Resummation of large transverse logarithms in the high energy evolution of color dipoles

Dionysios Triantafyllopoulos
ECT* / FBK, Trento, Italy

Outline

- NLO BK equation and large transverse logarithms
- Unphysical solutions
- Resummation of logarithms to all orders
- Restoration of stability, numerical solutions
- Fits to HERA data
CGC and BK in Heavy Ion Collisions

- CGC: high energy evolution of hadronic wave function
- Best d.o.f.: Wilson lines for projectile partons scattering
- Cross section in DIS, single/double particle production at forward rapidity in pA collisions, energy density just after an AA collision, ...:
  Local in rapidity observables: correlators of Wilson lines.
- To excellent accuracy, all such correlators expressed in terms of dipole scattering. BK equation.
Diagrams for dipole evolution

- **LO**

- **NLO $N_f$**

- **NLO $N_c$**

- First emission soft, second non-soft in general
The BK equation at NLO

\[ \frac{dS_{12}}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 + \bar{\alpha}_s \left( \frac{\bar{b} \ln z_{12}^2 \mu^2 - \bar{b} \frac{z_{13}^2 - z_{23}^2}{z_{12}^2} \ln \frac{z_{13}^2}{z_{23}^2} + \frac{67}{36} -\frac{\pi^2}{12} - \frac{5}{18} N_f}{N_c} \right) \right] (S_{13}S_{32} - S_{12}) \]

\[ + \frac{\bar{\alpha}_s^2}{8\pi^2} \int d^2z_3 d^2z_4 \frac{z_{34}^4}{z_{34}^4} \left[ -2 + \frac{z_{12}^2 z_{24}^2 + z_{14}^2 z_{23}^2 - 4z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right. \]

\[ \left. + \frac{z_{12}^2 z_{24}^2}{z_{13}^2 z_{24}^2} \left( 1 + \frac{z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \right) \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] \]

\[ \left[ S_{13}S_{34}S_{42} - \frac{1}{2N_c^3} \text{tr}(V_1 V_3^t V_4 V_2^t V_3 V_4^t) - \frac{1}{2N_c^3} \text{tr}(V_1 V_4^t V_3 V_2^t V_4 V_3^t) - S_{13}S_{32} + \frac{1}{N_c^2} S_{12} \right] \]

\[ + \frac{\bar{\alpha}_s^2}{8\pi^2} \frac{N_f}{N_c} \int d^2z_3 d^2z_4 \frac{z_{34}^4}{z_{34}^4} \left[ 2 - \frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2 - 4z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] \]

\[ \left[ S_{14}S_{32} - \frac{1}{N_c^3} \text{tr}(V_1 V_2^t V_3 V_4^t) - \frac{1}{N_c^3} \text{tr}(V_1 V_4^t V_3 V_2^t) + \frac{1}{N_c^2} S_{12}S_{34} - S_{13}S_{32} + \frac{1}{N_c^2} S_{12} \right] \]

\[ z_{ij} = z_i - z_j \quad S_{ij} = \frac{1}{N_c} \text{tr}(V_i^t V_j) \quad V_i^t = \text{P} \exp \left[ ig \int dz^+ A_a^-(z^+, z_i) t^a \right] \quad \bar{b} = \frac{11}{12} - \frac{1}{6} \frac{N_f}{N_c} \]

Resummation of large transverse logarithms

D. Triantafyllopoulos, ECT*/FBK
Large transverse logs

- Strongly ordered large "perturbative" dipoles (DLA)

\[
\frac{1}{Q_s} \gg z_{14} \simeq z_{24} \simeq z_{34} \gg z_{13} \simeq z_{23} \gtrsim z_{12} \exp \left( \bar{\alpha}_s^{-1/2,-1} \right) \gg z_{12}
\]

\[
\frac{dT_{12}}{dY} = \bar{\alpha}_s \int_{z_{12}^2}^{1/Q_s^2} d\bar{z}_{13} \frac{\bar{z}_{12}^2}{\bar{z}_{13}^4} \left( 1 - \bar{\alpha}_s \frac{1}{2} \ln^2 \frac{z_{13}^2}{z_{12}^2} - \bar{\alpha}_s \frac{11}{12} \ln \frac{z_{13}^2}{z_{12}^2} \right) T_{13}
\]

- Large dipoles interact stronger, real terms dominate

- NLO correction > LO term, unstable expansion in \( \alpha_s \)

