Refined Cauchy/Littlewood identities and partition functions of the six-vertex model

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Disclaimer: the word "Baxterize" does not appear in this talk.
Aim of talk

- The ASM conjectures were discovered by Mills, Robbins and Rumsey. They express the number of ASMs (with additional symmetries) as simple products.

- After Zeilberger’s complicated proof of the original conjecture, Kuperberg found a much simpler proof using the six-vertex model.

- Later on, in a real tour de force, Kuperberg computed partition functions of the six-vertex model on a large set of domains. All partition functions were expressed in terms of determinants and Pfaffians.

- Given their determinant and Pfaffian form, it is not surprising that they expand nicely in terms of Schur functions.

- What is much more surprising is that they expand nicely in non determinantal symmetric functions as well.

- The results in this talk allow these partition functions to be written, for example, in the form

\[
\frac{\langle 0 | \Gamma_+(x_1; t) \cdots \Gamma_+(x_n; t) \mathcal{O}(t; u) \Gamma_-(y_n; t) \cdots \Gamma_-(y_1; t) | 0 \rangle}{\langle 0 | \Gamma_+(x_1; t) \cdots \Gamma_+(x_n; t) \Gamma_-(y_n; t) \cdots \Gamma_-(y_1; t) | 0 \rangle}
\]
**Schur polynomials and SSYT**

- The Schur polynomials \( s_\lambda(x_1, \ldots, x_n) \) are the characters of irreducible representations of \( GL(n) \). They are given by the Weyl formula:

  \[
  s_\lambda(x_1, \ldots, x_n) = \frac{\det_{1 \leq i,j \leq n} [x_i^{\lambda_j-j+n}]}{\prod_{1 \leq i < j \leq n} (x_i - x_j)}
  \]

- A semi-standard Young tableau of shape \( \lambda \) is an assignment of one symbol \( \{1, \ldots, n\} \) to each box of the Young diagram \( \lambda \), such that

  1. The symbols have the ordering \( 1 < \cdots < n \).
  2. The entries in \( \lambda \) increase weakly along each row and strictly down each column.

- The Schur polynomial \( s_\lambda(x_1, \ldots, x_n) \) is also given by a weighted sum over semi-standard Young tableaux \( T \) of shape \( \lambda \):

  \[
  s_\lambda(x_1, \ldots, x_n) = \sum_T \prod_{k=1}^{n} x_k^{\#(k)}
  \]
SSYT and sequences of interlacing partitions

- Two partitions $\lambda$ and $\mu$ *interlace*, written $\lambda \succ \mu$, if

$$\lambda_i \geq \mu_i \geq \lambda_{i+1}$$

across all parts of the partitions. It is the same as saying $\lambda/\mu$ is a *horizontal strip*.

- One can interpret a SSYT as a sequence of interlacing partitions:

$$T = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(n)} \equiv \lambda\}$$

- The correspondence works by “peeling away” partition $\lambda^{(k)}$ from $T$, for all $k$:

```
1 1 2 2 4
2 2 3
3 3 4
4
T =  \lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \lambda^{(4)}
```
Plane partitions

- A plane partition is a set of non-negative integers $\pi(i, j)$ such that for all $i, j \geq 1$
  \[ \pi(i, j) \geq \pi(i + 1, j) \quad \text{and} \quad \pi(i, j) \geq \pi(i, j + 1) \]

- Plane partitions can be viewed as an increasing then decreasing sequence of interlacing partitions. They are equivalent to *conjoined* SSYT.

- We define the set
  \[
  \pi_{m,n} = \{ \emptyset \equiv \lambda^{(0)} < \lambda^{(1)} < \cdots < \lambda^{(m)} \equiv \mu^{(n)} \geq \cdots \geq \mu^{(1)} \geq \mu^{(0)} \equiv \emptyset \}
  \]
The Cauchy identity for Schur polynomials,

$$\sum_{\lambda} s_{\lambda}(x_1, \ldots, x_m)s_{\lambda}(y_1, \ldots, y_n) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j}$$

can thus be viewed as a generating series of plane partitions:

$$\sum_{\pi \in \Pi_{m,n}} \prod_{i=1}^{m} x_i^{\lambda(i)} \prod_{j=1}^{n} y_j^{\mu(j)} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - x_i y_j}$$

