A brief survey of high-energy evolution in pQCD

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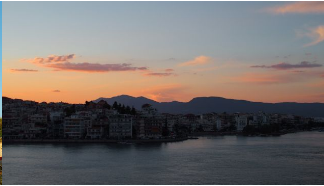

Once upon a time ...

- High-energy evolution in pQCD: a topic almost as old as pQCD itself
- It all started with the BFKL equation ...
  (Balitsky, Fadin, Kuraev, and Lipatov, 75-78)
  - conceptually interesting but reputedly tricky phenomenology
- Since then, two major directions of evolution (of the evolution ☺)

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- A. Including the non-linear effects associated with high gluon density:
  - gluon saturation, multiple scattering, color glass condensate
  - from pioneering papers ...
    L. Gribov, Levin, and Ryskin, 1983; Mueller and Qiu, 1985; McLerran and Venugopalan, 1993.
  - ... to correct equations
    - the Balitsky hierarchy *(Balitsky, 96)*
    - the Balitsky-Kovchegov (BK) equation *(Kovchegov, 99)*
    - the functional JIMWLK equation (1997-2000)
      (Jalilian-Marian, Iancu, McLerran, Weigert, Leonidov, Kovner)

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  - conceptually interesting but reputedly tricky phenomenology
- B. Extending the leading-order (LO) evolution equations to next-to-leading order (NLO) accuracy and beyond
  - the NLO BFKL equation *(Fadin, Lipatov; Camici, Ciafaloni, 98)*
  - running coupling corrections to BK *(Kovchegov, Weigert; Balitsky, 06)*
  - full NLO version of the BK equation *(Balitsky and Chirilli, 2008)*
  - Balitsky hierarchy at NLO *(Balitsky and Chirilli, 2013)*
  - JIMWLK evolution at NLO *(Kovner, Lublinsky, and Mulian, 2013)*

Once upon a time ...

- Some good surprises ...
- The effectiveness of gluon saturation in solving important conceptual issues and explaining remarkable aspects of the phenomenology
  - unitarization of scattering amplitudes at fixed impact parameter
  - first principle calculations of the bulk of particle production in pp, pA, and AA collisions at RHIC and the LHC

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- ... and some not so good ones
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  'DGLAP strikes back'
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  - taken at face value, the strict NLO result is meaningless
- All order resummations of the large logarithms are compulsory

Coming to this talk

- For the linear, BFKL equation, resummation methods have been devised quite early (in Mellin space) ...
  - collinear improvement *(Salam, Ciafaloni, Colferai, Stasto, 98-03)*
  - small–x resummation of DGLAP *(Altarelli, Ball, Forte, 00-03)*
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- Alternative resummations, which can deal with multiple scattering *(E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, 2015)*
  - direct calculation of Feynman graphs *(arXiv:1502.05642, PLB)*
  - promising phenomenology (so far, only DIS) *(arXiv:1507.03651, PLB)* *(see also J. Albacete, arXiv:1507.07120)*

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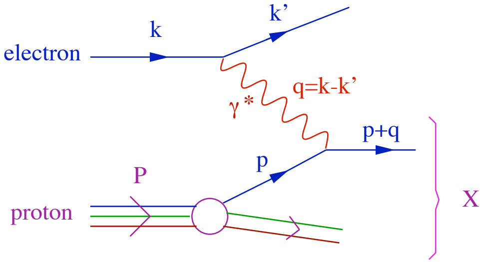
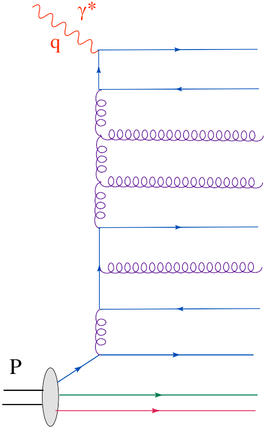
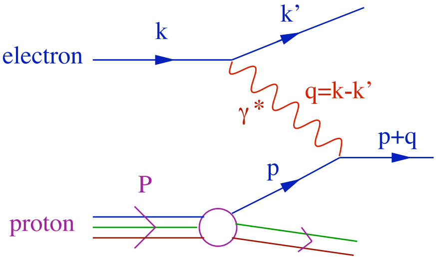
- This talk: an introduction to BFKL and BK, aiming at pedagogy
  - LO, NLO, all-order collinear resummations
- *For more details and phenomenology, see the talk by Dionysis on Friday*

Deep inelastic scattering

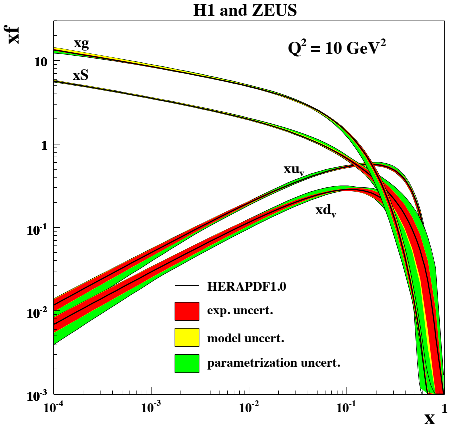




- The virtual photon γ^* couples to the (anti)quarks inside the proton
- Gluons are measured indirectly, via their effect on quark distribution
- Parton evolution: change in the partonic content when changing the resolution scales x and Q^2, due to additional radiation



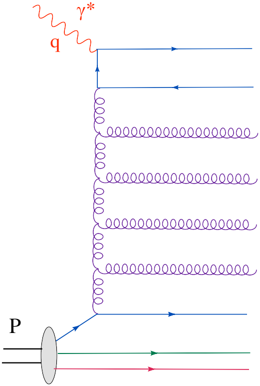
$$Q^2 = -q^\mu q_\mu > 0\,, \quad x = \frac{Q^2}{s}$$

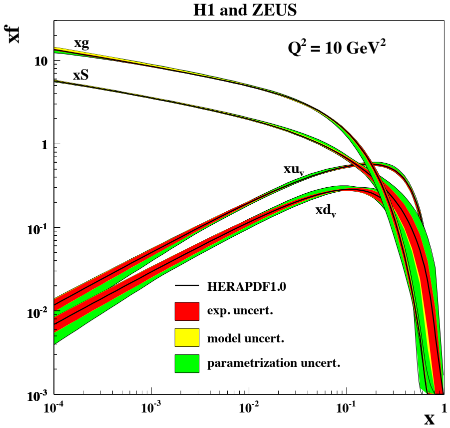


- Parton distribution functions: $xq(x,Q^2)$, $xG(x,Q^2)$

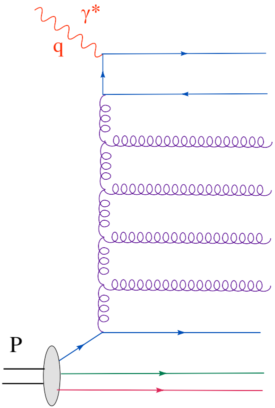
  ▷ number of partons (quark, gluons) with transverse size $\Delta x_\perp \sim 1/Q$ and longitudinal momentum fraction $x \sim Q^2/s$