- Simplest thing: one step for GBW-type initial condition

\[
T_{12} = \begin{cases} 
  z_{12}^2 Q_{s0}^2 & \text{for } z_{12} Q_{s0} \ll 1 \\
  1 & \text{for } z_{12} Q_{s0} \gg 1
\end{cases}
\]

\[
\Delta T_{12} \simeq \Delta Y \bar{\alpha}_s z_{12}^2 Q_{s0}^2 \left( \ln \frac{1}{z_{12}^2 Q_{s0}^2} - \bar{\alpha}_s \frac{1}{6} \ln^3 \frac{1}{z_{12}^2 Q_{s0}^2} - \bar{\alpha}_s \frac{11}{24} \ln^2 \frac{1}{z_{12}^2 Q_{s0}^2} \right)
\]

- Expect unstable numerical solution, as in BFKL
Unstable numerical solutions

Resummation of large transverse logarithms
Two gluons and time ordering (kinematics)

- Take $k \ll p$ and $k^+ \ll p^+$

$$\Delta T_{12} = -\frac{g^4 N_c^2}{(2\pi^2)} \int \frac{dp^+}{p^+} \int \frac{dk^+}{k^+} \int d^2 z_3 d^2 z_4 \int_{p k \bar{p} \bar{k}} \frac{p \cdot \bar{p} \ k \cdot \bar{k}}{p^2 \bar{p}^2 k^2 \bar{k}^2} \frac{e^{ip \cdot z_3 + i\bar{p} \cdot z_{13} + ik \cdot z_{42} + i\bar{k} \cdot z_{34}}}{1 + \frac{k^+ p^2}{p^+ k^2}} \frac{1}{1 + \frac{k^+ (\bar{p} - k)^2}{k^2}} 2T(z_4)$$

- Largest logs when time ordering $\tau_k = \frac{k^+}{k^2} \ll \tau_p = \frac{p^+}{p^2}$
- Add all diagrams: WW kernels, phases and trans. mom. int. $\rightarrow$ two dipole kernels. For $z_4 \gg z_3 \gg z_{12}$

$$\Delta T_{12} = \bar{\alpha}_s^2 \int \frac{dp^+}{p^+} \frac{dk^+}{k^+} \Theta\left(p^+ \frac{z_3^2}{z_4^2} - k^+\right) dz_3^2 dz_4^2 \frac{z_{12}^2}{z_3^2 z_4^2} T(z_4) \rightarrow -\frac{\bar{\alpha}_s^2 \Delta Y}{2} \int_{z_{12}^2} \frac{dz_4^2}{z_4^4} \ln^2 \frac{z_4^2}{z_{12}^2} T(z_4)$$
Resummation of double logs (DLA)

- Resum double logs to all orders in **non-local** equation

\[
T(Y, \rho) = T(0, \rho) + \bar{\alpha}_s \int_0^\rho d\rho_1 \int_0^{Y-\rho+\rho_1} dY_1 \, e^{-(\rho-\rho_1)} T(0, \rho_1), \quad \rho = \ln \frac{1}{z_{12}^2 Q_0^2}
\]

- Mathematically equivalent to **local** equation

\[
\tilde{T}(Y, \rho) = \tilde{T}(0, \rho) + \bar{\alpha}_s \int_0^Y dY_1 \int_0^\rho d\rho_1 \, e^{-(\rho-\rho_1)} \frac{J_1(2\sqrt{\bar{\alpha}_s (\rho-\rho_1)^2})}{\sqrt{\bar{\alpha}_s (\rho-\rho_1)^2}} \tilde{T}(0, \rho_1)
\]

- I.C. \( T(0, \rho) = e^{-\rho} \) \rightarrow \( \tilde{T}(0, \rho) = e^{-\rho} J_1(2\sqrt{\bar{\alpha}_s \rho^2}) \)

- Leads to physical amplitude \( T \) for \( Y > \rho \)
Double logs resummed in BK

\[ \frac{d\tilde{T}_{12}}{dY} = \frac{\bar{\alpha}_s}{2\pi} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \frac{J_1\left(2\sqrt{\bar{\alpha}_s L_{13} L_{23}}\right)}{\sqrt{\bar{\alpha}_s L_{13} L_{23}}} \left(\tilde{T}_{13} + \tilde{T}_{23} - \tilde{T}_{12} - \tilde{T}_{13}\tilde{T}_{23}\right) \]

