Refined Cauchy/Littlewood identities and partition functions of the six-vertex model

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Disclaimer: the word *Baxterize* does not appear in this talk.

Aim of talk

- The ASM conjectures were discovered by Mills, Robbins and Rumsey. They express the number of ASMs (with additional symmetries) as simple products.
- After Zeilberger's complicated proof of the original conjecture, Kuperberg found a much simpler proof using the six-vertex model.
- Later on, in a real *tour de force*, Kuperberg computed partition functions of the six-vertex model on a large set of domains. All partition functions were expressed in terms of determinants and Pfaffians.
- Given their determinant and Pfaffian form, it is not surprising that they expand nicely in terms of Schur functions.
- What is much more surprising is that they expand nicely in non determinantal symmetric functions as well.
- The results in this talk allow these partition functions to be written, for example, in the form

$$\frac{\langle 0|\Gamma_+(x_1;t)\dots\Gamma_+(x_n;t)\mathcal{O}(t;u)\Gamma_-(y_n;t)\dots\Gamma_-(y_1;t)|0\rangle}{\langle 0|\Gamma_+(x_1;t)\dots\Gamma_+(x_n;t)\Gamma_-(y_n;t)\dots\Gamma_-(y_1;t)|0\rangle}$$

Schur polynomials and SSYT

- The Schur polynomials $s_\lambda(x_1, \ldots, x_n)$ are the characters of irreducible representations of $GL(n)$. They are given by the Weyl formula:

$$s_\lambda(x_1, \ldots, x_n) = \frac{\det_{1 \leqslant i,j \leqslant n} \left[x_i^{\lambda_j - j + n} \right]}{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)}$$

- A semi-standard Young tableau of shape λ is an assignment of one symbol $\{1, \ldots, n\}$ to each box of the Young diagram λ, such that

  1. The symbols have the ordering $1 < \cdots < n$.
  2. The entries in λ increase weakly along each row and strictly down each column.

- The Schur polynomial $s_\lambda(x_1, \ldots, x_n)$ is also given by a weighted sum over semi-standard Young tableaux T of shape λ:

$$s_\lambda(x_1, \ldots, x_n) = \sum_T \prod_{k=1}^{n} x_k^{\#(k)}$$

SSYT and sequences of interlacing partitions

- Two partitions λ and μ *interlace*, written $\lambda \succ \mu$, if

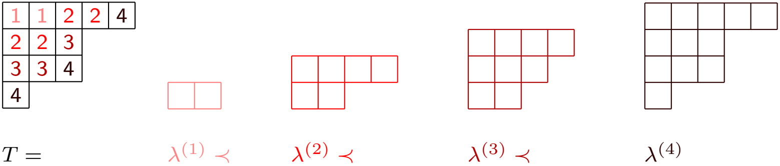
$$\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$$

across all parts of the partitions. It is the same as saying λ/μ is a *horizontal strip*.

- One can interpret a SSYT as a sequence of interlacing partitions:

$$T = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} \equiv \lambda\}$$

- The correspondence works by "peeling away" partition $\lambda^{(k)}$ from T, for all k:



Plane partitions

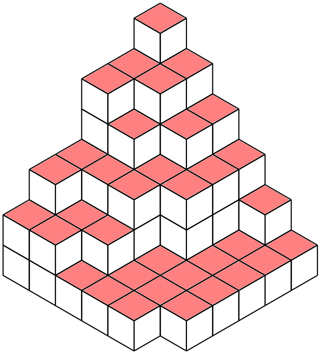
- A plane partition is a set of non-negative integers $\pi(i,j)$ such that for all $i,j \geqslant 1$

$$\pi(i,j) \geqslant \pi(i+1,j) \qquad \pi(i,j) \geqslant \pi(i,j+1)$$

- Plane partitions can be viewed as an increasing then decreasing sequence of interlacing partitions. They are equivalent to *conjoined* SSYT.

