega(n + \frac{1}{2}))\frac{\partial W}{\partial y}(x_0, y) \tag{53}$$

To simplify the discussion, let us assume that the impurity potential vanishes on the edges (for $y = 0$ and $y = L_y$). The above integral is easily computed, and the result distinguishes between four cases:

1)$-eW(x_0, y) + \hbar\omega(n + \frac{1}{2}) \equiv E_n(x_0, y) > \mu$ for both $y = 0$ and $y = L_y$.

This means that the n^{th} generalized Landau level is unoccupied in the presence of the external driving voltage, in the limit where the impurity potential vanishes. In this case, $I_{x,n}(x_0) = 0$. Note that this value is independent of the strength of the local impurity potential. In particular, if $U_{\rm imp}(\mathbf{R})$ has deep local minima, it may happen that $E_n(\mathbf{R}) < \mu$ in some finite areas, meaning that there are occupied bound states in the n^{th} Landau level localized near impurities. But in this very large field limit, we see that such localized states do not contribute to the global Hall current.

2)$E_n(x_0, y) < \mu$ for both $y = 0$ and $y = L_y$.

In the limit of vanishing impurity potential, the corresponding Landau level is then fully occupied. We obtain:

$$I_{x,n}(x_0) = \frac{e^2}{h}\left(W(x_0, L_y) - W(x_0, 0)\right) = \frac{e^2}{h}\left(V(L_y) - V(0)\right) \tag{54}$$

Again, this result is independent of the strength of the impurity potential. Such a fully occupied level provides therefore a contribution equal to $\frac{e^2}{h}$ to the total Hall conductance.

3)$E_n(x_0, 0) > \mu$ and $E_n(x_0, L_y) < \mu$.

Then:

$$I_{x,n}(x_0) = \frac{e^2}{h}\frac{\mu - E_n(x_0, L_y)}{e} \tag{55}$$

4)$E_n(x_0, 0) < \mu$ and $E_n(x_0, L_y) > \mu$.

Then:

$$I_{x,n}(x_0) = \frac{e^2}{h}\frac{E_n(x_0, 0) - \mu}{e} \tag{56}$$

These last two cases correspond to Landau levels which are partially filled in the absence of impurity potential, but in the presence of the driving field. They destroy the quantization of σ_{xy}. In order to avoid them, one has to fix the chemical potential in a gap of the unperturbed Landau level spectrum, and impose a weak enough driving electric field, typically such that $e|V(x_0, L_y) - V(x_0, 0)| < \hbar\omega$. If these conditions are satisfied, we have an integer number p of filled Landau levels which contribute to the Hall current, so that:

$$I_x(x_0) = p\frac{e^2}{h}\left(V(L_y) - V(0)\right) \tag{57}$$

in perfect agreement with:

$$\sigma_{xy} = p\frac{e^2}{h} \tag{58}$$

When does this simple and appealing picture break down? Clearly, it is problematic when the typical scale of the impurity potential becomes comparable to the magnetic length l. In this case,

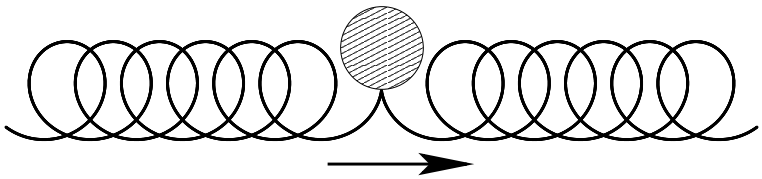

Figure 3: Illustration of cyclotron motion for a classical charged particle in the presence of uniform perpendicular magnetic and electric fields. A hard circular point scatterer is depicted as a dashed-filled circle.

it is no longer possible to preserve such a simple description of quantum energy eigenstates. Nevertheless, as shown on Fig. 3 illustrating the effect of a strong scatterer modeled as an impenetrable disk of radius a, even when a is small compared to the cyclotron radius, we may expect that such a scatterer does not disturb the shape of a strip-like eigenstate excepted in its immediate vicinity. In particular, the overall direction of propagation of a wave-packet is not modified by the presence of such impurities. These qualitative expectations are confirmed by more detailed perturbative calculations [8].

More serious problems arise when strong localized scatterers are densely packed, namely with an average nearest-neighbor spacing of the order of l. In this case, classical trajectories become very complicated. Quantum-mechanically, we expect that such a strong potential induces strong mixing between different Landau levels, and therefore, a perturbative analysis is not very helpful. Fortunately, a very interesting and famous argument has been given by Laughlin [5] which shows that nevertheless a strict quantization of σ_{xy} is still possible.

3.3 The Laughlin argument

Let us consider the same strip as before, defined by $0 \leq y \leq L_y$, but let us fold it into a cylinder by identifying points (x, y) and $(x + L_x, y)$. The magnetic field $\mathbf{B}$ is still normal to this finite domain, and a driving electric field is still applied along the y direction. To evaluate the current $j_x(\mathbf{r})$ in the quantum mechanical ground-state of this system, Laughlin uses the following exact relation:

$$j_x(\mathbf{r}) = -\frac{\partial \langle H \rangle}{\partial A_x(\mathbf{r})} \tag{59}$$

where $\mathbf{A}(\mathbf{r})$ is the external magnetic vector potential. Let us now impose spacial variations $\delta\mathbf{A}(\mathbf{r})$ of the form: $\delta A_x(\mathbf{r}) = \frac{\delta\Phi}{L_x}$ and $\delta A_y(\mathbf{r}) = 0$. Such variations do *not* modify the gauge-invariant electric and magnetic fields, but they introduce an Aharonov-Bohm flux through any closed path winding once around the cylinder in the positive x direction. The corresponding infinitesimal variation of the system average energy is:

$$\delta\langle H \rangle = -\frac{\delta\Phi}{L_x} \int_0^{L_x} dx \int_0^{L_y} dy\, j_x(\mathbf{r}) = -\delta\Phi\, I_x \tag{60}$$