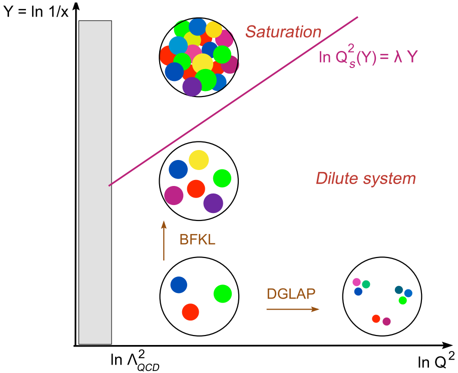
The small–x partons are mostly gluons





- For $x \leq 0.01$ the hadron wavefunction contains mostly gluons !
- Small–x gluons are abundantly produced via bremsstrahlung
- Their number increases rapidly with decreasing x : $xG(x, Q^2) \sim 1/x^{\lambda}$



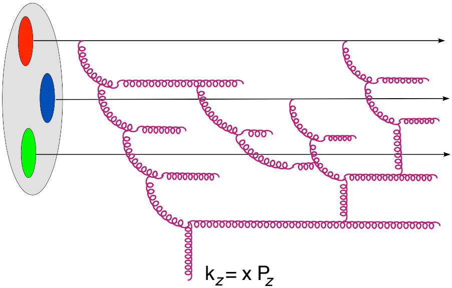


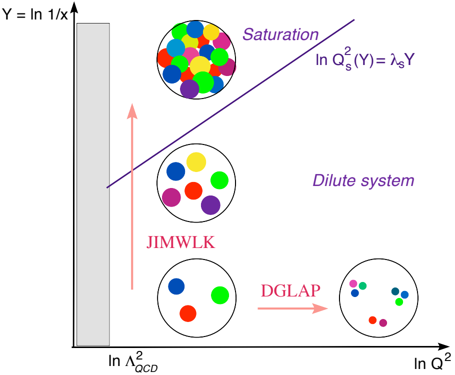
- The gluon occupation number rises rapidly with $1/x$:

$$n(x,Q^2)\,\equiv\,\frac{xG(x,Q^2)}{Q^2\,\pi R^2}\propto\frac{1}{x^\lambda}\,,\qquad \lambda=0.2\div0.3$$

- Cannot exceed a value $n\sim1/\alpha_s$ (would violate unitarity)

Gluon saturation





- $\alpha_s n \sim 1$: strong overlapping which compensates small coupling
- The evolution becomes non–linear, leading to gluon saturation
- BFKL gets replaced by JIMWLK (infinite hierarchy)
- At large N_c, JIMWLK ≈ BK (see below)

The saturation momentum

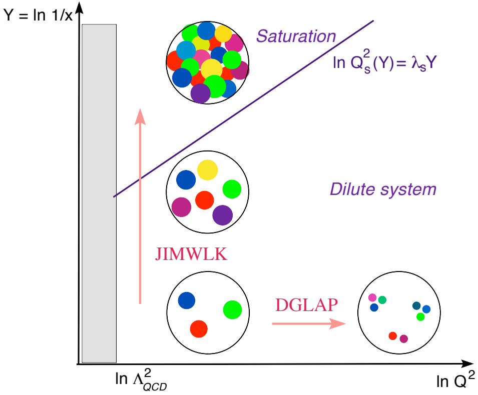
- Non–linear physics promotes the gluon density to an intrinsic transverse momentum scale, which rises with the energy $(1/x)$

$$n(x, Q^2) = \frac{\pi}{Q^2} \times \frac{xG(x, Q^2)}{\pi R^2}$$

$$n(x, Q_s^2(x)) \sim \frac{1}{\alpha_s}$$

$$Q_s^2(x) \simeq \alpha_s \frac{xG(x, Q_s^2)}{\pi R^2} \sim \frac{1}{x^{\lambda_s}}$$

λ_s : saturation exponent



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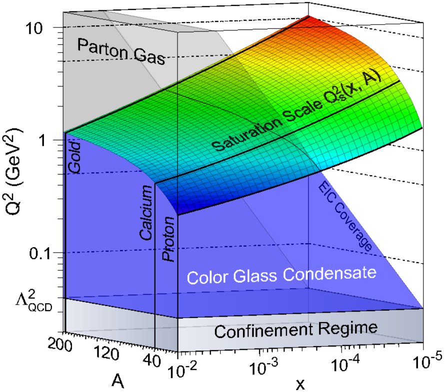
$$Q_s^2(x) \simeq \;\alpha_s \frac{xG_A(x,Q_s^2)}{\pi R_A^2} \;\sim\; \frac{A^{1/3}}{x^{\lambda_s}}$$



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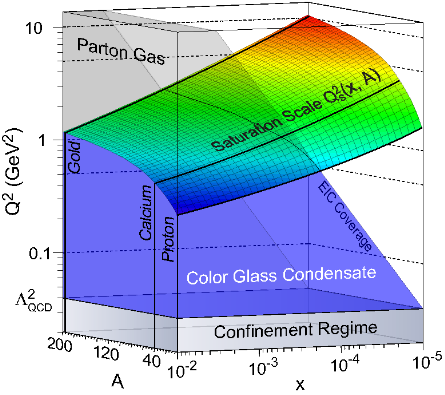
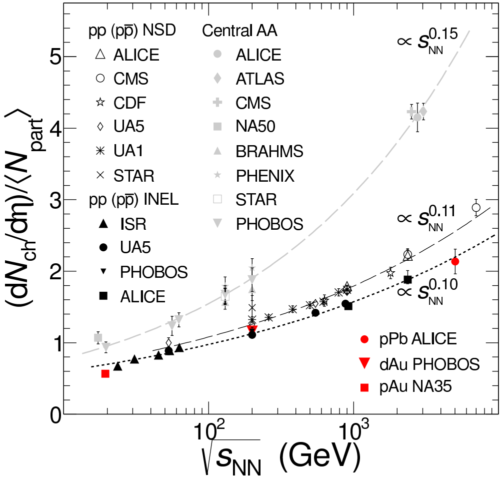
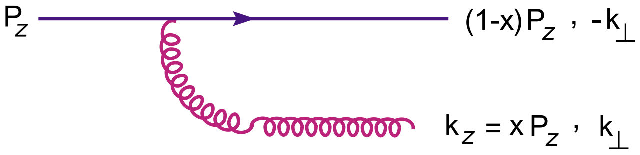


- ... and also with the atomic number A for a large nucleus $(A \gg 1)$
- $x \sim 10^{-5}$: $Q_s \sim 1$ GeV for proton and ~ 3 GeV for Pb or Au

