These are lectures which I gave at LPTHE in January-Febrary 2009. The lectures are based on our recent works with H. Boos, M. Jimbo, T. Miwa and Y. Takeyama. All the material presented in these lectures can be found in [1, 2, 3]. However, in my opinion. these lectures are worth publishing because they provide certain logical organisation of the results obtained in these papers.

1. Lecture 1.

1.1. **Introductory remarks.** In this lectures I shall consider the XXZ spin chain. Let us begin with the model in the finite volume with periodic boundary conditions. The Hamiltonian is given by

$$H_N = \tfrac{1}{2}\sum_{k=-N+1}^{N}\left(\sigma^1_k\sigma^1_{k+1}+\sigma^2_k\sigma^2_{k+1}+\Delta\sigma^3_k\sigma^3_{k+1}\right),\quad \Delta=\tfrac{1}{2}(q+q^{-1})\,, \tag{1.1}$$

where

$$\sigma^a_{N+1}=\sigma^a_{-N+1}\,.$$

I shall consider only the disordered regime:

$$q=e^{\pi i\nu},\quad \nu\in\mathbb{R}\,.$$

The integrability is based on Yang-Baxter relations. Consider the R-matrix

$$R_{1,2}(\zeta)=q^{\frac{1}{2}(\sigma^3_1\sigma^3_2+1)}\zeta-q^{-\frac{1}{2}(\sigma^3_1\sigma^3_2+1)}\zeta^{-1}+(q-q^{-1})(\sigma^+_1\sigma^-_2+\sigma^-_1\sigma^+_2)\,.$$

This R-matrix satisfies:

- Yang-Baxter equations:
$$R_{1,2}(\zeta_1/\zeta_2)R_{1,3}(\zeta_1/\zeta_3)R_{2,3}(\zeta_2/\zeta_3)=R_{2,3}(\zeta_2/\zeta_3)R_{1,3}(\zeta_1/\zeta_3)R_{1,2}(\zeta_1/\zeta_2)\,.$$
- Unitarity
$$R_{1,2}(\zeta)R_{1,2}(\zeta^{-1})=(\zeta q-\zeta^{-1}q^{-1})(\zeta q^{-1}-\zeta^{-1}q)\,.$$
- Crossing
$$R_{1,2}(\zeta^{-1})=-\sigma^2_2R^{t_2}_{1,2}(\zeta q^{-1})\sigma^2_2\,.$$

Two additional properties are

$$R_{1,2}(1)=(q-q^{-1})P_{1,2},\quad R_{1,2}(q^{-1})=2(q-q^{-1})\mathcal{P}^-_{1,2}\,,$$

where $P_{1,2}$ is the permutation and $\mathcal{P}^-_{1,2}$ is the antisymmetriser.

We have

$$R_{1,2}(\zeta)=\zeta^{\frac{1}{2}\sigma^3_2}\left(1+\frac{\zeta^2-1}{q\zeta^2-q^{-1}}\,K_{1,2}\right)P_{1,2}\,\zeta^{-\frac{1}{2}\sigma^3_2}(q\zeta^2-q^{-1})\,,$$

where

$$K_{1,2}=\left(\sigma^+_1\sigma^-_2+\sigma^-_1\sigma^+_2+\frac{q+q^{-1}}{4}(\sigma^3_1\sigma^3_2-1)+\frac{q-q^{-1}}{4}(\sigma^3_1-\sigma^3_2)\right),$$

is the generator of Tempeley-Lieb algebra, one of its properties is

$$K^2_{1,2}=-(q+q^{-1})K_{1,2}\,.$$

Introduce the notation

$$\widetilde{R}(\zeta)=1+\frac{\zeta^2-1}{q\zeta^2-q^{-1}}\ K_{1,2}\,.$$

Then define and evaluate the transfer-matrix

$$\begin{aligned}&t_N(\zeta)=\mathrm{Tr}_a\left(R_{-N+1,a}(\zeta)\cdots R_{N,a}(\zeta)\right)\\&=(q\zeta^2-q^{-1})^{2N}\mathrm{Tr}_a\left(\widetilde{R}_{-N+1,a}(\zeta)\widetilde{R}_{-N+2,-N+1}(\zeta)\cdots\widetilde{R}_{N,N-1}(\zeta)P_{-N+1,a}\cdots P_{N,a}\right)\\&=\mathrm{Tr}_a\left(\widetilde{R}_{-N+1,a}(\zeta)\widetilde{R}_{-N+2,-N+1}(\zeta)\cdots\widetilde{R}_{N,N-1}(\zeta)P_{N,a}\right)U\,,\end{aligned}$$

where U is the shift by one site along the lattice:

$$U\sigma^a_k=\sigma^a_{k-1}U\,.$$

Campbell-Hausdorf formula implies that

$$t_N(\zeta)=e^{\sum\limits_{p=1}^{\infty}(\zeta^2-1)^pI_p}\ U\,.$$

I_p are local integrals of motion:

$$I_p=\sum_{k=-N+1}^{N}d_{p,[k,k+p]}\,,$$

where the local density $d_{p,[k,k+p]}$ acts non-trivially only on the interval $[k,k+p]$, periodical boundary conditions are implied. In particular,

$$I_1=\frac{1}{q-q^{-1}}\ H_N\,.$$

Notice also that

$$t_N(q^{-1})=(q-q^{-1})^{2N}\mathrm{Tr}_a\left(2\mathcal{P}^-_{-N+1,a}(\zeta)\cdots2\mathcal{P}^-_{N,a}(\zeta)\right)=U^{-1}\,,\tag{1.2}$$

where I used

$$A_i\mathcal{P}^-_{i,j}=\theta_j(A_j)\mathcal{P}^-_{i,j}\,,$$

with

$$\theta(A)=\sigma^2A^t\sigma^2\,.$$

Obviously, $\theta_i(2\mathcal{P}_{i,j})=P_{i,j}$.

Combining the above formulae we find

$$\frac{t(\zeta)t(q^{-1})}{(q\zeta^2-q^{-1})(q-q^{-1})}=e^{\sum\limits_{p=1}^{\infty}(\zeta^2-1)^pI_p}\,.\tag{1.3}$$

1.2. **Formulation of the problem.** Consider the limit $N \to \infty$. Denote by $|\text{vac}\rangle$ the ground state of the Hamiltonian. Consider the Vacuum Expectation Values (VEV)

$$\langle q^{2\alpha S(0)} \mathcal{O} \rangle_{XXZ} = \frac{\langle \text{vac}| q^{2\alpha S(0)} \mathcal{O} |\text{vac}\rangle}{\langle \text{vac}| q^{2\alpha S(0)} |\text{vac}\rangle}\,, \tag{1.4}$$

where $S(k) = \frac{1}{2} \sum_{j=-\infty}^{k} \sigma_j^3$, and $\mathcal{O}$ is a local operator. From [5] integral formulae were known for these VEV's for a long time. However, we were not satisfied with these formulae, and put many efforts in their simplification (see [6] and references therein). We have shown that all the integrals can be evaluated, and the VEV's are expressed in terms of one transcendental function. I shall not go into details because our results allow significant generalisation which will be discussed in these lectures.

In the paper [7] an evidence was given that formulae similar to ours exist for the Temperature Expectation Values (TEV)

$$\langle q^{2\alpha S(0)} \mathcal{O} \rangle_{XXZ,\ \beta,h} \;=\; \frac{\mathrm{Tr}_{\mathrm{S}} \left(e^{-\beta H + hS} q^{2\alpha S(0)} \mathcal{O} \right)}{\mathrm{Tr}_{\mathrm{S}} \left(e^{-\beta H + hS} q^{2\alpha S(0)} \right)}\,, \tag{1.5}$$

where Tr_{S} stands for the trace on $\mathfrak{H}_{\mathrm{S}}$. For $\beta \to \infty$ and $h = 0$, the expectation value (1.5) reduces to (1.4).

At this point we ask ourselves a question: why don't we consider instead of $\exp(-\beta H)$ the general linear combination $\exp(-\sum_{p=1}^{\infty} \beta_p I_p)$. For an integrable model such a generalisation looks very natural even if its physical meaning is not very clear.

After some consideration one comes with the following most general case. Our main concern is the limit $N \to \infty$. This limit tas to be treated carefully, as is explained above. Having all that in mind we shall formally use the space

$$\mathfrak{H}_{\mathrm{S}} = \bigotimes_{j=-\infty}^{\infty} \mathbb{C}^2\,.$$

Let us consider also the space

$$\mathfrak{H}_{\mathbf{M}} = \bigotimes_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}} \mathbb{C}^{2s_{\mathbf{j}}+1}\,,$$

where $\mathbf{M}$ stands for Matsubara. With every space $\mathbb{C}^{2s_{\mathbf{j}}+1}$ I associate the parameter $\tau_{\mathbf{j}}$. Introduce

$$T_{j,\mathbf{M}}(\zeta) = R_{j,\mathbf{n}}(\zeta/\tau_{\mathbf{n}}) \cdots R_{j,\mathbf{1}}(\zeta/\tau_{\mathbf{1}})\,,$$

and

$$T_{j,\mathbf{M}} = T_{j,\mathbf{M}}(1)\,.$$

Further,

$$T_{\mathrm{S},\mathbf{M}} = \lim_{N \to \infty} T_{-N+1,\mathbf{M}} \cdots T_{N,\mathbf{M}}\,.$$

In other words, using obvious notations we can rewrite:

$$T_{\mathrm{S},\mathbf{M}} = T_{\mathrm{S},\mathbf{n}} \cdots T_{\mathrm{S},\mathbf{1}}\,.$$

The generalisation of TEV which we shall study is

$$Z^{\kappa}\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\}=\frac{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\mathcal{O}\Big)}{\mathrm{Tr}_{\mathrm{S}}\mathrm{Tr}_{\mathbf{M}}\Big(T_{\mathrm{S},\mathbf{M}}q^{2\kappa S+2\alpha S(0)}\Big)}\,. \tag{1.6}$$

Using (1.3) it is easy to see that if all $s_{\mathbf{j}}=\frac{1}{2}$, and

$$(\tau_{2\mathbf{j}}^2-1)=\frac{\beta}{\mathbf{n}},\quad \tau_{2\mathbf{j}-\mathbf{1}}=q^{-1}$$

and $\pi i\nu\kappa=h$ then in the limit $\mathbf{n}\to\infty$ the linear functional Z^{κ} coincides with TEV. This is the original idea of the method of"Quantum transfer-matrix" which was proposed in [9, 12]. Making clever use of parameters $\tau_{\mathbf{j}}$ we can obtain from Z^{κ} all the generalisations of TEV discussed above.

It is convenient to present the numerator of $Z^{\kappa}\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\}$ graphically:

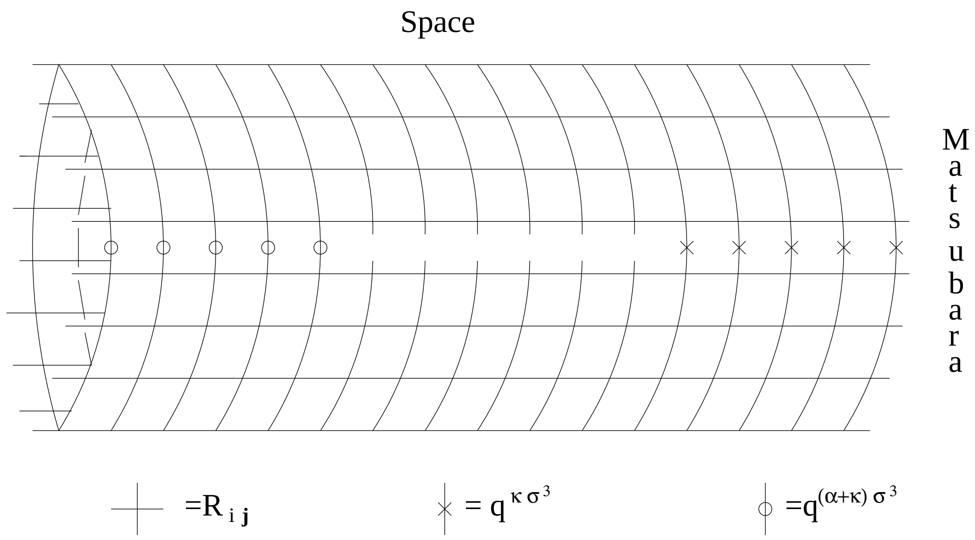

fig. 1

1.3. **Formulation of the result of our computations.** Let me announce the main result which I shall explain in these lectures.

