Six-vertex model partition functions and symmetric polynomials of type BC

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Outline

- Symplectic Schur polynomials
- Symplectic Cauchy identity and plane partitions
- Refined symplectic Cauchy identity
- BC Hall–Littlewood polynomials and another refined conjectural Cauchy identity
- Six-vertex model with reflecting boundary (UASMs, UUASMs)
- Putting it all together
- Conclusion

Symplectic Schur polynomials (aka symplectic characters)

The symplectic Schur polynomials $sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)$ are the irreducible characters of $Sp(2n)$. Weyl gives ($\bar{x} = \frac{1}{x}$)

$$sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \frac{\det\left(x_i^{\lambda_j - j + n + 1} - \bar{x}_i^{\lambda_j - j + n + 1}\right)_{1 \leqslant i,j \leqslant n}}{\prod_{i=1}^{n}(x_i - \bar{x}_i)\prod_{1 \leqslant i < j \leqslant n}(x_i - x_j)(1 - \bar{x}_i \bar{x}_j)}$$

A *symplectic tableau* of shape λ on the alphabet $1 < \bar{1} < \cdots < n < \bar{n}$ is a SSYT with the extra condition that all entries in row k of λ are at least k.

1	1	$\bar{1}$	$\bar{2}$	3
2	$\bar{2}$	3		
$\bar{3}$	$\bar{3}$	4		
4				

$$sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) = \sum_{T} \prod_{k=1}^{n} x_k^{\#(k) - \#(\bar{k})}$$

Symplectic tableaux as interlacing sequences of partitions

$$\overline{T} = \{\emptyset \equiv \bar{\lambda}^{(0)} \prec \lambda^{(1)} \prec \bar{\lambda}^{(1)} \prec \cdots \prec \lambda^{(n)} \prec \bar{\lambda}^{(n)} \equiv \lambda \mid \ell(\bar{\lambda}^{(i)}) \leqslant i\}$$

1	1	$\bar{1}$	$\bar{2}$	3
2	$\bar{2}$	3		
$\bar{3}$	$\bar{3}$	4		
4				

Example:

$$\overline{T} = \{\emptyset \prec (2) \prec (3) \prec (3,1) \prec (4,2) \prec (5,3) \prec (5,3,2) \prec (5,3,3,1) \prec (5,3,3,1)\}$$

$$\begin{aligned} sp_\lambda(x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n) &= \sum_{\overline{T}} \prod_{i=1}^{n} x_i^{|\lambda^{(i)}| - |\bar{\lambda}^{(i-1)}|} \prod_{j=1}^{n} x_j^{|\lambda^{(j)}| - |\bar{\lambda}^{(j)}|} \\ &= \sum_{\overline{T}} \prod_{i=1}^{n} x_i^{2|\lambda^{(i)}| - |\bar{\lambda}^{(i)}| - |\bar{\lambda}^{(i-1)}|} \end{aligned}$$

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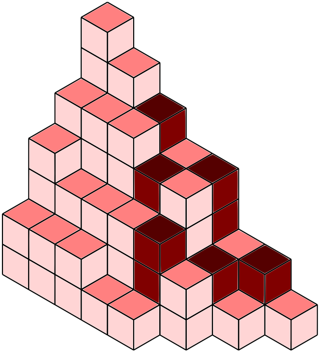
Plane partitions from SSYT + symplectic tableaux

The set of such plane partitions is (Schur left, symplectic Schur right):

$$\overline{\boldsymbol{\pi}}_{m,2n} = \{\emptyset \equiv \lambda^{(0)} \prec \lambda^{(1)} \prec \cdots \prec \lambda^{(m)} \equiv \bar{\mu}^{(n)} \succ \mu^{(n)} \succ \cdots \succ \bar{\mu}^{(1)} \succ \mu^{(1)} \succ \bar{\mu}^{(0)} \equiv \emptyset\}$$



$$(2) \prec (4,2) \prec (5,3,2) \prec (7,5,3,1) \succ (6,5,3,1) \succ (5,4,3) \succ (4,4,2) \succ (4,2) \succ (2,1) \succ (2) \succ (1)$$

