PROCEEDINGS OF THE SECOND MARCEL GROSSMANN MEETING ON GENERAL RELATIVITY
R. Ruffini (editor)

SURFACE EFFECTS IN BLACK HOLE PHYSICS

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This contribution reviews briefly the various analogies which have been drawn between black holes and ordinary physical objects. It is shown how, by concentrating on the properties of the surface of a black hole, it is possible to set up a sequence of tight analogies allowing one to conclude that a black hole is, qualitatively and quantitatively, similar to a fluid bubble possessing a negative surface tension and endowed with finite values of the electrical conductivity and of the shear and bulk viscosities. These analogies are valid simultaneously at the levels of electromagnetic, mechanical and thermodynamical laws. Explicit applications of this framework are worked out (eddy currents, tidal drag). The thermostatic equilibrium of a black hole electrically interacting with its surroundings is discussed, as well as the validity of a minimum entropy production principle in black hole physics.

I. INTRODUCTION.

The "gravitational field of a point mass" (Schwarzschild 1916)

$$ds^2 = -(1-\frac{2M}{r})\, dt_s^{\,2} + (1-\frac{2M}{r})^{-1}\, dr^2 + r^2(d\,\theta^2 + \sin^2\theta d\phi^2) \qquad (1)^*$$

seems to be plagued not only with the (expected) singularity at the "center": r=0 but also with a singularity at the "Schwarzschild radius" : r=2M. The spurious character of this "Schwarzschild singularity" was first realized by Lemaître (1933): the singularity at r=2M is due to a bad choice of coordinates. Using for instance the transformation (Eddington 1924, Finkelstein 1958):

$$t = t_S + r + 2M \ln (r - 2M), \qquad (2)$$

the line element (1) reads :

$$ds^2 = -(1-2M/r)\, dt^2 + 2dtdr + r^2(d\,\theta^{\,2} + \sin^2\theta\, d\,\phi^2), \qquad (3)$$

which is manifestly regular (in fact analytic) near r=2M.

Much later it was realized that the hypersurface r = 2M was endowed with very special properties : it is a stationary null hypersurface, neither expanding nor contracting as wave fronts usually do. Moreover, after the discovery by Kerr (1963) of another exact solution of the (vacuum) Einstein equations similar to eqn.(3) which was rewritten by Boyer and Lindquist (1967) in a form similar to (1), it was conjectured that the solutions of Schwarzschild and Kerr, as well as the solu-

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* We use units such that $G = c = 1$

tions of Reissner (1916) and its Kerr-like generalization (Newman et al 1965) were members of a large family of solutions where some "central singularities" are hidden behind a stationary null hypersurface (a "Killing horizon" (Carter 1969)). The discovery of neutron stars(Hewish et al 1968) together with the known existence of an upper limit to the mass of a neutron star (Chandrasekhar 1931, Landau 1932, Oppenheimer and Volkoff 1938, Rhoades and Ruffini 1974) strengthened the idea (Oppenheimer and Snyder 1938) that the Schwarzschild solution (1) was a possible final state for heavy stars, and prompted a period of intensive studies of the properties of what was soon known under the name of "black hole" : an asymptotically flat solution of Einstein equations where singularities are hidden behind a "horizon". The horizon being a null hypersurface defined as the boundary of the region from which particles and photons can escape to infinity (Penrose 1969, Carter 1971 , Hawking 1971). Therefore it became usual to consider a black hole as a new kind of <u>physical object</u> the properties of which were investigated by letting it interact with external matter, external fields or other black holes. Remarkably it was found that some of the properties of a black hole, although deriving ultimately from its very strong gravitational field, were very analogous to the properties of "ordinary bodies". By "ordinary body" we mean a body described in the frame of 19^{th} century physics, that is before the discovery of quantum and relativistic concepts.

One of the first examples of such a classical analogy for black holes is found in the work of Christodoulou(1970) and Christodoulou and Ruffini(1971). They showed that it was meaningful to use some <u>thermodynamical</u> concepts when describing the behaviour of black holes perturbed by external influences. The concept of reversible and irreversible transformations of black holes was introduced and it was proved that only a certain fraction of the mass-energy of a black hole was "free", i.e. could be extracted. This thermodynamical analogy was later extended by Bekenstein (1973) who, using a theorem of Hawking (1971), introduced the concept of black hole entropy.

On the other hand a <u>mechanical</u> and <u>electrical</u> analogy was introduced by Carter(1972) (1973) who showed that, in its state of equilibrium, the surface of a black hole, embedded in external gravitational and electromagnetic fields, possesses a uniform angular velocity of rotation and a constant (comoving) electric potential. In view of these mechanical and electrical "rigidities" at equilibrium, a black hole appeared as "analogous to an ordinary body (with finite viscosity and electrical conductivity)". Moreover all these analogies discovered about global features have been extended to <u>local</u> characteristics of black holes:

For instance Hawking and Hartle (1972) and Hartle (1973) suggested an analogy between the slowing down of a rotating black hole under the influence of a satellite and the local dissipation in the tides raised in the "shallow sea of incompressible viscous fluid" of a rotating planet. Bekenstein (1972) suggested to interpret the "ringing modes" of a black hole (Press 1971) as vibration modes of a "soap bubble model" of a black hole. We shall propose a related "bubble" model of a black hole but, contrarily to Bekenstein's model, our "bubble" will have a positive surface pressure (i.e. a negative surface tension), moreover it will be endowed with viscosity in accordance with the suggestion of Hawking and Hartle. Specially important for the following is the work of Hanni and Ruffini (1973). Analyzing the adiabatic swallowing of an electric charge by a black hole they introduced the concept of a "charge induced", on the surface of the black hole, by the external charge, by analogy with the surface charge density induced on an electric conductor. This type of approach has been extended to electrodynamical phenomena by Znajek (1978b) and Damour (1978) who introduced the concept of surface current density of a black hole and the related notion of surface resistivity of a black hole. Then Damour (1979) showed how to encompass all the mechanical and thermodynamical phenomena which can be associated with the surface of a black hole, by introducing the concepts of surface pressure, surface density of momentum and surface viscosities. In summary this approach is characterized by the introduction of a set of (fictitious) <u>surface densities</u> (of charge, electric current, momentum, entropy) and a corresponding set of local <u>intensive</u> quantities (electric potential, surface pressure, temperature). What is remarkable is that the laws connecting these quantities in the most general non equilibrium state of a black hole, as deduced from Einstein-Maxwell equations, become identical to some well known laws of prerelativistic physics : Ohm's law, Joule effect, Navier-Stokes equation, ... In this sense there is a precise analogy between a black hole and a fluid bubble endowed with shear and bulk viscosity and electrical conductivity.

This paper is organized as follows: In section II we shall discuss the mathematical concepts used in describing the " kinematics of horizons" and we shall introduce the new concept of surface velocity of a black hole. In section III the electromagnetic properties of the surface of a black hole will be studied with the conclusion that the surface resistivity of a black hole is equal to 377 ohms. In section IV we shall describe the local mechanics of the surface of a black hole, introduce the concepts of surface pressure and surface momentum density with the conclusion that the surface shear (resp. bulk) viscosity of a black hole is equal to $(16\pi)^{-1}$ (resp. $-(16\pi)^{-1}$). In section V we shall check that the analogies put forward in the preceding sections are consistent with the usual ideas about black hole thermodynamics and we shall briefly discuss the electrical equilibrium of a black hole. Finally in the appendix, the geometrical tools needed in the rest of the paper are provided.