Multiplicity : energy dependence

- Particle multiplicity $\mathrm{d}N/\mathrm{d}\eta \propto Q_s^2 \sim s^{\lambda_s/2}$
- $\lambda_s \simeq 0.2 \div 0.3$ at NLO accuracy *(Triantafyllopoulos, 2003)*





$$\mathrm{d}\mathcal{P}_{\rm Brem} \equiv \sum_{a,\lambda} \Big| \mathcal{M}^a_\lambda(k_z, \boldsymbol{k}_\perp) \Big|^2 \simeq \frac{\alpha_s C_R}{\pi^2} \frac{\mathrm{d}^2 k_\perp}{k_\perp^2} \frac{\mathrm{d}x}{x}$$

- Phase–space enhancement for the emission of
  - collinear $(k_\perp \to 0)$
  - and/or soft (low–energy) $(x \to 0)$ gluons
- The parent parton can be either a quark or a gluon
- The enhancement at small x occurs only for gluon emissions

The Double Logarithmic Approximation

- The zeroth order gluon distribution produced by a single quark

$$xG^{(0)}(x,Q^2) = \frac{\alpha_s C_F}{\pi}\int_{\Lambda^2}^{Q^2}\frac{\mathrm{d}k_\perp^2}{k_\perp^2} = \frac{\alpha_s C_F}{\pi}\ln\frac{Q^2}{\Lambda^2}$$

- The simplest evolution equation: DLA
  - gluon emissions strongly ordered both in longitudinal momentum and in transverse momentum (or virtuality)

$$1 \gg x_1 \gg x_2 \gg \cdots \gg x\,, \qquad Q^2 \gg k_{\perp 1}^2 \gg k_{\perp 2}^2 \cdots \gg \Lambda^2$$

$$\frac{\partial}{\partial \ln(1/x)}\frac{\partial}{\partial \ln Q^2}\, xG(x,Q^2) \,=\, \frac{\alpha_s N_c}{\pi}\, xG(x,Q^2)$$

- Easy to solve (below, for fixed coupling $\bar{\alpha} = \alpha_s N_c/\pi$)

$$xG(x,Q^2) \,\propto\, \exp\left\{\sqrt{4\bar{\alpha}\,\ln\frac{1}{x}\,\ln\frac{Q^2}{\Lambda^2}}\,\right\}$$

The DGLAP equation *(by Yuri et al)*

- Transverse momenta (or virtualities) are strongly ordered
- Longitudinal momenta are dealt with exactly (energy conservation)

$$\mathrm{d}\mathcal{P}_{gg} = \frac{\alpha_s(Q^2)\, N_c}{\pi} \left[\frac{1-x}{x} + \frac{x}{1-x} + x(1-x)\right] \mathrm{d}x\, \frac{\mathrm{d}Q^2}{Q^2}$$

- Gluons only ($N_f = 0$) for simplicity:

$$Q^2 \frac{\partial}{\partial Q^2} G(x, Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{\mathrm{d}z}{z} P_{gg}(z)\, G\left(\frac{x}{z}, Q^2\right)$$

  - local in Q^2, non-local in x, running coupling at LO
  - the $g \to gg$ splitting function $P_{gg}(z)$ also includes 'virtual' terms expressing probability conservation

- In general, matrix equation for coupled evolution of quarks and gluons
- Currently known up to NNLO, good convergence

The BFKL equation

- Longitudinal momentum fractions are strongly ordered
- Transverse momenta are assumed to be comparable

$$1 \gg x_1 \gg x_2 \gg \cdots \gg x\,, \qquad Q^2 \sim k_{\perp 1}^2 \sim k_{\perp 2}^2 \cdots \sim Q_0^2$$

- Usually formulated for the unintegrated gluon distribution $f(x, k_\perp^2)$

$$f(x, k_\perp^2) \equiv x \frac{\mathrm{d}N_{\rm gluon}}{\mathrm{d}x \mathrm{d}k_\perp^2} \quad \Longrightarrow \quad xG(x, Q^2) = \int_{\Lambda^2}^{Q^2} \mathrm{d}k_\perp^2 \, f(x, k_\perp^2)$$

  - proportional to the occupation number: $n(x, k_\perp^2) \propto f(x, k_\perp^2)/\pi R^2$

- Local in $Y \equiv \ln(1/x)$ ('rapidity'), non-local in $k_\perp$

$$\frac{\partial}{\partial Y} f(Y, k_\perp^2) = \bar{\alpha} \int \mathrm{d}^2\boldsymbol{k}_\perp \, \mathcal{K}_{\rm BFKL}(\boldsymbol{k}_\perp, \boldsymbol{p}_\perp) \, f(Y, p_\perp^2)$$

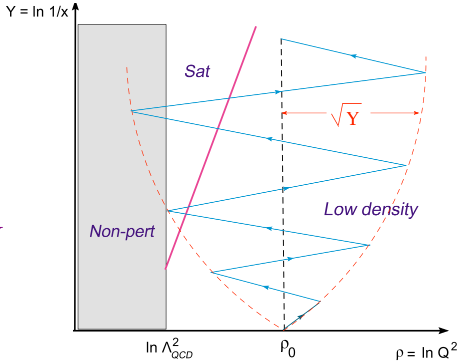
- Currently known up to NLO, very bad convergence

The diffusion approximation

- The non-locality can be studied by expanding in $\rho \equiv \ln(k_\perp^2/p_\perp^2)$

$$\partial_Y n \simeq \bar{\alpha}\omega_0 n + \bar{\alpha}\beta_0 \partial_\rho^2 n$$

- $\omega_0 = 4\ln 2 \simeq 2.77,\ \beta_0 \simeq 30$
- exponential growth: $n \propto \mathrm{e}^{\bar{\alpha}\omega Y}$
- diffusion in ρ : $(\Delta\rho)^2 \sim \bar{\alpha}\beta Y$



- Unitarity violation: typical scattering amplitude $T \sim \alpha_s n$
- Infrared diffusion: excursion through soft $(\sim \Lambda_{\rm QCD})$ momenta
- The growth is anyway too fast to match phenomenology: $\bar{\alpha}\omega_0 \simeq 1$

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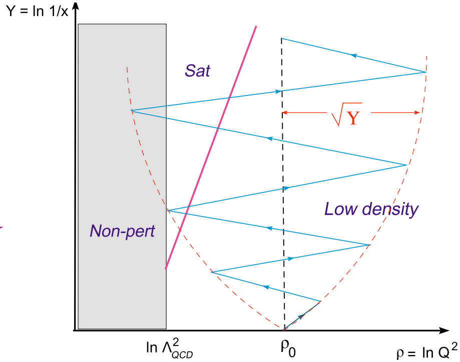
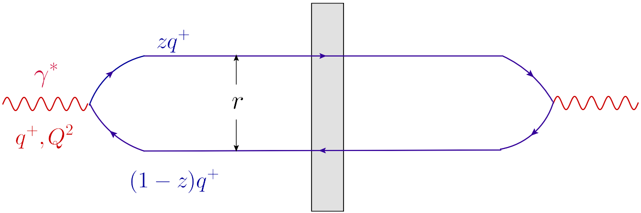