So the Hall current I_x is simply expressed as:

$$I_x = -\frac{d\langle H \rangle}{d\Phi} \tag{61}$$

Now Laughlin assumes that as Φ varies, the ground-state wave-function $|\Psi_0(\Phi)\rangle$ undergoes a smooth evolution, Φ being considered as an external parameter of the system Hamiltonian. A sufficient condition for this to occur is when the ground-state is unique, and well separated by a finite energy gap from excited states created in the *bulk* of the system. This happens for instance for non-interacting electrons with an integer filling factor in the limit of a weak impurity potential.

Let us now vary Φ by a *finite* quantity $\Phi_0 = h/e$. Note that the effect of changing Φ is simply the same as changing the periodic boundary condition along the x direction, so its effect

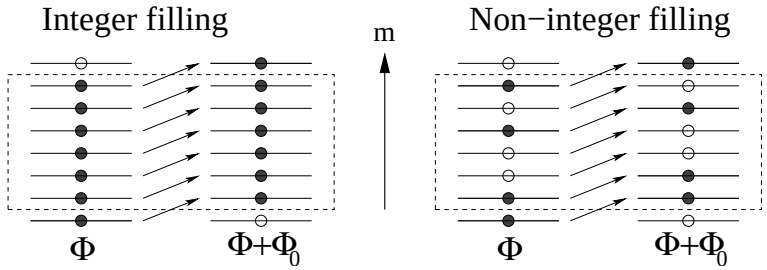

Figure 4: Illustration of the change in the spatial distribution of occupied energy states within a single Landau level, as flux Φ is changed into $\Phi + \Phi_0$. For integer filling, no change occurs in the bulk of the system (depicted by the dashed parallelogram), and the global effect is to transfer one electron from the lower to the upper boundary. For non-integer filling factor, this adiabatic process also implies a change of the level occupancy pattern in the bulk of the system.

on a macroscopic system is expected to be small. One way to see this is to consider single particle eigenstates in the Landau gauge ($A_x = -By$, $A_y = 0$). These states are localized in narrow strips centered around horizontal lines such that $y_m = \frac{2\pi l^2}{L_x}(m + \Phi/\Phi_0)$, m integer. So changing Φ into $\Phi+\Phi_0$ amounts simply to changing m into $m+1$ and the single electron spectrum is invariant in this operation. This in fact expresses the gauge-invariance of quantum-mechanics, as first emphasized by Aharonov and Bohm. In particular, this periodicity of the spectrum as a function of Φ with period Φ_0 holds for interacting electron systems such as those described by the Hamiltonian (45). Denoting by $\Delta\langle H\rangle$ the variation of the system energy during such process, Laughlin assumes that we may still write:

$$I_x = -\frac{\Delta\langle H\rangle}{\Phi_0} \tag{62}$$

Suppose now the chemical potential is such that the ground-state is well separated from excited states by an energy gap, at least when the driving electric field vanishes. Again, this is the case for non-interacting electrons with an integer filling factor in the limit of weak impurity potential. Then upon changing Φ into $\Phi + \Phi_0$, we cannot modify the wave-function in the bulk of the system. However, as the example of non-interacting electrons suggests (see Fig. 4), we may still transfer an integer number p of electrons (since the quantum number m is shifted into $m + 1$) from the lower edge to the upper edge. More precisely, for non-interacting electrons with an integer filling factor ν, then $p = \nu$. In this situation, the energy variation $\Delta\langle H\rangle$ during the shift from Φ to $\Phi + \Phi_0$ is of purely electrostatic origin, so that:

$$\Delta\langle H\rangle = -pe(V(L_y) - V(0)) \tag{63}$$

From Eq. (62), this yields:

$$I_x = p\frac{e^2}{h}(V(L_y) - V(0)) \tag{64}$$

or equivalently, $\sigma_{xy} = p\frac{e^2}{h}$.

The strength of this argument is that it is also valid for interacting systems, in the presence of a random potential, as long as the excitation gap present for the pure non-interacting system at integer ν is preserved. To some extent, we may even drop the weak disorder assumption. To see this, let us consider a non-interacting system, but with a possibly large disorder. The density of states has schematically the shape shown on Fig. 5 where the gaps of the Landau level spectrum have been partially filled under the influence of the random impurity potential. In two dimensions, (and in absence of magnetic field), it is very likely that all energy eigenstates are spacially localized [9]. Such localized wave-functions are mostly insensitive to changing boundary conditions, and therefore do not contribute to the adiabatic charge transport process involved in Laughlin's argument. This should yield a vanishing Hall conductance. However, there are theoretical arguments [10, 11] and substantial numerical evidence [12] that in the presence of a uniform magnetic