II. KINEMATICS OF HORIZONS.

Before discussing some aspects of the physics of a black hole it is convenient to present the mathematical tools which are needed to describe the evolution of the intrinsic, and extrinsic, geometry of the "horizon" or "surface of the black hole"*. The result which we shall take as a basis for our derivations is that the horizon is a null hypersurface admitting compact sections and generated by non terminating null geodesics (Penrose 1964, Hawking and Ellis 1973). In order to study the evolution of the horizon (see also Carter 1979) we introduce an arbitrary "time" coordinate t (similar to the regular time coordinate t appearing in I(3), and two arbitrary "surface" coordinates x^A (A =2,3) on each section S (t =const.) of the horizon (similar to θ and ϕ in I(3)). At this stage "t" is just a label in a slicing of space-time by spatial hypersurfaces, later in the applications t will be normalized in function of the Killing vector of time translations (when it exists). Therefore t can be thought of as the "time at infinity". If $\vec{\ell}$ denotes the 4-vector normal, and therefore tangent, to the horizon, we can parametrize the trajectories of $\vec{\ell}$, the "generators", by t. Hence in a 4-dimensional coordinate system** x^a the equations for the generators will be :

$$x^a = x^a(t) \quad . \tag{1}$$

Then we can normalize $\vec{\ell}$ such that :

$$\ell^a = \frac{d x^a(t)}{dt} \quad . \tag{2}$$

In the following we will continuously split the spacetime structure of the horizon in time (the parameter t) plus space (the twodimensional surface S). In such a "newtonian" description the horizon appears as a compact 2-surface S (a "bubble") which moves and changes with time t. We shall consider the generators (1) as the trajectories of the "particles" constituting the "bubble". Hence we can introduce (Damour 1978) the concept of surface velocity $\vec{v} = v^A\, \partial/\partial x^A$ of a black hole as

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* We shall call indinstinctly "surface of the black hole" the absolute event horizon or one of its time sections.

** The coordinates are such that $x^0 \equiv t$ and $x^1 = 0$ on the horizon. Spacetime indices : a,b = 0,1,2,3 ; Horizon indices : $\bar{A}$, $\bar{B}$ = 0,2,3 ; 2-surface indices : A,B=2,3. See the appendix for details.

the newtonian velocity of these "particles" :

$$V^A = \frac{dx^A(t)}{dt} \quad . \tag{3}$$

As a direct consequence of this definition the normal vector reads :

$$\vec{\ell} = \ell^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial t} + V^A \frac{\partial}{\partial x^A} \quad . \tag{4}$$

In the axisymmetric case , i.e. when there exists a rotational Killing vector $\vec{m} = m^a\ \partial/\partial\ x^a = \partial\ /\ \partial\phi$, one can choose the time slices such that the velocity is :

$$\vec{V} = V^A \frac{\partial}{\partial x^A} = \Omega_H \frac{\partial}{\partial \varphi} \tag{5}$$

where Ω_H is the angular velocity of the hole.* Moreover we shall use the well known concept of a 2-dimensional metric γ_{AB} induced by the 4-dimensional metric g_{ab} on each section S :

$$ds^2\big|_S = g_{ab}\, dx^a dx^b \Big|_{\substack{x^0 = const. \\ x^1 = 0}} = \gamma_{AB}\, dx^A dx^B \tag{6}$$

The coefficients γ_{AB}** define a time dependent riemannian metric on each section S (see e.g.Smarr 1973 for the Kerr-Newman case). Introducing the inverse γ^{AB} and the determinant $\gamma = \det \gamma_{AB}$, we can express the area of a surface element of S as :

$$dS_H = \sqrt{\gamma}\ dx^2 \wedge dx^3 \quad . \tag{7}$$

We can compute the squared magnitude of the surface velocity (3) :

$$V^2 = \gamma_{AB}\, V^A V^B \quad . \tag{8}$$

In the case of Kerr-Newman black holes, $\partial/\partial t$ being taken as the time Killing vector, the velocity satisfies :

$$V^2 \leqslant 1 \tag{9}$$

the equality being reached only on the equator of an extreme Kerr hole (a = M). It is tempting to conjecture that the inequality (9) holds for general externally perturbed stationary holes.

Now we notice that we have normalized $\vec{\ell}$ in such a way that it transports, in the sense of Lie, the sections S .

Therefore we can define the Lie derivative, with respect to $\vec{\ell}$, of any tensorial quantity defined on the sections S . By analogy with the mechanics of continuous

* for a clear discussion of the physical meaning of this concept see Ruffini(1973)

** numerically $\gamma_{AB} = g_{AB}$ (A,B = 2,3) in the chosen coordinate system but it is convenient to use a new notation for this induced metric.

media we call this Lie derivative the "convective derivative" and we denote it by the symbol D/dt (see the appendix for the general expression of the convective derivativeof a tensor or of a tensor density). In particular half the Lie derivative of the metric γ_{AB} measures how much the distances between "particles" (i.e. between generators) change during a time interval dt. As usual (Sachs 1964) we decompose this deformation rate : $1/2\ D\ \gamma_{AB}/dt$ in its irreductible parts. The trace :

$$\theta \equiv \frac{1}{2} \gamma^{AB} \frac{D\gamma_{AB}}{dt} \tag{10}$$

is the expansion rate and the trace free part:

$$\sigma_{AB} \equiv \frac{1}{2} \frac{D\gamma_{AB}}{dt} - \frac{1}{2} \theta\, \gamma_{AB} \tag{11}$$

is the shear tensor. The expansion θ measures the rate of change of the area of a surface element (7) under a Lie transport ; indeed we have :

$$\frac{D\, dS_H}{dt} = \frac{D\sqrt{\gamma}}{dt}\, dx^2 \wedge dx^3 = \frac{1}{2} \gamma^{AB} \frac{D\gamma_{AB}}{dt} \sqrt{\gamma}\, dx^2 \wedge dx^3 = \theta\, dS_H \tag{12}$$

This completes our account of the intrinsic geometry of the horizon, the elements of its extrinsic geometry which are needed in the following are to be found in the appendix.

III. ELECTRICAL CONDUCTIVITY OF BLACK HOLES.

A well known electrical characteristic of a black hole is its total charge Q_H. It is defined in the following way : The total charge Q of the system consisting of the black hole and the external matter is well defined by the " $Q\,\vec{r}/r^3$ " asymptotic behaviour (at spatial infinity) of the electric field. This behaviour allow us to express Q as a flux integral on a surface at infinity S_∞ :

$$Q = \frac{1}{4\pi} \oint_{S_\infty} \frac{1}{2} F^{ab}\, dS_{ab} \tag{1}$$

where

$$F_{ab} = \frac{\partial A_b}{\partial x^a} - \frac{\partial A_a}{\partial x^b} \tag{2}$$

is the electromagnetic tensor and where $d\,S_{ab} = 1/2\ \varepsilon_{abcd}\, dx^c \wedge dx^d$ is the surface element of S_∞ . Let Σ be a spatial 3-surface (of volume element $d\Sigma_a$) extending from S_∞ down to a section S of the horizon; using Gauss'theorem one can transform the 2-integral over S_∞ into a 3-integral over Σ and a 2-integral over the inner boundary S :

$$Q = \frac{1}{4\pi} \int_\Sigma F^{ab}{}_{;b}\, d\Sigma_a + \frac{1}{4\pi} \oint_S \frac{1}{2} F^{ab} dS_{ab} \tag{3}$$