- The first 2 problems are solved by gluon saturation
- The 3rd problem is solved by resumming higher-order corrections

$$\omega_{NLO} \simeq 4\ln 2\big(1 - 6.47\bar{\alpha}\big)$$



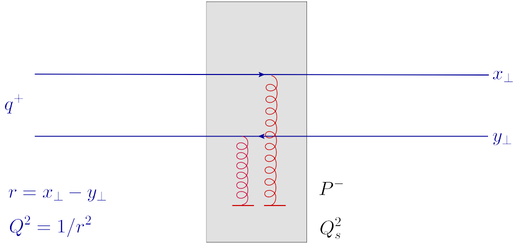
$$\sigma_{\gamma^* p}(Q^2, x) = 2\pi R_p^2 \sum_f \int \mathrm{d}^2 r \int_0^1 \mathrm{d}z \left|\Psi_f(r, z; Q^2)\right|^2 T(r, x)$$

$$x = \frac{Q^2}{2P \cdot q} \simeq \frac{Q^2}{s} \ll 1 \quad \text{(Bjoerken' } x\text{)}$$

- $T(r, x)$: scattering amplitude for a $q\bar{q}$ color dipole with transverse size r
  - $r^2 \sim 1/Q^2$: the resolution of the dipole in the transverse plane
  - x : longitudinal fraction of a gluon from the target that scatters

Dipole–hadron scattering

- A small color dipole: quark-antiquark pair in a color singlet state
  - a good probe of the gluon distribution in the target
- 'Duality': gluon saturation in the dense target $\leftrightarrow$ multiple scattering



- A single (elastic) scattering: 2 gluon exchange

$$T_0(r,x) \simeq \alpha_s C_F \, r^2 \, \frac{xG(x,1/r^2)}{\pi R^2} \sim \alpha_s C_F \, n(x, k_\perp \sim 1/r)$$

  - color transparency : $T_0(r,x) \propto r^2$
  - unitarity of the S–matrix requires $T(r,x) \leq 1$

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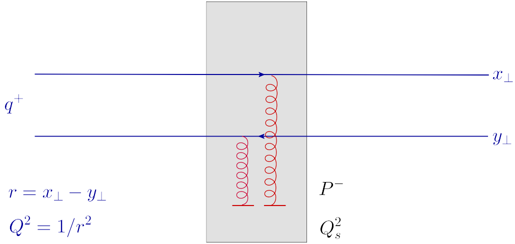

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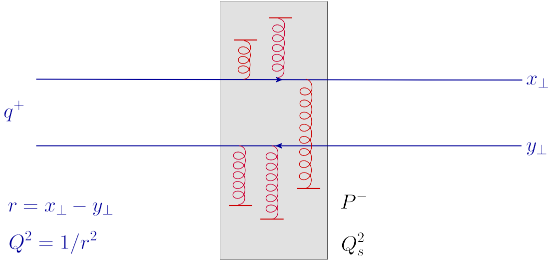
  - color transparency : $T_0(r,x) \propto r^2$
  - yet, $T_0(r,x) > 1$ for large enough r/small enough x

Multiple scattering

- When $T_0(r,x) \sim 1$, multiple scattering becomes important

$$T_0(r,x) \simeq 1 \quad \Longrightarrow \quad \frac{1}{r^2} \simeq \bar{\alpha}\, \frac{xG(x,1/r^2)}{\pi R^2} = Q_s^2(x)$$



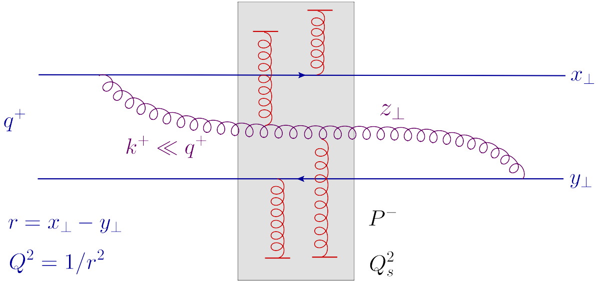
- Assuming independent collisions (McLerran-Venugopalan model)

$$T(r,x) \;\simeq\; 1 - \mathrm{e}^{-T_0(r,x)} \;\leq\; 1 \;: \quad \text{unitarization}$$

- Reasonable so long as x is not too small: $\bar{\alpha}\ln(1/x) \ll 1$

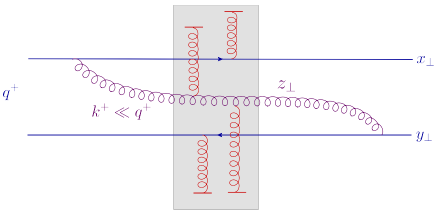
High energy evolution

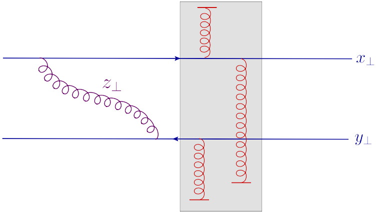
- Probability $\sim \bar{\alpha} \ln \frac{1}{x}$ to radiate a soft gluon with $x \ll 1$
- High-energy evolution is conceptually simpler for the dilute projectile



- The emitted gluon can multiply scatter as well
  - BFKL evolution of the dipole in the background of the dense target
  - multiple scattering $\Longrightarrow$ non-linear evolution $\Longrightarrow$ Balitsky-JIMWLK
  - non-linear generalizations of BFKL equation

- Both 'real' graphs (the soft gluon crosses the target) and 'virtual'





The BK equation *(Balitsky, '96; Kovchegov, '99)*

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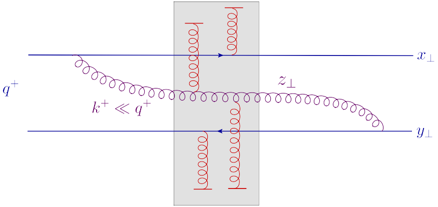
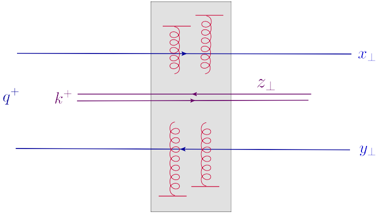




- Large N_c : the original dipole splits into two new dipoles
- Evolution equation for the dipole S–matrix $S_{\boldsymbol{xy}}(Y) \equiv 1 - T_{\boldsymbol{xy}}(Y)$

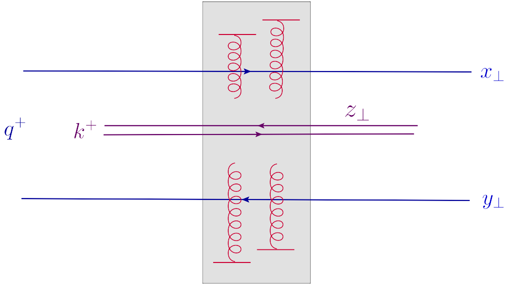
$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \mathcal{M}_{\boldsymbol{xyz}} \big[S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}} \big]$$