Symplectic Cauchy identity and associated plane partitions

The Cauchy identity for symplectic Schur polynomials,

$$\sum_{\lambda} s_\lambda(x_1,\ldots,x_m) sp_\lambda(y_1,\bar{y}_1,\ldots,y_n,\bar{y}_n) = \frac{\prod_{1\leqslant i<j\leqslant m}(1-x_ix_j)}{\prod_{i=1}^m \prod_{j=1}^n (1-x_iy_j)(1-x_i\bar{y}_j)}$$

can now be regarded as a generating series for the plane partitions defined:

$$\sum_{\pi\in\overline{\boldsymbol{\pi}}_{m,2n}} \prod_{i=1}^m x_i^{|\lambda^{(i)}|-|\lambda^{(i-1)}|} \prod_{j=1}^n y_j^{2|\mu^{(j)}|-|\bar{\mu}^{(j)}|-|\bar{\mu}^{(j-1)}|} =$$

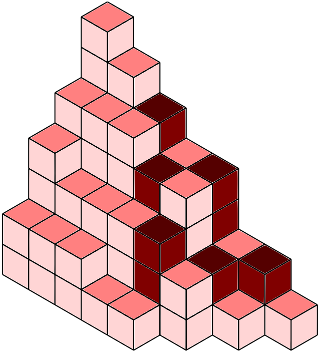
$$\frac{\prod_{1\leqslant i<j\leqslant m}(1-x_ix_j)}{\prod_{i=1}^m \prod_{j=1}^n (1-x_iy_j)(1-x_i\bar{y}_j)}$$

What is a "good" q-specialization? We choose $x_i = q^{m-i+3/2}, y_j = q^{1/2}$, giving

$$\sum_{\pi\in\overline{\boldsymbol{\pi}}_{m,2n}} q^{|\pi_{\leqslant}|} q^{|\pi^o_{>}|-|\pi^e_{>}|} = \frac{\prod_{1\leqslant i<j\leqslant m}(1-q^{i+j+1})}{\prod_{i=1}^m (1-q^i)^n(1-q^{i+1})^n}$$

Measure on symplectic plane partitions

Left is q^{Volume} (rose), right alternates between q^{Volume} (odd slices, in rose) and $q^{-\text{Volume}}$ (even positive slices, in coagulated blood).



$$\sum_{\pi \in \overline{\boldsymbol{\pi}}_{m,2n}} wt(\pi) = \frac{\prod_{1 \leqslant i < j \leqslant m} (1 - q^{i+j+1})}{\prod_{i=1}^{m} (1 - q^i)^n (1 - q^{i+1})^n}$$

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Refined Cauchy identity for symplectic Schur polynomials

Theorem (DB,MW)

$$\sum_{\lambda}\prod_{i=1}^{n}(1-t^{\lambda_i-i+n+1})s_{\lambda}(x_1,\ldots,x_n)sp_{\lambda}(y_1,\bar{y}_1,\ldots,y_n,\bar{y}_n)=$$

$$\frac{\prod_{i=1}^{n}(1-tx_i^2)}{\Delta(x)_n\Delta(y)_n\prod_{i<j}(1-\bar{y}_i\bar{y}_j)}\det\left\{\frac{(1-t)}{(1-tx_iy_j)(1-tx_i\bar{y}_j)(1-x_iy_j)(1-x_i\bar{y}_j)}\right\}_{1\leqslant i,j\leqslant n}$$

Proof.