But Maxwell equations :

$$F^{ab}{}_{;b} = 4\pi\, J^a \tag{4}$$

where J^a is the 4-current density, imply that the volume integral in (3) : $\int J^a \, d\Sigma_a$ is nothing but the usual electric charge Q_M of the matter outside the horizon. This leads to the definition of the charge of the section S of the hole as :

$$Q_H \equiv Q - Q_M = \frac{1}{4\pi} \oint_S \frac{1}{2} F^{ab} dS_{ab} \ . \tag{5}$$

Here Q_H is given a meaning only as a global characteristic of the hole. This is for instance the charge parameter appearing in the solutions of Reissner (1916) and Newman et al (1965). But Hanni and Ruffini (1973) introduced the concept of a " charge induced " on the surface of the black hole which amounts to interpret Q_H as being smeared over the surface S with a density given precisely by the differential element in (5) (that is the flux of the electric field). If $\vec{n}$ denotes the null vector orthogonal to S and normalized by :

$$n_a \ell^a = +1 \tag{6}$$

the tensorial surface element of S reads :

$$dS_{ab} = (n_a \ell_b - n_b \ell_a) \ dS_H \tag{7}$$

where dS_H is the usual area element of S (see II(7)). This leads to the definition of the surface charge density of the black hole :

$$\sigma_H \equiv \frac{1}{4\pi} F^{ab} n_a \ell_b \tag{8}$$

which implies :

$$Q_H = \oint_S \sigma_H \, dS_H \ . \tag{9}$$

It is possible to go further and to introduce the concept of a surface current density $\vec{K} = K^A \, \partial/\partial x^A$. This concept has been introduced by Znajek (1978b) (for the azimuthal component K^θ in the restricted case of axial symmetry) and by Damour (1978, 1979) in the general case. In fact the 4-vector $\sigma_H \, \partial/\partial t + K^A \partial/\partial x^A$ (4-current density of the hole) can be introduced at once, but here we wish to remark how, σ_H being given, it is natural to introduce $\vec{K}$: Indeed, in a non-stationary situation and/or if some external current is injected into the hole, the local charge content $\sigma_H \sqrt{\gamma} \, dx^2 \wedge dx^3$ of a surface element will vary with time, and a straight forward use of Maxwell equations (see appendix) yields for its time derivative an equation of the type :

$$\frac{1}{\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\,\sigma_H)}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial(\sqrt{\gamma}\,K^A)}{\partial t} = \text{injected current} = -J^a \ell_a \tag{10}$$

The existence of such a "local law of conservation of electricity" (of the type $\partial\rho/\partial t + \operatorname{div} \vec{j}$) allow us to call K^A (an explicit function of the electromagnetic field (see appendix)) the surface current density of the section S . Moreover by restraining the electromagnetic 2-form (i.e. the magnetic flux) to the horizon ($x^1 = 0$) we get :

$$\left(\frac{1}{2}F_{ab}\,dx^a\wedge dx^b\right)_{x^1=0} = \left(E_A\,dx^A\right)\wedge dt \;+\; B_\perp\,dS_H\;. \tag{11}$$

This formula gives a natural definition for the tangential electric field E_A and the normal magnetic induction $B_\perp$ on a section S of the hole. These fields satisfy the Faraday law :

$$\mathrm{curl}\,\vec{E} = -\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\,B_\perp\right) \tag{12}$$

as a consequence of the existence of the potential A_a (see(2)). A remarkable result is that the quantities so introduced are linked by an equation wich reads :

$$E_A + \epsilon_{AB}\,B_\perp V^B = 4\pi\,\gamma_{AB}\left(K^B - \sigma_H V^B\right) \tag{13a}$$

where V^B is the surface velocity of the black hole (see II(3)) and where ϵ_{AB} is the antisymmetric Levi Civita tensor on S (i.e. $\epsilon_{AB} = \pm\sqrt{\gamma}$ according to the parity of the permutation $(2,3) \rightarrow (A,B)$). Raising the index A and using self explanatory 2-dimensional vectorial notations we can rewrite (13a) as :

$$\vec{E} + \vec{V}\times\vec{B}_\perp = 4\pi\left(\vec{K} - \sigma_H\vec{V}\right)\;. \tag{13b}$$

Equations (13) asserts that the "comoving" electric field $\vec{E} + \vec{V}\times\vec{B}_\perp$ is proportional to the surface conduction current (the total current $\vec{K}$ minus the convection current $\sigma_H\vec{V}$). This reads exactly as the usual (non-relativistic)* Ohm's law. In other words the simultaneous validity of equations (10-13) allows us to say that the horizon of a black hole is endowed with a surface electrical resistivity equal to 4π (i.e. 377 ohm, the impedance of the vacuum).

Moreover we shall see in section V that the analogy with the ordinary laws of conductors can even be extended to dissipative effects : that is the presence of a "conduction current" at the surface of the black hole is associated with a Joule effect : $(\vec{K} - \sigma_H\vec{V})^2\times 4\pi$. Therefore an exact equilibrium is possible only in absence of conduction current, hence :

$$\vec{E} + \vec{V}\times\vec{B}_\perp = \vec{0} \quad \text{at equilibrium.} \tag{14}$$

In particular for non rotating black holes ($\vec{V} = 0$) this yields the constancy of the electric potential ϕ (such that $\vec{E} = -\vec{\nabla}\phi$) on the horizon. This result was first noticed by Whittaker (1927). In the case of rotating black holes ($\vec{V} = \Omega_H\,\partial/\partial\phi$) the equilibrium condition (14) was found by Carter (1973) and shown by him to imply the constancy of the comoving electric potential $\tilde{\phi}$:

$$\tilde{\phi} = \phi - \vec{A}\cdot\vec{V} = \phi - \Omega_H A_\varphi \tag{15}$$

* But eq.(13) is general-relativistically exact!

on the horizon. We shall discuss in section V the role of $\tilde{\phi}$ in the thermostatic equilibrium of a black hole electrically interacting with its surroundings. As an example of how the concept of black hole conductivity works quantitatively, we shall deduce from eqns (10-13) the "eddy currents" (Damour 1978) generated by the slow rotation of a black hole in a weak tilted uniform magnetic field $\vec{B}$. This is an instance of a non-equilibrium state of a black hole (eqn (14) is not satisfied) but the deviation from equilibrium is of second order in $\vec{B}$, therefore the system can still be in a quasi-stationary state which means that we can neglect the partial time derivatives in (10) and (12). Hence we find the system of equations :

Definition of the potential : $$E_A = -\frac{\partial \phi}{\partial x^A} \qquad (16a)$$

Conservation of electricity : $$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^A}\left(\sqrt{\gamma}\, K^A\right) = 0 \qquad (16b)$$

Ohm's law : $$E_A + \epsilon_{AB} V^B B_\perp = 4\pi \gamma_{AB}\left(K^B - \sigma_H V^B\right) . \qquad (16c)$$