First, we forget about Matsubara direction and describe the space of operators in space direction. The main idea is similar to that of Conformal Field Theory: we have to consider the space $\mathcal{W}_{\alpha,0}$ of operators of the form $q^{2\alpha S(0)}\mathcal{O}$ with spinless $\mathcal{O}$, and to introduce certain operators acting on this space. To be precise we have to consider bigger space:

$$\mathcal{W}^{(\alpha)}=\bigoplus_{s=-\infty}^{\infty}\mathcal{W}_{\alpha-s,s}\,.$$

On this space we defined the creation operators $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ and annihilation operators $\mathbf{b}(\zeta)$, $\mathbf{c}(\zeta)$. These are one-parameter families of operators of the form

$$\mathbf{t}^*(\zeta)=\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{t}^*_p\,,$$
$$\mathbf{b}^*(\zeta)=\zeta^{\alpha+2}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{b}^*_p\,,\ \mathbf{c}^*(\zeta)=\zeta^{-\alpha-2}\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{c}^*_p\,,$$
$$\mathbf{b}(\zeta)=\zeta^{-\alpha}\sum_{p=0}^{\infty}(\zeta^2-1)^{-p}\mathbf{b}_p\,,\ \mathbf{c}(\zeta)=\zeta^{\alpha}\sum_{p=0}^{\infty}(\zeta^2-1)^{-p}\mathbf{c}_p\,.$$

The operator $\mathbf{t}^*(\zeta)$ is in the center of our algebra of creation-annihilation operators,

$$[\mathbf{t}^*(\zeta_1),\mathbf{t}^*(\zeta_2)]=[\mathbf{t}^*(\zeta_1),\mathbf{c}^*(\zeta_2)]=[\mathbf{t}^*(\zeta_1),\mathbf{b}^*(\zeta_2)]=0,$$
$$[\mathbf{t}^*(\zeta_1),\mathbf{c}(\zeta_2)]=[\mathbf{t}^*(\zeta_1),\mathbf{b}(\zeta_2)]=0\,.$$

The rest of the operators $\mathbf{b}$, $\mathbf{c}$, $\mathbf{b}^*$, $\mathbf{c}^*$ are fermionic. The only non-vanishing anti-commutators are

$$[\mathbf{b}(\zeta_1),\mathbf{b}^*(\zeta_2)]_+=-\psi(\zeta_2/\zeta_1,\alpha)\,,\quad [\mathbf{c}(\zeta_1),\mathbf{c}^*(\zeta_2)]_+=\psi(\zeta_1/\zeta_2,\alpha)\,,$$

where

$$\psi(\zeta,\alpha)=\zeta^{\alpha}\frac{\zeta^2+1}{2(\zeta^2-1)}\,. \tag{1.7}$$

Each Fourier mode has the block structure

$$\begin{aligned}&\mathbf{t}^*_p\ :\ \mathcal{W}_{\alpha-s,s}\ \to\ \mathcal{W}_{\alpha-s,s}\\ &\mathbf{b}^*_p,\mathbf{c}_p\ :\ \mathcal{W}_{\alpha-s+1,s-1}\ \to\ \mathcal{W}_{\alpha-s,s}\,,\quad \mathbf{c}^*_p,\mathbf{b}_p\ :\ \mathcal{W}_{\alpha-s-1,s+1}\ \to\ \mathcal{W}_{\alpha-s,s}\,.\end{aligned} \tag{1.8}$$

Among them, $\boldsymbol{\tau}=\mathbf{t}^*_1/2$ plays a special role. It is the right shift by one site along the chain. Consider the set of operators

$$\boldsymbol{\tau}^m\mathbf{t}^*_{p_1}\cdots\mathbf{t}^*_{p_j}\mathbf{b}^*_{q_1}\cdots\mathbf{b}^*_{q_k}\mathbf{c}^*_{r_1}\cdots\mathbf{c}^*_{r_k}\Bigl(q^{2\alpha S(0)}\Bigr), \tag{1.9}$$

where $m\in\mathbb{Z}$, $j,k\in\mathbb{Z}_{\geq 0}$, $p_1\geq\cdots\geq p_j\geq 2$, $q_1>\cdots>q_k\geq 1$ and $r_1>\cdots>r_k\geq 1$. It has been shown [4] that (1.9) constitutes a basis of $\mathcal{W}_{\alpha,0}$. In the next two lectures I shall explain how these operators are constructed.

Second step consists in using the above description of $\mathcal{W}_{\alpha,0}$ for computation of Z^κ. We prove the following:

$$Z^\kappa\bigl\{\mathbf{t}^*(\zeta)(X)\bigr\}=2\rho(\zeta)Z^\kappa\{X\}\,, \tag{1.10}$$

$$Z^\kappa\bigl\{\mathbf{b}^*(\zeta)(X)\bigr\}=\frac{1}{2\pi i}\oint_\Gamma\omega(\zeta,\xi)Z^\kappa\bigl\{\mathbf{c}(\xi)(X)\bigr\}\frac{d\xi^2}{\xi^2}\,, \tag{1.11}$$

$$Z^\kappa\bigl\{\mathbf{c}^*(\zeta)(X)\bigr\}=-\frac{1}{2\pi i}\oint_\Gamma\omega(\xi,\zeta)Z^\kappa\bigl\{\mathbf{b}(\xi)(X)\bigr\}\frac{d\xi^2}{\xi^2}\,, \tag{1.12}$$

where Γ goes around $\xi^2 = 1$.

Form these formulae we derive

$$Z^{\kappa}\Big\{\mathbf{t}^*(\zeta_1^0)\cdots\mathbf{t}^*(\zeta_k^0)\mathbf{b}^*(\zeta_1^+)\cdots\mathbf{b}^*(\zeta_l^+)\mathbf{c}^*(\zeta_l^-)\cdots\mathbf{c}^*(\zeta_1^-)\big(q^{2\alpha S(0)}\big)\Big\} \tag{1.13}$$
$$= \prod_{p=1}^{k} 2\rho(\zeta_p^0) \times \det\big(\omega(\zeta_i^+, \zeta_j^-)\big)_{i,j=1,\cdots,l} \,.$$

Taking the Taylor coefficients in $(\zeta_i^{\epsilon})^2 - 1$ in both sides, one obtains the value of Z^{κ} on an arbitrary element of the basis (1.9).

2. Lecture 2.

2.1. **Quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_2)$.** We shall need some information about the quantum affine algebra $U_q'(\widehat{\mathfrak{sl}}_2)$. So, I have to apologise for presenting some formal mathematical facts in this lecture.

Consider the affine algebra $\widehat{\mathfrak{sl}}_2{}^0$ with central charge equal to zero. It is generated by Chevalley generators

$$e_0, \quad e_1, \quad f_0, \quad f_1, \quad h = h_1 = -h_0 \,.$$

They are subject to two kinds of relations. The trivial commutation relations:

$$[e_i, f_j] = \delta_{i,j} h_i \,,$$

and the Serre relations

$$ad^3_{e_0} e_1 = ad^3_{e_1} e_0 = 0, \quad ad^3_{f_0} f_1 = ad^3_{f_1} f_0 = 0 \,.$$

We have decomposition into direct sum of two Borel subalgebras:

$$\widehat{\mathfrak{sl}}_2{}^0 = \mathfrak{b}^+ \oplus \mathfrak{b}^-$$

which are generated respectively by e_i, h and f_i, h. The algebra $\widehat{\mathfrak{sl}}_2{}^0$ allows evaluation representation:

$$ev_\zeta(e_0) = \zeta F, \quad ev_\zeta(e_1) = \zeta E, \quad ev_\zeta(f_0) = \zeta^{-1} E, \quad ev_\zeta(f_1) = \zeta^{-1} F, \quad ev_\zeta(h) = H \,.$$

I shall need the q-deformation of the universal enveloping algebra of loop algebra. Let me remind formal definitions concerning the quantum groups. Quantum group is a Hopf algebra which means that it allows two operations: multiplication with unit 1:

$$m: \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \,,$$

and comultiplication

$$\Delta: \ \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} \,,$$

with the requirement that Δ is a homomorphism:

$$\Delta(xy) = \Delta(x)\Delta(y) \,.$$

Additional requirement consists in existence of antipode and counit. Antipode is an anti-homomorphism $s: \ \mathcal{A} \to \mathcal{A}$, it is a deformation of inverse for Lie algebra. The counit is a homomorphism $\epsilon: \ \mathcal{A} \to \mathcal{A}$, roughly speaking it projects $\mathcal{A}$ as linear space on 1 in a way compatible with the multiplication. The antipode and counit satisfy

$$m \circ (s \otimes id) \circ \Delta(x) = m \circ (id \otimes s) \circ \Delta(x) = \epsilon(x) .$$

Let σ be the permutation of two copies of $\mathcal{A}$ in the tensor product:

$$\sigma : \ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A} , \quad \sigma(x \otimes y) = y \otimes x .$$

The main requirement which distinguishes quantum groups among other Hopf algebras is the quisi-triangularity. Let $\Delta' = \sigma \circ \Delta$. Obviously, Δ' is also a comultiplication. The quasi-triangularity requires existence of universal R-matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ which intertwines two comultiplications:

$$\Delta' = \mathcal{R} \Delta \mathcal{R}^{-1} .$$

The universal R-matrix satisfy the Yang-Baxter equations:

$$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2} , \tag{2.1}$$

where usual notations for different embedding of $\mathcal{R}$ into $\mathcal{A}^{\otimes 3}$ are used. Another property is

$$(id \otimes s)\mathcal{R} = \mathcal{R}^{-1} . \tag{2.2}$$

We shall be interested in a particular example of quantum group which is $U'_Q(\widehat{\mathfrak{sl}}_2)$. Similarly to the loop algebra $U'_Q(\widehat{\mathfrak{sl}}_2)$ is generated by e_i, f_i, h_i $(i = 0, 1)$. We consider the case of central charge equal to zero: $h_1 = -h_0 \equiv h$. Two Borel subalgebras $U_q(\mathfrak{b}^+)$ and $U_q(\mathfrak{b}^-)$ are generated respectively by e_i, h and f_i, h. We have the commutation relations:

$$[e_i, f_j] = \delta_{i,j} \frac{t_i - t_i^{-1}}{q - q^{-1}} ,$$

where $t_i = q^{h_i}$. The deformed Serre relations are

$$\begin{aligned} & e_i^3 e_j + (q^2 + q^{-2} + 1)(e_i^2 e_j e_i - e_i e_j e_i^2) - e_j e_i^3 = 0 , \\ & f_i^3 f_j + (q^2 + q^{-2} + 1)(f_i^2 f_j f_i - f_i f_j f_i^2) - f_j f_i^3 = 0 \end{aligned} \tag{2.3}$$

The comultiplication and antipode are given by

$$\begin{aligned} & \Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i , \\ & s(e_i) = -t_i^{-1} e_i, \quad \Delta(f_i) = f_i t_i, \quad s(t_i) = t_i^{-1} . \end{aligned}$$

The comultiplication looks quite simple, but the universal R-matrix intertwining Δ and Δ' is complicated. It can be written as follows:

$$\begin{aligned} & \mathcal{R} = \overline{\mathcal{R}} q^{-\frac{h \otimes h}{2}} , \\ & \overline{\mathcal{R}} = 1 - (q - q^{-1}) \sum_{i=0}^{1} e_i \otimes f_j + \cdots \ \in \ U_q(\mathfrak{b}^+) \otimes U_q(\mathfrak{b}^-) , \end{aligned}$$

where the $\cdots$ stands for terms of higher degree in generators. A formula for the general term of the series is not known, only first several terms were calculated directly.

2.2. **Representations.** The evaluation representation is exactly as in undeformed case. Consider the $(2s+1)$-dimensional representations of $U_q(\mathfrak{sl}_2)$ (explicit formulae are given later). Its composition with the evaluation representation gives rise to the representation $\pi_\zeta^{(2s)}$ of $U_q'(\widehat{\mathfrak{sl}}_2)$. We have the following formula:

$$(ev_{\zeta_1}\otimes\pi^{(1)}_{\zeta_2})(\mathcal{R})=\tau(\zeta)L^\circ(\zeta)\,,\quad \zeta=\zeta_1/\zeta_2\,,$$

$$L^\circ(\zeta)=\begin{pmatrix}1-\zeta^2q^{H+1} & -(q-q^{-1})\zeta F\\ -(q-q^{-1})\zeta E & 1-\zeta^2q^{-H+1}\end{pmatrix}t_0^{\sigma^3/2}\,, \tag{2.4}$$

where in the case under consideration ($|q|=1$) the multiplier $\tau(\zeta)$ is some transcendental function depending on ζ and on the Casimir of $U_q(\mathfrak{sl}_2)$. The formula (2.4) is rather a result of consistency with Yang-Baxter, crossing and unitarity then of an honest computation. First of all, as it has been said, the formula for the universal R-matrix is not completely known. This is unpleasant, but can be fixed in principle. But knowing the full series for the universal R-matrix would not completely solve the problem. The expression through the universal R-matrix can provide only power series in ζ, while the function $\tau(\zeta)$ is such that this power series do not converge being only asymptotical ones. That is why I shall use the relation to $U_q'(\widehat{\mathfrak{sl}}_2)$ only as an intuitive idea, all the algebraic formulae which I shall use can be verified directly.

We shall use the following representations for $U_q(\mathfrak{sl}_2)$ in the L-operator (2.4):

- Finite-dimensional of dimension $2s+1$:

$$\begin{aligned}&Fv_j=v_{j+1},\quad Hv_j=(-2s+2j)v_j,\quad t_0=q^{-H}\,,\\ &Ev_j=(q^j-q^{-j})(q^{2(s-2s-1)}-q^{-2(j-2s-1)})v_{j-1},\quad j=0,\cdots,2s\,.\end{aligned}$$

- Shifted Verma module with lowest weight Λ and shift m are denoted by $V_{\eta,m}(\Lambda)$. They are defined by

$$\begin{aligned}&Fv_j=v_{j+1},\quad Hv_j=(\Lambda+2j)v_j,\\ &Ev_j=q^{-\Lambda+1}(q^{\Lambda-H-2}-1)(q^{\Lambda+H}-1)v_{j-1},\quad j=0,\cdots,\infty\,.\\ &t_0v_j=q^{-H-m}v_j\,,\end{aligned}$$

  where $v_{-1}=0$.