Cauchy-Binet

$$\prod_{i=1}^{n}(1-tx_i^2)(y_i-\bar{y}_i)\det\{\cdots\}_{1\leqslant i,j\leqslant n}=\det\left\{\sum_{k=0}^{\infty}(1-t^{k+1})x_i^k(y_j^{k+1}-\bar{y}_j^{k+1})\right\}_{1\leqslant i,j\leqslant n}$$

$$=\sum_{k_1>\cdots>k_n\geqslant 0}\prod_{i=1}^{n}(1-t^{k_i+1})\ \det\left\{x_i^{k_j}\right\}_{1\leqslant i,j\leqslant n}\det\left\{y_j^{k_i+1}-\bar{y}_j^{k_i+1}\right\}_{1\leqslant i,j\leqslant n}$$

The proof follows after the change of indices $k_i=\lambda_i-i+n$.

□

Is there a Hall–Littlewood analogue?

Macdonald extended his theory of symmetric functions to other root systems. We will use Hall–Littlewood polynomials of type BC. They have a combinatorial definition (Venkateswaran):

$$K_\lambda(y_1, \bar{y}_1, \dots, y_n, \bar{y}_n; t) = \frac{1}{v_\lambda(t)} \sum_{\omega \in W(BC_n)} \omega \left(\prod_{i=1}^{n} \frac{y_i^{\lambda_i}}{(1-\bar{y}_i^2)} \prod_{1 \leqslant i < j \leqslant n} \frac{(y_i - t y_j)(1 - t\bar{y}_i \bar{y}_j)}{(y_i - y_j)(1 - \bar{y}_i \bar{y}_j)} \right)$$

- Koornwinder (or BC Macdonald) with $q = 0$.
- $t = 0 \Rightarrow$ symplectic Schur polynomials.
- No known interpretation as a sum over tableaux!

Main conjecture in type BC

Conjecture (DB,MW)

$$\sum_{\lambda}\prod_{i=0}^{\infty}\prod_{j=1}^{m_i(\lambda)}(1-t^j)P_\lambda(x_1,\dots,x_n;t)K_\lambda(y_1,\bar{y}_1,\dots,y_n,\bar{y}_n;t)=$$

$$\frac{\prod_{i,j=1}^{n}(1-tx_iy_j)(1-tx_i\bar{y}_j)}{\prod_{1\leqslant i<j\leqslant n}(x_i-x_j)(y_i-y_j)(1-tx_ix_j)(1-\bar{y}_i\bar{y}_j)}$$

$$\times\det\left\{\frac{(1-t)}{(1-tx_iy_j)(1-tx_i\bar{y}_j)(1-x_iy_j)(1-x_i\bar{y}_j)}\right\}_{1\leqslant i,j\leqslant n}$$

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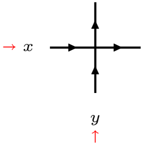
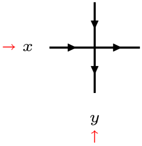
The six-vertex model

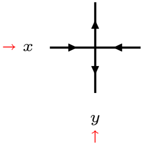
The vertices of the six-vertex model are



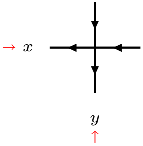
$a_+(x,y)$



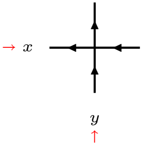
$b_+(x,y)$



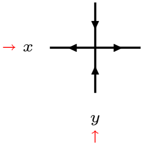
$c_+(x,y)$



$a_-(x,y)$



$b_-(x,y)$



$c_-(x,y)$

The six-vertex model

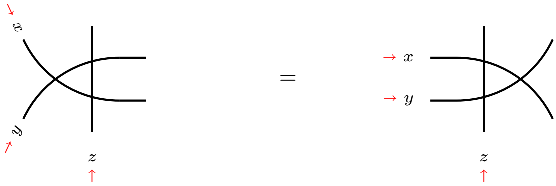
The Boltzmann weights are given by

$$a_+(x,y) = \frac{1 - tx/y}{1 - x/y} \qquad a_-(x,y) = \frac{1 - tx/y}{1 - x/y}$$

$$b_+(x,y) = 1 \qquad b_-(x,y) = t$$

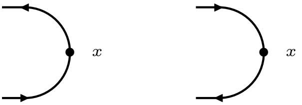
$$c_+(x,y) = \frac{(1-t)}{1 - x/y} \qquad c_-(x,y) = \frac{(1-t)x/y}{1 - x/y}$$