If we neglect the gravitational influence of the magnetic field and if we work in the first order in the velocity $\vec{V}$ we can take for γ_{AB} the metric on the horizon of a Schwarzschild solution (i.e. the metric on a sphere or radius 2M see I(3)) :

$$\gamma^{(0)}_{AB}\, dx^A dx^B = (2M)^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right) \qquad (17)$$

we can neglect σ_H (no zeroth order electricfield), and we can take $\vec{V} = \Omega_H \partial/\partial\phi$ where Ω_H is the angular velocity of the hole. Taking the divergence of Ohm's law we get an equation for ϕ :

$$\Delta^{(0)}\phi = \frac{\Omega_H}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\, B_\perp\right) \qquad (18)$$

where $\Delta^{(0)}$ is the laplacian on the sphere (17) : $\frac{1}{(2M)^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right)$
If $B_\perp(\theta, \phi)$ is known, the equation (18) determines $\phi(\theta, \phi)$ uniquely (modulo an additive constant). In particular taking for $\vec{B}$ a uniform magnetic field around a Schwarzschild black hole making an angle γ with the axis $\theta = 0$ (see e.g. Hanni and Ruffini 1976) it is easy to compute the component of $\vec{B}$ normal to the horizon in function of the field at infinity :

$$B_\perp = B_\infty\left(\cos\gamma\cos\theta + \sin\gamma\sin\theta\cos\varphi\right) \qquad (19)$$

Inserting (19) into (18) we get :

$$\phi = \frac{1}{2}(2M)^2\Omega_H B_\infty\left[\cos\gamma\sin\theta - \sin\gamma\cos\theta\cos\varphi\right] \qquad (20)$$

Knowing ϕ the problem is solved, we can compute the current $\vec{K}$ (by Ohm's law), the Joule dissipation and the negative torque acting on the hole and due to the electrodynamic forces $\vec{K}\times\vec{B}_\perp$ (see Damour 1978 for details).

We can conclude this section by saying that from an electromagnetic point of view the surface of a black hole behaves as a "thin shell" or a "bubble" endowed with a surface resistivity equal to 4π.

IV. SURFACE STRESSES OF A BLACK HOLE.

We shall now extend the approach of the preceding section to mechanical properties of black holes. To start with it is well known that the total mass M and the total angular momentum J of the system consisting of a black hole and some external matter and fields can be defined by the asymptotic behaviour at spatial infinity of the components of the metric tensor (see e.g. Papapetrou 1948). As in the case of the total electric charge Q it is possible to express M and J as surface integrals at infinity but contrarily to the electromagnetic case it does not seem meaningful in general to transform these integrals into volume integrals because the energy and the momentum are not localizable in General Relativity. Yet there is one situation where some components of the angular momentum are certainly localizable, this is when there exists a rotational Killing vector :

$$\vec{m} = m^a \frac{\partial}{\partial x^a} = \frac{\partial}{\partial \varphi} \tag{1}$$

In this case the total angular momentum $J(=J_\phi)$ is given by a Komar-type expression (for a review of these properties see Carter 1973, 1979) :

$$J = -\frac{1}{8\pi} \oint_{S_\infty} \frac{1}{2} m^{a;b} \, dS_{ab} \tag{2}$$

As in section III, equation (2) can be transformed in,a volume integral on the (axially symmetric) hypersurface Σ , and, a surface integral on the axially symmetric 2-surface S (the intersection of Σ with the horizon) :

$$J = J_\Sigma + J_H \tag{3}$$

where

$$J_\Sigma = -\frac{1}{8\pi} \int_\Sigma m^{a;b}{}_{;b} \, d\Sigma_a = \frac{1}{8\pi} \int R^a{}_b \, m^b \, d\Sigma_a \tag{4}$$

represents, by Einstein equations :

$$R_{ab} - \frac{1}{2} R \, g_{ab} = 8\pi \, T_{ab} \tag{5}$$

the angular momentum $\int T^a{}_b m^b d\Sigma_a$ of the external matter and fields contributing to the stress energy tensor T_{ab} , and where :

$$J_H = -\frac{1}{8\pi} \oint_S \frac{1}{2} m^{a;b} \, dS_{ab} \tag{6}$$

can be meaningfully defined as the angular momentum of the black hole itself.

Using the expression III(7) for dS_{ab} and the fact that the section S is axially symmetric (and therefore that $\vec{\ell}$ can be taken as commuting with $\vec{m}$: $(\ell^a{}_{;b} m^b = m^a{}_{;b} \ell^b)$ we obtain :

$$J_H = -\frac{1}{8\pi} \oint_S n_a \, \ell^a{}_{;b} \, m^b \, dS_H \tag{7}$$

Going beyond this global definition we wish to introduce now the concept of surface density of momentum of a black hole (Damour 1979) : a covector π_A (A=2,3)* defined on the section S by :

$$\pi_A = -\frac{1}{8\pi}\, n_a\, \ell^a{}_{;A} \quad . \tag{8}$$

By definition the toroïdal component $\pi_\phi = \pi_A m^A$ fullfills :

$$J_H = \oint_S \pi_\varphi \, dS_H \quad . \tag{9}$$

The covector π_A is proportional to some components of the "Weingarten map" of the horizon (see appendix) which is a kind of extrinsic curvature for a null hypersurface (of normal ℓ^a). In the particular case of a totally geodesic horizon, i.e. when $\sigma_{AB} = 0 = \theta$, π_A is proportional to the " gravitomagnetic field " of Hajicek (1975, 1977) but here we are precisely interested in the general case of shearing and expanding horizons.

Encouraged by the success of the approach we took in the electromagnetic case, we now look for a local "conservation law of momentum" that is for an equation relating the time derivative of π_A to the divergence of a "stress tensor" and to an external flux of momentum through the horizon. This aim is achieved by contracting Einstein equations(5) with $\ell^a \partial x^b/\partial x^A$ and by using the contracted Codazzi equation whose demonstration, in the case of a null hypersurface, will be found in the appendix. We get :

$$\frac{D\pi_A}{dt} = -\frac{\partial}{\partial x^A}\left(\frac{g}{8\pi}\right) + 2\,\frac{1}{16\pi}\,\sigma_A{}^B{}_{|B} - \frac{1}{16\pi}\frac{\partial\theta}{\partial x^A} - \ell^a T_{aA} \tag{10}$$

where g is the surface gravity ($\nabla_\ell \ell^a = g\,\ell^a$ see appendix) and where the vertical bar (in $\sigma_A{}^B{}_{|B}$) denotes the riemannian covariant derivative associated with γ_{AB} . Equation (10) has precisely the form of a Navier-Stokes equation relating the time derivative of the impulsion density of a viscous fluid to the negative gradient of the pressure, to the divergence of the viscous stresses and to the external force. Therefore the surface of a black hole appears analogous to a "viscous fluid bubble" of surface <u>pressure</u> p.

$$p = +\, g/(8\pi) \tag{11}$$

acted upon by the external force density f_A :

$$f_A = -\,\ell^a T_{aA} \tag{12}$$

(which is, as expected, the flux of external impulsion through the horizon per unit area and per unit time "t") and endowed with a surface shear viscosity :

$$\eta_s = +\, 1/(16\pi) \tag{13}$$

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* Discarding here the zero component linked to the surface gravity. see appendix.

and a surface bulk viscosity :

$$\zeta_s = - 1/(16\pi) \tag{14}$$

The total stress tensor whose divergence appears in (10) is :

$$- p\, \delta_A^B + 2\eta_s \sigma_A^B + \zeta_s \theta \delta_A^B$$

If the external flux of impulsion entering the hole is due to the stress energy tensor of an electromagnetic field we find the usual electrodynamic force $\vec{f} = \sigma_H \vec{E} + \vec{K} \times \vec{B}_\perp$ acting tangentially on the surface charge and current densities introduced in section III.