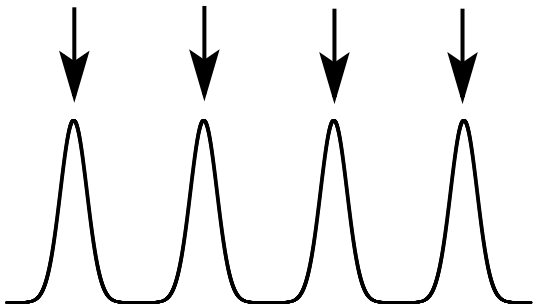

Figure 5: Schematic plot of the density of single particle energy levels, as a function of energy in the presence of a static random impurity potential. The arrows correspond to positions of Landau levels for a pure system, and to extended states in the disordered case. The random potential lifts the huge degeneracy of each Landau level, and most of energy eigenstates become spatially localized.

field, some delocalized eigenstates exist for a discrete set of energies, in one to one correspondence with the original Landau levels (see arrows on Fig. 5). Understanding precisely the onset of such extended states as the energy is tuned towards one of these critical values still remains a theoretical challenge [13]. A recent review on these magnetic-field induced delocalization transitions may be found in a paper by Kramer et al. [14]. Combining this picture of the single-particle spectrum (mostly localized states, but isolated energies allowing for extended states) with Laughlin's argument shows that the quantized Hall conductance $\sigma_{xy} = p\frac{e^2}{h}$ may still exist in a relatively strong disorder regime.

Finally, let us mention briefly the case of arbitrary filling factors ν. We may still use Eq. (62) to evaluate the Hall conductance. In the case of non-interacting electron for a pure system, we have a partially filled Landau level crossing the Fermi energy. Therefore, the process of adding one flux quantum Φ_0 in the system translates the pattern of occupied and empty horizontal strip-like single particle states by the amount $\Delta y = \frac{2\pi l^2}{L_x}$, as shown on Fig. 4. Another way to say this is that this adiabatic process induces particle-hole excitations inside the partially occupied Landau level, which affect now the bulk of the system, and not only its boundaries. The corresponding change in electrostatic energy is then proportional to the total number of electrons, and this implementation of the Laughlin argument yields the *classical, unquantized* value $\sigma_{xy} = \nu\frac{e^2}{h}$. The situation changes dramatically in the presence of electron-electron interactions, and indeed plateau values of the form $\sigma_{xy} = \frac{p}{q}\frac{e^2}{h}$ have been observed, where p is an integer and q an odd integer [15]. In a pioneering insight [16], Laughlin explained the appearance of these fractional values as a consequence of two remarkable properties of the system:
i)Interactions are lifting completely the degeneracy of the partially filled Landau level, at least for filling factors $\nu = 1/q$, q odd. The corresponding ground-state is liquid-like, isotropic and translationally-invariant.
ii)The elementary locally charged excitations correspond to collective reorganizations of the electron fluid producing a fractional charge $e^* = 1/q$.
These two surprising properties of the energy spectrum and of elementary excitations have been incorporated by Laughlin in his 1981 argument to account for the existence of quantized plateaus in σ_{xy} with a fractional value. Since the theory of this fractional effect is the subject of S. Girvin's contribution to this seminar, we shall not discuss it further here.

3.4 The Hall conductivity as a topological invariant

Let us further modify the geometry used for the Laughlin argument by gluing together the lower ($y = 0$) and upper ($y = L_y$) edges of the cylinder, thus forming a torus. The small external driving field is still uniform, directed along $\hat{\boldsymbol{y}}$, and we shall still measure the current density $\langle j_x \rangle$

along the $\hat{\boldsymbol{x}}$ direction. As we wish to apply linear response theory, it is convenient to work with a time-dependent electric field:

$$E_y(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{E}_y(\omega). \tag{65}$$

The frequency-dependent Hall conductivity $\sigma_{xy}(\omega)$ is defined by:

$$\tilde{j}_x(\omega) = \sigma_{xy}(\omega)\tilde{E}_y(\omega). \tag{66}$$

Quantum-mechanically, the simplest way to introduce an electric field is through a time-dependent vector-potential $\delta\mathbf{A}(t)$ such that $E_y(t) = -\frac{\partial \delta A_y(t)}{\partial t}$, or equivalently: $\tilde{E}_y(\omega) = i\omega\delta\tilde{A}_y(\omega)$. If we define $K_{xy}(\omega)$ to be such that $\tilde{j}_x(\omega) = K_{xy}(\omega)\delta\tilde{A}_y(\omega)$, then: $\sigma_{xy}(\omega) = \frac{K_{xy}(\omega)}{i\omega}$. For a uniform electric field, δA_y is also spatially uniform. In real space and time, the response kernel K_{xy} is given by the standard Kubo linear response formula:

$$K_{xy}(\mathbf{r}, t; t') = \frac{i}{\hbar}\langle[\frac{\delta H}{\delta A_x(\mathbf{r})}, \int \frac{\delta H}{\delta A_y(\mathbf{r}')} d^2\mathbf{r}']\rangle, \tag{67}$$

where we have used again $j_x(\mathbf{r}) = -\frac{\delta H}{\delta A_x(\mathbf{r})}$, and the fact that δA_y is uniform. The quantum-mechanical expectation values are taken in the ground state of the system, since we are assuming a very low temperature. In the absence of impurities, we expect a uniform current, but if impurities are present, only the total current $I_x(x) = \int_0^{L_y} dy\; j_x(x, y)$ is independent of x (because of current conservation) in the static limit. It is therefore natural to average the above response function over the "probe" position $\mathbf{r}$. Introducing fluxes Φ_x and Φ_y as in the previous section, but now along the two main directions of the torus, we may write this space-averaged response function as:

$$K_{xy}(\mathbf{r}, t; t') = \frac{i}{\hbar}\frac{1}{L_x L_y}\langle[\int \frac{\delta H}{\delta A_x(\mathbf{r})} d^2\mathbf{r}, \int \frac{\delta H}{\delta A_y(\mathbf{r}')} d^2\mathbf{r}']\rangle = \frac{i}{\hbar}\langle[\frac{\partial H}{\partial \Phi_x}, \frac{\partial H}{\partial \Phi_y}]\rangle. \tag{68}$$