- Dipole kernel: BFKL kernel in the dipole picture *(Al Mueller, 1990)*

$$\mathcal{M}_{\boldsymbol{xyz}} = \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} = \left[\frac{z^i - x^i}{(\boldsymbol{z}-\boldsymbol{x})^2} - \frac{z^i - y^i}{(\boldsymbol{z}-\boldsymbol{y})^2} \right]^2$$

Deconstructing the BK equation

- Non-linear equation for the dipole scattering amplitude $T_{\boldsymbol{xy}} \equiv 1 - S_{\boldsymbol{xy}}$

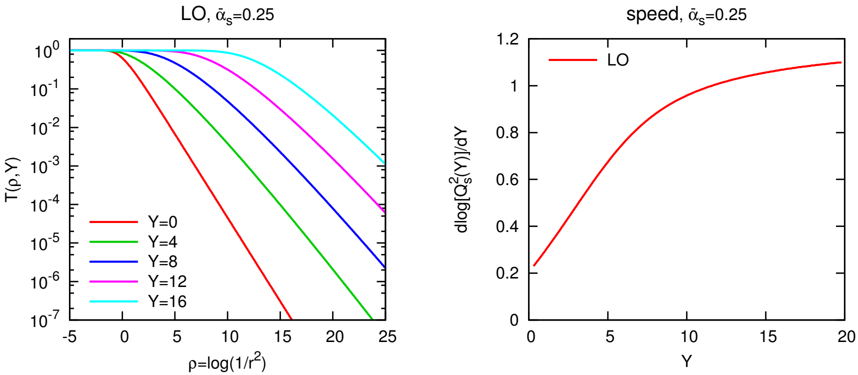


$$\frac{\partial T_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2} \left[T_{\boldsymbol{xz}} + T_{\boldsymbol{zy}} - T_{\boldsymbol{xy}} - T_{\boldsymbol{xz}} T_{\boldsymbol{zy}}\right]$$

- respects unitarity bound: $T(r, Y) \leq 1$
- weak scattering (dilute target): $T(r, Y) \ll 1 \Rightarrow$ BFKL equation
- saturation momentum $Q_s(Y)$: $T(r, Y) = 0.5$ when $r = 2/Q_s(Y)$

The saturation front (numerical solutions to BK)

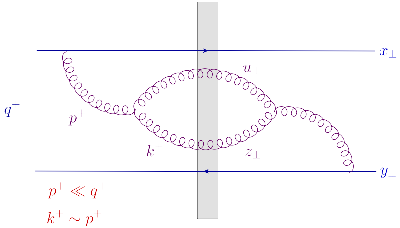
- $T(\rho, Y)$ as a function of $\rho = \ln(1/r^2 Q_0^2)$ with increasing Y

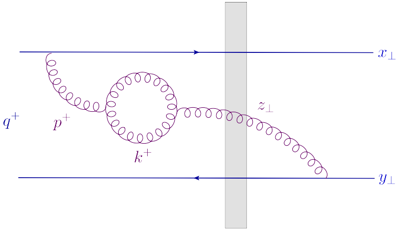


- color transparency at large ρ (small r) : $T \propto r^2 = \mathrm{e}^{-\rho}$
- unitarization at small ρ (large r) : $T = 1$ (black disk limit)
- transition point $\rho_s \equiv \ln[Q_s^2(Y)/Q_0^2]$ rises with Y
- saturation exponent: $\lambda_s \equiv \mathrm{d}\rho_s/\mathrm{d}Y \simeq 1$ for $Y \gtrsim 5$: still too large

Next-to-leading order

- Any effect of $\mathcal{O}(\bar{\alpha}^2 Y) \Longrightarrow \mathcal{O}(\bar{\alpha})$ correction to the BFKL kernel





- The prototype: two successive emissions, one soft and one non-soft
- The maximal correction thus expected: $\mathcal{O}(\bar{\alpha}\rho)$ with $\rho \equiv \ln(Q^2/Q_s^2)$
- But one finds an even larger effect: $\mathcal{O}(\bar{\alpha}\rho^2)$ ('double collinear log')
- Originally found as a NLO correction to the BFKL kernel

  (Fadin, Lipatov, Camici, Ciafaloni ... 95-98; Balitsky, Chirilli, 07)

- Reasonably simple (= it fits into one slide)
- Note however: $N_f = 0$, large N_c, tiny fonts

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, (S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}) \Big\{ 1 +$$

$$+ \bar{\alpha} \Big[\bar{b}\, \ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^2 - (\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}$$

$$+ \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \Big] \Big\}$$

$$+ \frac{\bar{\alpha}^2}{8\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{u}\, \mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\, (S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}})$$

$$\Big\{ -2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+ \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2} \Big[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \Big] \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \Big\}$$

Deconstructing NLO BK

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi}\int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\,\left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)\Bigg\{1 +$$

$$+\,\bar{\alpha}\left[\bar{b}\,\ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\,\frac{(\boldsymbol{x}-\boldsymbol{z})^2-(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left.+\,\frac{67}{36}-\frac{\pi^2}{12}-\frac{1}{2}\ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+\,\frac{\bar{\alpha}^2}{8\pi^2}\int\frac{\mathrm{d}^2\boldsymbol{u}\,\mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4}\left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

$$\Bigg\{-2+\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2+(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2-4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+\,\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1+\frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2-(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right]\ln\frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- blue : leading-order (LO) terms
- red : NLO terms enhanced by (double or single) transverse logarithms
- black : pure $\bar{\alpha}$ corrections (no logarithms)

Deconstructing NLO BK

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2\boldsymbol{z}\, \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{z})^2}\, \left(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right) \Bigg\{ 1 +$$

$$+ \bar{\alpha}\left[\bar{b}\, \ln(\boldsymbol{x}-\boldsymbol{y})^2\mu^2 - \bar{b}\, \frac{(\boldsymbol{x}-\boldsymbol{z})^2 - (\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{y}-\boldsymbol{z})^2}\right.$$

$$\left. + \frac{67}{36} - \frac{\pi^2}{12} - \frac{1}{2} \ln \frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln \frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}\right]\Bigg\}$$

$$+ \frac{\bar{\alpha}^2}{8\pi^2} \int \frac{\mathrm{d}^2\boldsymbol{u}\, \mathrm{d}^2\boldsymbol{z}}{(\boldsymbol{u}-\boldsymbol{z})^4} \left(S_{\boldsymbol{xu}}S_{\boldsymbol{uz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xu}}S_{\boldsymbol{uy}}\right)$$