The mechanical equation (10) strengthens the thermodynamical considerations of Hawking and Hartle (1972) who compared the slowing down of a distorted rotating black hole to the "tidal drag" in a shallow sea of a planet. On the other hand a word is in order to discuss the fact that we find a positive surface pressure $p = + g/8\pi$ in contradiction with Bekenstein (1972) who, in his "soap bubble" model of a black hole, introduced the notion of a surface tension $g/8\pi$ which means a negative surface pressure $-g/8\pi$. We interpret the positive surface pressure (11) as necessary for sustaining the gravitational self-attraction of the " bubble " modelling the black hole, in the same way as Poincaré introduced negative pressures (tensions) for counteracting the electric self-repulsion of an extended model of the electron.

Moreover we shall see in section V that the analogy with viscous fluids can be consistently extended to dissipative effects in the sense that the presence of viscous stresses imply a heat dissipation given by the usual formula : $2\eta_s \sigma_{AB} \sigma^{AB} + \zeta_s \theta^2$. Therefore a state of exact equilibrium can exist only in absence of any viscous stresses. From this condition and equation (10) it is possible to deduce that the surface pressure $p = g/8\pi$ must be uniform on the horizon at equilibrium. This is the famous zeroth law of black hole dynamics (Bardeen, Carter and Hawking 1973) but here it is interpreted as a consequence of the mechanical equilibrium and not of the thermal equilibrium (uniformity of $g/8\pi$ thought of as a pressure not as a temperature). Let us describe an example of a non equilibrium, but quasi-stationary, state where the gradient of the non uniform pressure is opposed by the viscous stresses. This example consists of a slowly rotating black hole, tidally distorted by a weak stationary external gravitational field : such a system has been studied by Hartle (1973) using very different methods; we shall show here how the consideration of equation (10) together with the definition of σ_{AB} and θ as functions of the velocity V^A will suffice to find the quasistationary solution of this problem.

Firstly we suppose that we know the "tide" K raised by the stationary external gravitational field on a Schwarzschild black hole. We can choose the coordinates on the horizon such that the distorted metric reads :

$$\gamma_{AB} = (1 - K)\, \gamma_{AB}^{(0)} \tag{15}$$

where $\gamma_{AB}^{(0)}$ is the metric on a sphere of radius 2M (see III(17)). Then we notice that, as we are looking for a quasistationary state, the expansion of the horizon: θ will (contrarily to σ_{AB}) be of second order in the perturbation K (see Hawking and Hartle 1972). From the appendix we have the following expression for θ :

$$\theta = \frac{1}{2}\, \gamma^{AB} \frac{\partial \gamma_{AB}}{\partial t} + \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^A}\left(\sqrt{\gamma}\, V^A\right) \tag{16}$$

Therefore, working at the first order in K, and neglecting the partial time derivative (again because of the quasistationary character) we find the following condition of "incompressibility" :

$$0 = \frac{1}{\sqrt{\gamma}} \frac{\partial}{\partial x^A}\left(\sqrt{\gamma}\, V^A\right) \equiv V^A{}_{|A} \tag{17}$$

The other equations we need are : equation (10) where we neglect the partial time derivative as well as the convective terms contained in $D\pi_A / dt$ (because they are second order in V) :

$$\frac{1}{8\pi}\, \sigma_A{}^B{}_{|B} = \frac{\partial}{\partial x^A}\left(g/8\pi \right) \tag{18}$$

and the expression of σ_{AB} in function of the velocity (see appendix) :

$$\sigma_{AB} = \frac{1}{2}\left(V_{A|B} + V_{B|A} - V^C{}_{|C}\, \gamma_{AB} \right) \tag{19}$$

The three equations (17-19) are sufficient to solve the problem at the approximation considered : we can, for instance, deduce from (17) that $\vec{V}$ (or rather the perturbation of $\vec{V}$ away from the Kerr velocity II (5)) can be written as the curl of a (pseudo) scalar N ; then taking the curl of (18) to eliminate g we get a fourth order partial differential equation for N. It happens that this equation can be factorized and thereby reduced to a second order equation linking N to "perturbing tide" K. Remarkably this last equation is identical to eqn III(18) linking the electric potential $\emptyset$ to the "perturbing magnetic induction" $B_\perp$ (see Damour 1979 for the details).

As a conclusion to this section we can say that the mechanical behaviour of the surface of a black hole is closely analogous to the behaviour of a fluid bubble of surface pressure $p = g / 8\pi$ and endowed with shear and bulk viscosity.

V. THERMODYNAMICS OF BLACK HOLES.

The topic of black hole thermodynamics was initiated by the work of Christodoulou (1970), and of Christodoulou and Ruffini (1971), who showed that a black hole could undergo two kinds of transformations : the irreversible ones where the "irreducible mass" increases and the reversible ones which are the limiting case of constant irreducible mass. This result was extended by a theorem of Hawking (1971) proving in general that the sum of the total areas of a time section of several interacting black holes, that is 16π times the sum of their squared irreducible mass, could only increase in the course of time. Then Bekenstein (1972,1973) conjectured that a certain multiple of the total area S_H of a section of a black hole :

$$S = \alpha S_H = \alpha \oint_S dS_H \tag{1}$$

could be conceived as the "entropy" of the blak hole in the precise sense that the sum of the "entropy" of the black hole and of the usual entropy of the external matter would never decrease (Generalized Second Law). This conjecture has been proved to be consistent with the phenomenon of quantum evaporation of a black hole (Hawking 1975) thereby fixing the proportionality constant α to the value $(4\pi)^{-1}$ (For reviews of the quantum evaporation process see e.g.Gibbons 1977, 1979 and Damour 1977a ; For studies of its thermodynamical implications see e.g. Bekenstein 1975, Hawking 1976 and Wilkins 1977). We shall check later in this section the consistency of a local generalization of Bekenstein proposal (1) with the electromagnetic and mechanical analogies introduced in the preceding sections.

By analogy with usual thermodynamics, black hole thermodynamics can be divided into two categories :

A. black hole thermostatics : that is the study of equilibrium states of black holes and of the global changes between two neighbouring equilibrium states (for reviews see Carter 1973,1979).

B. black hole irreversible thermodynamics : that is the study of states of black holes where a continuous irreversible transformation takes place. This last category concerns necessarily non equilibrium states although they can sometimes be described as quasi-stationary. (These states could, in ordinary physics, be stationary by removal of the created entropy, but in black hole physics there is no way to remove the "entropy"(1)).