Taking the $q$-specialization $x_i = q^{m-i+1/2}$ and $y_j = q^{n-j+1/2}$, we recover volume-weighted plane partitions:

$$\sum_{\pi \in \Pi_{m,n}} q^{\pi} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - q^{m+n-i-j+1}} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1}{1 - q^{i+j-1}}$$
Symmetric plane partitions

- Symmetric plane partitions satisfy the condition that $\pi(i, j) = \pi(j, i)$ for all $i, j \geq 1$.
- A symmetric plane partition is determined by an increasing sequence of interlacing partitions. (The decreasing part is obtained from the symmetry.)
- They are in one-to-one correspondence with SSYT.
Littlewood identities and symmetric plane partitions

- The three (simplest) Littlewood identities for Schur polynomials

\[ \sum_{\lambda} s_{\lambda}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \prod_{i=1}^{n} \frac{1}{1 - x_i} \]

\[ \sum_{\lambda \text{ even}} s_{\lambda}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \prod_{i=1}^{n} \frac{1}{1 - x_i^2} \]

\[ \sum_{\lambda' \text{ even}} s_{\lambda}(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} \frac{1}{1 - x_i x_j} \]

can each be viewed as generating series for symmetric plane partitions, with a (possible) constraint on the partition forming the main diagonal.
Hall–Littlewood polynomials

- Hall–Littlewood polynomials are $t$-generalizations of Schur polynomials. They can be defined as a sum over the symmetric group:

\[
P_\lambda(x_1, \ldots, x_n; t) = \frac{1}{v_\lambda(t)} \sum_{\sigma \in S_n} \sigma \left( \prod_{i=1}^{n} x_i^{\lambda_i} \prod_{1 \leq i < j \leq n} \frac{x_i - tx_j}{x_i - x_j} \right)
\]

- Alternatively, the Hall–Littlewood polynomial $P_\lambda(x_1, \ldots, x_n; t)$ is given by a weighted sum over semi-standard Young tableaux $T$ of shape $\lambda$:

\[
P_\lambda(x_1, \ldots, x_n; t) = \sum_{T} \prod_{k=1}^{n} \left( x_k^{\#(k)} \psi_{\lambda(k)/\lambda(k-1)}(t) \right)
\]

where the function $\psi_{\lambda/\mu}(t)$ is given by

\[
\psi_{\lambda/\mu}(t) = \prod_{i \geq 1, m_i(\mu) = m_i(\lambda) + 1} \left( 1 - t^{m_i(\mu)} \right)
\]
As Vuletić discovered, the effect of the $t$-weighting in tableaux has a nice combinatorial interpretation on plane partitions.

The refinement is that all paths at level $k$ receive a weight of $1 - t^k$.

Example of a plane partition with weight $(1 - t)^3(1 - t^2)^4(1 - t^3)^2$ shown below:
Hall–Littlewood Cauchy identity and path-weighted plane partitions

- The Cauchy identity for Hall–Littlewood polynomials,

\[
\sum_{\lambda} \prod_{i=1}^{m} \prod_{j=1}^{\infty} (1 - t^j) P_\lambda(x_1, \ldots, x_m; t) P_\lambda(y_1, \ldots, y_n; t) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 - tx_i y_j}{1 - x_i y_j}
\]

is thus a generating series of (path-weighted) plane partitions:

\[
\sum_{\pi \in \pi_{m, n}} \prod_{i \geq 1} \left(1 - t^i\right)^{p_i(\pi)} \prod_{i=1}^{m} x_i^{d(\pi) - d(\pi - 1)} \prod_{j=1}^{n} y_j^{d(\pi) - d(\pi - 1)} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 - tx_i y_j}{1 - x_i y_j}
\]

- Taking the same \(q\)-specialization as earlier, we obtain

\[
\sum_{\pi \in \pi_{m, n}} \prod_{i \geq 1} \left(1 - t^i\right)^{p_i(\pi)} q^{d(\pi)} = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{1 - t q^{i+j-1}}{1 - q^{i+j-1}}
\]
The \( t \)-analogues of the previously stated Littlewood identities are

\[
\sum_{\lambda} P_\lambda(x_1, \ldots, x_n; t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j} \prod_{i=1}^{n} \frac{1}{1 - x_i} \\
\sum_{\lambda \text{ even}} P_\lambda(x_1, \ldots, x_n; t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j} \prod_{i=1}^{n} \frac{1}{1 - x_i^2} \\
\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}} (1 - t^{j-1}) P_\lambda(x_1, \ldots, x_n; t) = \prod_{1 \leq i < j \leq n} \frac{1 - tx_i x_j}{1 - x_i x_j}
\]

These can be regarded as generating series for path-weighted symmetric plane partitions.