Transforming to Fourier-space, we now obtain:

$$\sigma_{xy}(\omega) = \frac{i}{\hbar\omega}\sum_\alpha \left\{ \frac{\langle 0|\partial_x H|\alpha\rangle\langle\alpha|\partial_y H|0\rangle}{\omega - \omega_{\alpha 0}} - \frac{\langle 0|\partial_y H|\alpha\rangle\langle\alpha|\partial_x H|0\rangle}{\omega + \omega_{\alpha 0}} \right\} \tag{69}$$

where $|0\rangle$ is the ground-state and $|\alpha\rangle$ denotes a complete orthonormal basis of energy eigenstates of H, with energies E_α. The Bohr frequencies $\omega_{\alpha 0}$ are equal to $(E_\alpha - E_0)/\hbar$. In this expression, H is the *full* Hamiltonian of the system in the absence of driving electric field. It may therefore include both impurity potentials and interaction effects. To simplify notations, $\partial_x H$ and $\partial_x H$ stand respectively for $\frac{\partial H}{\partial \Phi_x}$ and $\frac{\partial H}{\partial \Phi_y}$. Gauge-invariance requires that the current vanishes when a static uniform vector potential is applied. This enables us to replace the above expression by:

$$\sigma_{xy}(\omega) = \frac{i}{\hbar}\sum_\alpha \left\{ \frac{\langle 0|\partial_x H|\alpha\rangle\langle\alpha|\partial_y H|0\rangle}{\omega_{\alpha 0}(\omega - \omega_{\alpha 0})} + \frac{\langle 0|\partial_y H|\alpha\rangle\langle\alpha|\partial_x H|0\rangle}{\omega_{\alpha 0}(\omega + \omega_{\alpha 0})} \right\} \tag{70}$$

which has a well-defined static limit $\omega \to 0$ provided the system has a finite energy gap, so that denominators are not vanishing in this limit. We may then write the static Hall conductance as:

$$\sigma_{xy} = \frac{\hbar}{i}\sum_\alpha \left\{ \frac{\langle 0|\partial_x H|\alpha\rangle\langle\alpha|\partial_y H|0\rangle}{(E_0 - E_\alpha)^2} - \frac{\langle 0|\partial_y H|\alpha\rangle\langle\alpha|\partial_x H|0\rangle}{(E_0 - E_\alpha)^2} \right\} \tag{71}$$

It is now convenient to view the Aharonov-Bohm fluxes Φ_x and Φ_y as external parameters. The ground-state $|0\rangle$ becomes a function of $(\Phi_x, \Phi_y) \equiv \mathbf{\Phi}$ and we shall denote it by $|\mathbf{\Phi}\rangle$. This allows us to recast the previous equation as:

$$\sigma_{xy}(\mathbf{\Phi}) = \frac{\hbar}{i}\left(\frac{\partial\langle\mathbf{\Phi}|}{\partial\Phi_x}\frac{\partial|\mathbf{\Phi}\rangle}{\partial\Phi_y} - \frac{\partial\langle\mathbf{\Phi}|}{\partial\Phi_y}\frac{\partial|\mathbf{\Phi}\rangle}{\partial\Phi_x} \right). \tag{72}$$

This may be regarded as the curl of a two-dimensional vector:

$$\frac{1}{i}\left(\langle\mathbf{\Phi}|\frac{\partial|\mathbf{\Phi}\rangle}{\partial\Phi_x},\langle\mathbf{\Phi}|\frac{\partial|\mathbf{\Phi}\rangle}{\partial\Phi_y}\right). \tag{73}$$

Since $\langle\mathbf{\Phi}|\mathbf{\Phi}\rangle = 1$, this vector has purely real components. The above expression depends on a choice of two Aharonov-Bohm fluxes (Φ_x, Φ_y), which by gauge transformations is equivalent to choosing the following boundary conditions for the wave-functions (for simplicity of notation, we consider just one electron here, since generalization to N electrons is obvious):

$$\Psi(x+L_x,y) = e^{i2\pi\frac{\Phi_x}{\Phi_0}}\Psi(x,y) \tag{74}$$

$$\Psi(x,y+L_y) = e^{i2\pi\frac{\Phi_y}{\Phi_0}}\Psi(x,y) \tag{75}$$

This is of course connected to the idea that all physical quantities, like $\sigma_{xy}(\mathbf{\Phi})$ are periodic functions of both Φ_x and Φ_y with period Φ_0. So the $\mathbf{\Phi}$-plane may be folded onto a two-dimensional torus.