A. BLACK HOLE THERMOSTATICS.

As an instance of black hole thermostatics, and in order to complete the results of section III, let us show the role of the comoving electric potential in the equilibrium of a rotating black hole surrounded by circularly symmetric matter and electric currents. It was shown by Carter (1973) that the change of the total mass M of the system (including the matter, the electromagnetic field and the gravitational field) between two neighbouring circularly symmetric equilibrium states was equal to :

$$\begin{aligned}\delta M = {} & \Omega_H(\delta J_H + \delta J_F) + \tilde{\phi}_H\,\delta Q_H + (8\pi)^{-1} g\,\delta S_H + \\ & + \int \Omega\,\delta(d^3 J_M) - \int \ell^c A_c\,\delta(d^3 Q) + \int \bar{T}\,\delta(d^3 S_M) + \\ & + \int \bar{\mu}\,\delta(d^3 N) + \int (J^b \ell^a - J^a \ell^b)\,\delta A_b\, d\Sigma_a \,.\end{aligned} \qquad (2)$$

The notations of eqn(2) are as follows : the central black hole is characterized by its angular velocity Ω_H (II.5), its proper angular momentum $J_H = \int \pi\phi\, dS_H$ (IV.6) its comoving electric potential $\tilde{\phi}_H$ (III.15), its charge $Q_H = \int \sigma_H dS_H$ (III.9), its surface gravity g(IV.10), its total area S_H and its normal vector :

$$\vec{\ell} = \ell^a \frac{\partial}{\partial x^a} = \vec{k} + \Omega_H \vec{m} = \frac{\partial}{\partial t} + \Omega_H \frac{\partial}{\partial \varphi} \qquad (3)$$

$\vec{k}$ being the time translations Killing vector and $\vec{m}$ the rotational Killing vector, eqn(3) allow to define $\vec{\ell}$ in the whole spacetime. The external matter is described as a perfect fluid in circular motion possessing a local angular velocity Ω , a stress-energy tensor $T_M{}^{ab}$ giving rise to a local angular momentum element $d^3 J_M = m_b T_M{}^{ba}\, d\Sigma_a$, a local charge distribution $d^3Q = J^a d\Sigma_a$ a renormalized local temperature $\bar{T} = [-(\vec{k}+\Omega\vec{m})^2]^{1/2}\ T$, a local entropy distribution $d^3 S_M$, a local renormalized chemical potential $\bar{\mu} = \mu\ \bar{T}/T$ corresponding to a local distribution of a conserved chemical species d^3N . Finally the electromagnetic field is described by Maxwell equations with a 4 potential A_a a field $F_{ab} = \partial_a A_b - \partial_b A_a$ a circular current $J^a = F^{ab}{}_{;b}/4\pi$, a Maxwell stress energy tensor $T_F{}^{ab} = (F^{ac} F_{cb} - 1/4\,\delta^a{}_b\, F^2)/(4\pi)$ entailing a <u>total</u> electromagnetic angular momentum :

$$J_F = \int_\Sigma m_b\, T_F{}^{ab}\, d\Sigma_a \qquad (4)$$

integrated over the whole space outside the hole i.e. on a spatial hypersurface Σ extending from the section S of the hole to spatial infinity.

What is striking in the formula(2), and what seems to destroy the analogy with an electrically conducting rotating shell in interaction with external matter and currents (consisting for instance of other conducting rotating shells, or rings) is the fact that Ω_H appears multiplied, not only by the variation of the "local mechanical angular momentum of the hole" $J_H = \oint \pi\phi d\, S_H$, but also by the variation of the total angular momentum J_F (4) of the electromagnetic field outside the hole. Moreover the interpretation of the last term of (2) is unclear. We wish to point out here that these two difficulties compensate each other after a suitable rewriting of(4). Using the identity valid for axisymmetric fields :

$$m^b F_{bc} F^{ac} \equiv (m^b A_b) F^{ac}{}_{;c} - (m^b A_b F^{ac})_{;c} \tag{5}$$

we can redistribute the total angular momentum of the electromagnetic field J_F on its sources $J^a = F^{ac}{}_{;c}/(4\pi)$, including the surface charge distribution of the black hole $Q_H = \oint \sigma_H\, dS_H$:

$$J_F = \int_\Sigma A_\varphi\, d^3Q + \oint_S A_\varphi \sigma_H\, dS_H \tag{6}$$

where A_ϕ denotes the invariant : $A_b m^b$. Then after some transformations we can rewrite (2) as :

$$\delta M = \Omega_H\, \delta\left(J_H + \int A_\varphi \sigma_H\, dS_H\right) + \tilde{\phi}_H\, \delta Q_H + (8\pi)^{-1} g\, \delta S_H + \\ + \int \Omega\, \delta\left(d^3J_M + A_\varphi\, d^3Q\right) + \int \tilde{\phi}\, \delta(d^3Q) + \int \bar{T}\, \delta(d^3S_M) + \int \bar{\mu}\, \delta(d^3N). \tag{7}$$

where $\tilde{\phi} = -(A_t + \Omega A_\phi)$ is the comoving potential of the matter and where the mechanical angular momentum $J^{mec.}$ is everywhere replaced by the "conserved" angular momentum $J^{cons.}$:

$$J^{cons.} = J^{mec.} + A_\phi \times \text{(electric charge)}.$$

It is well known that $J^{cons.}$ is conserved when a piece of charged matter is transferred from one point to another.

Now the mass variation (7) is exactly analogous to what one could expect in the case of an electrically conducting material shell (with comoving potential $\tilde{\phi}_H$ mechanical angular momentum J_H , surface charge distribution σ_H , entropy $\alpha\, S_H$ (1) and temperature$(g/(8\pi\alpha))$) in equilibrium with circularly symmetric external matter and currents(Damour 1979). Moreover, if me suppose the validity of Bekenstein's Generalized Second Law i.e. that the total entropy :

$$S^{tot.} = \int d^3S_M + \alpha S_H$$

can only increase, the system will be in equilibrium against the transfer of particles, charge and entropy between its different parts only if the intensive quantities ($\Omega, \tilde{\phi}, \bar{T}, \bar{\mu}$) associated to the "conserved" extensive quantities ($J^{cons.}$, Q, S, N) are equal on the parts which admit the corresponding transfer. In particular we find that $\tilde{\phi}_H$ must be equal to the (necessarily uniform) comoving electric potential of the bodies (rings or shells) with which the black hole can exchange some charge. This is the condition which was used by several authors (Wald 1974, Petterson 1975, Damour 1977b, 1979, Linet 1977, Znajek 1978a) in the case of equilibrium with infinity (where $\phi = 0$), but we must notice that such a condition neglects the corresponding condition of mechanical equilibrium ($\Omega_H = \Omega$) on the intuitive ground that it takes place on a much longer time scale.

B. BLACK HOLE IRREVERSIBLE THERMODYNAMICS.

If we wish to extend the preceding electromagnetic mechanical and thermostatic analogies to a full thermodynamical analogy we must take into account the heat dissipated by the electrical conductivity (Joule effect) and by the shear and bulk viscosities. According to the usual laws of dissipative thermodynamics (see e.q.Landau and Lifshitz 1959) we expect a "heat" production rate in each surface element dS_H equal to :

$$\dot{q} = dS_H \left[2 \frac{1}{16\pi} \sigma_{AB} \sigma^{AB} - \frac{1}{16\pi} \theta^2 + 4\pi \left(\vec{K} - \sigma_H \vec{v} \right)^2 \right] \tag{8}$$