The parameter t from Hall–Littlewood is now the crossing parameter of the model. The Boltzmann weights obey the *Yang–Baxter* equations:



Boundary vertices

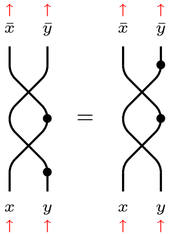
In addition to the bulk vertices, we need U-turn vertices



$1/(1-x^2)$ $\quad\quad$ $1/(1-x^2)$

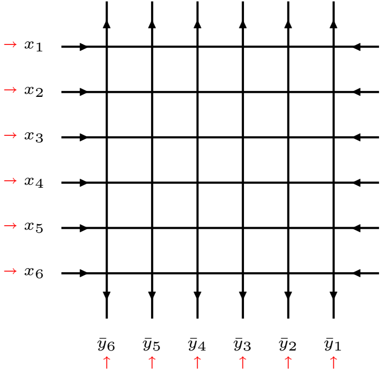
which depend on a single spectral parameter and are spin-conserving.

Boundary vertices satisfy the Sklyanin reflection equation



Domain wall boundary conditions

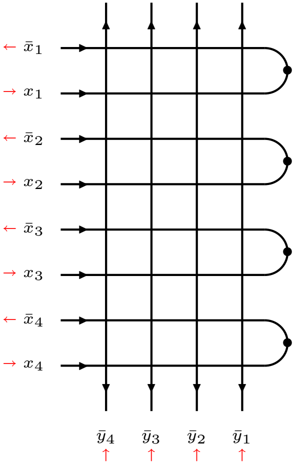
The six-vertex model on a lattice with *domain wall* boundary conditions:



This partition function (the IK determinant) is of fundamental importance in periodic quantum spin chains and combinatorics.

Reflecting domain wall boundary conditions

Interested in the following:



This quantity is important in quantum spin-chains with open boundary conditions.

Reflecting domain wall boundary conditions

Configurations on this lattice are in one-to-one correspondence with U-turn ASMs (UASMs):

$$\begin{pmatrix} 0 & + & 0 \\ + & - & 0 \\ 0 & 0 & + \\ 0 & + & - \\ 0 & 0 & 0 \\ 0 & 0 & + \end{pmatrix}$$

The partition function is also a determinant (Tsuchiya):

$$Z_{\mathrm{UASM}}(x_1, \dots, x_n; y_1, \bar{y}_1, \dots, y_n, \bar{y}_n; t) = \frac{\prod_{i,j=1}^{n} (1 - t x_i y_j)(1 - t x_i \bar{y}_j)}{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(y_i - y_j)(1 - t x_i x_j)(1 - \bar{y}_i \bar{y}_j)} \times \det \left[\frac{(1-t)}{(1 - t x_i y_j)(1 - t x_i \bar{y}_j)(1 - x_i y_j)(1 - x_i \bar{y}_j)} \right]_{1 \leqslant i,j \leqslant n}$$

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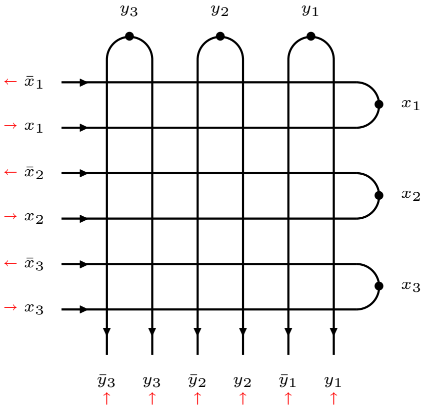
Putting it together