Let us now make the assumption that $\sigma_{xy}(\mathbf{\Phi})$ is only very weakly modified upon changing these boundary conditions. In the case where the ground-state is well separated from excited states by a finite energy gap, arguments have been given to show that $\sigma_{xy}(\mathbf{\Phi})$ becomes constant for a large system, up to corrections of order l/L_x, l/L_y [6]. We may therefore replace $\sigma_{xy}(\mathbf{\Phi})$ by its average over the $\mathbf{\Phi}$-torus and then transform the two-dimensional integral of a curl into a line-integral along the boundary of the square $[0,\Phi_0]\times[0,\Phi_0]$:

$$\sigma_{xy} = \frac{e^2}{h}\frac{1}{2\pi i}\int_{\square}\langle\mathbf{\Phi}|d|\mathbf{\Phi}\rangle. \tag{76}$$

But as shown in Appendix, the quantity $\frac{1}{2\pi i}\int_{\square}\langle\mathbf{\Phi}|d|\mathbf{\Phi}\rangle$ is equal to $2\pi n_{\mathrm{C}}$ where n_{C} is an integer called a Chern number. Finally, we get [17, 6]:

$$\sigma_{xy} = n_{\mathrm{C}}\frac{e^2}{h}. \tag{77}$$

Note that this formula seems to be valid in great generality, so we may wonder how we might explain fractional values for the Hall conductance. In fact, the above derivation requires the ground-state to be *unique* for all values of $\mathbf{\Phi}$. For fractional filling factors of the form $\nu = p/q$, Tao and Haldane have shown that the ground-state is q-fold degenerate on a torus in the absence of impurities [18]. Furthermore, since these degeneracies are associated only to the center of motion of the electron fluid, the effect of impurities on the energy spectrum is rather small whenever there is an energy gap for internal excitations. These authors have shown how such degeneracies induce values of the form p/q in unity of e^2/h for the Hall conductivity.

We have not covered here all the aspects related to this description of the Hall conductivity as a topological invariant. A recent very accessible review may be found in [19]. We also mention briefly that such topological ideas have generated a rigorous proof of the integer Hall conductance quantization for an infinite disordered system of non-interacting electrons [20]. This work uses a rather elaborate mathematical apparatus (K-theory for C^* algebras) which we will not try to describe here, so the interested reader is invited to consult the original paper [20].

3.5 Edge-state picture of the Quantum Hall effect

Starting from an influential paper by Halperin [21], this viewpoint has been emphasized by Büttiker [7], and plays a crucial role in experiments aimed at showing the existence of fractionally charged quasi-particles in the fractional regime. These experiments will be discussed in detail by C. Glattli. Here, we shall just present a brief introduction to this approach.

Usually, experiments and metrological applications of the quantum Hall effect involve rectangular-shaped samples schematized on Fig. 6. Current is injected along the long axis of the Hall bar, and voltage probes located on the sides of the sample are used to measure longitudinal and Hall resistivities. In such systems, there is a strong but smooth lateral confining potential

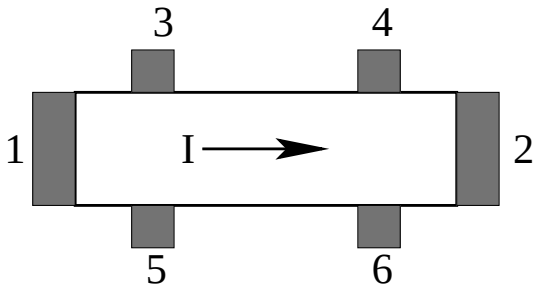

Figure 6: Typical geometry of a Hall bar. The current I is injected in the sample through contacts labelled 1 and 2. Lateral contacts labelled from 3 to 6 are used to measure the longitudinal voltage as for instance $V_3 - V_4$ or the Hall voltage as for instance $V_3 - V_5$.

for electron approaching outer boundaries. The corresponding semi-classical spectrum for non-interacting electron has the shape shown on Fig. 7(a) inspired from [21]. From our description of the dynamics in a large magnetic field, we expect that even in the absence of external electric field, the strong confining potential gradients will induce static currents along boundaries. With a positive magnetic field along the $\hat{\boldsymbol{z}}$ direction, electrons acquire a positive group velocity along the upper boundary ($y \simeq L_y$) and a negative group velocity along the lower one ($y \simeq 0$). The same reasoning as in section 3.2 applies here showing that the total current across a section of the system vanishes, so there is no global current along the sample in equilibrium. What happens when the system is driven out of equilibrium by a non-zero average longitudinal current injected in the sample by external contacts (labeled 1 and 2 on Fig. 6)? This simply means that the population of edge states will be increased (resp. decreased) with respect to their equilibrium values when edge currents move in the same (resp. opposite) direction as the injected current. In other words, the chemical potentials $\mu(L_y)$ and $\mu(0)$ on both edges are now different. The difference $\mu(L_y) - \mu(0)$ is precisely equal to $-e(V(L_y) - V(0))$ which is the energy cost to transfer an electron from the Fermi level at $y = 0$ to the Fermi level at $y = L_y$. Adapting Eq. (53), we now have:

$$I_{x,n}(x_0) = \frac{e^2}{h} \int_{-\infty}^{\infty} dy\, \theta(\mu(y) + eW(x_0, y) - \hbar\omega(n + \frac{1}{2})) \frac{\partial W}{\partial y}(x_0, y). \tag{78}$$

Here, $W(x, y)$ is the sum of random impurity and confining electrostatic potentials. By contrast to the discussion in section 3.2, it does not include an external driving field, since the current is viewed here as the result of imposing an out of equilibrium distribution of single-particle states along edges. The integral has now been extended to $[-\infty, \infty]$ since the precise locations of the sample edges do depend on the populations of edges states as illustrated on Fig. 7(b). Each filled Landau level produces now an integrated current:

$$I_{x,n}(x_0) = \frac{e^2}{h}(W(L_y) - W(0)). \tag{79}$$

But we have:

$$-e(W(L_y) - W(0)) = \mu(L_y) - \mu(0) = -e(V(L_y) - V(0)) \tag{80}$$

So finally:

$$I_{x,n}(x_0) = \frac{e^2}{h}(V(L_y) - V(0)) \tag{81}$$

for each filled Landau level.

