Yuri Stroganov

1944 – 2011

What is now the status of $\Delta = -1/2$?

Alexander Razumov

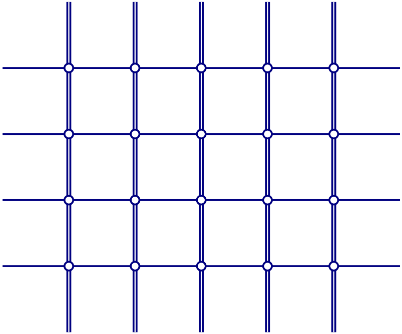
Presqu'île de Giens, June, 23, 2014

Contents

Contents

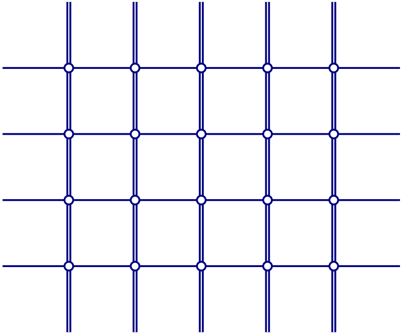
Contents

Vertex models



General definitions

- The playground is a two dimensional square $n \times m$ lattice.
- Each horizontal edge can be in one of ℓ states, and each vertical edge — in one of k states.
- The goal is to find the partition function.
- The partition function is the sum of the Boltzmann weights of all possible states of the lattice.
- Boltzmann weight of a state is the product of the Boltzmann weights of the vertices.



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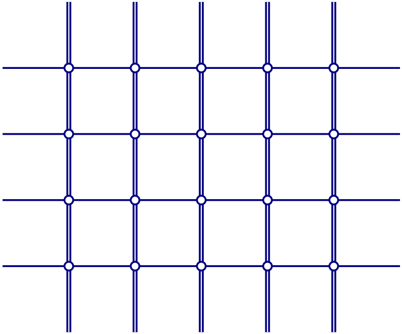

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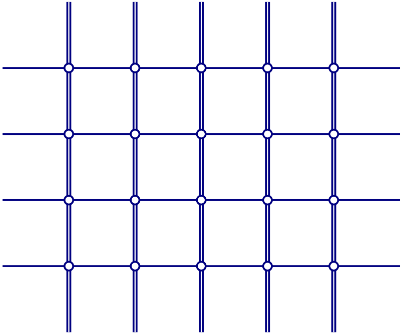

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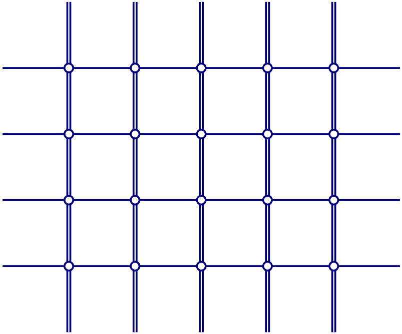

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Vertex models

Weight of a vertex

- Boltzmann weigth of a vertex is determined by the states of the adjacent edges.
- Associate with the states of horizontal edges the indices $(a, b, \ldots)$ taking ℓ values, and with the states of vertical edges the indices $(i, j, \ldots)$ taking k values.
- The summation over the states is now the summation over the indices.

Monodromy matrix

- From the weights of the vertices an $(\ell\, k \times \ell\, k)$ matrix in accordance with the picture

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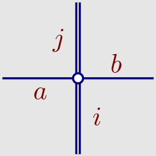
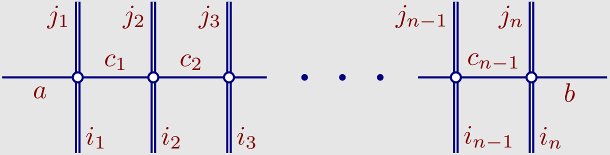


$$= \qquad M_{ai|bj}$$

Vertex models

Monodromy matrix for n vertices

- Sum over the states of internal horizontal edges.