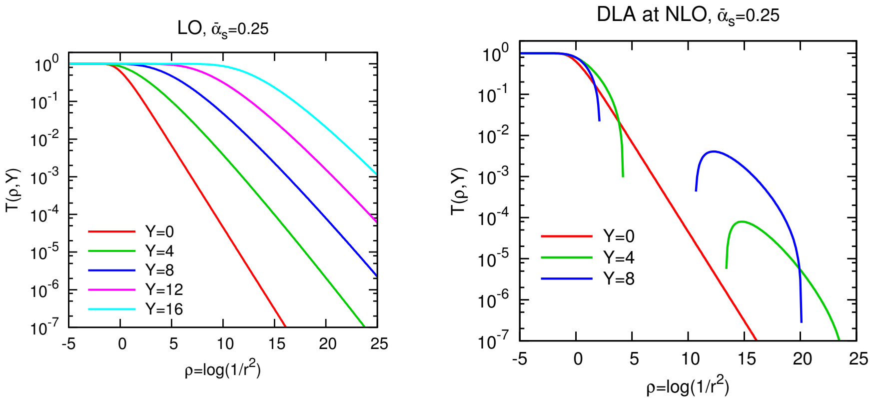
$$\Bigg\{ -2 + \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 + (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2 - 4(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2} \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}$$

$$+ \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}\left[1 + \frac{(\boldsymbol{x}-\boldsymbol{y})^2(\boldsymbol{u}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2 - (\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\right] \ln \frac{(\boldsymbol{x}-\boldsymbol{u})^2(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{y}-\boldsymbol{u})^2}\Bigg\}$$

- Keeping just the logarithmically enhanced terms ($z \gg r$, weak scattering)

$$\frac{\partial T(r)}{\partial Y} \simeq \bar{\alpha} \int \mathrm{d}z^2\, \frac{r^2}{z^4} \left\{1 - \bar{\alpha}\left(\frac{1}{2}\ln^2\frac{z^2}{r^2} + \frac{11}{12}\ln\frac{z^2}{r^2} - \bar{b}\, \ln r^2\mu^2\right)\right\} T(z)$$

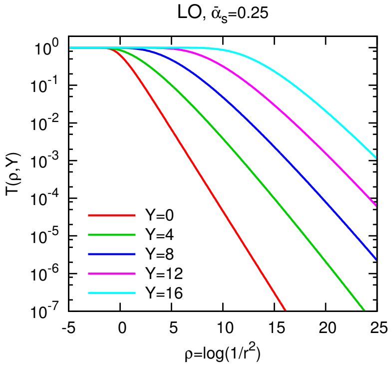
NLO : unstable numerical solutions

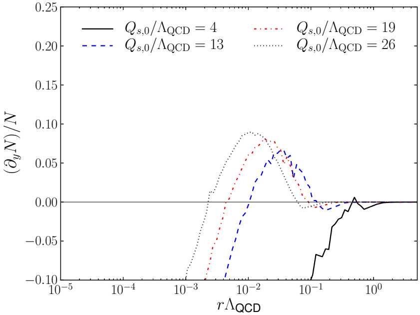


- Left: the saturation front $T(\rho, Y)$ from leading-order BK
- Right: LO BK + the double collinear logarithm at NLO

  (our calculation, arXiv:1502.05642)

NLO : unstable numerical solutions

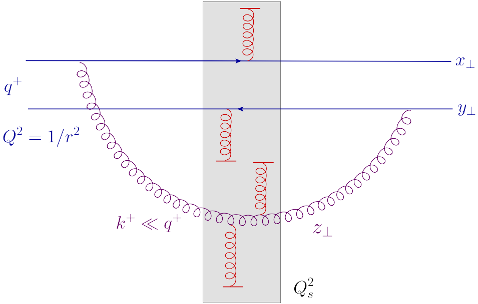




- Left: the saturation front $T(\rho, Y)$ from leading-order BK
- Right: full NLO BK : evolution speed $(\partial_Y T)/T$
  (Lappi, Mäntysaari, arXiv:1502.02400)
- The main source of instability: the double collinear logarithm

Revisiting DLA

- Large transverse separation between projectile and target: $Q^2 \gg Q_s^2$

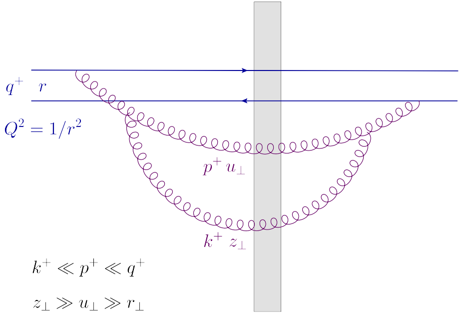


- Large transverse phase-space for gluon emission at $r \ll z \ll 1/Q_s$
  - the daughter dipoles scatter stronger (since larger): $T(r) \ll T(z) \ll 1$

$$\frac{\partial}{\partial Y}\frac{T(r^2)}{r^2} \simeq \bar{\alpha}\int_{r^2}^{1/Q_s^2}\frac{\mathrm{d}z^2}{z^2}\frac{T(z^2)}{z^2} \Longrightarrow \frac{\Delta T}{T} \sim \bar{\alpha} Y \ln\frac{Q^2}{Q_s^2} = \bar{\alpha} Y \rho$$

The double collinear logarithm

- Two successive emissions which are strongly ordered in both ...

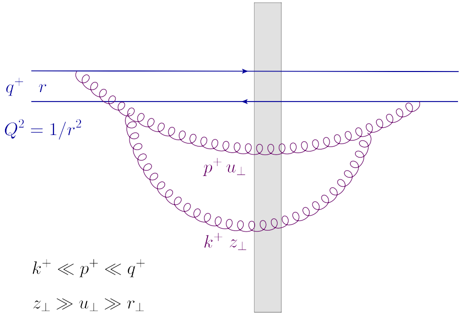


  - longitudinal momenta : $q^+ \gg p^+ \gg k^+$
  - ... and transverse sizes (or momenta): $r_\perp^2 \ll u_\perp^2 \ll z_\perp^2 \ll 1/Q_s^2$

- "Two iterations of DLA $\Longrightarrow (\bar{\alpha} Y \rho)^2$" ... not exactly !
  - additional constraint due to time ordering: $\tau_p > \tau_k$

Time ordering

- Heisenberg: fluctuations have a finite lifetime $\tau_k \sim \frac{k^+}{k_\perp^2} \sim k^+ z_\perp^2$



$$p^+ u_\perp^2 > k^+ z_\perp^2 \implies \Delta Y \equiv \ln \frac{p^+}{k^+} > \Delta\rho \equiv \ln \frac{z_\perp^2}{u_\perp^2}$$

- Additional restriction on the DLA phase-space $\Rightarrow$ double collinear logs

$$Y > \rho \implies \bar{\alpha} Y \rho \longrightarrow \bar{\alpha}(Y - \rho)\rho = \bar{\alpha} Y \rho - \bar{\alpha}\rho^2$$