Paths which intersect the main diagonal might not have a \( t \)-weight.
$t$-weighting of symmetric plane partitions

\[
\sum_{\lambda' \text{ even}} \prod_{i=1}^{\infty} \prod_{j \text{ even}} m_i(\lambda) (1 - t^{j-1}) P_\lambda(x_1, \ldots, x_n; t)
\]
**Example 1(a): Refined Cauchy identity for Schur polynomials**

**Theorem**

\[
\sum_{\lambda} \prod_{i=1}^{n} (1 - ut^{\lambda_i - i + n}) s_{\lambda}(x_1, \ldots, x_n) s_{\lambda}(y_1, \ldots, y_n) = \frac{1}{\Delta(x)_n \Delta(y)_n} \det_{1 \leq i, j \leq n} \left[ \frac{1 - u + (u - t)x_i y_j}{(1 - tx_i y_j)(1 - x_i y_j)} \right]
\]

**Proof.**

Expand the entries of the determinant as formal power series, and use Cauchy–Binet:

\[
\det_{1 \leq i, j \leq n} \left[ \frac{1 - u + (u - t)x_i y_j}{(1 - tx_i y_j)(1 - x_i y_j)} \right] = \det_{1 \leq i, j \leq n} \left[ \sum_{k=0}^{\infty} (1 - ut^k)x_i^k y_j^k \right]
\]

\[
= \sum_{k_1 > \cdots > k_n \geq 0} \prod_{i=1}^{n} (1 - ut^{k_i}) \det_{1 \leq i, j \leq n} \left[ x_i^{k_j} \right] \det_{1 \leq i, j \leq n} \left[ y_j^{k_i} \right]
\]

The proof follows after the change of indices \( k_i = \lambda_i - i + n \).
Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Define

\[ C_n(t; u) = \sum_{\lambda} \prod_{k=1}^{m_0(\lambda)} (1 - ut^{k-1}) \prod_{i=1}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1 - t^j) P_{\lambda}(x_1, \ldots, x_n; t) P_{\lambda}(y_1, \ldots, y_n; t) \]

**Theorem**

\[ C_n(t; u) = \frac{\prod_{i,j=1}^{n} (1 - tx_iy_j)}{\Delta(x)_n \Delta(y)_n} \det_{1 \leq i, j \leq n} \left[ \frac{1 - u + (u-t)x_iy_j}{(1 - tx_iy_j)(1 - x_iy_j)} \right] \]

- The specialization \( u = t \) is particularly nice:

\[ \sum_{\lambda} \prod_{i=0}^{\infty} \prod_{j=1}^{m_i(\lambda)} (1 - t^j) P_{\lambda}(x_1, \ldots, x_n; t) P_{\lambda}(y_1, \ldots, y_n; t) = \frac{\prod_{i,j=1}^{n} (1 - tx_iy_j)}{\Delta(x)_n \Delta(y)_n} \det_{1 \leq i, j \leq n} \left[ \frac{(1 - t)}{(1 - tx_iy_j)(1 - x_iy_j)} \right] \]
Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Question: What does the refinement do at the level of plane partitions?
Example 1(b): Refined Cauchy identity for Hall–Littlewood polynomials

- Answer: The zero-height entries are treated like the rest.
Example 1(c): Refined Cauchy identity for Macdonald polynomials

- The Cauchy identity for Macdonald polynomials is

\[
\sum_{\lambda} b_{\lambda}(q,t)P_{\lambda}(x_1, \ldots, x_n; q, t)P_{\lambda}(y_1, \ldots, y_n; q, t) = \prod_{i,j=1}^{n} \frac{(tx_i y_j; q)}{(x_i y_j; q)}
\]

where

\[
(x; q) = \prod_{k=0}^{\infty} (1 - q^k x), \quad b_{\lambda}(q,t) = \prod_{s \in \lambda} \frac{1 - q^{a(s)} t^{l(s)} + 1}{1 - q^{a(s) + 1} t^{l(s)}}
\]

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Theorem (Kirillov–Noumi,Warnaar)

\[
\sum_{\lambda} \prod_{i=1}^{n} (1 - uq^{\lambda_i} t^{n-i})b_{\lambda}(q,t)P_{\lambda}(x_1, \ldots, x_n; q, t)P_{\lambda}(y_1, \ldots, y_n; q, t) =
\]

\[
\prod_{i,j=1}^{n} \frac{(tx_i y_j; q)}{(x_i y_j; q)} \prod_{i,j=1}^{n} \frac{(1 - x_i y_j)}{(1 - tx_i y_j)(1 - x_i y_j)} \det_{1 \leq i,j \leq n} \left[ \frac{1 - u + (u - t)x_i y_j}{(1 - tx_i y_j)(1 - x_i y_j)} \right]
\]

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Proof.