Recall that $\overline{\mathcal{R}}\in\ U_q(\mathfrak{b}^+)\otimes U_q(\mathfrak{b}^-)$. Suppose we have two homomorphisms

$$U_q(\mathfrak{b}^+)\ \to\ A^+,\quad U_q(\mathfrak{b}^-)\ \to\ A^-\,.$$

I shall use the term L-operator for the image of the universal R-matrix under these maps. This is an observation of Bazhanov, Lukyanov and Zamolodchikov that for constructing L-operator it is not necessary to represent the entire affine algebra as we did before, but only its Borel sublagebras.

Consider the important example. The q-oscillator algebra Osc is an associative with generators $\mathbf{a},\mathbf{a}^*,q^D$ and defining relations

$$\begin{aligned}&q^D\mathbf{a}\,q^{-D}=q^{-1}\mathbf{a},\quad q^D\,\mathbf{a}^*q^{-D}=q\,\mathbf{a}^*,\\ &\mathbf{a}\,\mathbf{a}^*=1-q^{2D+2},\quad \mathbf{a}^*\mathbf{a}=1-q^{2D}\,.\end{aligned}$$

Representations of Osc relevant to us are $\rho^{\pm}: Osc \to \mathrm{End}(W^{\pm})$ defined by

$$
\begin{aligned}
&W^{+} = \oplus_{k\geq 0}\mathbb{C}|k\rangle, \quad W^{-} = \oplus_{k<0}\mathbb{C}|k\rangle, \\
&q^{D}|k\rangle = q^{k}|k\rangle,\ \mathbf{a}|k\rangle = (1-q^{2k})|k-1\rangle,\ \mathbf{a}^{*}|k\rangle = (1-\delta_{k,-1})|k+1\rangle\,.
\end{aligned} \tag{2.5}
$$

We shall use the trace functional $\mathrm{Tr}(q^{2\alpha D}\cdot): Osc \to \mathbb{C}(q^{\alpha})$ given as follows. For each $x \in Osc$ and $y \in \mathbb{C}$, the ordinary trace $\pm\mathrm{Tr}_{W^{\pm}}(y^{D}x)$ on $W^{\pm}$ is well-defined for sufficiently small $|y|^{\pm 1}$, and gives the same rational function $g_x(y)$ in y. By definition, $\mathrm{Tr}(q^{2\alpha D}x)$ means $g_x(q^{2\alpha}) \in \mathbb{C}(q^{\alpha})$. Notice that $\mathrm{Tr}(q^{2\alpha D}\cdot)$ is a purely algebraic operation characterized as the unique linear map with the properties

$$
\begin{aligned}
&\mathrm{Tr}(q^{2\alpha D}XY) = \mathrm{Tr}(q^{2\alpha D}q^{2\alpha d(X)}YX) \quad (X, Y \in Osc, q^{D}Xq^{-D} = q^{d(X)}X), \\
&\mathrm{Tr}(q^{2\alpha D}q^{mD}) = \frac{1}{1-q^{2\alpha+m}} \quad (m \in \mathbb{Z}).
\end{aligned}
$$

There is a homomorphism of algebras $o_{\zeta}: U_q\mathfrak{b}^{+} \to Osc$ given by

$$
o_{\zeta}(e_0) = \frac{\zeta}{q-q^{-1}}\mathbf{a}, \quad o_{\zeta}(e_1) = \frac{\zeta}{q-q^{-1}}\mathbf{a}^{*}, \quad o_{\zeta}(t_0) = q^{-2D}, \quad o_{\zeta}(t_1) = q^{2D}\,.
$$

We define representations $o_{\zeta}^{\pm}: U_q\mathfrak{b}^{+} \to \mathrm{End}(W^{\pm})$ by

$$
o_{\zeta}^{+} = \rho^{+} \circ o_{\zeta}, \quad o_{\zeta}^{-} = \rho^{-} \circ o_{\zeta} \circ \iota\,,
$$

where ι denotes the involution $e_i \to e_{1-i}$, $t_i \to t_{1-i}$ of $U_q\mathfrak{b}^{+}$.

We define

$$
(o_{\zeta}^{\pm} \otimes \pi_{\xi})\mathcal{R} = \sigma(\zeta/\xi) \cdot L^{\circ}_{Aj}{}^{\pm}(\zeta/\xi)\,.
$$

Then by self-consistence one finds:

$$
L^{\circ}_{A,j}(\zeta) := \begin{pmatrix} 1-\zeta^{2}q^{2D_A+2} & -\zeta\mathbf{a}_A \\ -\zeta\mathbf{a}^{*}_A & 1 \end{pmatrix}_j \begin{pmatrix} q^{-D_A} & 0 \\ 0 & q^{D_A} \end{pmatrix}_j, . \tag{2.6}
$$

Here I started to put indices counting different algabras. They are often tautological, but many formulae are unreadable if we do not use them. The rule is: $A, B, \cdots$ for q-oscillators, $a, b, \cdots$ for auxiliary two-dimensional spaces, $j, k, \cdots$ for space direction, $\mathbf{j}, \mathbf{k}, \cdots$ for Matsubara direction.

The main property of the q-oscillator representation is that its tensor product with usual evaluation representation has reducible but not decomposible structure. Without going into much details let me write directly the manifestation of this fact:

$$
\begin{aligned}
L_{\{a,A\},j}(\zeta) &= (F_{a,A})^{-1}L_{a,j}(\zeta)L_{A,j}(\zeta)F_{a,A} \\
&= \begin{pmatrix} 1 & 0 \\ \frac{q-q^{-1}}{\zeta-\zeta^{-1}}\,\sigma_j^{+} & 1 \end{pmatrix}_a \begin{pmatrix} L_{A,j}(q\zeta)q^{-\sigma_j^{3}/2} & 0 \\ 0 & L_{A,j}(q^{-1}\zeta)q^{\sigma^{3}_j/2} \end{pmatrix}_a,
\end{aligned} \tag{2.7}
$$

where $F_{a,A} = 1 - \mathbf{a}_A\sigma_a^{+}$.

The explicit formula for $\sigma(\zeta)$ is not important, but the functional equation and relation to $\rho(\zeta)$ are given by:

$$
\rho(\zeta) = q^{-1/2}\frac{\sigma(q^{-1}\zeta)}{\sigma(\zeta)}, \quad \sigma(\zeta)\sigma(q^{-1}\zeta) = \frac{1}{1-\zeta^{2}}\,. \tag{2.8}
$$

3. Lecture 3.

3.1. Evaluation of Z^κ. Let us return to the functional Z^κ. It is defined as

$$Z^\kappa\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\}=\lim_{l\to\infty}\frac{\mathrm{Tr}_{[-l+1,l]}\mathrm{Tr}_{\mathbf{M}}\Big(T_{[-l+1,l],\mathbf{M}}q^{2\kappa S_{[-l+1,l]}+2\alpha S_{[-l+1,0]}}\mathcal{O}\Big)}{\mathrm{Tr}_{[-l+1,l]}\mathrm{Tr}_{\mathbf{M}}\Big(T_{[-l+1,l],\mathbf{M}}q^{2\kappa S_{[-l+1,l]}+2\alpha S_{[-l+1,0]}}\Big)}\,, \tag{3.1}$$

where I used obvious notations for traces over finite chain $[-l+1,l]$. Consider the transfer-matrices in Matsubara direction:

$$T_{\mathbf{M}}(\zeta,\kappa)=\mathrm{Tr}_j\left(T_{\mathbf{j},\mathbf{M}}(\zeta)q^{\kappa\sigma^3_j}\right),\quad T_{\mathbf{M}}(\zeta,\kappa+\alpha)=\mathrm{Tr}_j\left(T_{\mathbf{j},\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma^3_j}\right),$$

where

$$T_{\mathbf{j},\mathbf{M}}(\zeta)=L^\circ_{j,\mathbf{n}}(\zeta/\tau_{\mathbf{n}})\cdots L^\circ_{j,\mathbf{1}}(\zeta/\tau_{\mathbf{1}})\,.$$

These are two commutative (but not mutually commutative) families of operators. Since $\mathcal{O}$ is local in the expression for Z^κ there are infinitely many $T_{\mathbf{M}}(1,\kappa)$ to the right of $\mathcal{O}$ and $T_{\mathbf{M}}(1,\kappa+\alpha)$ to the left of $\mathcal{O}$. Suppose that $T_{\mathbf{M}}(\zeta,\kappa)$ $(T_{\mathbf{M}}(\zeta,\kappa+\alpha))$ has unique eigen- (co)vector $|\kappa\rangle$ $(\langle\kappa+\alpha|$) such that

- If $T(\zeta,\kappa)$ $(T(\zeta,\kappa+\alpha))$ is corresponding eigenvalue, $T(1,\kappa)$ $(T(1,\kappa+\alpha))$ is of maximal absolute value.
- Nondegeneracy:
$$\langle\kappa+\alpha|\kappa\rangle\neq 0\,.$$

Suppose that

$$q^{2\alpha S(0)}\mathcal{O}=q^{2\alpha S(k-1)}X_{[k,m]}\,,$$

where $X_{[k,m]}$ acts non-trivially only on the space interval $[k,m]$. Then it is quite clear that

$$Z^\kappa\Big\{q^{2\alpha S(0)}\mathcal{O}\Big\}=\rho(1)^{k-1}\frac{\langle\kappa+\alpha|\mathrm{Tr}_{[k,m]}\Big(T_{[k,m],\mathbf{M}}q^{2\kappa S_{[k,m]}}X_{[k,m]}\Big)|\kappa\rangle}{T(1,\kappa)^{m-k+1}\langle\kappa+\alpha|\kappa\rangle}\,,$$

where

$$\rho(\zeta)=\frac{T(\zeta,\kappa+\alpha)}{T(\zeta,\kappa)}\,. \tag{3.2}$$

Let us discuss the diagonalisation of $T_{\mathbf{M}}(\zeta,\kappa)$ using Q-operators and Baxter equations . One can say that the diagonilasation is solved simpler by Bethe Ansatz, but later we shall need the algebraic structure described during the previous lecture for more serious goals, so, the diagonilasation will serve a good exercise.

3.2. Spectral problem in Matsubara direction. Introduce the Q-operator (transfer-matrix with q-oscillator as auxiliary space):

$$Q_{\mathbf{M}}(\zeta,\kappa)=\zeta^{\kappa-\mathbf{S}}\mathrm{Tr}^+_A\left(T_{\mathbf{A},\mathbf{M}}(\zeta)q^{2\kappa D_A}\right),$$

where Tr^+ is trace over W^+,

$$T_{\mathbf{A},\mathbf{M}}(\zeta)=L^\circ_{A,\mathbf{n}}(\zeta/\tau_{\mathbf{n}})\cdots L^\circ_{A,\mathbf{1}}(\zeta/\tau_{\mathbf{1}})\,,$$

$\mathbf{S}$ is the operstor of total spin in Matsubara direction. Recall that we allow arbitrary spins in Matsubara direction, corresponding L-operators $L^{\circ}_{A,\mathbf{j}}(\zeta/\tau_{\mathbf{j}})$ are obtained from (2.6) by standard fusion procedure. Using the fusion relation (2.7) we immediately obtain the Baxter equation:

$$T_{\mathbf{M}}(\zeta,\kappa)Q_{\mathbf{M}}(\zeta,\kappa)=a(\zeta)Q_{\mathbf{M}}(\zeta q^{-1},\kappa)+d(\zeta)Q_{\mathbf{M}}(\zeta q,\kappa)\,, \tag{3.3}$$

where

$$a(\zeta)=\prod_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}\left(\left(\frac{q\zeta}{\tau_{\mathbf{j}}}\right)^2-1\right),\quad d(\zeta)=\prod_{\mathbf{j}=\mathbf{1}}^{\mathbf{n}}\left(\left(\frac{\zeta}{\tau_{\mathbf{j}}}\right)^2-1\right).$$

By construction $T_{\mathbf{M}}(\zeta,\kappa)$ and $\zeta^{-\kappa+\mathbf{S}}Q_{\mathbf{M}}(\zeta,\kappa)$ are polynomials of ζ^2. That is why the equations (3.3) imply Bethe equations and, hence, defines the spectrum of transfer-matrices.