- We define the set

$$\boldsymbol{\pi}_{m,n} = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(m)} \equiv \mu^{(n)} \succ \cdots \succ \mu^{(1)} \succ \mu^{(0)} \equiv \emptyset\}$$



Cauchy identity and plane partitions

- The Cauchy identity for Schur polynomials,

$$\sum_{\lambda} s_\lambda(x_1,\dots,x_m)s_\lambda(y_1,\dots,y_n) = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j}$$

can thus be viewed as a generating series of plane partitions:

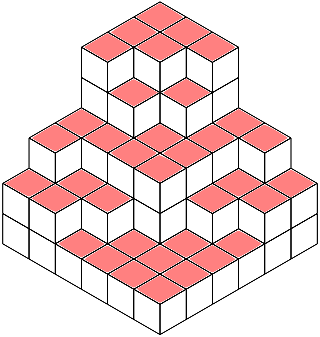
$$\sum_{\pi\in\boldsymbol{\pi}_{m,n}} \prod_{i=1}^{m} x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^{n} y_j^{|\mu^{(j)}|-|\mu^{(j-1)}|} = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-x_iy_j}$$

- Taking the q-specialization $x_i = q^{m-i+1/2}$ and $y_j = q^{n-j+1/2}$, we recover volume-weighted plane partitions:

$$\sum_{\pi\in\boldsymbol{\pi}_{m,n}} q^{|\pi|} = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-q^{m+n-i-j+1}} = \prod_{i=1}^{m}\prod_{j=1}^{n}\frac{1}{1-q^{i+j-1}}$$

Symmetric plane partitions

- Symmetric plane partitions satisfy the condition that $\pi(i,j) = \pi(j,i)$ for all $i, j \geqslant 1$.
- A symmetric plane partition is determined by an increasing sequence of interlacing partitions. (The decreasing part is obtained from the symmetry.)
- They are in one-to-one correspondence with SSYT.



Littlewood identities and symmetric plane partitions

- The three (simplest) Littlewood identities for Schur polynomials

$$\sum_{\lambda} s_{\lambda}(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j} \prod_{i=1}^{n} \frac{1}{1-x_i}$$

$$\sum_{\lambda \text{ even}} s_{\lambda}(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j} \prod_{i=1}^{n} \frac{1}{1-x_i^2}$$

$$\sum_{\lambda' \text{ even}} s_{\lambda}(x_1,\dots,x_n) = \prod_{1\leqslant i<j\leqslant n} \frac{1}{1-x_i x_j}$$

can each be viewed as generating series for symmetric plane partitions, with a (possible) constraint on the partition forming the main diagonal.

Hall–Littlewood polynomials

- Hall–Littlewood polynomials are t-generalizations of Schur polynomials. They can be defined as a sum over the symmetric group:

$$P_\lambda(x_1,\dots,x_n;t) = \frac{1}{v_\lambda(t)} \sum_{\sigma \in S_n} \sigma \left(\prod_{i=1}^{n} x_i^{\lambda_i} \prod_{1 \leqslant i < j \leqslant n} \frac{x_i - t x_j}{x_i - x_j} \right)$$

- Alternatively, the Hall–Littlewood polynomial $P_\lambda(x_1,\dots,x_n;t)$ is given by a weighted sum over semi-standard Young tableaux T of shape λ:

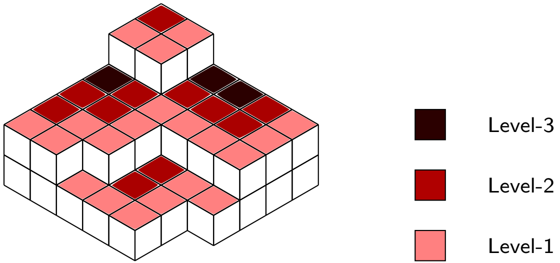
$$P_\lambda(x_1,\dots,x_n;t) = \sum_{T} \prod_{k=1}^{n} \left(x_k^{\#(k)} \psi_{\lambda^{(k)}/\lambda^{(k-1)}}(t) \right)$$

where the function $\psi_{\lambda/\mu}(t)$ is given by

$$\psi_{\lambda/\mu}(t) = \prod_{\substack{i \geqslant 1 \\ m_i(\mu) = m_i(\lambda)+1}} \left(1 - t^{m_i(\mu)} \right)$$

Path-weighted plane partitions

- As Vuletić discovered, the effect of the t-weighting in tableaux has a nice combinatorial interpretation on plane partitions.
- The refinement is that all *paths* at level k receive a weight of $1 - t^k$.
- Example of a plane partition with weight $(1-t)^3(1-t^2)^4(1-t^3)^2$ shown below:



Hall–Littlewood Cauchy identity and path-weighted plane partitions

- The Cauchy identity for Hall–Littlewood polynomials,

$$\sum_{\lambda} \prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1-t^j) P_\lambda(x_1,\dots,x_m;t) P_\lambda(y_1,\dots,y_n;t) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1-tx_iy_j}{1-x_iy_j}$$

  is thus a generating series of (path-weighted) plane partitions:

$$\sum_{\pi \in \boldsymbol{\pi}_{m,n}} \prod_{i \geqslant 1} (1-t^i)^{p_i(\pi)} \prod_{i=1}^{m} x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^{n} y_j^{|\mu^{(j)}|-|\mu^{(j-1)}|} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1-tx_iy_j}{1-x_iy_j}$$

- Taking the same q-specialization as earlier, we obtain

$$\sum_{\pi \in \boldsymbol{\pi}_{m,n}} \prod_{i \geqslant 1} (1-t^i)^{p_i(\pi)} q^{|\pi|} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1-tq^{i+j-1}}{1-q^{i+j-1}}$$

Littlewood identities for Hall–Littlewood polynomials

- The t-analogues of the previously stated Littlewood identities are

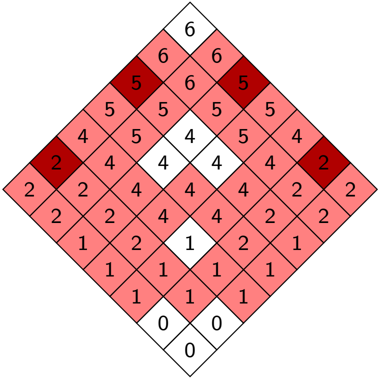
$$\sum_{\lambda} P_\lambda(x_1,\dots,x_n;t) = \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j} \prod_{i=1}^{n} \frac{1}{1-x_i}$$

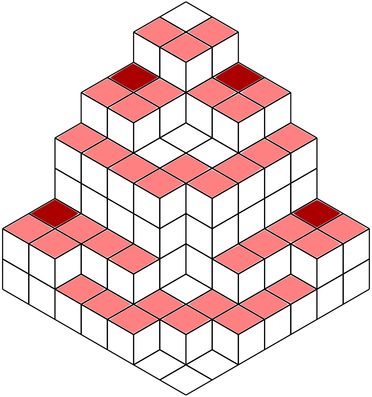
$$\sum_{\lambda \text{ even}} P_\lambda(x_1,\dots,x_n;t) = \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j} \prod_{i=1}^{n} \frac{1}{1-x_i^2}$$

$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1-t^{j-1}) P_\lambda(x_1,\dots,x_n;t) = \prod_{1\leqslant i<j\leqslant n} \frac{1-tx_ix_j}{1-x_ix_j}$$

- These can be regarded as generating series for path-weighted symmetric plane partitions.
- Paths which intersect the main diagonal might not have a t-weight.

t-weighting of symmetric plane partitions





$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_\lambda(x_1, \ldots, x_n; t)$$

Example 1(a): Refined Cauchy identity for Schur polynomials

Theorem

$$\sum_{\lambda} \prod_{i=1}^{n} (1 - ut^{\lambda_i - i + n}) s_\lambda(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n)$$
$$= \frac{1}{\Delta(x)_n \Delta(y)_n} \det_{1 \leqslant i,j \leqslant n} \left[\frac{1 - u + (u-t) x_i y_j}{(1 - t x_i y_j)(1 - x_i y_j)} \right]$$

Proof.

Expand the entries of the determinant as formal power series, and use Cauchy–Binet:

$$\det_{1 \leqslant i,j \leqslant n} \left[\frac{1 - u + (u-t) x_i y_j}{(1 - t x_i y_j)(1 - x_i y_j)} \right] = \det_{1 \leqslant i,j \leqslant n} \left[\sum_{k=0}^{\infty} (1 - ut^k) x_i^k y_j^k \right]$$
$$= \sum_{k_1 > \cdots > k_n \geqslant 0} \prod_{i=1}^{n} (1 - ut^{k_i}) \det_{1 \leqslant i,j \leqslant n} \left[x_i^{k_j} \right] \det_{1 \leqslant i,j \leqslant n} \left[y_j^{k_i} \right]$$