- Agrees with NLO BK (double log term) when truncated to order $\bar{\alpha}_s^2$
- Resums double log terms to all orders
- Solution should behave nicely
- Initial condition: $\tilde{T}(0, \rho) = 1 - e^{-\tilde{T}_{2g}(0,\rho)}$
Numerical solution

BFKL on $z_{12}^{2\gamma} \equiv r^{2\gamma}$

\[ \omega_{\text{LO}} = \frac{\bar{\alpha}_s}{\gamma} + \frac{\bar{\alpha}_s}{1 - \gamma} + \text{finite} \]
\[ \omega_{\text{NLO}} = \frac{\bar{\alpha}_s}{\gamma} + \frac{\bar{\alpha}_s}{1 - \gamma} - \frac{(\bar{\alpha}_s)^2}{(1 - \gamma)^3} + \text{finite} \]
\[ \omega_{\text{NLO}} = \omega_{\text{NLO}} - \frac{\bar{\alpha}_s}{1 - \gamma} + \frac{(\bar{\alpha}_s)^2}{(1 - \gamma)^3} + \frac{1}{2} \left[ -(1 - \gamma) + \sqrt{(1 - \gamma)^2 + 4\bar{\alpha}_s} \right] + \text{finite} \]
Numerical solution

- Considerable speed reduction, roughly factor of $1/2$
Single log in quark contribution (dynamics)

- Take $k \ll p$ and $\zeta = k^+/p^+ \ll 1$

\[
\sum A_{ij} = 8\alpha_s^2 N_f \int \frac{dp^+}{p^+} \int_0^1 d\zeta \int_{p k \tilde{p} k} \frac{p \cdot k \tilde{p} \cdot k + p \cdot \tilde{p} k \cdot \tilde{k} - p \cdot \tilde{k} k \cdot \tilde{p}}{p^2 \tilde{p}^2 (\zeta p^2 + k^2)(\zeta \tilde{p}^2 + \tilde{k}^2)} (e^{i p \cdot \hat{z}_{31} + i k \cdot \hat{z}_{41}} - e^{i p \cdot \hat{z}_{32} + i k \cdot \hat{z}_{42}}) (e^{i \tilde{p} \cdot \hat{z}_{13} + i \tilde{k} \cdot \hat{z}_{14}} - e^{i \tilde{p} \cdot \hat{z}_{23} + i \tilde{k} \cdot \hat{z}_{24}})
\]

- Integrate transverse momenta

\[
\sum A_{ij} = \frac{\alpha_s^2 N_f}{2\pi^4} \Delta Y \int_0^1 d\zeta \frac{z_{12}^2 z_{3}^2 + \zeta^2 z_{4}^4}{(z_{3}^2 + \zeta z_{4}^2)^4} \approx \frac{\alpha_s^2 N_f}{3\pi^4} \Delta Y \frac{z_{12}^2}{z_{3}^2 z_{4}^4}
\]

$z_3^2$ scales like $\zeta z_4^2$, no ordering in time

- Insert color structure and Wilson lines

\[
\frac{\Delta T_{12}}{\Delta Y} = \frac{\bar{\alpha}_s^2}{6} \frac{N_f}{N_c} \left(1 - \frac{1}{N_c^2}\right) z_{12}^2 \int_{z_{12}^2}^{\infty} \frac{dz_4^2}{z_{12}^2} \ln \frac{z_4^4}{z_{12}^4} T(z_4)
\]

Resummation of large transverse logarithms

D. Triantafyllopoulos, ECT*/FBK
Relationship to splitting functions

- Pole residues related to coefficients in splitting functions

\[ \int_0^1 dz \, z^\omega P_{gg}(z) = \frac{1}{\omega} - \frac{11}{12} - \frac{N_f}{6N_c} + \mathcal{O}(\omega) \]

\[ \frac{C_F}{N_c} \int_0^1 dz \, z^\omega P_{qg}(z) = \frac{N_f}{6N_c} - \frac{N_f}{6N_c^3} + \mathcal{O}(\omega) \]

- Starting from DGLAP, write in Mellin space

\[ T_{12} \approx \int \frac{d\gamma}{2\pi i} \frac{d\omega}{2\pi i} \frac{e^{\omega Y - \gamma \rho}}{1 - \gamma + \omega - \bar{\alpha}_s \tilde{P}(\omega)} \approx \frac{1}{\omega} - A_1 \]