- Time ordering enters perturbation theory via energy denominators

DLA 2.0

- The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within the naive DLA

  (E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)

$$\frac{\partial T(Y,r^2)}{\partial Y} = \bar{\alpha}\int_{r^2}^{1/Q_0^2} \frac{\mathrm{d}z^2}{z^2}\,\frac{r^2}{z^2}\,T(Y,z^2)$$

DLA 2.0

- The double-collinear logs can be systematically resummed to all orders by enforcing time-ordering within the naive DLA

  (E.I., Madrigal, Mueller, Soyez, and Triantafyllopoulos, arXiv:1502.05642)

$$\frac{\partial T(Y,r^2)}{\partial Y} = \bar{\alpha}\int_{r^2}^{1/Q_s^2} \frac{\mathrm{d}z^2}{z^2}\,\frac{r^2}{z^2}\,\Theta\left(Y-\ln\frac{z^2}{r^2}\right) T\left(Y-\ln\frac{z^2}{r^2}, z^2\right)$$

- Non-local in Y

- Resums all the powers of $\bar{\alpha}Y\rho$ and $\bar{\alpha}\rho^2$

- The importance of enforcing 'kinematical constraints' within LO BFKL had already been recognized ...

  Ciafaloni (88), CCFM (90), Lund group (Andersson et al, 96), Kwiecinski et al (96), Salam (98) ... Motyka, Stasto (09), G. Beuf (14)

- Its diagrammatic foundation in pQCD was not properly appreciated

- So far, no change in the kernel: double-logs come from non-locality

The collinearly improved BK equation

- Equivalently: a local equation but with an all-order resummed kernel
- The argument extends beyond DLA, that is, to BFKL/BK equations

$$\frac{\partial S_{\boldsymbol{xy}}}{\partial Y} = \bar{\alpha} \int \frac{\mathrm{d}^2\boldsymbol{z}}{2\pi} \frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2} \mathcal{K}_{\rm DLA}\big(\rho^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})\big) \big(S_{\boldsymbol{xz}}S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\big)$$

... with the all-order resummed kernel: *(see also Sabio-Vera, 2005)*

$$\mathcal{K}_{\rm DLA}(\rho^2) \equiv \frac{\mathrm{J}_1\big(2\sqrt{\bar{\alpha}\rho^2}\big)}{\sqrt{\bar{\alpha}\rho^2}} = 1 - \frac{\bar{\alpha}\rho^2}{2} + \frac{(\bar{\alpha}\rho^2)^2}{12} + \cdots$$

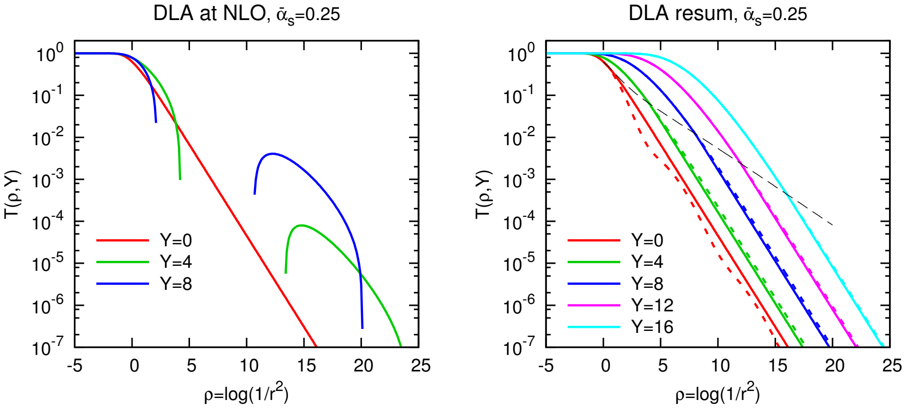
... and the symmetrized version of the collinear double-log:

$$\rho^2(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}) \equiv \ln\frac{(\boldsymbol{x}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2} \ln\frac{(\boldsymbol{y}-\boldsymbol{z})^2}{(\boldsymbol{x}-\boldsymbol{y})^2}$$

- The first correction, of $\mathcal{O}(\bar{\alpha}\rho^2)$, coincides with the NLO double-log

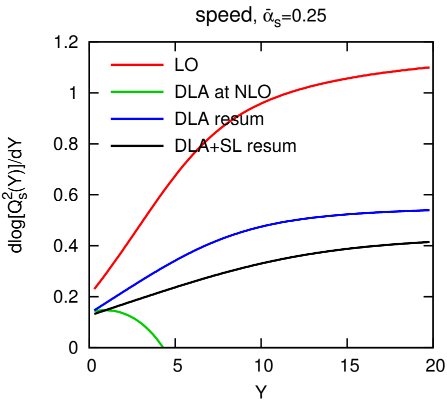
Numerical solutions: saturation front

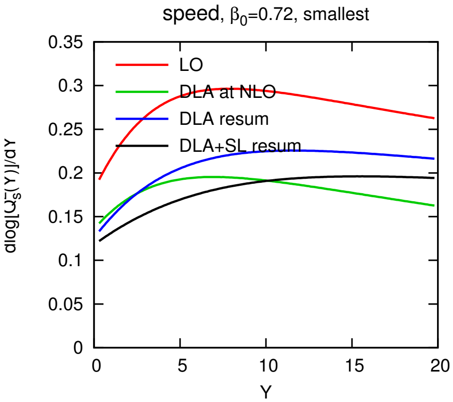
- The resummation stabilizes & slows down the evolution



- Fixed coupling $\bar{\alpha} = 0.25$, resummed kernel
  - left: $\mathcal{K}_{\rm DLA}$ expanded to NLO
  - right: full kernel, double collinear logs resummed to all orders

Saturation exponent $\lambda_s \equiv \mathrm{d}\ln Q_s^2/\mathrm{d}Y$

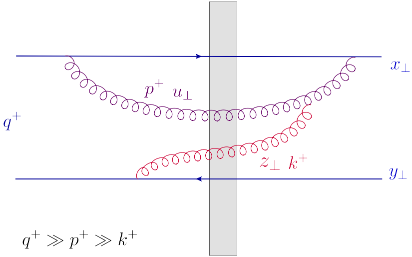




- Fixed coupling
  - LO: $\lambda_s \simeq 1$
  - resummed DL: $\lambda_s \simeq 0.5$
  - DL + SL: $\lambda_s \simeq 0.4$
- Running coupling
  - LO: $\lambda_s = 0.25 \div 0.30$
  - DL + SL: $\lambda_s \simeq 0.2$
  - better convergence

Time-ordered (light-cone) perturbation theory

- Mixed Fourier representation: p^+ and x^+, with $p^- = p_\perp^2/2p^+ = 1/\tau_p$
- All possible time orderings for successive, soft, emissions
- The time (x^+) integrals yield energy denominators