These edge states have been the subject of intense research during the last fifteen years. Many directions have been explored, including the precise modeling of electron transport through mesoscopic coherent samples [22, 23, 24], the theoretical and experimental investigation of interaction effects [25, 26], the generalization of edge states to the fractional quantum Hall regime [27], and possible applications of edge channels to quantum information processing [28, 29]...

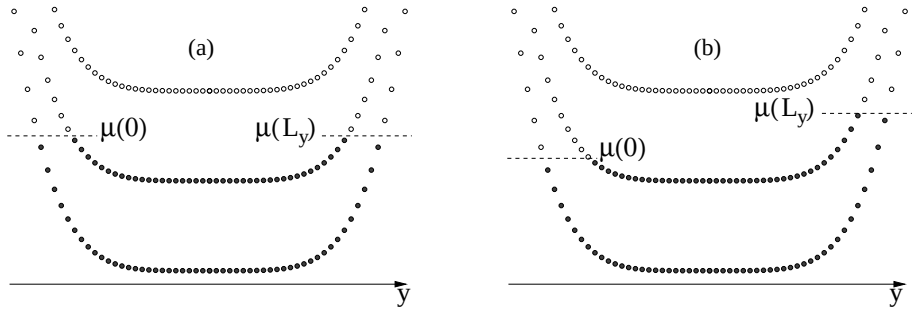

Figure 7: Semi-classical energy spectrum for three Landau levels, as a function of the position y across a Hall bar. In (a), an equilibrium situation is depicted, where the chemical potential is uniform. Filled (resp. empty) circles represent occupied (resp. empty) single-particle states. Currents flow along the x direction in regions where these levels depend strongly on y, that is near the edges. In (b), a non-equilibrium state is depicted, with a smaller chemical for $y = 0$ than for $y = L_y$. As a result, the sum of currents flowing along both edges is non-zero.

4 Interactions, a preview

This section has no ambition to be exhaustive, it reflects the author's understanding of the role of interactions in the Hall effect. We have seen earlier that the main feature of electrons in a strong magnetic field is the large degeneracy of the Landau levels. It is therefore natural to expect that in this regime, the physics can be understood through degenerate perturbation theory. In all phenomena where the filling factor is less than one, the projection on the lowest Landau level should therefore give an accurate description of the physics. Here, we indicate how the projection mechanism results in rigid properties of the interacting electron system, which are fairly independent of interactions involved. Also, the properties of the quasiparticles which emerge, such as their charge, are completely different from those of the original electrons.

It is instructive to consider the dynamics of two particles within the lowest Landau level. The two particles interact through a potential $V(\mathbf{r_1} - \mathbf{r_2})$ which is supposed to be both translation and rotation invariant. In a physical situation the potential is the Coulomb interaction between the electrons, but it can in principle be any potential. For reasons that will become clear in the text we consider particles with a charge respectively equal to q_1 and q_2 times the charge of the electron. Our aim is to show that, to a large extent, the properties of the dynamics are independent of the detailed shape of the potential. More precisely, the potential interaction is a two body operator which can be projected into the lowest Landau level. Up to normal ordering ambiguities, the projection consists in replacing the coordinates, $\mathbf{r_1}$, $\mathbf{r_2}$, with the guiding center coordinates, $\mathbf{R_1}$, $\mathbf{R_2}$. After the projection is taken, the potential becomes an operator which is the effective Hamiltonian for the lowest Landau level dynamics. By choosing conveniently a basis, we can see that the eigenstates of the potential do not depend on it, as long as it is invariant under the isometries of the plane. In other words, the two body wave functions of the Hall effect are independent of the interactions. By extension, we are led to expect that the many body wave functions have some universality properties, and do not depend on the details of the potential.

In this section, we use the symmetric gauge, and l denotes the magnetic length (29). The guiding center coordinates for a particle of charge $q > 0$ times the charge of the electron have the expression:

$$b = \sqrt{2}(l\partial_{\bar{z}} + q\frac{z}{4l}), \quad b^+ = \sqrt{2}(-l\partial_z + q\frac{\bar{z}}{4l}). \tag{82}$$

Together with the angular momentum, L, they generate a central extension of the algebra of the isometries of the plane:

$$[b, b^+] = q, \quad [L, b^+] = -b^+, \quad [L, b] = b. \tag{83}$$

This algebra commutes with the Hamiltonian H, and therefore acts within the lowest Landau level. It plays a role similar to the angular momentum in quantum mechanics, and the operators b, b^+, L are the analogous of the angular momentum operators J^-, J^+, J^z . The Landau level index n plays the same role as the representation index j in the rotation group, and it can be recovered as the eigenvalue of a Casimir operator: $C = 2b^+b/q + L$. The states within each Landau level can be labeled by their angular momentum $m \leq n$.