- This gives the matrix with the matrix entries

$$M_{ai_1i_2\ldots i_n|bj_1j_2\ldots j_n} = \sum_{c_1,c_2,\ldots,c_{n-1}} M_{ai_1|c_1j_1} M_{c_1i_2|c_2j_2} \ldots M_{i_nc_{n-1}|j_nb}.$$

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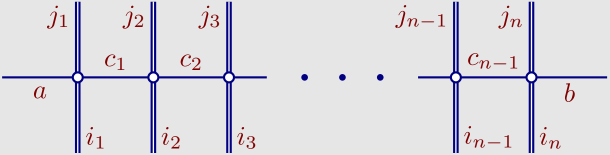

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Vertex models

Transfer matrix

- Apply the periodic boundary condition in the horizontal direction.
- Sum over the states of the boundary horizontal edges
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$$T_{i_1 i_2 \dots i_n | j_1 j_2 \dots j_n} = \sum_{a, c_1, c_2, \dots, c_{n-1}} M_{a i_1 | c_1 j_1} M_{c_1 i_2 | c_2 j_2} \dots M_{i_n c_{n-1} | j_n a}.$$

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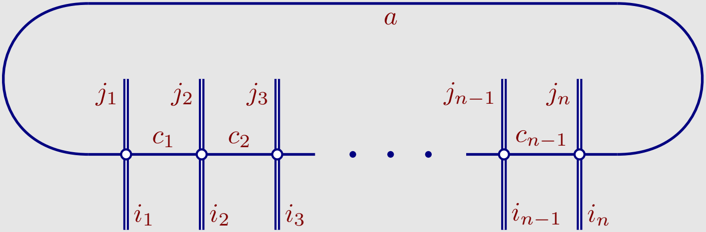

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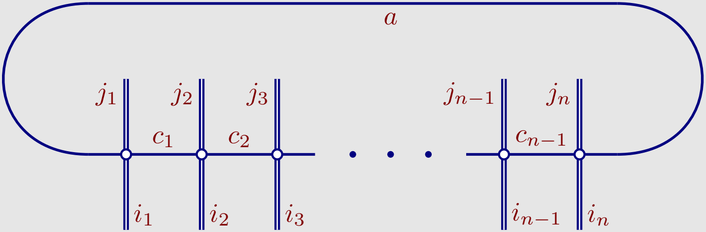

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Partition function

- Applying the periodic boundary condition in the vertical direction we see that

$$Z = \operatorname{tr} T^m = \lambda_0^m + \lambda_1^m + \dots,$$

where $\lambda_0 > \lambda_1 > \dots$ are the eigenvalues of the transfer matrix T.

Thermodynamic limit

- The term thermodynamic limit in the case under consideration means that $n, m \to \infty$. For the free energy for a vertex we have

$$\mathcal{F} = \frac{1}{mn} \ln Z = \frac{1}{n} \ln \lambda_0 + \frac{1}{mn} \ln \left(1 + \left(\frac{\lambda_1}{\lambda_0}\right)^m + \dots\right) \approx \frac{1}{n} \ln \lambda_0.$$

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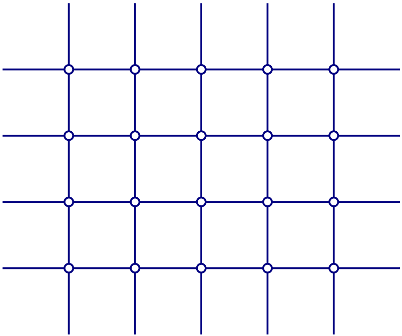
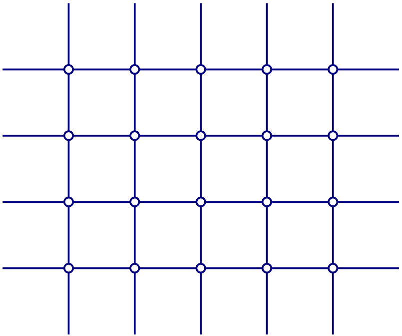
Six vertex model



Ice rule

- For each vertex there are exactly two arrows pointing in and exactly two arrows pointing out.

Allowed configurations



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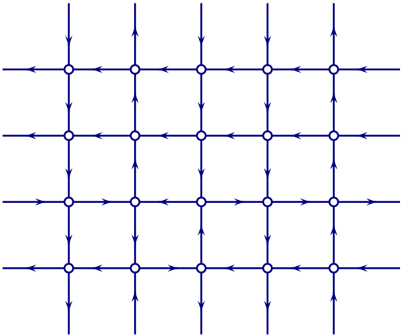

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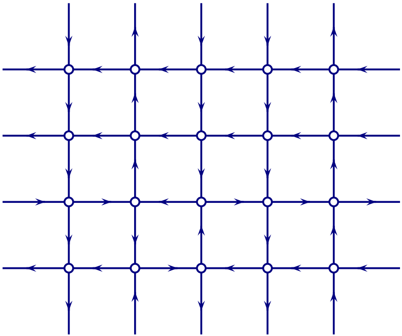

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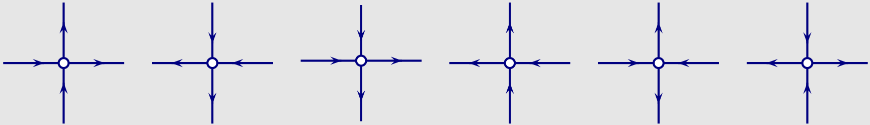

Six vertex model

Boltzmann weights

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- The Boltzmann weight of a configuration is the product of the Boltzmann weights of the vertices.