Denote by $\mathbf{J}$ the operation of spin reversal in Matsubara direction. It is easy to see that

$$\mathbf{J}T_{\mathbf{M}}(\zeta,\kappa)\mathbf{J}=T_{\mathbf{M}}(\zeta,-\kappa)\,.$$

The Q-operator $Q_{\mathbf{M}}(\zeta,\kappa)$ is originally defined for $q^{\kappa}<1$, but then it is analytically continued. I shall often denote $Q_{\mathbf{M}}(\zeta,\kappa)$ by $Q^{+}_{\mathbf{M}}(\zeta,\kappa)$. We have another solution of Baxter equation

$$Q^{-}_{\mathbf{M}}(\zeta,\kappa)=\mathbf{J}Q^{+}_{\mathbf{M}}(\zeta,\kappa)\mathbf{J}\,.$$

This Q-operator is directly defined by

$$Q_{\mathbf{M}}(\zeta,\kappa)=\zeta^{-\kappa+\mathbf{S}}\mathrm{Tr}^{-}_{A}\left(T^{-}_{\mathbf{A},\mathbf{M}}(\zeta)q^{-2\kappa D_A}\right)\,,$$

and

$$L^{-}_{A,\mathbf{j}}(\zeta)=\sigma^1_{\mathbf{j}}L^{+}_{A,\mathbf{j}}(\zeta)\sigma^1_{\mathbf{j}}\,.$$

3.3. **Commutation relations. BLZ construction.** I want to prove the commutation relations:

$$[T(\zeta_1,\kappa),T(\zeta_2,\kappa)]=0\,, \tag{3.4}$$

$$[T(\zeta_1,\kappa),Q^{+}(\zeta_2,\kappa)]=[T(\zeta_1,\kappa),Q^{-}(\zeta_2,\kappa)]=0\,, \tag{3.5}$$

$$[Q^{+}(\zeta_1,\kappa),Q^{+}(\zeta_2,\kappa)]=[Q^{-}(\zeta_1,\kappa),Q^{-}(\zeta_2,\kappa)]=0\,, \tag{3.6}$$

$$[Q^{+}(\zeta_1,\kappa),Q^{-}(\zeta_2,\kappa)]=0\,. \tag{3.7}$$

The equation (3.4) we already know. The equations (3.5) follow from

$$L_{A,a}(\zeta_2/\zeta_1)L_{A,\mathbf{j}}(\zeta_2)L_{a,\mathbf{j}}(\zeta_1)=L_{a,\mathbf{j}}(\zeta_1)L_{A,\mathbf{j}}(\zeta_2)L_{A,a}(\zeta_2/\zeta_1)\,.$$

The equations (3.6) are more complicated. The point is that *a priori* we cannot expect that the R-matrix intertwining $L_{A,\mathbf{j}}(\zeta_1)$ and $L_{B,\mathbf{j}}(\zeta_2)$ because both A and B correspond to representations of the same Borel subalgebra. However, by direct computation we find that the R-matrix satisfying

$$R_{A,B}(\zeta_1/\zeta_2)L_{A,\mathbf{j}}(\zeta_1)L_{B,\mathbf{j}}(\zeta_2)=L_{B,\mathbf{j}}(\zeta_2)L_{A,\mathbf{j}}(\zeta_1)R_{A,B}(\zeta_1/\zeta_2)\,.$$

does exist. It is given by

$$R_{A,B}(\zeta)=P_{A,B}h(\zeta,u_{A,B})\zeta^{D_A+D_B}, \tag{3.8}$$

where we have set $u_{A,B}=\mathbf{a}_A^* q^{-2D_A}\mathbf{a}_B$, and $h(\zeta,u)$ is the unique formal power series in u satisfying

$$(1+\zeta u)h(\zeta,u)=(1+\zeta^{-1}u)h(\zeta,q^2u), \tag{3.9}$$

$$h(\zeta,u)=(1+\zeta^{-1}u)(1+q^{-2}\zeta u)h(q^{-2}\zeta,u) \tag{3.10}$$

and $h(\zeta,0)=1$.

Finally, let us consider the most complicated equation (3.7). Trying to find an R-matrix intertwining $L^+_{A,\mathbf{j}}(\zeta_1)$ and $L^-_{B,\mathbf{j}}(\zeta_2)$ one immediately finds a contradiction.

Let us introduce the following elements of $Osc^{\otimes 2}$ which will play a role in the sequel.

$$U_{A,B}(\zeta)=\zeta\mathbf{a}_A^*+\mathbf{a}_Bq^{2D_A}\,, \tag{3.11}$$

$$V_{A,B}(\zeta)=\zeta\mathbf{a}_B^*+\mathbf{a}_Aq^{2D_B}\,, \tag{3.12}$$

$$Y_{A,B}(\zeta)=(\zeta q^2-\mathbf{a}_A\mathbf{a}_B)q^{2D_A}\,, \tag{3.13}$$

$$Z_{A,B}(\zeta)=\zeta^{-1}q^{2D_B+2}-\mathbf{a}_A^*\mathbf{a}_B^*q^{-2D_A}\,. \tag{3.14}$$

These operators appear as matrix elements of products of L-operators,

(3.15)

$$L^{\circ\,+}_A(\zeta_1)L^{\circ\,-}_B(\zeta_2)$$
$$=\begin{pmatrix}1-\zeta_1\zeta_2Y_{A,B}(\zeta) & -\zeta_1(1-\zeta^{-1}q^{-2}Z_{A,B}(\zeta))\mathbf{a}_A-c(\zeta_1,\zeta_2)V_{A,B}(\zeta)\\ -\zeta_2U_{A,B}(\zeta) & 1-\zeta_1\zeta_2Z_{A,B}(\zeta)\end{pmatrix}q^{-(D_A-D_B)\sigma^3}\,,$$

(3.16)

$$L^{\circ\,-}_B(\zeta_2)L^{\circ\,+}_A(\zeta_1)$$
$$=\begin{pmatrix}1-\zeta_1\zeta_2Z_{B,A}(\zeta^{-1}) & -\zeta_1U_{B,A}(\zeta^{-1})\\ -\zeta_2(1-\zeta q^{-2}Z_{B,A}(\zeta^{-1}))\mathbf{a}_B-c(\zeta_2,\zeta_1)V_{B,A}(\zeta^{-1}) & 1-\zeta_1\zeta_2Y_{B,A}(\zeta^{-1})\end{pmatrix}q^{(D_B-D_A)\sigma^3}\,.$$

Here we have set $\zeta=\zeta_1/\zeta_2$, and $c(\zeta_1,\zeta_2)=\zeta_1^{-1}\zeta_2^2(1-\zeta_1^2q^2)$.

Let us list the commutation relations that are relevant to us.

- $U_{A,B}$, $Y_{A,B}$, $Z_{A,B}$, $\mathbf{a}_A$ among themselves:

$$\begin{aligned}
&Y_{A,B}(\zeta)U_{A,B}(\zeta)=q^2U_{A,B}(\zeta)Y_{A,B}(\zeta)\,,\\
&Z_{A,B}(\zeta)U_{A,B}(\zeta)=q^{-2}U_{A,B}(\zeta)Z_{A,B}(\zeta)\,,\\
&\mathbf{a}_AU_{A,B}(\zeta)-q^2U_{A,B}(\zeta)\mathbf{a}_A=\zeta(1-q^2),\\
&Y_{A,B}(\zeta)\mathbf{a}_A=q^{-2}\mathbf{a}_AY_{A,B}(\zeta)\\
&Z_{A,B}(\zeta)\mathbf{a}_A=q^2\mathbf{a}_AZ_{A,B}(\zeta)+q^2(1-q^2)\zeta^{-1}V_{A,B}(\zeta)\,,\\
&Y_{A,B}(\zeta)Z_{A,B}(\zeta)=q^2-\zeta^{-1}q^4U_{A,B}(\zeta)V_{A,B}(\zeta)\,,\\
&Z_{A,B}(\zeta)Y_{A,B}(\zeta)=q^2-\zeta^{-1}q^2U_{A,B}(\zeta)V_{A,B}(\zeta)\,.
\end{aligned} \tag{3.17}$$

- $U_{A,B}$, $Y_{A,B}$, $Z_{A,B}$, $\mathbf{a}_A$, $\mathbf{a}_B$ with $V_{A,B}$:

$$\begin{aligned}
&[V_{A,B}(\zeta),X]=0\quad\text{for } X=U_{A,B}(\zeta),Y_{A,B}(\zeta),Z_{A,B}(\zeta),\mathbf{a}_A,\\
&V_{A,B}(\zeta)\mathbf{a}_B-q^{-2}\mathbf{a}_BV_{A,B}(\zeta)=\zeta(1-q^{-2})\,.
\end{aligned} \tag{3.18}$$

• $U_{A,B}$, $Y_{A,B}$, $Z_{A,B}$, $\mathbf{a}_A$ with $\mathbf{a}_B$:

$$\begin{aligned}(3.19)\qquad &[U_{A,B}(\zeta),\mathbf{a}_B]=[Y_{A,B}(\zeta),\mathbf{a}_B]=[\mathbf{a}_A,\mathbf{a}_B]=0,\\ &Z_{A,B}(\zeta)\mathbf{a}_B=\mathbf{a}_B Z_{A,B}(\zeta)+\zeta^{-1}(1-q^2)U_{A,B}(\zeta)q^{2(D_B-D_A)}\,.\end{aligned}$$

The following result can be extracted from [8].

Lemma. *Set*

$$\zeta=\frac{\zeta_1}{\zeta_2},\quad q^{\Lambda}=q\zeta\,. \tag{3.20}$$

The tensor product $W^+_{\zeta_1}\otimes W^-_{\zeta_2}$ has an increasing filtration by $U_q\mathfrak{b}^+$-submodules

$$\begin{aligned}(3.21)\qquad &\{0\}=W_L^{(-1)}\subset W_L^{(0)}\subset W_L^{(1)}\subset\cdots\subset W_L^{(m)}\subset\cdots\subset W^+_{\zeta_1}\otimes W^-_{\zeta_2}\,,\\ &\bigcup_{m=-1}^{\infty}W_L^{(m)}=W^+_{\zeta_1}\otimes W^-_{\zeta_2},\end{aligned}$$

such that each subquotient is isomorphic to a shifted Verma module

$$\iota_L: W_L^{(m)}/W_L^{(m-1)}\xrightarrow{\sim}V_{\sqrt{\zeta_1\zeta_2},2m}(\Lambda)\,. \tag{3.22}$$

The tensor product $W^-_{\zeta_2}\otimes W^+_{\zeta_1}$ in the opposite order has a decreasing filtration by $U_q\mathfrak{b}^+$-submodules

$$\begin{aligned}(3.23)\qquad &W^-_{\zeta_2}\otimes W^+_{\zeta_1}=W_R^{(-1)}\supset W_R^{(0)}\supset\cdots\supset W_R^{(m)}\supset\cdots\,,\\ &\bigcap_{l=-1}^{\infty}W_R^{(m)}=0,\end{aligned}$$

such that each subquotient is isomorphic to a shifted Verma module

$$\iota_R: W_R^{(m-1)}/W_R^{(m)}\xrightarrow{\sim}V_{\sqrt{\zeta_1\zeta_2},2m}(\Lambda)\,. \tag{3.24}$$

Proof. The vector space $W^+\otimes W^-$ has the following basis

$$e_{j,p}=U_{A,B}(\zeta)^j\mathbf{a}_B^p\ |0\rangle\otimes|-1\rangle\quad(j,p\in\mathbb{Z}_{\geq0})\,.$$

Let $W_L^{(m)}$ denote the linear span of $e_{j,p}$ with $j\geq 0$ and $p\leq m$. Introduce the operator H by $q^He_{j,m}=\zeta q^{2j+1}e_{j,m}$. A direct calculation using (3.17)–(3.19) shows that (with $\star$ denoting an irrelevant constant)

$$\begin{aligned}&U_{A,B}(\zeta)e_{j,m}=e_{j+1,m}\,,\\ &Y_{A,B}(\zeta)e_{j,m}=q^{H+1}e_{j,m}\,,\\ &Z_{A,B}(\zeta)e_{j,m}=q^{-H+1}e_{j,m}+\star\ e_{j+1,m-1}\,,\\ &(1-\zeta^{-1}q^{-2}Z_{A,B}(\zeta))\mathbf{a}_Ae_{j,m}=\zeta^{-1}(\zeta q^{-H-1}-1)(\zeta q^{H+1}-1)e_{j-1,m}+\star\ e_{j,m-1}\,,\\ &V_{A,B}(\zeta)e_{j,m}=\star\ e_{j,m-1}\\ &q^{2(D_A-D_B)}e_{j,m}=\zeta^{-1}q^{H+2m+1}e_{j,m}\,.\end{aligned}$$

In view of the relations

$$(q-q^{-1})\Delta(e_0)=\zeta_1(1-q^{-2}\zeta^{-1}Z_{A,B}(\zeta))\mathbf{a}_A+\zeta_1^{-1}\zeta_2^2V_{A,B}(\zeta),$$
$$(q-q^{-1})\Delta(e_1)=\zeta_2U_{A,B}(\zeta),$$
$$\Delta(t_0)^{-1}=\Delta(t_1)=q^{2(D_A-D_B)},$$

we see that $W_L^{(m)}$ are $U_q\mathfrak{b}^+$-submodules, and the factors coincide with the Verma modules.

Similarly, for $W^-_{\zeta_2}\otimes W^+_{\zeta_1}$ we introduce a basis

$$f_{j,p}=\left((1-\zeta q^{-2}Z_{B,A}(\zeta^{-1}))\mathbf{a}_B\right)^jV_{B,A}(\zeta^{-1})^p|-1\rangle\otimes|0\rangle\,.$$

Let $W_R^{(m)}$ be the linear span of $f_{j,p}$ with $j\geq 0$ and $p>m$. Setting $q^Hf_{j,m}=\zeta q^{2j+1}f_{j,m}$, we have

$$\begin{aligned}
&(1-\zeta q^{-2}Z_{B,A}(\zeta^{-1}))\,\mathbf{a}_Bf_{j,m}=f_{j+1,m}\,,\\
&U_{B,A}(\zeta^{-1})f_{j,m}=\zeta^{-1}(\zeta q^{-H-1}-1)(\zeta q^{H+1}-1)f_{j-1,m}\,,\\
&Z_{B,A}(\zeta^{-1})f_{j,m}=q^{H+1}f_{j,m}+\star f_{j-1,m+1}\,,\\
&Y_{B,A}(\zeta^{-1})f_{j,m}=q^{-H+1}f_{j,m}\,,\\
&V_{B,A}(\zeta^{-1})f_{j,m}=f_{j,m+1}\,,\\
&q^{2(D_A-D_B)}f_{j,m}=\zeta^{-1}q^{H+2m+1}f_{j,m}\,.
\end{aligned}\tag{3.25}$$

The second statement follows from these. **QED**

Let us say that an operator $\mathcal{X}^L(\zeta)\in\mathrm{End}(W^+\otimes W^-)$ (resp. $\mathcal{X}^R(\zeta)\in\mathrm{End}(W^-\otimes W^+)$) is left (resp. right) admissible if it preserves the filtration (3.21) (resp. (3.23)). The operators

$$U_{A,B}(\zeta),V_{A,B}(\zeta),Y_{A,B}(\zeta),Z_{A,B}(\zeta),\mathbf{a}_A,q^{2(D_A-D_B)}$$

are left admissible, and

$$U_{B,A}(\zeta^{-1}),V_{B,A}(\zeta^{-1}),Y_{B,A}(\zeta^{-1}),Z_{B,A}(\zeta^{-1}),\mathbf{a}_B,q^{2(D_A-D_B)}$$

are right admissible. By the isomorphisms (3.22),(3.24), we have the correspondence of operators on each subquotient,

$$\iota_L\circ\mathcal{X}^L(\zeta)\circ\iota_L^{-1}=\mathcal{X}(\zeta)=\iota_R\circ\mathcal{X}^R(\zeta)\circ\iota_R^{-1},$$

where $\mathcal{X}^L(\zeta)$, $\mathcal{X}^R(\zeta)$ and $\mathcal{X}(\zeta)$ are related to each other via the following table 1.