When two particles of positive charge q_1 and q_2 are restricted to their respective lowest Landau level, we can form the operators $b^+ = b_1^+ + b_2^+$, $b = b_1 + b_2$ and the total angular momentum $L = L_1 + L_2$. These operators obey the commutation relations of the algebra (83) with the charge $q = q_1 + q_2$. Thus, as for the angular momentum, a product of two representations decomposes into representations of the isometry of the plane (83). The physically interesting case is when the two charges are equal to the electron charge ($q_1 = q_2 = 1$). It is easy to verify that each representation is constructed from a generating state state annihilated by b: $(b_1^+ - b_2^+)^n|0\rangle$, and the value of the Casimir operator is $C = -n$. The corresponding wave functions are:

$$\Psi_n(\bar{z}_1, \bar{z}_2) = (\bar{z}_1 - \bar{z}_2)^n \exp\left(-(\bar{z}_1 z_1 + \bar{z}_2 z_2)/4l^2\right), \tag{84}$$

an expression that plays an important role in the theory of the fractional Hall effect. The potential being invariant under the displacements, it is a number V_n in each representation. Conversely, the information about the V_n is all the information about the potential that is retained by the lowest Landau level physics. The numbers V_n are called pseudopotentials, and turn out to be extremely useful to characterize the different phases of the fractional Hall effect [30].

A case of even more interest is when the two particles have charges of opposite sign, $q_1 > 0$ and $q_2 < 0$, $|q_2| < q_1$. Because of the sign of the second charge, b_2^+ and b_2 become respectively annihilation and creation operators and the lowest Landau level wave functions are polynomials in z_2 instead of $\bar{z}_2$. The same analysis can be repeated, but now the Casimir operator has a positive value n exactly as for the Landau levels. The physical interpretation is that a couple of charges with opposite sign behaves exactly like a bound state of charge $q^* = q_1 - |q_2|$. The states annihilated by b have a wave function independent of the precise expression of the potential, given by:

$$\Psi_n(\bar{z}_1, z_2) = z_2^n \exp\left(-q_1 \bar{z}_1 z_1/4l^2 - |q_2|\bar{z}_2 z_2/4l^2 + |q_2|\bar{z}_1 z_2/2l^2\right), \tag{85}$$

and they are the n^{th} Landau level's wave functions with the largest possible angular momentum $L = n$.

Heuristically, let us indicate how a scenario involving these composite particles enables to apprehend the region of magnetic field between 21 and 27 Tesla on Fig.1. For this, we use the well established theoretical fact that a quantized Hall effect (for bosons) develops at the filling factor $\nu_0 = 1/2$. We assume that the ground state is the $\nu_0 = 1/2$ quantum Hall liquid made of particles of charge $-q_1 e$. The physical motivation to start from this bosonic ground-state is that it exhibits a rather low Coulomb energy, since the probability for two particles to come close from each other is small in this state. More details on these correlated ground-states are presented in S. Girvin's contribution to which we direct the reader. To recover fermionic statistics, we add on top of this ground state a sea of quasiparticles which are the bound-states introduced above, made of an electron of charge $-e$ and a hole (of charge $q_1 e$) in the $\nu_0 = 1/2$ ground state . The charge of the quasiparticles is thus $-q^* e$, with:

$$q^* = 1 - q_1. \tag{86}$$

To obtain the values of the filling factor that give rise to a Hall effect, using (29) and (31) generalized to particles of an arbitrary charge, we see that for a fixed magnetic field and a fixed density, the following proportionality relation between the charge and the filling factor holds:

$$\text{charge} \propto \frac{1}{\text{filling factor}}, \tag{87}$$

When the magnetic field is varied, the charge $q_1 e$ adjusts itself so that the filling factor of the ground-state is always equal to 1/2. Thus, $q_1 e \propto 2$. Then, an integer quantum Hall effect will

develop when the filling factor of the quasiparticles is an integer p. So, when $q^*e \propto 1/p$. Finally, we can recover the normalization coefficient through the relation between the filling factor of the electrons and their charge: $e \propto 1/\nu$. Substituting these relations in (86) we obtain the following expression for the filling factors giving rise to a Hall effect:

$$\frac{1}{\nu} = 2 + \frac{1}{p}. \tag{88}$$

These filling factors are those predicted by Jain [31], and fit well with the Hall effect observed at $\nu = 3/7, 4/9, 5/11, 6/13$ in Fig.1. The fractions on the other side of $1/2$ are the complement to one of the previous ones, and the corresponding states can be obtained through a particle hole transformation. In the region, close to $\nu = 1/2$, the quasiparticles have practically zero charge, and therefore see a weak magnetic field. They should therefore behave very much like a neutral Fermi liquid. This has been confirmed by several experiments. One of them measures directly the charge q^* of the quasiparticles through the cyclotron radius of their trajectory [32] (see the footnote of section 2.1).