Boltzmann weights of the vertices for the six vertex model

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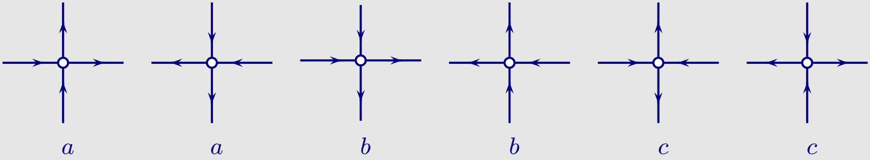

Commuting transfer matrices

- One can show that
$$[T(a,b,c),T(a',b',c')]=0$$
if
$$\frac{a^2+b^2-c^2}{2ab}=\frac{a'^2+b'^2-c'^2}{2a'b'}$$

Sectral parameter

- Standard parametrization
$$a=\rho(q\zeta-q^{-1}\zeta^{-1}),\qquad b=\rho(\zeta-\zeta^{-1}),\qquad c=\rho(q-q^{-1}).$$
- For this parametrization
$$\Delta=(q+q^{-1})/2.$$

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Baxter's Q-operator

- Transfer matrices for different values of the spectral parameter commute:

$$[T(\zeta), T(\zeta')] = 0.$$

- Baxter proved the existence of the (matrix) operator $Q(\zeta)$ having the properties

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Baxter TQ-equation

- The operator equation:

$$T(\zeta)Q(\zeta) = a^n(\zeta)Q(q^{-1}\zeta) + b^n(\zeta)Q(q\zeta).$$

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The special value of the parameter q

- For $q = \exp(\pm 2\pi \mathrm{i}/3)$ ($\Delta = -1/2$) there are some reasonings for the existence of a solution to the TQ-equation with
$$\lambda(\zeta) = -(q^{-1}\zeta - q\zeta^{-1})^n.$$
- In this case for the function
$$\varphi(\zeta) = (q^{-1}\zeta - q\zeta^{-1})^n \theta(\zeta)$$
TQ-equation takes the form
$$\varphi(\zeta) + \varphi(q\zeta) + \varphi(q^2\zeta) = 0.$$

An explicit solution to this equation was found by Yuri Stroganov

- Yu. G. Stroganov, *The importance of being odd*, J. Phys. A: Math. Gen. **34** (2001) L179-L185.

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Connection to the six vertex model

- The Hamiltonian of the XXZ spin chain

$$H_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{k=1}^{n}[\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y + \Delta\sigma_k^z\sigma_{k+1}^z]$$

  is connected to the transfer matrix of the six vertex model as follows:

$$T(\zeta)\frac{\mathrm{d}T(\zeta)}{\mathrm{d}\zeta}\bigg|_{\zeta=1} = -\frac{2}{q-q^{-1}}\left(H_{\mathrm{XXZ}} - \frac{n\Delta}{2}\right).$$

- It follows from this equality that

$$[H_{\mathrm{XXZ}}, T(\zeta)] = 0$$

- At $\Delta = -1/2$, if an eigenvector of the transfer matrix with the eigenvalue $\lambda(\zeta) = -(q^{-1}\zeta - q\zeta^{-1})^n$ exists it is an eigenvector of the Hamiltonian H_{XXZ} with the eigenvalue

$$E = -3n/4.$$

Mathematica enters the game

- It was demonstrated that an eigenvector of the Hamiltonian H_{XXZ} with the eigenvalue $E = -3n/4$ exists for $\Delta = -1/2$ for odd $n = 1, 3, \ldots, 17$. For even $n = 2, 4, \ldots, 16$ there is no such vector.

These results are given in the paper

- A. V. Razumov and Yu. G. Stroganov, *Spin chains and combinatorics*, J. Phys. A: Math. Gen. **34** (2001) 31853190.
- In the same paper a few conjectures on the properties of the eigenvector are formulated.