The above Lemma has two corollaries which are important to us. We shall omit writing the intervals $[k,l]$.

If $\mathcal{X}^L(\zeta)$ is left admissible, then

$$\begin{aligned}
&N(\alpha-\mathbb{S})\mathrm{Tr}_{A,B}\left\{\mathcal{X}^L(\zeta)\;T^+_{A,\mathbf{M}}(\zeta_1,\alpha)T^-_{B,\mathbf{M}}(\zeta_2,\alpha)\right\}\zeta^{\alpha-\mathbb{S}}\\
&\quad=-\mathrm{Tr}_{V(\Lambda)}\left\{\mathcal{X}(\zeta)\;T_v(\sqrt{\zeta_1\zeta_2},\alpha)\right\}\\
&\quad=N(\alpha-\mathbb{S})\mathrm{Tr}_{A,B}\left\{\mathcal{X}^R(\zeta)\;T^-_{B,\mathbf{M}}(\zeta_2,\alpha)T^+_{A,\mathbf{M}}(\zeta_1,\alpha)\right\}\zeta^{\alpha-\mathbb{S}}\,.
\end{aligned}\tag{3.26}$$

TABLE 1. Correspondence of operators: $W^+_{\zeta_1}\otimes W^-_{\zeta_2}$ (left), $V_\zeta(\Lambda)$ (middle), $W^-_{\zeta_2}\otimes W^+_{\zeta_1}$ (right)

$\mathcal{X}^L$	$\mathcal{X}$	$\mathcal{X}^R$
$V_{A,B}(\zeta)$	0	$\star$
$\star$	0	$V_{B,A}(\zeta^{-1})$
$Y_{A,B}(\zeta)$	q^{H+1}	$Z_{B,A}(\zeta^{-1})$
$Z_{A,B}(\zeta)$	q^{-H+1}	$Y_{B,A}(\zeta^{-1})$
$\sqrt{\zeta}(1-\zeta^{-1}q^{-2}Z_{A,B}(\zeta))\mathbf{a}_A$	$(q-q^{-1})F$	$\sqrt{\zeta}\,U_{B,A}(\zeta^{-1})$
$\sqrt{\zeta}^{-1}U_{A,B}(\zeta)$	$(q-q^{-1})E$	$\sqrt{\zeta}^{-1}(1-\zeta q^{-2}Z_{B,A}(\zeta^{-1}))\mathbf{a}_B$

The operators $\mathcal{X}(\zeta)$, $\mathcal{X}^R(\zeta)$ are obtained from $\mathcal{X}^L(\zeta)$ via the table (1). In particular taking $\mathcal{X}^L(\zeta)=\mathcal{X}^R(\zeta)$ we get the commutation relation (3.7).

4. LECTURE 4.

4.1. **Construction of annihilation operators.** During the last lecture I explained several properties of the q-oscillator representation and applied it to diagonalisation of transfer-matrices in Matsubara direction. If this were the only application I would not bother you with it: there are simpler ways to diagonalise the transfer-matrix. Today I shall explain much more important application of q-oscillators. Here we shall forget about Matsubara and concentrate on the space direction.

Consider the operator $X_{[k,l]}$ which acts on $\mathbb{C}^{\otimes(l-k+1)}$. Define

$$T_{a,[k,l]}(\zeta)=L^\circ_{a,l}(\zeta)\cdots L^\circ_{a,k}(\zeta)\,.$$

The adjoint monodromy matrix $\mathbb{T}_a(\zeta,\alpha)$ is defined by

$$\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]})=T_{a,[k,l]}(\zeta)q^{\alpha\sigma^3_a}X_{[k,l]}T_{a,[k,l]}(\zeta)^{-1}\,,\,.$$

Here and later I use the abbreviation: in the left hand side the suffix $[k,l]$ is used only for the argument and not for $\mathbb{T}_a(\zeta,\alpha)$.

Define

$$\mathbb{S}(X_{[k,l]}):=[S_{[k,l]},X_{[k,l]}],\quad S_{[k,l]}:=\tfrac{1}{2}\textstyle\sum_{j\in[k,l]}\sigma^3_j.$$

Notice that $L^\circ_{a,j}(\zeta)^{-1}$ has poles at $\zeta^2=q^{\pm2}$. Hence $\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]})$ is a meromorphic function of ζ^2 which has poles of degree at most $l-k+1$ at the above points.

Define similarly the matrices $\mathbb{T}_A(\zeta,\alpha)$ and $\mathbb{T}_{a,A}(\zeta,\alpha)$. It is easy to compute $L^\circ_{A,j}(\zeta)^{-1}$ and to make sure that it has simple pole at $\zeta^2=1$. Hence $\mathbb{T}_A(\zeta,\alpha)$ has pole of degree at most $l-k+1$ at $\zeta^2=1$, and $\mathbb{T}_{a,A}(\zeta,\alpha)$ has poles of degree at most $l-k+1$ at $\zeta^2=1,q^{\pm2}$.

Due to the fusion relation (2.7) we have

$$\begin{aligned}\mathbb{T}_{\{a,A\}}(\zeta,\alpha)(X_{[k,l]}) &= (F_{a,A})^{-1}\left(\mathbb{T}_a(\zeta,\alpha)\mathbb{T}_A(\zeta,\alpha)(X_{[k,l]})\right)F_{a,A}\\ &= \begin{pmatrix}\mathbb{A}_A(\zeta,\alpha)(X_{[k,l]}) & 0\\ \mathbb{C}_A(\zeta,\alpha)(X_{[k,l]}) & \mathbb{D}_A(\zeta,\alpha)(X_{[k,l]})\end{pmatrix}_a,\end{aligned}\tag{4.1}$$

where

$$\mathbb{A}_A(\zeta,\alpha)(X_{[k,l]}) = \mathbb{T}_A(q\zeta,\alpha)q^{\alpha-\mathbb{S}}(X_{[k,l]}), \tag{4.2}$$

$$\mathbb{D}_A(\zeta,\alpha)(X_{[k,l]}) = \mathbb{T}_A(q^{-1}\zeta,\alpha)q^{-\alpha+\mathbb{S}}(X_{[k,l]}). \tag{4.3}$$

At the previous lecture we took in the relations similar to (4.1) trace of the diagonal element which led to Baxter equations. The main idea of our construction is to take the trace of the off diagonal element defining

$$\mathbf{k}(\zeta,\alpha)(X_{[k,l]}) := \mathrm{Tr}_A\Bigl\{\mathbb{C}_A(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}}\bigl(q^{-2S_{[k,l]}}\,X_{[k,l]}\bigr)\Bigr\}\,. \tag{4.4}$$

This operator raises spin of $X_{[k,l]}$. We introduce another operator which lowers the spin:

$$\phi(\mathbf{k})(\zeta,\alpha) = q^{-1}N(\alpha-\mathbb{S}-1)\circ\mathbb{J}\circ\mathbf{k}(\zeta,-\alpha)\circ\mathbb{J}\,,$$

where $\mathbb{J}$ is the adjoint of spin reversal, $N(x) = q^{-x}-q^{x}$.

In what follows I shall often use the q-difference operator:

$$\Delta_\zeta f(\zeta) = f(\zeta q) - f(\zeta q^{-1})\,.$$

I shall say that $\Delta_\zeta f(\zeta)$ is q-exact 1-form if $\zeta^{-\alpha}f(\zeta)$ or $\zeta^{\alpha}f(\zeta)$ is a meromorphic function os ζ^2, singular at $\zeta^2=1$ only.

Using the R-matrix for the first case and BLZ construction for the second one arrives after some calculations to the following relations:

$$\begin{aligned}\mathbf{k}(\zeta_1,\alpha)\mathbf{k}(\zeta_2,\alpha+1)+\mathbf{k}(\zeta_2,\alpha)\mathbf{k}(\zeta_1,\alpha+1)&\\ =\Delta_{\zeta_1}\mathbf{m}^{(++)}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(++)}(\zeta_2,\zeta_1,\alpha),&\end{aligned} \tag{4.5}$$

$$\begin{aligned}\mathbf{k}(\zeta_1,\alpha)\phi(\mathbf{k})(\zeta_2,\alpha+1)+\phi(\mathbf{k})(\zeta_2,\alpha)\mathbf{k}(\zeta_1,\alpha-1)&\\ =\Delta_{\zeta_1}\mathbf{m}^{(+-)}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(-+)}(\zeta_2,\zeta_1,\alpha)\,,&\end{aligned} \tag{4.6}$$

where for $\mathbf{m}^{(++)}$, $\mathbf{m}^{(+-)}$ we have rather frightening formulae.

$$\begin{aligned}&\mathbf{m}^{(++)}(\zeta_1,\zeta_2,\alpha)(X_{[k,l]})\\ &=\mathrm{Tr}_{b,A,B}\left(M_{b,A,B}(\zeta_1/\zeta_2)\mathbb{T}_A(\zeta_1,\alpha)\mathbb{T}_{\{b,B\}}(\zeta_2,\alpha)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}}(q^{-4S_{[k,l]}}X_{[k,l]})\right),\\ &M_{b,A,B}(\zeta)=\frac{\zeta^{-1}q^{-1}}{\zeta-\zeta^{-1}}\left(q^{2D_B-2D_A+\sigma_b^3+2}(\mathbf{a}_A^*)^2\sigma_b^3-\zeta^{-1}q^{D_B}(1+\zeta u_{A,B})\mathbf{a}_A^*q^{D_B}\sigma_b^+\right)\,,\\ &\mathbf{m}^{(+-)}(\zeta_1,\zeta_2,\alpha)(X_{[k,l]})\\ &=N(\alpha-\mathbb{S})\mathrm{Tr}_{b,A,B}\left(M'_{b,A,B}(\zeta_1/\zeta_2)\mathbb{T}_A^+(\zeta_1,\alpha)\mathbb{T}^-_{\{b,B\}}(\zeta_2,\alpha)(X_{[k,l]})\right)\eta^{\alpha-\mathbb{S}},\\ &M'_{b,A,B}(\zeta)=\frac{1}{\zeta-\zeta^{-1}}q^{\sigma_b^3D_A}\left(\tfrac12(\zeta+\zeta^{-1})\sigma_b^3+\zeta^{-1}U_{A,B}(\zeta)\sigma_b^-\right)q^{-\sigma_b^3D_A}\,.\end{aligned}$$

It can be shown that the singularity of $\mathbf{m}^{(++)}(\zeta_1,\zeta_2,\alpha)$ at $\zeta_1^2=\zeta_2^2$ is fictitious, so, the only singularities of $\mathbf{m}^{(++)}(\zeta_1,\zeta_2,\alpha)$ as function of ζ_1^2 are situated at $\zeta_1^2=1$. Hence, the right hand side of (4.5) can be called q-exact 2-form. Having an equation like (4.5) with classical exact form in the right hand side we would integrate it over closed cycles in both variables to find operators whose anti-commutator vanishes. Similar trick applies to our case, namely, if we define

$$\bar{\mathbf{c}}(\zeta,\alpha)(X_{[k,l]}) := \tfrac{1}{2\pi i}\oint_\Gamma \psi(\zeta/\xi,\alpha+\mathbb{S})\mathbf{k}(\xi,\alpha)(X_{[k,l]})\tfrac{d\xi^2}{\xi^2}\,,$$
$$\mathbf{c}(\zeta,\alpha)(X_{[k,l]}) := \tfrac{1}{4\pi i}\oint_\Gamma \psi(\zeta/\xi,\alpha+\mathbb{S})\left\{\mathbf{k}(q\xi,\alpha)+\mathbf{k}(q^{-1}\xi,\alpha)\right\}(X_{[k,l]})\tfrac{d\xi^2}{\xi^2}\,,$$

where Γ goes clockwise around $\zeta^2=1$, then integrating (4.5) one easily gets

$$\mathbf{c}(\zeta_1,\alpha)\mathbf{c}(\zeta_2,\alpha+1)+\mathbf{c}(\zeta_2,\alpha)\mathbf{c}(\zeta_1,\alpha+1)=0\,, \tag{4.7}$$

and the same if we replace one or both $\mathbf{c}$ by $\bar{\mathbf{c}}$. Actually, as we shall see $\mathbf{c}$ and $\bar{\mathbf{c}}$ are not independent, so, in practice I shall use only $\mathbf{c}$. This is the first important property of $\mathbf{c}(\zeta,\alpha)$.