At $\nu = 1/2$ exactly, let us introduce a simple model to grasp the physics of these composite particles. At this value of the magnetic field, the particle and the hole have exactly opposite charge so that the quasiparticle has exactly zero charge (86). Let us assume for simplicity that the particle and the hole are linked by a spring of strength K. In a strong magnetic field, we disregard the kinetic energy term in the action (14). Thus, after we include the interaction term, the action becomes:

$$S = \int \left(-e\mathbf{A}(\mathbf{r_1}).\dot{\mathbf{r}}_1 + e\mathbf{A}(\mathbf{r}_2).\dot{\mathbf{r}}_2 - \frac{K}{2}\left(\mathbf{r}_1 - \mathbf{r}_2\right)^2 \right) dt \tag{89}$$

If we denote by $\mathbf{P} = \mathbf{p_1} + \mathbf{p_2}$ the total momentum of the system, straightforward quantization leads to:

$$\mathbf{P} = \hbar \hat{\mathbf{z}} \wedge \frac{\mathbf{r}_1 - \mathbf{r}_2}{l^2}, \tag{90}$$

and

$$H = \frac{\mathbf{P}^2}{2m^*}, \tag{91}$$

where the quasiparticle mass is $m^* = (Be)^2/4K$. Thus, these neutral quasiparticles behave like free particles, and do not feel the external magnetic field. Note that their effective mass m^* is independent of the true electron mass, and reflects the properties of the interactions. Eq. (90) tells us that the quasiparticles are dipoles oriented perpendicularly to $\mathbf{P}$, with a dipole size proportional to the momentum. At this moment, no experimental evidence of their dipolar structure has yet been given.

Finally, let us say a few words about the second quantized formalism in the lowest Landau level. If $V(\mathbf{r})$ denotes the interacting potential, the dynamics is governed by the Hamiltonian:

$$H = \int \tilde{V}(\mathbf{q}) \rho_{\mathbf{q}} \rho_{-\mathbf{q}} d^2\mathbf{q}, \tag{92}$$

where $\tilde{V}(\mathbf{q})$ are the Fourier modes of the potential multiplied by the short distance cut-off factor $e^{-\mathbf{q}^2 l^2/2}$, and $\rho_{\mathbf{q}}$ are the Fourier modes of the density. Again, due to the projection to the lowest Landau level, the density Fourier modes do not commute as in the usual case. Instead, they obey the commutation relations analogous to (23) ($\rho_{\mathbf{q}} \equiv \sum_{i=1}^{N_e} e^{i\mathbf{q}.\mathbf{R_i}}$):

$$[\rho_{\mathbf{q}}, \rho_{\mathbf{q}'}] = \frac{1}{i} \sin \frac{l\mathbf{q} \wedge l\mathbf{q}'}{2} \rho_{\mathbf{q}+\mathbf{q}'}. \tag{93}$$

It can be verified that for a finite size system, $\mathbf{q}$ can take only N^2 values where $N = \text{area}/2\pi l^2$ is the degeneracy of the lowest Landau level, and the algebra (93) is the Lie algebra of the group $U(N)$. In the limit of strong magnetic field, $l \to 0$, and one recovers the algebra of area preserving diffeomorphisms (23) as the classical limit of (93).

A Chern number for a two-dimensional torus

In section 3.4, we have introduced the ground-states $|\mathbf{\Phi}\rangle$, where $\mathbf{\Phi} \equiv (\Phi_x, \Phi_y)$ denotes two Aharonov-Bohm fluxes associated to the two main topologically non-trivial closed loops winding around the real-space torus defining our electron system. These states are periodic functions of $\mathbf{\Phi}$, up to possible phase-factors, so we may write:

$$|\Phi_0, \Phi_y\rangle = e^{i\lambda(\Phi_y)}|0, \Phi_y\rangle \quad (94)$$

$$|\Phi_x, \Phi_0\rangle = e^{i\mu(\Phi_x)}|\Phi_x, 0\rangle \quad (95)$$

$$(96)$$

In particular, we get:

$$|\Phi_0, \Phi_0\rangle = e^{i(\lambda(0)+\mu(\Phi_0))}|0,0\rangle = e^{i(\mu(0)+\lambda(\Phi_0))}|0,0\rangle \quad (97)$$

Consequently:

$$\lambda(\Phi_0) - \lambda(0) - \mu(\Phi_0) + \mu(0) = 2\pi\, n \quad (98)$$

where n is an integer. Now:

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\square}\langle\mathbf{\Phi}|d|\mathbf{\Phi}\rangle &= \frac{1}{2\pi i}\int_0^{\Phi_0} d\Phi_x \left(\langle\mathbf{\Phi}|\frac{\partial}{\partial\Phi_x}|\mathbf{\Phi}\rangle(\Phi_x,0) - \langle\mathbf{\Phi}|\frac{\partial}{\partial\Phi_x}|\mathbf{\Phi}\rangle(\Phi_x,\Phi_0)\right) \\
&\quad + \frac{1}{2\pi i}\int_0^{\Phi_0} d\Phi_y \left(\langle\mathbf{\Phi}|\frac{\partial}{\partial\Phi_y}|\mathbf{\Phi}\rangle(\Phi_0,\Phi_y) - \langle\mathbf{\Phi}|\frac{\partial}{\partial\Phi_y}|\mathbf{\Phi}\rangle(0,\Phi_y)\right) \quad (99)\\
&= \frac{1}{2\pi i}\int_0^{\Phi_0} d\Phi_x \left(-i\frac{\partial\mu}{\partial\Phi_x}(\Phi_x)\right) + \frac{1}{2\pi i}\int_0^{\Phi_0} d\Phi_y \left(i\frac{\partial\lambda}{\partial\Phi_y}(\Phi_y)\right) \quad (100)\\
&= \frac{1}{2\pi}(\lambda(\Phi_0) - \lambda(0) - \mu(\Phi_0) + \mu(0)) \quad (101)\\
&= n \quad (102)
\end{aligned}$$

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