Act on the Cauchy identity with a generating series of Macdonald’s difference operators. The left hand side follows immediately. The right hand side follows after acting on the Cauchy kernel, and performing some manipulation.
Example 2(a): Refined Littlewood identity for Schur polynomials

**Theorem**

\[
\sum_{\lambda \text{ even}} \prod_{i=1}^{n} (1 - ut^{2i-2i+2n}) s_{\lambda}(x_1, \ldots, x_{2n}) = \prod_{1 \leq i < j \leq 2n} \frac{1}{(x_i - x_j)} \prod_{1 \leq i < j \leq 2n} \text{Pf} \left[ \frac{(1 - u + (u-t)x_i x_j)(x_i - x_j)}{(1 - tx_i x_j)(1 - x_i x_j)} \right]
\]

**Proof.**

Expand the entries of the Pfaffian and use a Pfaffian analogue of Cauchy–Binet:

\[
\text{Pf}_{1 \leq i < j \leq 2n} [\cdots] = \text{Pf}_{1 \leq i < j \leq 2n} \left[ \sum_{0 \leq k < l} \delta_{l,k+1} (1 - ut^k)(x_i^l x_j^k - x_i^k x_j^l) \right]
\]

\[
= \sum_{k_1 > \cdots > k_{2n} \geq 0} \text{Pf}_{1 \leq i < j \leq 2n} \left[ \delta_{k_i,k_j+1} (1 - ut^{k_i}) \right] \det_{1 \leq i, j \leq 2n} [x_i^{k_j}]
\]

The Pfaffian in the sum factorizes, to produce the correct (blue) factor and the restriction on the summation. □
Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- Define

\[ L_{2n}(t; u) = \sum_{\lambda' \text{ even}} m_0(\lambda) \prod_{k \text{ even}} (1 - ut^{k-2}) \prod_{i=1}^{\infty} \prod_{j \text{ even}} (1 - t^{j-1})P_\lambda(x_1, \ldots, x_{2n}; t) \]

**Theorem (DB,MW,PZJ)**

\[
L_{2n}(t; u) = \prod_{1 \leq i < j \leq 2n} \frac{(1 - tx_ix_j)}{(x_i - x_j)} \frac{\text{Pf}}{1 \leq i < j \leq 2n} \left[ \frac{(1 - u + (u - t)x_ix_j)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]
\]

- The specialization \( u = t \) is again especially nice:

\[
\sum_{\lambda' \text{ even}} \prod_{i=0}^{\infty} \prod_{j \text{ even}} (1 - t^{j-1})P_\lambda(x_1, \ldots, x_{2n}; t)
\]

\[
= \prod_{1 \leq i < j \leq 2n} \frac{(1 - tx_ix_j)}{(x_i - x_j)} \frac{\text{Pf}}{1 \leq i < j \leq 2n} \left[ \frac{(1 - t)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]
\]

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Refined Cauchy and Littlewood identities
Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- At the level of plane partitions, this is (again) a very simple refinement.
Example 2(b): Refined Littlewood identity for Hall–Littlewood polynomials

- At the level of plane partitions, this is (again) a very simple refinement.
Example 2(c): Refined Littlewood identity for Macdonald polynomials

- The most fundamental Littlewood identity for Macdonald polynomials is

\[
\sum_{\lambda' \text{ even}} b_{\lambda}^{el}(q,t) P_{\lambda}(x_1, \ldots, x_{2n}; q, t) = \prod_{1 \leq i < j \leq 2n} \frac{(tx_ix_j; q)}{(x_ix_j; q)}
\]

where

\[
b_{\lambda}^{el}(q,t) = \prod_{s \in \lambda \atop l(s) \text{ even}} \frac{1 - q^{a(s)} t^{l(s)+1}}{1 - q^{a(s)+1} t^{l(s)}}
\]