Define further

$$\mathbf{b}(\zeta,\alpha)=\phi(\mathbf{c})(\zeta,\alpha)\,.$$

The relation (4.7) implies

$$\mathbf{b}(\zeta_1,\alpha)\mathbf{b}(\zeta_2,\alpha-1)+\mathbf{b}(\zeta_2,\alpha)\mathbf{b}(\zeta_1,\alpha-1)=0\,. \tag{4.8}$$

In order to find commutation relations between $\mathbf{c}$ and $\mathbf{b}$ one has to use (4.6). Here there is one trouble: the right hand side of (4.6) is a singular q-exact 2-form: it has a simple pole at $\zeta_1^2=\zeta_2^2$ because

$$\mathbf{m}^{(+-)}(\zeta_1,\zeta_2,\alpha)=\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S})+O(1),\quad \zeta_1^2\sim\zeta_2^2\,.$$

This fact is very important for the commutation relations with creation operators which we shall consider later, but it does not affect the commutation relations between $\mathbf{c}$ and $\mathbf{b}$ which are

$$\mathbf{c}(\zeta_1,\alpha)\mathbf{b}(\zeta_2,\alpha+1)+\mathbf{b}(\zeta_2,\alpha)\mathbf{c}(\zeta_1,\alpha-1)=0\,. \tag{4.9}$$

Now I want to discuss one more miraculous property of operators $\mathbf{c}$ and $\mathbf{b}$. I shall concentrate on $\mathbf{c}$. By the very construction it is quite obvious that for an operator $X_{[k,l]}=q^{2(\alpha+1)S_{[k,m-1]}}\otimes Y_{[m,l]}$ with $k<m<l$ we have

$$\mathbf{k}(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}}\otimes Y_{[m,l]})=q^{2\alpha S_{[k,m-1]}}\otimes\mathbf{k}(\zeta,\alpha)(Y_{[m,l]})\,,$$

where the convention is that in the right hand side $\mathbf{k}(\zeta,\alpha)$ is constructed on the interval $[m,l]$. We call this relation the left reduction relation.

Really nontrivial relation occurs when we consider $X_{[k,l]}=Y_{[k,m]}\otimes I_{[m+1,l]}$. In that case we compute:

$$\mathbf{k}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})=\mathbf{k}(\zeta,\alpha)(Y_{[k,m]})\otimes I_{[m+1,l]}+\Delta_\zeta\mathbf{v}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})\,, \tag{4.10}$$

where $\mathbf{v}$ is rather messy operator, but its only property which interests us here is that it is singular at $\zeta^2=1$ only. So, the last term of (4.10) is a q-exact 1-form and we obtain for $\mathbf{c}(\zeta,\alpha)$:

$$\mathbf{c}(\zeta,\alpha)(q^{2(\alpha+1)S_{[k,m-1]}}\otimes Y_{[m,l]})=q^{2\alpha S_{[k,m-1]}}\otimes\mathbf{c}(\zeta,\alpha)(Y_{[m,l]}) \tag{4.11}$$

$$\mathbf{c}(\zeta,\alpha)(Y_{[k,m]}\otimes I_{[m+1,l]})=\mathbf{c}(\zeta,\alpha)(Y_{[k,m]})\otimes I_{[m+1,l]}\,. \tag{4.12}$$

These reduction relations allow to define the operator $\mathbf{c}(\zeta,\alpha)$ in the infinite volume as operator sending quasi-local operators of the type $q^{2(\alpha+1)S(0)}\mathcal{O}$ to operators of the type $q^{2(\alpha)S(0)}\mathcal{O}'$. Notice that the spin of $\mathcal{O}'$ is greater than the spin of $\mathcal{O}$ by 1. Now we construct the operator $\mathbf{c}(\zeta)$ acting on $\mathcal{W}^{(\alpha)}$ from blocks $\mathbf{c}(\zeta,\alpha-s)\ :\ \mathcal{W}_{\alpha-s+1,s-1}\to\mathcal{W}_{\alpha-s,s}$.

This non-violent construction of operators in infinite volume out of operators in finite volume is the essence of our approach. Notice the contrast with the thermodynamic limit for Bethe Ansatz which is a complicated and mathematically non-rigorous procedure.

One important property of $\mathbf{c}$, $\mathbf{b}$ is that they kill the primary field:

$$\mathbf{c}(\zeta)(q^{2\alpha S(0)})=0,\quad \mathbf{b}(\zeta)(q^{2\alpha S(0)})=0\,.$$

For that reason and for the fact that $\mathbf{c}$, $\mathbf{b}$ do not increase the length of operators we call them annihilation operators.

4.2. **Construction of creation operators.** Our next goal is to construct creation operators which produce the space $\mathcal{W}^{(\alpha)}$ acting on the primary field. I start with the simple one.

On the interval $[k,l]$ consider the operator

$$\mathbf{t}^*(\zeta,\alpha)(X_{[k,l]})=\mathrm{Tr}_a\mathbb{T}_a(\zeta,\alpha)(X_{[k,l]})\,.$$

For trivial reasons it satisfies the left reduction relation:

$$\mathbf{t}^*(\zeta,\alpha)(q^{2\alpha S_{[k,m-1]}}\otimes X_{[m,l]})=q^{2\alpha S_{[k,m-1]}}\otimes\mathbf{t}^*(\zeta,\alpha)(X_{[m,l]})\,,\tag{4.13}$$

which allows to define the inductive limit $k\to-\infty$.

Let us expand $\frac{1}{2}\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})$ in ζ^2-1. Recall the formulae from the first lectures and set

$$\widetilde{R}^{\vee}_{i,j}(\zeta^2)=\zeta^{\sigma^3_i/2}R_{i,j}(\zeta)P_{i,j}\zeta^{-\sigma^3_j/2},\quad \widetilde{\mathbb{R}}^{\vee}_{i,j}(\zeta^2)=\zeta^{\mathbb{S}_i}\mathbb{R}_{i,j}(\zeta)\mathbb{P}_{i,j}\zeta^{-\mathbb{S}_j}.$$

We have

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})=\mathrm{Tr}_a\{\widetilde{\mathbb{R}}^{\vee}_{a,l}(\zeta^2)\widetilde{\mathbb{R}}^{\vee}_{l,l-1}(\zeta^2)\cdots\widetilde{\mathbb{R}}^{\vee}_{k+1,k}(\zeta^2)(q^{\alpha\sigma^3_k}\boldsymbol{\tau}(X_{[k,m]}))\}.$$

Define an operator $\mathbf{r}_{i,j}(\zeta^2)$ by

$$\widetilde{\mathbb{R}}^{\vee}_{i,j}(\zeta^2)=1+(\zeta^2-1)\mathbf{r}_{i,j}(\zeta^2).$$

Note that $\mathbf{r}_{i,j}(\zeta^2)$ is regular at $\zeta^2=1$ and that $\mathbf{r}_{i,j}(\zeta^2)(Z)=0$ if Z is a local operator such that its action on the i-th and the j-th components is proportional to the identity operator or $q^{\alpha(\sigma^3_i+\sigma^3_j)}$. We define $\tilde{\mathbb{R}}^{\vee}_{[k,l]}(\zeta^2)$ acting on $M_{[k,l]}$ by

$$\tilde{\mathbb{R}}^{\vee}(\zeta^2)(X_{[k,l]}):=\widetilde{\mathbb{R}}^{\vee}_{l,l-1}(\zeta^2)\cdots\widetilde{\mathbb{R}}^{\vee}_{k+1,k}(\zeta^2)(X_{[k,l]}).$$

We have

$$\begin{aligned}
&\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})\\
&=2\sum_{j=m}^{l-1}(\zeta^2-1)^{j-m}\mathbf{r}_{j+1,j}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})\\
&+(\zeta^2-1)^{l-m}\mathrm{Tr}_a\left\{\mathbf{r}_{a,l}(\zeta^2)\mathbf{r}_{l,l-1}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]})\right\}.
\end{aligned}$$

where $Y_{[k,m+1]}=q^{\alpha\sigma^3_k}\boldsymbol{\tau}(X_{[k,m]})$. There are no gaps between $\mathbf{r}_{k+1,k}(\zeta^2)$ for $k>m$ due to the vanishing property of $\mathbf{r}_{i,j}(\zeta^2)(Z)$ discussed above. Therefore, the inductive limit $l\to\infty$ is well-defined as a formal power series in ζ^2-1. Namely, for $X\in\mathcal{W}^{(\alpha)}_s$ such that the support of X is contained in $[k,m]$ we define

$$\begin{aligned}
\mathbf{t}^*(\zeta)(X) &= \lim_{l\to\infty}q^{2(\alpha-s)S(k-1)}\mathbf{t}^*_{[k,l]}(\zeta,\alpha-s)(X_{[k,m]})\\
&= 2q^{2\alpha S(k-1)}\sum_{j=m}^{\infty}(\zeta^2-1)^{j-m}\mathbf{r}_{j+1,j}(\zeta^2)\cdots\mathbf{r}_{m+2,m+1}(\zeta^2)\tilde{\mathbb{R}}^{\vee}(\zeta^2)(Y_{[k,m+1]}).
\end{aligned}$$

The operators $\mathbf{t}^*_p$ are the coefficients of $\mathbf{t}^*(\zeta)$.

$$\mathbf{t}^*(\zeta)=\sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{t}^*_p.$$

It can be shown that $\mathbf{t}^*(\zeta)$ is a commutative family of operators. Moreover, $\mathbf{t}^*(\zeta_1)$ commute with $\mathbf{c}(\zeta_2)$, $\mathbf{b}(\zeta_2)$. As I said the operator $\bar{\mathbf{c}}(\zeta)$ is not independent:

$$\bar{\mathbf{c}}(\zeta)=-\frac{1}{2\pi i}\int_{\Gamma}\psi(\zeta/\xi,\boldsymbol{\alpha})\mathbf{t}^*(\xi)\mathbf{c}(\xi)\frac{d\xi^2}{\xi^2},$$

where $\boldsymbol{\alpha}$ takes value $\alpha-s$ on $\mathcal{W}_{\alpha-s,s}$. Actually, the operators $\mathbf{t}^*_p$ have rather simple nature: they are constructed from form shift $\boldsymbol{\tau}^*$ and adjoint action of local integrals of motion. Acting on the primary field they create operators from $\mathcal{W}_{\alpha,0}$, but certainly only a part of them. So, we have to find the rest of creation operators.

For an analogy with the theory of singular Riemann surfaces it is very tempting to consider $\Delta^{-1}_\zeta\mathbf{k}(\zeta,\alpha)$. I do not want to go into details of this analogy because I have never been able to explain it clearly, but let me say that taking Δ^{-1}_ζ is natural at least for the reduction relation (4.10). The operator $\mathbf{k}(\zeta,\alpha)$ has singularity, so, before applying Δ^{-1}_ζ it is convenient to make some subtractions in order that no transcendental function occur. These subtraction are done with operators which are already familiar:

$$\mathbf{f}(\zeta,\alpha)(X_{[k,l]})=\Delta^{-1}_\zeta\left(\left\{\mathbf{k}(\zeta,\alpha)-\bar{\mathbf{c}}(\zeta,\alpha)-\mathbf{c}(q\zeta,\alpha)-\mathbf{c}(q^{-1}\zeta,\alpha)\right\}(X_{[k,l]})\right).$$

It is easy to see that $\zeta^{-\alpha+\mathbb{S}}\mathbf{f}(\zeta,\alpha)(X_{[k,l]})$ is a meromorphic function of ζ^2 with singularities at $\zeta^2=1$ and $\zeta^2=\infty$.

Now I give the main definition. I cannot explain where these formulae come from, for us it was a result of long experimental work

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,l]}) := \left(\mathbf{f}(q\zeta,\alpha)+\mathbf{f}(q^{-1}\zeta,\alpha)-\mathbf{t}^*(\zeta,\alpha)\mathbf{f}(\zeta,\alpha)\right)(X_{[k,l]})\,.\tag{4.14}$$

We define one more operator in familiar way

$$\mathbf{c}^*(\zeta,\alpha)(X_{[k,l]}) := -\phi(\mathbf{b}^*)(\zeta,\alpha)(X_{[k,l]}).\tag{4.15}$$

The first property of $\mathbf{b}^*(\zeta,\alpha)$ is the right reduction relation:

$$\mathbf{b}^*(\zeta,\alpha)(X_{[k,m]}\otimes I_{[m+1,l]}) = \mathrm{Tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left(\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]})\otimes I_{[m+1,l]}\right)\right\},\tag{4.16}$$

where

$$\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]}) = \Big(\tfrac{1}{2}\mathbf{f}(q\zeta,\alpha)+\tfrac{1}{2}\mathbf{f}(q^{-1}\zeta,\alpha)-\mathbb{T}_c(\zeta,\alpha)\mathbf{f}(\zeta,\alpha)+\mathbf{u}_c(\zeta,\alpha)\Big)(X_{[k,m]}),$$
$$\mathbf{u}_c(\zeta,\alpha)(X_{[k,m]}) = \mathrm{Tr}_{A,a}\left\{Y_{a,c,A}\mathbb{T}_{\{a,A\}}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\},$$
$$Y_{a,c,A} = -\tfrac{1}{2}\sigma^3_c\sigma^+_a+\sigma^+_c\sigma^3_a-\mathbf{a}_A\sigma^+_c\sigma^+_a\,.$$

The formula for $\mathbf{g}_c(\zeta,\alpha)$ is given for completeness, here we need only to properties:

- The operator $\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]})$ is localised on the interval $[k,m]$.
- As a function of ζ^2 the operator $\mathbf{g}_c(\zeta,\alpha)(X_{[k,m]})$ regular at $\zeta^2=1$.