- Integrate \( \omega \rightarrow \) BFKL
Resum single logs

- Keep \( \frac{1}{\omega} - A_1 \rightarrow \text{local} \) in \( Y \) equation

\[
\omega_{\text{coll}}(\gamma) = \frac{1}{2} \left[ -(1 - \gamma + \tilde{\alpha}_s A_1) + \sqrt{(1 - \gamma + \tilde{\alpha}_s A_1)^2 + 4\tilde{\alpha}_s} \right]
\]

\[e^{-A_1 \tilde{\alpha}_s (\rho - \rho_1)}\] extra factor in kernel: anomalous dimension

- Extend to BK as in double logs case

\[
\frac{d\tilde{T}_{12}}{dY} = \frac{\tilde{\alpha}_s}{2\pi} \int d^2 z_{13} \left( \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \right)^{\pm A_1 \tilde{\alpha}_s} J_1 \left( \frac{2\sqrt{\tilde{\alpha}_s L_{13} L_{23}}}{\sqrt{\tilde{\alpha}_s L_{13} L_{23}}} \right) (\tilde{T}_{13} + \tilde{T}_{23} - \tilde{T}_{12} - \tilde{T}_{13} \tilde{T}_{23})
\]

\[z_\lessgtr = \min\{z_{13}, z_{23}\}. \text{ - sign when } z_\lessgtr < z_{12}, \text{ resums logs}\]
Running coupling

\[
\frac{dT_{12}}{dY} = \frac{\bar{\alpha}_s(\mu)}{2\pi} \int d^2z_3 \frac{z_{12}^2}{z_{13}^2 z_{23}^2} \left[ 1 + \bar{\alpha}_s(\mu) \left( \bar{b} \ln z_{12}^2 \mu^2 - \bar{b} \frac{z_{13}^2 - z_{23}^2}{z_{12}^2} \ln \frac{z_{13}^2}{z_{23}^2} \right) \right] (T_{13} + T_{23} - T_{12} - T_{13}T_{23})
\]

- Choose \( \mu \) to cancel potentially large log in all regions
  - Large daughter dipoles: \( \mu \approx 1/z_{12} \)
  - Small daughter dipoles: \( \mu \approx 1/\min\{z_{13}, z_{23}\} \)
  - In general: \( \mu \approx 1/\min\{z_{i,j}\} \)  ✓ Hardest scale

- Balitsky-prescription: ✓, albeit unphysical slow

- Choose coefficient of \( \bar{b} \) to vanish: ✓, better choice

\[
\alpha_s = \left[ \frac{1}{\alpha_s(z_{12})} + \frac{z_{13}^2 - z_{23}^2}{z_{12}^2} \frac{\alpha_s(z_{13}) - \alpha_s(z_{23})}{\alpha_s(z_{13}) \alpha_s(z_{23})} \right]^{-1}
\]
Couplings comparison

Resummation of large transverse logarithms

D. Triantafyllopoulos, ECT*/FBK
Numerical solution (fixed)

DLA+SL resum, $\tilde{\alpha}_s=0.25$

$T(\rho,Y)$

- $Y=0$
- $Y=4$
- $Y=8$
- $Y=12$
- $Y=16$

$d\log[Q^2(Y)]/dY$

- LO
- DLA at NLO
- DLA resum
- DLA+SL resum

speed, $\tilde{\alpha}_s=0.25$

Resummation of large transverse logarithms
Numerical solution (prescription: small)

DLA+SL resum, $\beta_0=0.72$, smallest

$T(\rho, Y) = \log(1/r^2)$

speed, $\beta_0=0.72$, smallest

- LO
- DLA at NLO
- DLA resum
- DLA+SL resum

Resummation of large transverse logarithms
Resummation of large transverse logarithms

Fit

\begin{align*}
\chi^2 & \approx 1.13 \\
\lambda_s & \approx 0.22
\end{align*}
No anomalous dimension in initial condition

Including single logs: more physical parameters

rcMV model: can be extrapolated to higher $Q^2$

Small and ``new'' prescription: good fit
B-presentation: ``doesn’t make it’’
Backup slides
Resummation of double logs in BK

Extend DLA to match BK

- Let $2\tilde{T}(Y, \rho_1) \to \tilde{T}_{13} + \tilde{T}_{23}$
- Restore dipole kernel $d\rho_1 e^{-(\rho-\rho_1)} = \frac{d z_{13}^2 z_{12}^2}{z_{13}^4} \to \frac{1}{\pi} \frac{d z_3^2 z_{12}^2}{z_{13}^2 z_{23}^2}$
- Insert virtual term $-\tilde{T}_{12}$, non-linear term, remove cutoffs
- Replace argument in argument of DLA kernel to switch off resummation for small daughter dipoles