Conjecture (DB,MW,PZJ)

\[
\sum_{\lambda' \text{ even}} \prod_{i=1}^{n} (1 - uq^{\lambda_2}t^{2n-2i}) b_{\lambda}^{el}(q,t) P_{\lambda}(x_1, \ldots, x_{2n}; q, t) =
\]

\[
\prod_{1 \leq i < j \leq 2n} \frac{(tx_ix_j; q)}{(x_ix_j; q)} \prod_{1 \leq i < j \leq 2n} \frac{1 - x_ix_j}{(x_i - x_j)} \Pf_{1 \leq i < j \leq 2n} \left[ \frac{(1 - u + (u - t)x_ix_j)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]
\]
The six-vertex model

The vertices of the six-vertex model are

- $a_+(x, y)$
- $b_+(x, y)$
- $c_+(x, y)$
- $a_-(x, y)$
- $b_-(x, y)$
- $c_-(x, y)$
The six-vertex model

- The Boltzmann weights are given by
  \[
  a_+(x, y) = \frac{1 - tx/y}{1 - x/y}, \quad a_-(x, y) = \frac{1 - tx/y}{1 - x/y}
  \]
  \[
  b_+(x, y) = \sqrt{t}, \quad b_-(x, y) = \sqrt{t}
  \]
  \[
  c_+(x, y) = \frac{(1 - t)}{1 - x/y}, \quad c_-(x, y) = \frac{(1 - t)x/y}{1 - x/y}
  \]

- The parameter \( t \) from Hall–Littlewood is now the crossing parameter of the model.
- The Boltzmann weights obey the Yang–Baxter equations (the \( \mathcal{U}_q(\widehat{sl}_2) \) solution):

![Diagram](image-url)
Boundary vertices

- We also require corner vertices

![Diagram of boundary vertices]

which do not depend on a spectral parameter and behave like sources/sinks.

- The corner vertices satisfy a reflection equation:

![Diagram of reflection equation]
The six-vertex model on a lattice with domain wall boundary conditions was first considered by Korepin:

This partition function is of fundamental importance in periodic quantum spin chains based on $\mathcal{Y}(sl_2)$ and $\mathcal{U}_q(sl_2)$. 
Domain wall boundary conditions

- Configurations on this lattice are in one-to-one correspondence with alternating sign matrices:

\[
\begin{pmatrix}
0 & 0 & + & 0 & 0 & 0 \\
0 & + & - & 0 & + & 0 \\
+ & - & 0 & + & 0 & 0 \\
0 & + & 0 & - & 0 & + \\
0 & 0 & 0 & + & 0 & 0 \\
0 & 0 & + & 0 & 0 & 0
\end{pmatrix}
\]

- The domain wall partition function was evaluated in determinant form by Izergin:

\[
Z_{ASM}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) = \prod_{i,j=1}^{n} \frac{(1 - tx_i y_j)}{(1 - t x_i y_j) (1 - x_i y_j)} \det \left[ \frac{(1 - t)}{(1 - t x_i y_j) (1 - x_i y_j)} \right]_{1 \leq i, j \leq n}
\]

- The DWPF is equal to the right hand side of a refined Cauchy identity:

\[
Z_{ASM}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) = C_n(t; t)
\]
Half-turn symmetry

- One can consider those configurations under domain wall boundary conditions which have $180^\circ$ rotational symmetry.
- The fundamental domain is given by:
Half-turn symmetry

• Configurations on this lattice are in one-to-one correspondence with half-turn symmetric alternating sign matrices:

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & + & 0 \\
0 & 0 & 0 & + & - & + \\
0 & 0 & + & - & + & 0 \\
0 & + & - & + & 0 & 0 \\
+ & - & + & 0 & 0 & 0 \\
0 & + & 0 & 0 & 0 & 0
\end{pmatrix}
\]

• Kuperberg evaluated this partition function as a product of determinants:

\[
Z_{HT}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) = \frac{\prod_{i,j=1}^{n} (1 - tx_i y_j)^2}{\prod_{1 \leq i < j \leq n} (x_i - x_j)^2 (y_i - y_j)^2} \times \det_{1 \leq i,j \leq n} \left[ \frac{(1-t)}{(1-t x_i y_j)(1-x_i y_j)} \right] \det_{1 \leq i,j \leq n} \left[ \frac{(1+\sqrt{t})(1-\sqrt{tx_i y_j})}{(1-t x_i y_j)(1-x_i y_j)} \right]
\]

• In other words,

\[
Z_{HT}(x_1, \ldots, x_n; y_1, \ldots, y_n; t) = C_n(t; t)C_n(t; -\sqrt{t})
\]
Off-diagonally symmetric boundary conditions

Off-diagonally symmetric boundary conditions were introduced by Kuperberg. One considers domain wall configurations with reflection symmetry about a diagonal axis, and which have no $c$ vertices on that diagonal.