Conjecture (DB,MW)

$$Z_{\mathrm{UASM}}(x_1,\ldots,x_n;y_1,\bar{y}_1,\ldots,y_n,\bar{y}_n;t) = \sum_{\lambda}\prod_{i=0}^{\infty}\prod_{j=1}^{m_i(\lambda)}(1-t^j)P_\lambda(x_1,\ldots,x_n;t)K_\lambda(y_1,\bar{y}_1,\ldots,y_n,\bar{y}_n;t)$$

We can do more (doubly reflecting domain wall)



is a product $\det_1 \times \det_2$ (Kuperberg) with $\det_1$ already described (with appropriate vertex weights).

The missing determinant

A general version of $\det_2$ is:

Conjecture (DB,MW,PZJ)

$$\sum_{\lambda} \prod_{i=1}^{m_0(\lambda)} (1 - ut^{i-1}) b_\lambda(t) P_\lambda(x_1, \dots, x_n; t) \tilde{K}_\lambda(y_1^{\pm 1}, \dots, y_n^{\pm 1}; 0, t, ut^{n-1}; t_0, t_1, t_2, t_3) =$$

$$\prod_{i=1}^{n} \frac{(1 - t_0 x_i)(1 - t_1 x_i)(1 - t_2 x_i)(1 - t_3 x_i)}{(1 - t x_i^2)} \frac{\prod_{i,j=1}^{n} (1 - t x_i y_j)(1 - t x_i \bar{y}_j)}{\prod_{1 \leqslant i < j \leqslant n} (x_i - x_j)(y_i - y_j)(1 - t x_i x_j)(1 - \bar{y}_i \bar{y}_j)}$$

$$\times \det_{1 \leqslant i,j \leqslant n} \left[\frac{1 - u + (u - t)(x_i y_j + x_i \bar{y}_j) + (t^2 - u) x_i^2}{(1 - x_i y_j)(1 - t x_i y_j)(1 - x_i \bar{y}_j)(1 - t x_i \bar{y}_j)} \right] := \det{}_2(t, t_0, t_1, t_2, t_3, u)$$

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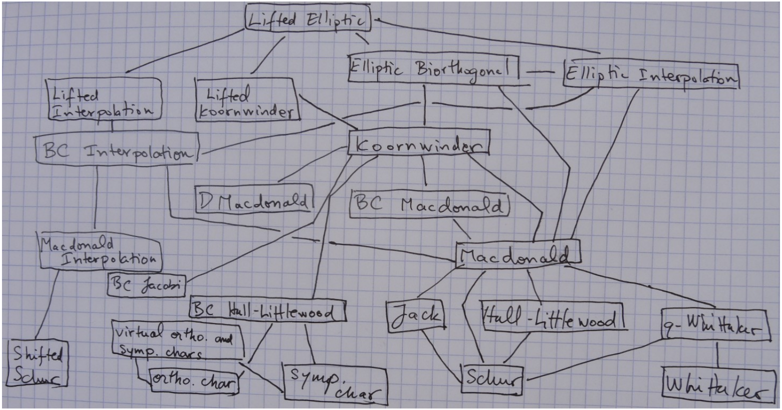
Open problems and further investigations

1. How to prove the conjecture(s)?
2. Is there a reasonable branching rule for Hall–Littlewood polynomials of type BC?
3. Are symplectic plane partitions interesting in their own right? Can one obtain correlations and asymptotics by using half-vertex operators?
4. Is there anything gained by going from Hall–Littlewood to Macdonald or Koornwinder level?
5. Do the corresponding identities at the elliptic level (which seem to exist according to Rains) connect to the 6VSOS model?
6. Are these identities just mere coincidences? Could one investigate one side by using the other?

A simple integrable and combinatorial graph



"Formulae are smarter than we are!" (Y. Stroganov)

Thank you!