$\rho - \rho_1 = \ln \frac{z_{13}^2}{z_{12}^2} \to \sqrt{\ln \frac{z_{13}^2}{z_{12}^2} \ln \frac{z_{23}^2}{z_{12}^2}} \equiv \sqrt{L_{13} L_{23}}$
In DLA need $x^+$ ordering $\rightarrow p^-$ ordering

Proper variable $Y^- = \ln \frac{p^-_{\text{max}}}{p^-} = \ln \frac{p^+}{p^+_{\text{min}}} + \ln z_{12}^2 Q_0^2 = Y - \rho$

Double Mellin representation of BFKL solution

$$T_{12} = \int \frac{d\xi}{2\pi i} \frac{d\omega}{2\pi i} \frac{\bar{T}^0(\xi)}{\omega - \bar{\alpha}_s \chi_0(\xi)} e^{\omega Y^- (z_{12}^2 Q_0^2)^{\xi}} \Rightarrow$$

$$T_{12} = \int \frac{d\gamma}{2\pi i} \frac{d\omega}{2\pi i} \frac{\bar{T}^0(\gamma - \omega)}{\omega - \bar{\alpha}_s \chi_0(\gamma - \omega)} e^{\omega Y (z_{12}^2 Q_0^2)^{\gamma}} \Rightarrow$$

$$\omega = \frac{\bar{\alpha}_s}{1 - \gamma + \omega} \quad \text{Leads to } \sqrt{ } \text{ seen earlier}$$
Large single logs from large dipoles

- Easily seen in BFKL, “some effort” to uncover in BK

\[
\frac{d T_{12}}{d Y} = \frac{\alpha_s^2}{8\pi^2} \int \frac{d^2 z_3 \, d^2 z_4}{z_{34}^4} \left[ -2 + \frac{z_{13}^2 z_{24}^2 + z_{14}^2 z_{23}^2 - 4 z_{12}^2 z_{34}^2}{z_{13}^2 z_{24}^2 - z_{14}^2 z_{23}^2} \ln \frac{z_{13}^2 z_{24}^2}{z_{14}^2 z_{23}^2} \right] T_{34}
\]

- Ordered dipoles \( z_{14} \simeq z_{24} \simeq z_{34} \gg z_{13} \simeq z_{23} \gg z_{12} \)

\[
K_{5L}^g \simeq -\frac{6 - \cos^2 \phi}{12} \frac{\alpha_s^2}{\pi^2} \frac{z_1^2}{z_3^2 z_4^4} \rightarrow -\frac{11}{24} \frac{\alpha_s^2}{\pi^2} \frac{z_1^2}{z_3^2 z_4^4}.
\]

\[
\Delta T(z_{12}) \bigg|_{SL}^g \simeq -\frac{11}{12} \frac{\alpha_s^2 z_{12}^2}{z_4^4} \int_{z_{12}^2}^{\infty} \frac{dz_4^2}{z_4^4} \ln \frac{z_4^2}{z_{12}^2} T(z_4)
\]

- Dominates the LO term
Numerical solution (Balitsky prescription)

DLA+SL resum, $\beta_0 = 0.72$, Balitsky

speed, $\beta_0 = 0.72$, Balitsky
Some fitting expressions

**GBW**

\[ T(Y_0, r) = \left\{ 1 - \exp \left[ - \left( \frac{r^2 Q_s^2}{4} \right)^p \right] \right\}^{1/p} \]

**rcMV**

\[ T(Y_0, r) = \left\{ 1 - \exp \left[ - \left( \frac{r^2 Q_s^2}{4} \bar{\alpha}_s(C_{MV} r) \left[ 1 + \log \left( \frac{\bar{\alpha}_{\text{sat}}}{\bar{\alpha}_s(C_{MV} r)} \right) \right] \right]^p \right\} \]

**RC**

\[ \alpha_s(r) = \frac{1}{b_{N_f} \ln \left[ 4 C^2_\alpha / (r^2 \Lambda_{N_f}^2) \right]} \quad (\text{plus freezing}) \]
Fit

mv initial condition
\[ m_v = 0.10 \text{ GeV} \]
\[ m_c = 1.3 \text{ GeV} \]
\[ \chi^2 / n_{\text{pts}} = 1.13 \]