- time-ordered graphs

$$\frac{1}{p^- + k^-} = \frac{\tau_p \tau_k}{\tau_p + \tau_k}$$

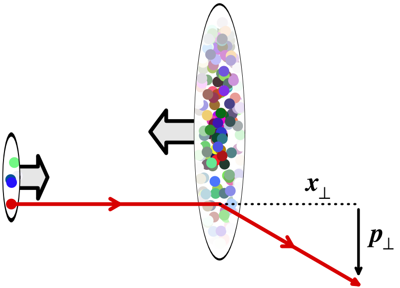
- to have double logs, one needs

$$\tau_p \gg \tau_k$$

- Integrate out the harder gluon ($p^+, u_\perp$) to DLA :

$$\bar{\alpha} \int_{r^2}^{z^2} \frac{\mathrm{d}u^2}{u^2} \int_{k^+}^{q^+} \frac{\mathrm{d}p^+}{p^+} \, \Theta(p^+ u^2 - k^+ z^2) \, = \, \bar{\alpha} Y \rho - \frac{\bar{\alpha}\rho^2}{2}$$

- A quark (or gluon) from the proton scatters off the dense gluon distribution inside the nucleus: $\mathcal{L}_{\rm int}(x) = j_a^\mu(x) A_\mu^a(x)$



- the quark color current density

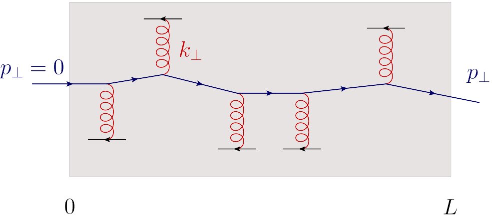
$$j_a^\mu(x) = g\bar\psi(x)\gamma^\mu t^a \psi(x)$$

- the quark S–matrix operator :

$$\hat S = \mathrm{T}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(x)}$$

Multiple scattering in pA : **Wilson lines**

- A quark (or gluon) from the proton scatters off the dense gluon distribution inside the nucleus: $\mathcal{L}_{\rm int}(x) = j_a^\mu(x) A_\mu^a(x)$



- the quark color current density

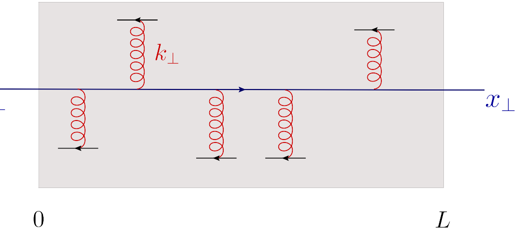
$$j_a^\mu(x) = g\bar{\psi}(x)\gamma^\mu t^a \psi(x)$$

- the quark S–matrix operator :

$$\hat{S} = \mathrm{T}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4 x\, \mathcal{L}_{\rm int}(x)}$$

- View the process in the nucleus rest frame : the quark is very energetic
- Quark energy $E \gg$ typical $k_\perp \lesssim Q_s \Longrightarrow$ small deflection angle $\theta \ll 1$

- A quark (or gluon) from the proton scatters off the dense gluon distribution inside the nucleus: $\mathcal{L}_{\rm int}(x) = j_a^\mu(x) A_\mu^a(x)$



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- the quark S–matrix operator :

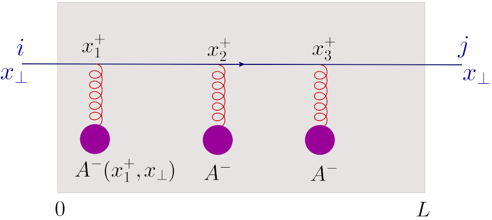
$$\hat S = \mathrm{T}\, \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\, \mathcal{L}_{\rm int}(x)}$$

- View the process in the nucleus rest frame : the quark is very energetic
- Quark energy $E \gg$ typical $k_\perp \lesssim Q_s \Longrightarrow$ small deflection angle $\theta \ll 1$
- The quark transverse position is unchanged: eikonal approximation

$$j_a^\mu(x) \simeq \delta^{\mu +} g t^a \delta(x^-)\delta^{(2)}(\boldsymbol{x} - \boldsymbol{x}_0)$$

Multiple scattering in pA : **Wilson lines**

- A quark (or gluon) from the proton scatters off the dense gluon distribution inside the nucleus: $\mathcal{L}_{\rm int}(x) \,=\, j^\mu_a(x) A^a_\mu(x)$



- the quark color current density

$$j^\mu_a(x) \,=\, g\bar\psi(x)\gamma^\mu t^a\psi(x)$$

- the quark S–matrix operator :

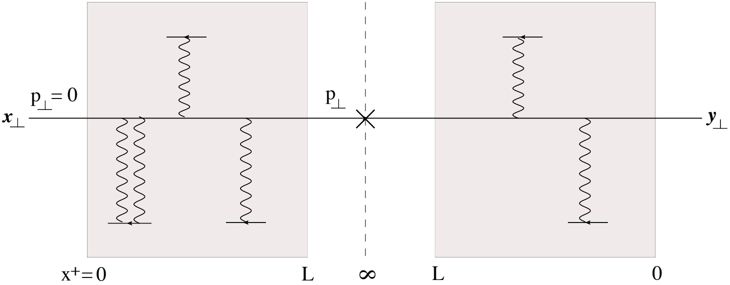
$$\hat S \,=\, \mathrm{T}\,\mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,\mathcal{L}_{\rm int}(x)}$$

- View the process in the nucleus rest frame : the quark is very energetic
- Quark energy $E \gg$ typical $k_\perp \lesssim Q_s \Longrightarrow$ small deflection angle $\theta \ll 1$
- The S–matrix reduces to a Wilson line (color rotation)

$$\Psi_i(x_\perp) \,\to\, V_{ji}(x_\perp)\,\Psi_i(x_\perp), \quad V(x_\perp) = \mathrm{T}\exp\left\{i\int \mathrm{d}x^+ A^-_a(x^+, x_\perp)t^a\right\}$$

Dipole factorization for pA

- The $p_\perp$–spectrum of the quark after crossing the medium $(\boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y})$



$$\frac{\mathrm{d}N}{\mathrm{d}^2\boldsymbol{p}} = \int \frac{\mathrm{d}^2\boldsymbol{r}}{(2\pi)^2}\, \mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{r}} \langle S_{\boldsymbol{xy}} \rangle\,, \qquad S_{\boldsymbol{xy}} \equiv \frac{1}{N_c}\,\mathrm{tr}\big(V_{\boldsymbol{x}} V_{\boldsymbol{y}}^{\dagger}\big)$$

▷ sum over the final color indices, average over the initial ones

▷ average over the distribution of the medium field A_a^-

- S–matrix for effective color dipole: $q\bar{q}$ pair in a color singlet state

Semi-hard intrinsic $k_\perp$

- $Q_s^2(x) \propto$ the gluon density per unit transverse area
- $Q_s(x)$: the typical transverse momentum of the gluons with a given x

$$xG(x,Q^2) = \int d^2b_\perp \int^Q dk_\perp\, k_\perp\, n(x,b_\perp,k_\perp)$$