The proof follows after the change of indices $k_i = \lambda_i - i + n$. □

Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Define

$$\mathcal{C}_n(t;u) = \sum_{\lambda} \prod_{k=1}^{m_0(\lambda)} (1-ut^{k-1}) \prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1-t^j) P_\lambda(x_1,\dots,x_n;t) P_\lambda(y_1,\dots,y_n;t)$$

Theorem

$$\mathcal{C}_n(t;u) = \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)}{\Delta(x)_n\Delta(y)_n} \det_{1\leqslant i,j\leqslant n}\left[\frac{1-u+(u-t)x_iy_j}{(1-tx_iy_j)(1-x_iy_j)}\right]$$

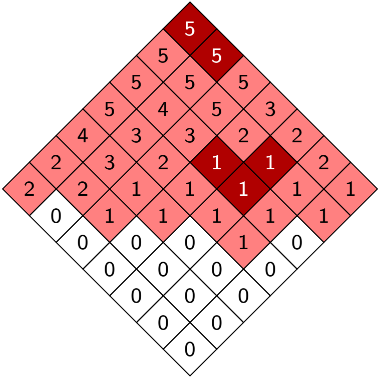
- The specialization $u=t$ is particularly nice:

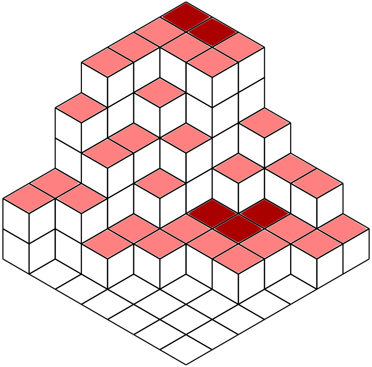
$$\sum_{\lambda} \prod_{i=0}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1-t^j) P_\lambda(x_1,\dots,x_n;t) P_\lambda(y_1,\dots,y_n;t)$$

$$= \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)}{\Delta(x)_n\Delta(y)_n} \det_{1\leqslant i,j\leqslant n}\left[\frac{(1-t)}{(1-tx_iy_j)(1-x_iy_j)}\right]$$

Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

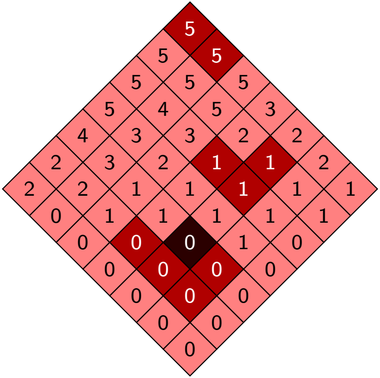
- Question: What does the refinement do at the level of plane partitions?

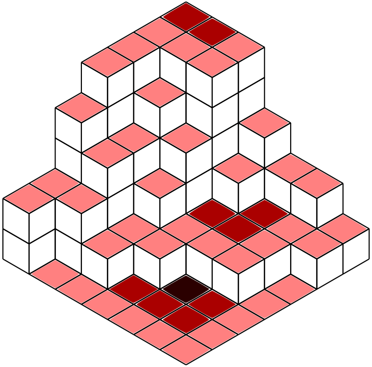




Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Answer: The zero-height entries are treated like the rest.





Example 1(c): Refined Cauchy identity for Macdonald polynomials

- The Cauchy identity for Macdonald polynomials is

$$\sum_{\lambda} b_{\lambda}(q,t) P_{\lambda}(x_1,\dots,x_n;q,t) P_{\lambda}(y_1,\dots,y_n;q,t) = \prod_{i,j=1}^{n} \frac{(tx_iy_j;q)}{(x_iy_j;q)}$$

where

$$(x;q) = \prod_{k=0}^{\infty}(1-q^k x), \qquad b_{\lambda}(q,t) = \prod_{s\in\lambda} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}$$

Theorem (Kirillov–Noumi,Warnaar)

$$\sum_{\lambda} \prod_{i=1}^{n}(1-uq^{\lambda_i}t^{n-i}) b_{\lambda}(q,t) P_{\lambda}(x_1,\dots,x_n;q,t) P_{\lambda}(y_1,\dots,y_n;q,t) =$$

$$\prod_{i,j=1}^{n} \frac{(tx_iy_j;q)}{(x_iy_j;q)} \frac{\prod_{i,j=1}^{n}(1-x_iy_j)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)} \det_{1\leqslant i,j\leqslant n}\left[\frac{1-u+(u-t)x_iy_j}{(1-tx_iy_j)(1-x_iy_j)}\right]$$

Proof.