So, using the reduction relation (4.16) and the above properties we realise that $\mathbf{b}^*(\zeta,\alpha)$ considered as power series in ζ^2-1 allows the inductive limit $k\to-\infty$, $l\to\infty$. Then from blocks we combine the operator $\mathbf{b}^*(\zeta)$. Similarly, the operator $\mathbf{c}^*(\zeta)$ is defined.

The operator $\mathbf{t}^*(\zeta_1)$ commutes with $\mathbf{c}^*(\zeta_2)$, $\mathbf{b}^*(\zeta_2)$. The operators $\mathbf{c}$, $\mathbf{b}$, $\mathbf{c}^*$ and $\mathbf{b}^*$ are fermions:

$$\begin{aligned}
&[\mathbf{b}(\zeta_1),\mathbf{c}^*(\zeta_2)]_+ = [\mathbf{c}(\zeta_1),\mathbf{b}^*(\zeta_2)]_+ = 0\,,\\
&[\mathbf{b}(\zeta_1),\mathbf{b}^*(\zeta_2)]_+ = -\psi(\zeta_2/\zeta_1,\boldsymbol{\alpha}+\mathbb{S})\,,\\
&[\mathbf{c}(\zeta_1),\mathbf{c}^*(\zeta_2)]_+ = \psi(\zeta_1/\zeta_2,\boldsymbol{\alpha}+\mathbb{S})\,.
\end{aligned}\tag{4.17}$$

Proof of these commutation relations is extremely complicated. It is based on (4.5) and (4.6) and uses a lot of algebra.

Finally, we prove the completeness (1.9).

5. Lecture 5.

5.1. **Functional Z^κ and creation operator $\mathbf{t}^*(\zeta)$.** In the previous lecture we introduced the operators which create the entire space $\mathcal{W}_{\alpha,0}$ from the primary field. The goal of this, last, lecture is to compute Z^κ on the descendants created by these operators.

Consider $X\in\mathcal{W}_{\alpha,0}$. Without loss of generality we set

$$X = X_{[1,l]}q^{2\alpha S(0)}\,.$$

The functional Z^κ is evaluated as follows:

$$Z^\kappa\Bigl\{\mathbf{t}^*(\zeta,\alpha)\bigl(X_{[1,m]}q^{2\alpha S(0)}\bigr)\Bigr\}$$
$$=\lim_{l\to\infty}\frac{\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\Bigl(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)(X_{[1,m]})\Bigr)|\kappa\rangle}{T(1,\kappa)^l\langle\kappa+\alpha|\kappa\rangle}\,.$$

An important consequence of the definition of $\mathbf{t}^*$ is that if we define for some matrix K

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha,K)(X_{[k,m]})=\frac{2}{\mathrm{Tr}(K)}\mathrm{Tr}_a\left(K_a\mathbb{T}_{a,[k,l]}(\zeta,\alpha)(X_{[k,m]})\right)\,,$$

then it is easy to conclude that

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha,K)(X_{[k,m]})=\mathbf{t}^*_{[k,l]}(\zeta,\alpha)(X_{[k,m]})\mod(\zeta^2-1)^{l-m}\,. \tag{5.1}$$

From this observation we derive

$$\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\Bigl(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}\mathbb{T}_{a,[1,l]}(\zeta)(X_{[1,m]})\Bigr)|\kappa\rangle$$
$$=\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\Bigl(T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)(X_{[1,m]})\Bigr)|\kappa\rangle$$
$$\mod(\zeta^2-1)^{l-m}\,.$$

The idea here is exactly as in (5.1). The monodromy matrix $T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}$ plays the role of K_a. The fact that it carries the additional structure as operator in the Matsubara space is not important. What is important is that the state $|\kappa\rangle$ is an eigenstate of $\mathrm{Tr}_a\bigl(T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}\bigr)$ with eigenvalue $T(\zeta,\kappa)$. Now we can proceed using the Yang-Baxter equation and the cyclicity of trace:

$$\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\Bigl(T_{[1,l],\mathbf{M}}\ q^{2\kappa S_{[1,l]}}T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma^3_a}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)(X_{[1,m]})\Bigr)|\kappa\rangle$$
$$=\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\Bigl(\mathbb{T}_{a,[1,l]}(\zeta)\Bigl(T_{a,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma^3_a}T_{[1,l],\mathbf{M}}\ q^{2\kappa S_{[1,l]}}X_{[1,m]}\Bigr)\Bigr)|\kappa\rangle$$
$$=\frac{2}{T(\zeta,\kappa)}\langle\kappa+\alpha|\mathrm{Tr}_{[1,l],a}\Bigl(T_{a,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma^3_a}T_{[1,l],\mathbf{M}}\ q^{2\kappa S_{[1,l]}}X_{[1,m]}\Bigr)|\kappa\rangle$$
$$=2\rho(\zeta)\langle\kappa+\alpha|\mathrm{Tr}_{[1,l]}\Bigl(T_{[1,l],\mathbf{M}}\ q^{2\kappa S_{[1,l]}}X_{[1,m]}\Bigr)|\kappa\rangle\,,$$

Which implies the first of our main relations:

$$Z^\kappa\Bigl\{\mathbf{t}^*(\zeta)\bigl(q^{2\alpha S(0)}\mathcal{O}\bigr)\Bigr\}=2\rho(\zeta)Z^\kappa\Bigl\{q^{2\alpha S(0)}\mathcal{O}\Bigr\}\,. \tag{5.2}$$

Notice the important property of this formula. The operator $\mathbf{t}^*(\zeta)$ was defined as formal power series in ζ^2-1, it did not make sense to talk about convergence of these series. However, when substituted under Z^κ the operator $\mathbf{t}^*(\zeta)$ provides the analytical function $\rho(\zeta)$.

5.2. **Functional Z^κ and creation operator $\mathbf{b}^*(\zeta)$. Preparation.** Now consider the operator $\mathbf{b}^*(\zeta)$. Using the reduction relation (4.16) and the arguments used when considering $\mathbf{t}^*$ we get

$$
\begin{aligned}
(5.3)\qquad & T(\zeta,\kappa)Z^\kappa\left(\mathbf{b}^*(\zeta,\alpha)(q^{2(\alpha+1)S(0)}X_{[1,m]})\right)\\
&=\frac{\mathrm{Tr}_{[1,m],c}\left(\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa)2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)(X_{[1,m]})|\kappa\rangle\right)}{T(1,\kappa)^m\langle\kappa+\alpha|\kappa\rangle}\,.
\end{aligned}
$$

The right hand side of this equation is of the form $\zeta^\alpha R(\zeta^2)$, where $R(\zeta^2)$ is a meromorphic function of ζ^2 with poles at $\zeta^2=q^{\pm 2}$. The first goal is to compute the singularities. This is not very complicated problem comparing to other computations which we had to do during this work. The result is

$$
(5.4)\qquad T(\zeta,\kappa)Z^\kappa\Bigl\{\Bigl(\mathbf{b}^*(\zeta,\alpha)-\frac{1}{2\pi i}\oint\limits_\Gamma\omega_{sym}(\zeta,\xi)\mathbf{c}(\xi,\alpha)\frac{d\xi^2}{\xi^2}\Bigr)(X)\Bigr\}=\zeta^\alpha P_{\mathbf{n}}(\zeta^2),
$$

where $X\in\mathcal{W}_{\alpha+1,-1}$, Γ encircles $\xi^2=1$,

$$
\begin{aligned}
\omega_{sym}(\zeta,\xi|\kappa,\alpha)=4\ &\frac{a(\xi)d(\zeta)\psi(q\zeta/\xi,\alpha)-a(\zeta)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha)}{T(\zeta,\kappa)T(\xi,\kappa)}\\
&-\Delta_\zeta\psi(\zeta/\xi,\alpha)+2\psi(\zeta/\xi,\alpha)\bigl(\rho(\zeta)-\rho(\xi)\bigr)\,.
\end{aligned}
$$

and $P_{\mathbf{n}}(\zeta^2)$ is a polynomial in ζ^2 of degree at most $\mathbf{n}$.

Inspired by (5.4) we conjecture that

$$
(5.5)\qquad T(\zeta,\kappa)Z^\kappa\bigl\{\mathbf{b}^*(\zeta,\alpha)(X)\bigr\}=\frac{1}{2\pi i}\oint\limits_\Gamma Z^\kappa\bigl\{\mathbf{c}(\xi,\alpha)(X)\bigr\}\ \omega(\zeta,\xi)\frac{d\xi^2}{\xi^2}\,,
$$

with some function $\omega(\zeta,\xi)$.

Applying the operattion ϕ we obtain similar equation for $\mathbf{c}^*$ and $\mathbf{b}$. There are two ways to compute

$$
Z^\kappa\bigl\{\mathbf{b}^*(\zeta_1)\mathbf{c}^*(\zeta_2)(q^{2\alpha S(0)})\bigr\}\,.
$$

Comparing the results and making explicit the dependance on α, κ we get the condition on $\omega(\zeta,\xi)$:

$$
(5.6)\qquad \omega(\zeta,\xi|\alpha,\kappa)=\omega(\xi,\zeta|-\alpha,-\kappa)\,.
$$

The case $\alpha=0$ is not very simple for our construction in general: some indefinitenesses should be uncovered using the L'Hopital rules. However, the function $\omega(\zeta,\xi|0,\kappa)$ is perfectly well defined, it satisfies the symmetry

$$
\omega(\zeta,\xi|0,\kappa)=\omega(\zeta,\xi|0,-\kappa)\,,
$$

and, hence,

$$
(5.7)\qquad \omega(\zeta,\xi|0,\kappa)=\omega(\xi,\zeta|0,\kappa)\,.
$$

Recall that $\mathbf{n}$ is the number of sites in Matsubara direction, so, the singularity at infinity described by (5.4) is same for all X. This is in contrast with the order of pole of $T(\zeta,\kappa)Z^\kappa\bigl\{\mathbf{b}^*(\zeta,\alpha)(X)\bigr\}$ which is given by (5.4) and which can be of order of any degree depending on X. Let me make a digression on a similar situation which takes place in classical mathematics. This digression will also explain the origin of certain class of symmetric functions of two complex variables.

5.3. **Digression on canonical second kind differential.** Consider a Riemann surface Σ. For definiteness the surface will be hyperelliptic given by equation

$$w^2 = P(z), \qquad \deg(P) = 2\mathbf{n}\,.$$

Genus of this Riemann surface equals $\mathbf{n-1}$. The holomorphic differentials are

$$\sigma_{\mathbf{j}}(z) = \frac{z^{\mathbf{j-1}}}{\sqrt{P(z)}}dz, \quad \mathbf{j = 1}, \cdots, \mathbf{n-1}\,.$$

The surface Σ contains two points which project on $z = \infty$, I denote them by $\infty^{\pm}$. I shall consider the differentials which have singularities only at $\infty^{\pm}$, and no residues (only first and second kind differentials). For such differentials we have canonical anti-symmetric pairing:

$$\omega_1 \circ \omega_2 = \sum_{\infty^{\pm}} \text{res}\ \omega_1 d^{-1}\omega_2\,,$$

where $d^{-1}\omega_2$ is the primitive function. Among those, singular at $\infty^{\pm}$ differentials there are exact forms:

$$\frac{d}{dz}(z^k w), \quad z^k dz, \quad k \geq 0\,.$$

The basis of remaining non-trivial second kind differential is

$$\tilde{\sigma}_{\mathbf{j}} == z^{\mathbf{j}} \left[\frac{d}{dz}\big(z^{-2\mathbf{j}}P(z)\big)\right]_+ \frac{dz}{2\sqrt{P(z)}}, \quad \mathbf{j = 1}, \cdots, \mathbf{n-1}\,.$$

It is easy to compute that these differentials together with the holomorphic ones constitute the canonical basis:

$$\sigma_{\mathbf{i}} \circ \tilde{\sigma}_{\mathbf{j}} = \delta_{\mathbf{i,j}}, \quad \sigma_{\mathbf{i}} \circ \sigma_{\mathbf{j}} = 0, \quad \tilde{\sigma}_{\mathbf{i}} \circ \tilde{\sigma}_{\mathbf{j}} = 0\,. \tag{5.8}$$

Construct the 2-form on $\Sigma \times \Sigma$:

$$\sigma(x, y) = \sum_{\mathbf{j=1}}^{\mathbf{n-1}} \big(\sigma_{\mathbf{j}}(x)\tilde{\sigma}_{\mathbf{j}}(y) - \sigma_{\mathbf{j}}(y)\tilde{\sigma}_{\mathbf{j}}(x)\big)\,. \tag{5.9}$$

Using the above formulae one computes

$$\sigma(x, y) = \Big(\frac{\partial}{\partial y}\Big(\frac{1}{y - x}\frac{\sqrt{P(y)}}{\sqrt{P(x)}}\Big) - \frac{\partial}{\partial x}\Big(\frac{1}{x - y}\frac{\sqrt{P(x)}}{\sqrt{P(y)}}\Big)\Big)dxdy\,. \tag{5.10}$$

This formula implies the Riemann bilinear identities:

$$\int_{g_1}\int_{g_2} \sigma(x, y) = 2\pi i\ g_{\mathbf{1}} \circ g_{\mathbf{2}}\,. \tag{5.11}$$

Consider a canonical homology basis $(a_{\mathbf{1}}, \cdots a_{\mathbf{n-1}}, b_{\mathbf{1}}, \cdots b_{\mathbf{n-1}})$, and construct the matrices:

$$\mathcal{A}_{\mathbf{i,j}} = \int_{a_{\mathbf{i}}} \sigma_{\mathbf{j}}, \quad \mathcal{B}_{\mathbf{i,j}} = \int_{a_{\mathbf{i}}} \tilde{\sigma}_{\mathbf{j}}\,.$$

Form the general theory we know that $\det \mathcal{A} \neq 0$, in other words, there is no first kind differential such that all its a-periods vanish. So, we can define the matrix

$$\mathcal{X} = \mathcal{A}^{-1}\mathcal{B}\,.$$

The Riemann bilinear relations imply that this matrix is symmetric.