The fundamental domain is

\[
\begin{array}{ccccccc}
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
\bar{x}_6 & \bar{x}_5 & \bar{x}_4 & \bar{x}_3 & \bar{x}_2 & \bar{x}_1 \\
\rightarrow & x_1 & & & & & \\
\rightarrow & x_2 & & & & & \\
\rightarrow & x_3 & & & & & \\
\rightarrow & x_4 & & & & & \\
\rightarrow & x_5 & & & & & \\
\rightarrow & x_6 & & & & & \\
\end{array}
\]
Off-diagonally symmetric boundary conditions

- Configurations on this lattice are in one-to-one correspondence with off-diagonally symmetric ASMs (OSASMs):

\[
\begin{pmatrix}
0 & 0 & + & 0 & 0 & 0 \\
+ & 0 & - & + & 0 & 0 \\
0 & 0 & 0 & 0 & + & 0 \\
0 & + & 0 & 0 & - & + \\
0 & 0 & + & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & + & 0
\end{pmatrix}
\]

- Kuperberg evaluated this partition function as a Pfaffian:

\[
Z_{\text{OSASM}}(x_1, \ldots, x_{2n}; t) = \prod_{1 \leq i < j \leq 2n} \frac{(1 - tx_ix_j)}{(x_i - x_j)} \text{Pf} \left[ \frac{(1 - t)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]_{1 \leq i < j \leq 2n}
\]

- The OSASM partition function is equal to the right hand side of a refined Littlewood identity:

\[
Z_{\text{OSASM}}(x_1, \ldots, x_{2n}; t) = \mathcal{L}_{2n}(t; t)
\]
Similarly, one can consider domain wall configurations with reflection symmetry in both diagonals, and with no c vertices on those diagonals.

The fundamental domain is

![Diagram of domain wall configurations with reflection symmetry in both diagonals, and with no c vertices on those diagonals.](image)
Off-diagonally/off-anti-diagonally symmetric boundary conditions

- Configurations on this lattice are in one-to-one correspondence with off-diagonally/off-anti-diagonally symmetric ASMs (OOSASMs):

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\
0 & 0 & 0 & + & - & + & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & + & 0 \\
0 & + & 0 & 0 & 0 & 0 & + & - \\
+ & - & 0 & 0 & 0 & 0 & + & 0 \\
0 & + & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & + & - & + & 0 & 0 & 0 \\
0 & 0 & 0 & + & 0 & 0 & 0 & 0
\end{pmatrix}
\]

- The partition function can be evaluated as a product of Pfaffians:

\[
Z_{\text{OOSASM}}(x_1, \ldots, x_{2n}; t) = \prod_{1 \leq i < j \leq 2n} \frac{(1 - tx_ix_j)^2}{(x_i - x_j)^2} \times \\
Pf_{1 \leq i < j \leq 2n} \left[ \frac{(1 - t)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right] \text{Pf}_{1 \leq i < j \leq 2n} \left[ \frac{(1 + \sqrt{t})(1 - \sqrt{t}x_ix_j)(x_i - x_j)}{(1 - tx_ix_j)(1 - x_ix_j)} \right]
\]

- In other words,

\[
Z_{\text{OOSASM}}(x_1, \ldots, x_{2n}; t) = \mathcal{L}_{2n}(t; t) \mathcal{L}_{2n}(t; -\sqrt{t})
\]
Open questions

- Expansion of other symmetry classes of ASMs.
- What is the missing operator needed to prove the conjecture?
- Do these correspondences have a combinatorial meaning? The similarity of the underlying domains on both sides of these correspondences is very curious.
- Can more general objects in the six-vertex/XXZ model (form factors/correlation functions) be expanded nicely in terms of Hall–Littlewood polynomials?
- What about more general models, such as the eight-vertex and 8VSOS models?