Act on the Cauchy identity with a generating series of Macdonald's difference operators. The left hand side follows immediately. The right hand side follows after acting on the Cauchy kernel, and performing some manipulation. □

Example 2(a): Refined Littlewood identity for Schur polynomials

Theorem

$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{n} (1 - ut^{\lambda_{2i}-2i+2n}) s_\lambda(x_1, \ldots, x_{2n})$$
$$= \prod_{1 \leqslant i < j \leqslant 2n} \frac{1}{(x_i - x_j)} \operatorname*{Pf}_{1 \leqslant i < j \leqslant 2n} \left[\frac{(1 - u + (u - t)x_i x_j)(x_i - x_j)}{(1 - tx_i x_j)(1 - x_i x_j)} \right]$$

Proof.

Expand the entries of the Pfaffian and use a Pfaffian analogue of Cauchy–Binet:

$$\operatorname*{Pf}_{1 \leqslant i < j \leqslant 2n} [\cdots] = \operatorname*{Pf}_{1 \leqslant i < j \leqslant 2n} \left[\sum_{0 \leqslant k < l} \delta_{l,k+1} (1 - ut^k)(x_i^l x_j^k - x_i^k x_j^l) \right]$$
$$= \sum_{k_1 > \cdots > k_{2n} \geqslant 0} \operatorname*{Pf}_{1 \leqslant i < j \leqslant 2n} \left[\delta_{k_i, k_j+1} (1 - ut^{k_i}) \right] \det_{1 \leqslant i,j \leqslant 2n} \left[x_i^{k_j} \right]$$

The Pfaffian in the sum factorizes, to produce the correct (blue) factor and the restriction on the summation. □

Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- Define

$$\mathcal{L}_{2n}(t;u) = \sum_{\lambda' \text{ even}} \prod_{k \text{ even}}^{m_0(\lambda)} (1 - ut^{k-2}) \prod_{i=1}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_\lambda(x_1, \dots, x_{2n}; t)$$

Theorem (DB,MW,PZJ)

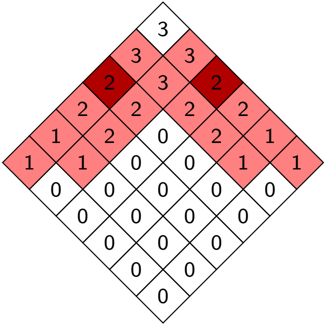
$$\mathcal{L}_{2n}(t;u) = \prod_{1 \leqslant i < j \leqslant 2n} \frac{(1 - tx_ix_j)}{(x_i - x_j)} \operatorname*{Pf}_{1 \leqslant i < j \leqslant 2n} \left[\frac{(1 - u + (u - t)x_ix_j)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]$$

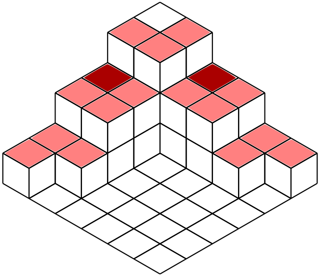
- The specialization $u = t$ is again especially nice:

$$\sum_{\lambda' \text{ even}} \prod_{i=0}^{\infty} \prod_{j \text{ even}}^{m_i(\lambda)} (1 - t^{j-1}) P_\lambda(x_1, \dots, x_{2n}; t) = \prod_{1 \leqslant i < j \leqslant 2n} \frac{(1 - tx_ix_j)}{(x_i - x_j)} \operatorname*{Pf}_{1 \leqslant i < j \leqslant 2n} \left[\frac{(1 - t)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]$$

Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

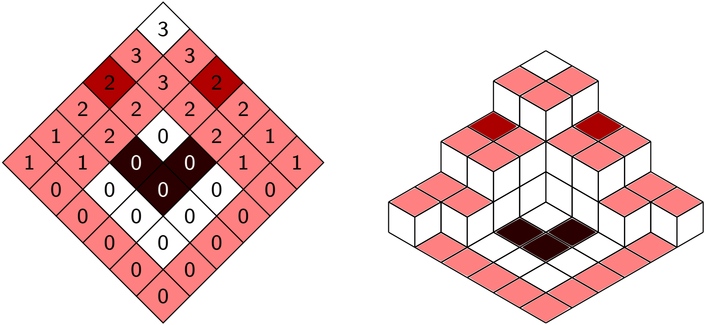
- At the level of plane partitions, this is (again) a very simple refinement.





Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- At the level of plane partitions, this is (again) a very simple refinement.



Example 2(c): Refined Littlewood identity for Macdonald polynomials

- The most fundamental Littlewood identity for Macdonald polynomials is

$$\sum_{\lambda' \text{ even}} b_\lambda^{\text{el}}(q,t) P_\lambda(x_1,\ldots,x_{2n};q,t) = \prod_{1\leqslant i<j\leqslant 2n} \frac{(tx_ix_j;q)}{(x_ix_j;q)}$$

where

$$b_\lambda^{\text{el}}(q,t) = \prod_{\substack{s\in\lambda \\ l(s) \text{ even}}} \frac{1-q^{a(s)}t^{l(s)+1}}{1-q^{a(s)+1}t^{l(s)}}$$

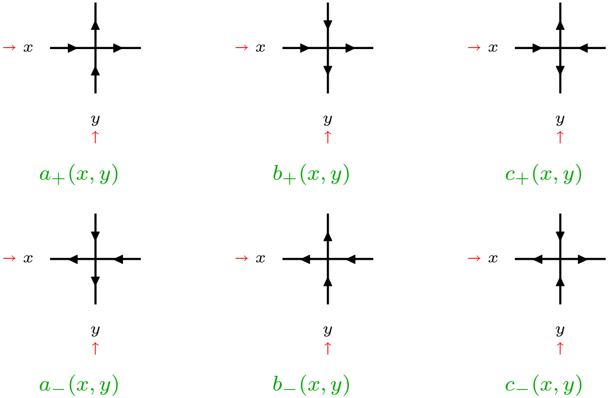
Conjecture (DB,MW,PZJ)

$$\sum_{\lambda' \text{ even}} \prod_{i=1}^{n} (1-uq^{\lambda_{2i}}t^{2n-2i}) b_\lambda^{\text{el}}(q,t) P_\lambda(x_1,\ldots,x_{2n};q,t) =$$

$$\prod_{1\leqslant i<j\leqslant 2n} \frac{(tx_ix_j;q)}{(x_ix_j;q)} \prod_{1\leqslant i<j\leqslant 2n} \frac{(1-x_ix_j)}{(x_i-x_j)} \operatorname*{Pf}_{1\leqslant i<j\leqslant 2n} \left[\frac{(1-u+(u-t)x_ix_j)(x_i-x_j)}{(1-tx_ix_j)(1-x_ix_j)} \right]$$

The six-vertex model

- The vertices of the six-vertex model are



The six-vertex model

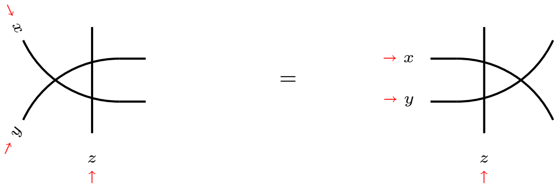
- The Boltzmann weights are given by

$$a_+(x,y) = \frac{1-tx/y}{1-x/y} \qquad a_-(x,y) = \frac{1-tx/y}{1-x/y}$$

$$b_+(x,y) = \sqrt{t} \qquad b_-(x,y) = \sqrt{t}$$

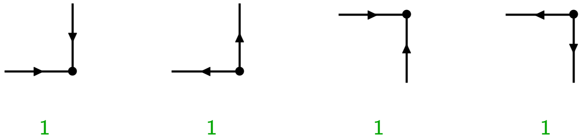
$$c_+(x,y) = \frac{(1-t)}{1-x/y} \qquad c_-(x,y) = \frac{(1-t)x/y}{1-x/y}$$