XXZ spin chain at $\Delta = -1/2$

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XXZ spin chain at $\Delta = -1/2$

Conjecture 1

- The ground state of the Hamiltonian $H_{\mathrm{XXZ}}|_{\Delta=-1/2}$ for odd n has the energy $-3n/4$.

Proof

- X. Yang and P. Fendley, *Non-local space-time supersymmetry on the lattice*, J. Phys. A: Math. Gen. **37** (2004) 8937-48;
- G. Veneziano and J. Wosiek, *A supersymmetric matrix model: III. Hidden SUSY in statistical systems*, JHEP **11** (2006) 030.

XXZ spin chain at $\Delta = -1/2$

Conjecture 2

- If one divides the components of the ground state vector by the component with minimal absolute value all other components become positive integers. Here the maximal component for $n = 2m + 1$ coincides with the number A_m of the alternating sign matrices of order m.

Proof

- A. V. Razumov, Yu. G. Stroganov and P. Zinn–Justin, *Polynomial solutions of qKZ equation and ground state of XXZ spin chain at $\Delta = -1/2$*, J. Phys. A: Math. Theor. **40** (2007) 11827-11847.

XXZ spin chain at $\Delta = -1/2$

Definition

An $m \times m$ matrix satisfying the conditions

- all matrix entries are either 0 or $+1$ or -1
- $+1$ and -1 alternates in every row and every column
- the first and the last nonzero entry in every row and every column is $+1$

is called an alternating sign matrix of order m.

All alternating-sign matrices of order 3

$$\begin{pmatrix} +1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & +1 \end{pmatrix} \quad \begin{pmatrix} +1 & 0 & 0 \\ 0 & 0 & +1 \\ 0 & +1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & +1 & 0 \\ 0 & 0 & +1 \\ +1 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & +1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & +1 \end{pmatrix}$$

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XXZ spin chain at $\Delta = -1/2$

Conjecture 3

- With the above normalization, the sum of squared components of the ground state vector is $\mathcal{N}_m^2$ and the the sum of the components is $3^{m/2}\mathcal{N}_m$, where

$$\mathcal{N}_m = \frac{3^{m/2}}{2^m} \frac{2 \cdot 5 \cdots (3m-1)}{1 \cdot 3 \cdots (2m-1)} A_m.$$

Proof

- P. Di Francesco, P. Zinn–Justin and J.–B. Zuber, *Sum rules for the ground states of the O(1) loop model on a cylinder and the XXZ spin chain*, J. Stat. Phys.: Theor. Exp. (2006) P08011.

XXZ spin chain at $\Delta = -1/2$

- To study the correlation functions it is convenient to use the operators

$$\alpha_k = (1 + \sigma_k^z)/2.$$

Conjecture 4

- The emptiness formation probabilities satisfy the equality

$$\frac{\langle \alpha_1 \cdots \alpha_{p-1} \rangle}{\langle \alpha_1 \cdots \alpha_{p-1} \alpha_p \rangle} = \frac{(2p-2)!(2p-1)!(2m+p)!(m-p)!}{(p-1)!(3p-2)!(2m-p+1)!(m+p-1)!}.$$

Proof

- N. Kitanine, J. M. Maillet, N. A. Slavnov and V. Terras, *Emptiness formation probability of the XXZ spin-1 / 2 Heisenberg chain at* $\Delta = 1/2$, J. Phys. A: Math. Gen. **35** (2002) L385-L388.
- L. Cantini, *Finite size emptiness formation probability of the XXZ spin chain at* $\Delta = -1/2$, J. Phys. A: Math. Theor. **45** (2012) 135207.

General XYZ spin chain

Hamiltonian of XYZ spin chain

$$H_{\mathrm{XYZ}} = -\frac{1}{2}\sum_{j=1}^{n}[J_x\sigma_k^x\sigma_{k+1}^x + J_y\sigma_k^y\sigma_{k+1}^y + J_z\sigma_k^z\sigma_{k+1}^z]$$

- The periodic boundary conditions

$$\sigma_{n+1}^x = \sigma_1^x, \qquad \sigma_{n+1}^y = \sigma_1^y, \qquad \sigma_{n+1}^z = \sigma_1^z.$$

Baxter's observation

- Baxter observed that the case

$$J_xJ_y + J_xJ_z + J_yJ_z = 0$$

  is very special.
- For XXZ spin chain

$$J_x = 1, \qquad J_y = 1, \qquad J_z = \Delta,$$

  and the above equality takes the form

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$$\Delta = -1/2.$$

References for the case of XYZ spin chain I

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