Then if we consider, in keeping with Bekenstein's suggestion (1), that each surface element dS_H possesses an entropy :

$$s = \alpha \, dS_H \tag{9}$$

we would expect a relation between heat and entropy of the type :

$$\frac{Ds}{dt} \stackrel{?}{=} \dot{q} / T \tag{10}$$

with a local temperature :

$$T = g/(8\pi\alpha) \tag{11}$$

Actually we get by rewriting the generalized Raychauduri equation (see appendix) :

$$\frac{Ds}{dt} - \frac{1}{g} \frac{D^2 s}{dt^2} = \dot{q} / T \tag{12}$$

A few words are in order to comment eqn(12) :
Firstly in the case of an "adiabatic transformation" (slow evolution as compared to the characteristic time scale g^{-1} of the hole) the term : $-g^{-1} D^2 S/ dt^2$ is negligible compared to DS/dt . Therefore (12) reduces to the usual law (10) in what anyway would be its classical domain of validity. In this sense it is ascertained that the shear viscosity and the electrical resistivity of a black hole are analogous to their classical counterparts both at the level of dynamical phenomena and at the level of thermodynamical (dissipative) phenomena where in fact they were first hinted at (Hawking and Hartle 1972, Znajek 1978b) .
We exclude here the bulk viscosity because a black hole is quasi "incompressible" in a slow transformation see (IV 17).
Secondly we can give arguments to justify and interpret (12) in the most general case. Let us recall Dirac (1938) interpretation of the equation of motion of a radiating electron :

$$\frac{dv}{dt} - \tau \frac{d^2 v}{dt^2} = \frac{1}{m} F(t) \tag{13}$$

written for simplicity in the one-dimensional non-relativistic case, where V is the velocity and m the mass of the electron, F being the external force and τ the quantity $2 e^2/ 3mc^3$.

Adding to (13) the final boundary condition :

$$\frac{dv}{dt} \longrightarrow 0 \quad \text{when } t \to +\infty \tag{14}$$

One can write the solution of (13-14) :

$$\frac{dv}{dt}(t) = \frac{1}{m\tau}\int_t^{+\infty} e^{+\frac{t-t'}{\tau}} F(t')\, dt' \qquad (15)$$

which is interpreted by saying that the electron "feels" the force F(t) with a negative response time $-\tau$ (pre-acceleration). In the same way the future boundary condition :

$$\frac{Ds}{dt} = \theta . s \longrightarrow 0 \quad \text{when } t \to +\infty \qquad (16)$$

is satisfied by the very definition of a black hole. We can therefore interpret (12) by saying that the "entropy" of a black hole responds to the "heat" dissipation, caused by the electrical conductivity, the shear and the bulk viscosities, not instantaneously but on a negative time scale*:

$$-\tau = -g^{-1} .$$

In this approach the "thermal conductivity" of a black hole is equal to zero. We consider too as thermodynamically justified the negative value $\zeta_S = -(16\pi)^{-1}$ of the bulk viscosity in contradistinction with recent proposals of a positive value $\zeta'_S = +(16\pi)^{-1}$ (Carter 1979). Indeed such a negative value, which classically means an instability against spontaneous contraction or expansion, is perfectly in keeping with the natural tendency of a null hypersurface to continually contract or expand, the horizon being precisely defined as the only null hypersurface which reach a stationary state in the far future.

Finally we wish to point out that the very general connection found by Prigogine (1968) between the dissipative phenomena and the dynamical equations of a classical stationary thermodynamical system are valid in the case of a weakly perturbed, slowly rotating, quasistationary black hole. In other words we have the remarkable result :

Consider the total "dissipation function" :

$$D = \oint_S (\text{heat production rate}) = \oint_S \dot{q} \qquad (17)$$

as a functional of the velocity field $V^A(x^2, x^3)$ and of the electric potential $\emptyset(x^2, x^3)$. Then the condition that $D[V^A, \emptyset]$ is a minimum with respect to functional variations of V^A and $\emptyset$ leads precisely to the dynamical equations satisfied by V^A and $\emptyset$ in a quasi-stationary state.

For instance in the case of the "eddy currents" generated by a slow rotation of the black hole in an external magnetic field (section III) minimizing :

$$D[\emptyset] = \frac{1}{4\pi}\oint \left(-\vec{\nabla}\emptyset + \vec{v}\times\vec{B}_\perp\right)^2 dS_H$$

with respect to $\emptyset$ leads to eqn(III.18), and in the case of the shearing velocity field generated by a slow rotation in an external gravitational field (section IV) minimizing :

$$D[v^A] = \frac{1}{32\pi}\oint \left(v_{A|B} + v_{B|A}\right)\left(v_{A'|B'} + v_{B'|A'}\right)\gamma^{AA'}\gamma^{BB'} dS_H$$

* This type of "a causal" behaviour of a black hole is well known and shows itself in various phenomena.

with the "incompressibility" constraint $V^c{}_{|c} = 0$ leads to (IV.18). Hence the entropy production rate is minimum in the actual quasistationary state.

CONCLUSION. We have shown that, with respect to both kinematical, dynamical and thermodynamical phenomena, the surface of a black hole behaves in close analogy with a "fluid bubble" endowed with electrical resistivity and with shear and bulk viscosities.

ACKNOWLEDGMENTS : It is a pleasure to acknowledge fruitful discussions with B.Carter, R.Ruffini and R.L.Znajek.

APPENDIX

For a full demonstration of the properties described here cf Damour (1979). We use a signature -+++ and units such that G = c = 1. Let x^a (a,b = 0,1,2,3) be a coordinate system for the space time V_4 endowed with a Lorentzian metric $ds^2 = g_{ab}dx^a dx^b$. We denote the covariant derivative by ∇_a or ; a, and the ordinary derivative by $\partial/\partial x^a$, ∂_a or , a . We consider in V_4 a null hypersurface H ("horizon") with normal vector $\vec{\ell} = \ell^a \partial_a$ satisfying $\vec{\ell}^2 = g_{ab}\ell^a \ell^b = 0$. For the sake of simplicity let us use a coordinate system adapted to H in the sense that on H : $x^1 = 0$. Then $x^{\bar{A}}$ ($\bar{A},\bar{B}$ = 0,2,3) are arbitrary coordinates on H, $\vec{e}_{\bar{A}} = \partial/\partial x^{\bar{A}}$ being the natural basis tangent to H.

The first fondamental form of H is just the metric ds^2 restricted to H :

$$ds^2|_H = g_{ab}\, dx^a dx^b|_{x^1=0} = g_{\bar{A}\bar{B}}\, dx^{\bar{A}} dx^{\bar{B}} \tag{1}$$

In order to define the analogue of the second fondamental form of a spatial hypersurface we notice that taking the derivative of the equality :

$$\vec{\ell}^{\,2} = 0 \tag{2}$$

along a vector tangent to H, say $\partial/\partial x^{\bar{A}}$, yields :

$$\vec{\ell} \cdot \nabla_{\bar{A}} \vec{\ell} = 0 . \tag{3}$$

Hence the vector $\nabla_{\bar{A}} \vec{\ell}$ is tangent to H and can therefore be expanded on the tangent basis $\vec{e}_{\bar{A}} = \partial_{\bar{A}}$:

$$\nabla_{\bar{A}} \vec{\ell} = \chi_{\bar{A}}^{\bar{B}}\, \vec{e}_{\bar{B}} . \tag{4}$$

By analogy with the case of spatial hypersurfaces (Hicks 1965), we call the mixed tensor $\chi_{\bar{A}}^{\bar{B}}$ defined on H the Weingarten map (The second fundamental form being $g_{\bar{A}\bar{C}}\, \chi_{\bar{B}}^{\bar{C}}$). Expressing the Ricci tensor ($R_{ab} = \partial_c \Gamma_{ab}^c - \dots$) in the coordinate system x^1 , $x^{\bar{A}}$ one can check the "contracted Codazzi equation" :

$$\ell^a e^b_{\bar{A}}\, R_{ab} \equiv R_{\ell \bar{A}} = \bar{\nabla}_{\bar{B}}\, \chi_{\bar{A}}^{\bar{B}} - \partial_{\bar{A}}\, \chi_{\bar{B}}^{\bar{B}} \tag{5}$$

where $\bar{\nabla}$ is any projection of the connection ∇ onto H (in the sense that X and Y being two vectors tangent to H, $\bar{\nabla}_X Y$ is the projection onto H of $\nabla_X Y$ along any transverse vector $\vec{n}$ ($\vec{\ell}\cdot\vec{n} \neq 0$)). The result (5) is independent of the choice of $\vec{n}$ (if $\vec{n} = \partial/\partial x^1$ the coefficients of $\bar{\nabla}$ are $\Gamma_{\bar{B}\bar{C}}^{\bar{A}}$).