- The parameter t from Hall–Littlewood is now the crossing parameter of the model.
- The Boltzmann weights obey the Yang–Baxter equations (the $\mathcal{U}_q(\widehat{sl_2})$ solution):



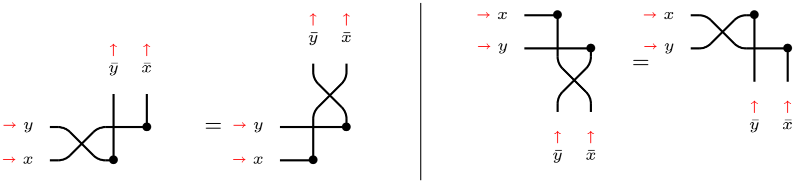
Boundary vertices

- We also require corner vertices



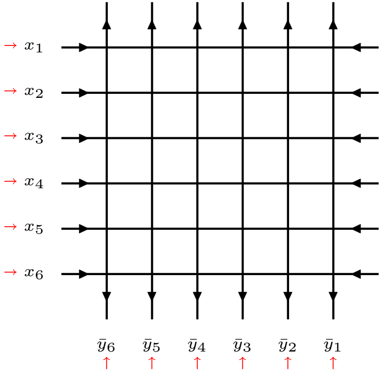
which do not depend on a spectral parameter and behave like sources/sinks.

- The corner vertices satisfy a reflection equation:



Domain wall boundary conditions

- The six-vertex model on a lattice with domain wall boundary conditions was first considered by Korepin:



- This partition function is of fundamental importance in periodic quantum spin chains based on $\mathcal{Y}(sl_2)$ and $\mathcal{U}_q(\widehat{sl_2})$.

Domain wall boundary conditions

- Configurations on this lattice are in one-to-one correspondence with alternating sign matrices:

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ 0 & + & - & 0 & + & 0 \\ + & - & 0 & + & 0 & 0 \\ 0 & + & 0 & - & 0 & + \\ 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \end{pmatrix}$$

- The domain wall partition function was evaluated in determinant form by Izergin:

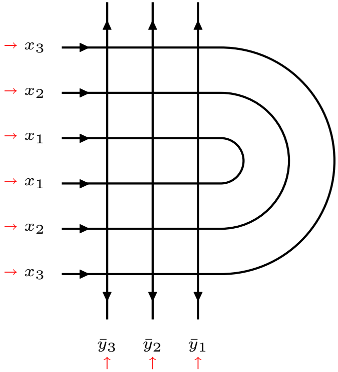
$$Z_{\mathrm{ASM}}(x_1,\dots,x_n;y_1,\dots,y_n;t) = \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)} \det\left[\frac{(1-t)}{(1-tx_iy_j)(1-x_iy_j)}\right]_{1\leqslant i,j\leqslant n}$$

- The DWPF is equal to the right hand side of a refined Cauchy identity:

$$Z_{\mathrm{ASM}}(x_1,\dots,x_n;y_1,\dots,y_n;t) = \mathcal{C}_n(t;t)$$

Half-turn symmetry

- One can consider those configurations under domain wall boundary conditions which have 180° rotational symmetry.
- The fundamental domain is given by:



Half-turn symmetry

- Configurations on this lattice are in one-to-one correspondence with half-turn symmetric alternating sign matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & + & - & + \\ 0 & 0 & + & - & + & 0 \\ 0 & + & - & + & 0 & 0 \\ + & - & + & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 \end{pmatrix}$$

- Kuperberg evaluated this partition function as a product of determinants:

$$Z_{\mathrm{HT}}(x_1,\dots,x_n;y_1,\dots,y_n;t) = \frac{\prod_{i,j=1}^{n}(1-tx_iy_j)^2}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)^2(y_i-y_j)^2}$$

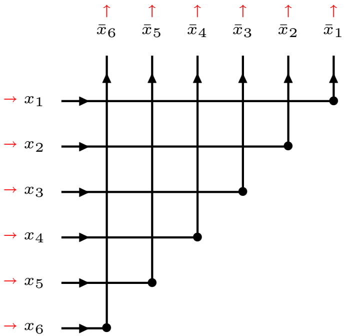
$$\times \det_{1\leqslant i,j\leqslant n}\left[\frac{(1-t)}{(1-tx_iy_j)(1-x_iy_j)}\right] \det_{1\leqslant i,j\leqslant n}\left[\frac{(1+\sqrt{t})(1-\sqrt{t}x_iy_j)}{(1-tx_iy_j)(1-x_iy_j)}\right]$$

- In other words,

$$Z_{\mathrm{HT}}(x_1,\dots,x_n;y_1,\dots,y_n;t) = \mathcal{C}_n(t;t)\mathcal{C}_n(t;-\sqrt{t})$$

Off-diagonally symmetric boundary conditions

- Off-diagonally symmetric boundary conditions were introduced by Kuperberg. One considers domain wall configurations with reflection symmetry about a diagonal axis, and which have no c vertices on that diagonal.
- The fundamental domain is



Off-diagonally symmetric boundary conditions

- Configurations on this lattice are in one-to-one correspondence with off-diagonally symmetric ASMs (OSASMs):

$$\begin{pmatrix} 0 & 0 & + & 0 & 0 & 0 \\ + & 0 & - & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & 0 & - & + \\ 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \end{pmatrix}$$

- Kuperberg evaluated this partition function as a Pfaffian:

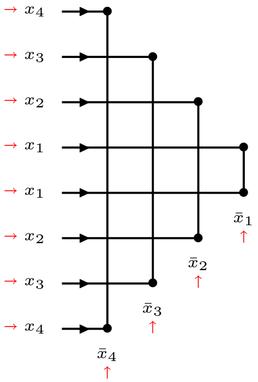
$$Z_{\mathrm{OSASM}}(x_1,\ldots,x_{2n};t) = \prod_{1\leqslant i<j\leqslant 2n} \frac{(1-tx_ix_j)}{(x_i-x_j)} \mathrm{Pf}\left[\frac{(1-t)(x_i-x_j)}{(1-tx_ix_j)(1-x_ix_j)}\right]_{1\leqslant i<j\leqslant 2n}$$

- The OSASM partition function is equal to the right hand side of a refined Littlewood identity:

$$Z_{\mathrm{OSASM}}(x_1,\ldots,x_{2n};t) = \mathcal{L}_{2n}(t;t)$$

Off-diagonally/off-anti-diagonally symmetric boundary conditions

- Similarly, one can consider domain wall configurations with reflection symmetry in both diagonals, and with no c vertices on those diagonals.
- The fundamental domain is



Off-diagonally/off-anti-diagonally symmetric boundary conditions

- Configurations on this lattice are in one-to-one correspondence with off-diagonally/off-anti-diagonally symmetric ASMs (OOSASMs):

$$\begin{pmatrix} 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & 0 & 0 & 0 & - & + \\ + & - & 0 & 0 & 0 & 0 & + & 0 \\ 0 & + & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & + & - & + & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & 0 \end{pmatrix}$$

- The partition function can be evaluated as a product of Pfaffians:

$$Z_{\mathrm{OOSASM}}(x_1,\ldots,x_{2n};t) = \prod_{1\leqslant i<j\leqslant 2n} \frac{(1-tx_ix_j)^2}{(x_i-x_j)^2} \times$$

$$\underset{1\leqslant i<j\leqslant 2n}{\mathrm{Pf}} \left[\frac{(1-t)(x_i-x_j)}{(1-tx_ix_j)(1-x_ix_j)} \right] \underset{1\leqslant i<j\leqslant 2n}{\mathrm{Pf}} \left[\frac{(1+\sqrt{t})(1-\sqrt{t}x_ix_j)(x_i-x_j)}{(1-tx_ix_j)(1-x_ix_j)} \right]$$

- In other words,

$$Z_{\mathrm{OOSASM}}(x_1,\ldots,x_{2n};t) = \mathcal{L}_{2n}(t;t)\mathcal{L}_{2n}(t;-\sqrt{t})$$

Open questions

- Expansion of other symmetry classes of ASMs.
- What is the missing operator needed to prove the conjecture?
- Do these correspondences have a combinatorial meaning? The similarity of the underlying domains on both sides of these correspondences is very curious.
- Can more general objects in the six-vertex/XXZ model (form factors/correlation functions) be expanded nicely in terms of Hall–Littlewood polynomials?
- What about more general models, such as the eight-vertex and 8VSOS models?