In the following, as in the text, it is convenient to use a more specialized coordinate system ($x^0 = t$, x^1, x^A; A = 2, 3) where $\vec{\ell} = \partial_0 + V^A \partial_A$. Hence denoting by γ_{AB} the riemannian metric on a section S ($x^1 = 0$, x^0= const., $\gamma_{AB} = g_{AB}$) the first fondamental form (1) reads

$$ds^2|_H = \gamma_{AB}\, (dx^A - V^A dx^0)(dx^B - V^B dx^0) . \tag{6}$$

As the sections S of H are Lie transported by $\vec{\ell}$, one can define the Lie derivative along $\vec{\ell}$ of any tensor $Q_{A\dots}^{B\dots}$ defined in the riemannian sections S . Denoting by D/ dt ("convective"derivative) this Lie derivative (which is by definition the ordinary derivative $\partial_0 = \partial/\partial t$ in a coordinate system where $V^A = 0$) we find :

$$\mathcal{L}_{\vec{\ell}}\, Q_{A\dots}^{B\dots} \equiv \frac{D Q_{A\dots}^{B\dots}}{dt} = \partial_0 Q_{A\dots}^{B\dots} + V^C \partial_C Q_{A\dots}^{B\dots} - \partial_C V^B Q_{A\dots}^{C\dots} - \dots + \partial_A V^C Q_{C\dots}^{B\dots} + \dots \tag{7}$$

For instance applying this definition to the metric γ_{AB} we get :

$$\frac{D\gamma_{AB}}{dt} = \partial_0 \gamma_{AB} + \partial_A V^C \gamma_{CB} + \partial_B V^C \gamma_{AC} \tag{8}$$

which can be rewritten as :

$$\frac{D\gamma_{AB}}{dt} = \partial_0 \gamma_{AB} + V_{A|B} + V_{B|A} \tag{9}$$

where a vertical bar denotes the riemannian covariant derivative associated to γ_{AB} and where the indices are moved by γ_{AB} and its inverse γ^{AB}. The decomposition of $1/2\, D\gamma_{AB}/dt$ in trace and trace free parts defines the expansion :

$$\theta = \frac{1}{2} \gamma^{AB} \frac{D\gamma_{AB}}{dt} = \frac{1}{2} \gamma^{AB} \partial_0 \gamma_{AB} + V^C{}_{|C} \tag{10}$$

and the shear

$$\sigma_{AB} = \frac{1}{2} \frac{D\gamma_{AB}}{dt} - \frac{1}{2}\, \theta\, \gamma_{AB} . \tag{11}$$

Now if $Q^{B\cdots}_{A\cdots}$ is considered as a <u>density</u> (like σ_H, K^A, π_A) we add to the expression defining $DQ^{B\cdots}_{A\cdots}/dt$ <u>the term</u> $\theta\, Q^{B\cdots}_{A\cdots}$ so that :

$$\frac{D}{dt}\left(Q^{B\cdots}_{A\cdots}\, dS_H \right) = \left(\frac{D Q^{B\cdots}_{A\cdots}}{dt} \right) dS_H . \tag{12}$$

Using as tangent basis $\vec{\ell}$ and $\vec{e}_A = \partial_A$ the Weingarten map (4) reads :

$$\nabla_\ell \vec{\ell} = g\, \vec{\ell} \tag{13a}$$

$$\nabla_A \vec{\ell} = \Omega_A \vec{\ell} + (\sigma_A{}^B + \tfrac{1}{2}\theta\, \delta_A^B)\, \vec{e}_B \tag{13b}$$

where g is the surface gravity and Ω_A the generalization, to non totally geodesic H, of the "gravitomagnetic field" of Hajicek (1975,1977). Contracting (13b) with a vector $\vec{n}$ orthogonal to S and such that $\vec{\ell}.\vec{n} = +1$ yields :

$$\Omega_A = \vec{n} . \nabla_A \vec{\ell} \tag{14}$$

Hence the impulsion density π_A (IV-8) is equal to $- \Omega_A/8\pi$. The explicit form of (5) is :

$$R_{\ell\ell} \equiv \ell^a \ell^b R_{ab} = -(\partial_0 + V^A \partial_A)\theta + g\,\theta - \sigma_{AB}\sigma^{AB} - \tfrac{1}{2}\theta^2 \tag{15a}$$

$$R_{\ell A} \equiv \ell^a e_A^b R_{ab} = (\partial_0 + \theta)\,\Omega_A + V^C \Omega_{A|C} + V^C_{|A} \Omega_C + \sigma_A{}^B{}_{|B} - \partial_A (g + \tfrac{1}{2}\theta) \tag{15b}$$

Equation (15b) yields (IV.10) (with D/dt appropriate to a density). Equation (15a) is the generalized Raychauduri equation and yields eqn (V.12) if one notices that in a Lie transport $D(dS_H)/dt = \theta dS_H$ and therefore $D^2(dS_H)/dt^2 = [(\partial_0 + V^A \partial_A)\,\theta + \theta^2]\, dS_H$.
Maxwell equations can be treated by noting that in our coordinate system :

$$g_{ab}\, \ell^b = \ell_a = (\bar{g}/\gamma)^{1/2}\, \partial x^1 / \partial x^a \tag{16}$$

where $\bar{g} = -\det g_{ab}$. Hence Maxwell equations

$$\left(\sqrt{\bar{g}}\, F^{ba} \right)_{,b} = -4\pi \sqrt{\bar{g}}\, J^a \tag{17}$$

imply :

$$- J^a \ell_a = -(\bar{g}/\gamma)^{1/2} J^1 = (4\pi\sqrt{\gamma})^{-1} \left(\sqrt{\bar{g}}\, F^{\bar{A}1}\right)_{,\bar{A}} \tag{18}$$

But it is easy to check that by definition of σ_H :

$$4\pi\, \sigma_H = (\bar{g}/\gamma)^{1/2} F^{01} = F^{0a} \ell_a \tag{19}$$

Hence the quantity K^A defined by :

$$4\pi\, K^A = (\bar{g}/\gamma)^{1/2} F^{A1} = F^{Ab} \ell_b \tag{20}$$

satisfies the conservation law (III.10). Lowering the indices A and b in F^{Ab} and replacing ℓ^b by its expression (II.4) we check easily that $\vec{K}$, defined by eqn(20), satisfies Ohm's law (III.13) (because $F_{A0} = E_A$ and $F_{AB} = B_\perp\, \varepsilon_{AB}$).

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