On the product of two copies of Riemann surface we have the canonical second kind differential $\rho(x, y)$ with the following properties.

- The differential $\rho(x, y)$ is holomorphic everywhere except the diagonal, where it has a double pole with no residue

$$\rho(x, y) = \left(\frac{1}{(x-y)^2} + O(1)\right) dxdy\,. \tag{5.12}$$

- The differential $\rho(x, y)$ is normalised with respect to x,

$$\int_{a_{\mathbf{m}}} \rho(x, y) = 0, \qquad \mathbf{m} = \mathbf{1}, \cdots, \mathbf{n-1}\,. \tag{5.13}$$

An important consequence of the Riemann bilinear relations is that this differential is automatically symmetric:

$$\rho(x, y) = \rho(y, x)\,. \tag{5.14}$$

Let us explain this by giving an explicit construction of $\rho(x, y)$. We start with an exact form in x,

$$-\frac{\partial}{\partial x}\left(\frac{\sqrt{P(x)}}{\sqrt{P(y)}(x-y)}\right) dxdy\,.$$

which obviously has the required singularity at $x = y$, but has also additional singularities at infinity. Because of (5.9) and (5.10), these singularities are cancelled in the following expression:

$$\rho(x, y) = -\frac{\partial}{\partial x}\left(\frac{\sqrt{P(x)}}{\sqrt{P(y)}(x-y)}\right) dxdy + \sum_{\mathbf{j}=1}^{\mathbf{n-1}} \tilde{\sigma}_{\mathbf{j}}(x)\sigma_{\mathbf{j}}(y) + \sum_{\mathbf{i},\mathbf{j}=1}^{\mathbf{n-1}} \sigma_{\mathbf{i}}(x) X_{\mathbf{i},\mathbf{j}} \sigma_{\mathbf{j}}(y)\,,$$

where the matrix $X_{\mathbf{i},\mathbf{j}}$ must be defined from the normalisation condition

$$\sum_{\mathbf{i}=1}^{\mathbf{n-1}} \int_{a_{\mathbf{k}}} \sigma_{\mathbf{i}}\ X_{\mathbf{i},\mathbf{j}} + \int_{a_{\mathbf{k}}} \tilde{\sigma}_{\mathbf{j}} = 0\,.$$

Hence $\mathcal{X} = \mathcal{A}^{-1}\mathcal{B}$, and the symmetry of $\rho(x, y)$ follows from Riemann bilinear relation and the formula for $\sigma(x, y)$.

Now suppose that we want to construct a normalised second kind differential with given singular part at some point. Namely, take the point $z = 1$ and the Laurent polynomial

$$f(z) = \sum_{k=1}^{N} \frac{c_k}{(z-1)^k}\,.$$

We want to find the differential $\eta(z)$ satisfying two requirements:

$$\eta(z) = df(z) + O(1), \quad z \sim 1\,,$$

and no other singularities,

$$\int_{a_{\mathbf{j}}} \eta = 0\,.$$

It is quite obvious that η can be constructed using ρ:

$$\eta(z) = -\int_{\Gamma} \rho(z,y) f(y)\,, \tag{5.15}$$

where the integration variable is y, the contour γ encircles $y = 1$, the point z is outside Γ.

Our guiding intuitive observation was the similarity between the formula (5.15) and (5.5). Roughly, the correspondence must be like:

$$\begin{aligned}
\zeta^2, \xi^2 \quad &\leftrightarrow \quad z, y\,,\\
T(\zeta,\kappa) Z^{\kappa}\bigl\{\mathbf{b}^*(\zeta,\alpha)(X)\bigr\} \quad &\leftrightarrow \quad \eta(z)\,,\\
Z^{\kappa}\bigl\{\mathbf{c}(\xi,\alpha)(X)\bigr\} \quad &\leftrightarrow \quad f(y)\,,\\
\omega(\zeta,\xi) \quad &\leftrightarrow \quad \rho(z,y)\,.
\end{aligned}$$

Let me explain how it works.

5.4. **Deformed Abelian integrals.** The Deformed Abelian integrals appeared in my works [15, 14, 13] as a result of writing the matrix elements for integrable models in the frameworks of the Method of Separation of Variables proposed by Sklyanin [16]. I shall not go into details of this method defining the Deformed Abelian Integrals formally, however, I recommend to look through the simple papers cited above.

We have the Baxter equation

$$T(\zeta,\lambda) Q^{\pm}_{\mathbf{M}}(\zeta,\lambda) = d(\zeta) Q^{\pm}(\zeta q,\lambda) + a(\zeta) Q^{\pm}(\zeta q^{-1},\lambda)\,, \tag{5.16}$$

for us λ equals either κ of $\kappa+\alpha$. For simplicity we consider the Bethe vectors of spin 0. The functions $T_{\mathbf{M}}(\zeta,\lambda)$ and $\zeta^{\mp\lambda\zeta} Q^{\pm}_{\mathbf{M}}(\zeta,\lambda)$ are polynomials of ζ^2. The analogy with the hyperelliptic curves goes as follows. Consider $z = \zeta^2$ and y which satisfies the commutation relations $yz = q^2 zy$. Let $T(z) = T(\zeta,\alpha)$, $Q(z) = Q(\zeta,\lambda)$ $\alpha(z) = a(\zeta)$, $\delta(z) = d(\zeta)$ then the Baxter eqaution reads

$$(\alpha(z) y^{-1} + \delta(z) y - T(z)) Q(z) = 0\,.$$

We interpret this as quantisation of the curve $\alpha(z)y^{-1} + \delta(z)y - T(z) = 0$ on the plane (z,y). This curve is brought to canonical form $w^2 = P(z)$ by $w = 2\delta(\zeta)y - T(z)$, $P(z) = T(z)^2 - 4\alpha(z)\delta(z)$.

For the eigenvalue such that $T(1,\lambda)$ has maximal absolute value we have

$$T(\zeta,\lambda) = T(\zeta,-\lambda), \quad Q^{+}(\zeta,\alpha) = Q^{-}(\zeta,-\alpha)\,.$$

Introduce the function $\varphi(\zeta)$ which satisfies the equation

$$a(\zeta q)\varphi(\zeta q)=d(\zeta)\varphi(\zeta)\,. \tag{5.17}$$

This function is elementary,

$$\varphi(\zeta)=\prod_{\mathbf{m}=\mathbf{1}}^{\mathbf{n}}\varphi_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}})\,,\quad \varphi_s(\zeta)=\prod_{k=0}^{2s}\frac{1}{\zeta^2q^{-2s+2k+1}-1}\,.$$

We had the contour Γ which encircles $\zeta^2=1$, now we consider additional $\mathbf{n}+\mathbf{1}$ contours in the ζ^2 plane: $\Gamma_{\mathbf{0}}$ which goes around 0, and $\Gamma_{\mathbf{m}}$ which encircles the poles $\zeta^2=\tau_{\mathbf{m}}^2q^{2s_{\mathbf{m}}-2k-1}$ $(k=0,\cdots,2s_{\mathbf{m}})$ of $\varphi_{s_{\mathbf{m}}}(\zeta/\tau_{\mathbf{m}})$.

Consider the functions $f^{\pm}(\zeta)$ such that $\zeta^{\mp\alpha}f^{\pm}(\zeta)$ are polynomials of ζ^2. The q-deformed Abelian integrals are defined by

$$\int\limits_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,. \tag{5.18}$$

It is rather easy to see that for $\mathbf{j}=\mathbf{1},\cdots\mathbf{n}-\mathbf{1}$ and for $\alpha=0$ in the quasiclassical limit $q\rightarrow 1$ these integrals go to

$$\int\limits_{a_{\mathbf{j}}}\frac{L(z)}{\sqrt{P(z)}}dz\,,$$

if $\zeta^{mp}f^{\pm}(\zeta)=\zeta^2L(\zeta^2)$. This explains the name "deformed Abelian integrals".

Using Baxter equations and moving contours it is easy to show that for $f^{\pm}$ as before

$$\begin{aligned}&\int\limits_{\Gamma_{\mathbf{m}}}\Bigl\{T(\zeta,\kappa)\Delta_\zeta^{-1}f^{\pm}(\zeta q)-T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}f^{\pm}(\zeta)\Bigr\}Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\\ &=\int\limits_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta)a(\zeta)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta q^{-1},\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,,\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\int\limits_{\Gamma_{\mathbf{m}}}\Bigl\{T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}f^{\pm}(\zeta)-T(\zeta,\kappa)\Delta_\zeta^{-1}f^{\pm}(\zeta q^{-1})\Bigr\}Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\\ &=\int\limits_{\Gamma_{\mathbf{m}}} f^{\pm}(\zeta)d(\zeta)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta q,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,.\end{aligned} \tag{5.20}$$

These identities serve two goals.

First, they allow to define the deformed Abelian integrals in slightly more general situation than before. Suppose we take $f^{\pm}(\zeta)=\psi((\zeta/\xi)^{\pm1},\alpha)$. The left hand side of (5.19), (5.20) are not obvious because it contain transcendental functions. However, using these relations we can define the left hand sides by the right hand side which is perfectly well defined if ξ^2 does not coincide with $\tau_{\mathbf{j}}q^{2n}$. Considering the classical limit one realises that this is the way to define the deformation of a-periods for the second kind differentials.

Second, consider the expression

(5.21)

$$\begin{aligned}
&E\bigl(f^{\pm}(\zeta)\bigr)\\
&= T(\zeta,\kappa)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta)T(\zeta,\kappa)\right) + T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta)T(\zeta,\kappa+\alpha)\right)\\
&- T(\zeta,\kappa)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta q)T(\zeta q,\kappa+\alpha)\right) - T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(f^{\pm}(\zeta q^{-1})T(\zeta q^{-1},\kappa)\right)\\
&+ a(\zeta q)d(\zeta)f^{\pm}(\zeta q) - d(\zeta q^{-1})a(\zeta)f^{\pm}(\zeta q^{-1})\,,
\end{aligned}$$

If $\zeta^{\mp\alpha}f^{\pm}(\zeta)$ is a polynomial in ζ^2 one easily finds using the (5.19), (5.20) that

$$\int\limits_{\Gamma_{\mathbf{m}}} E\bigl(f^{\pm}(\zeta)\bigr)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2} = 0\,.$$

We call $E\bigl(f^{\pm}(\zeta)\bigr)$ q-deformed exact form. For the same reason as before the q-deformed exact form is defined for $f^{\pm}(\zeta) = \psi((\zeta/\xi)^{\pm},\alpha)$, and its periods still vanish.

Finally, we have the following q-deformed version of Riemann bilinear relation: Consider the following function in two variables

$$r(\zeta,\xi) = r^{+}(\zeta,\xi) - r^{-}(\xi,\zeta)\,,$$

where

$$r^{+}(\zeta,\xi) = r^{+}(\zeta,\xi|\kappa,\alpha), \quad r^{-}(\xi,\zeta) = r^{+}(\xi,\zeta|-\kappa,-\alpha),$$

and

$$\begin{aligned}
r^{+}(\zeta,\xi|\kappa,\alpha) &= T(\zeta,\kappa)\Delta_\zeta^{-1}\left(\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa)-T(\xi,\kappa))\right)\\
&+ T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(\psi(\zeta/\xi,\alpha)(T(\zeta,\kappa+\alpha)-T(\xi,\kappa+\alpha))\right)\\
&- T(\zeta,\kappa)\Delta_\zeta^{-1}\left(\psi(q\zeta/\xi,\alpha)(T(\zeta q,\kappa+\alpha)-T(\xi,\kappa+\alpha))\right)\\
&- T(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(\psi(q^{-1}\zeta/\xi,\alpha)(T(\zeta q^{-1},\kappa)-T(\xi,\kappa))\right)\\
&+ \bigl(a(\zeta q)-a(\xi)\bigr)d(\zeta)\psi(q\zeta/\xi,\alpha) - \bigl(d(\zeta q^{-1})-d(\xi)\bigr)a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha)\,.
\end{aligned} \tag{5.22}$$

Then

(5.23)

$$\int\limits_{\Gamma_{\mathbf{i}}}\int\limits_{\Gamma_{\mathbf{j}}} r(\zeta,\xi)Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)Q^{+}(\xi,\kappa+\alpha)Q^{-}(\xi,\kappa)\varphi(\zeta)\varphi(\xi)\frac{d\zeta^2}{\zeta^2}\frac{d\xi^2}{\xi^2} = 0\,.$$

Actually, this is only one quarter of the Riemann bilinear relation because here we integrate only over the a-cycles. However, this is sufficient for our goals which consist, as in classical case, in constructing the canonical normalised second kind differential.

Like in classical case define

$$r^{+}(\zeta,\xi) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \zeta^{\alpha}p^{+}_{\mathbf{m}}(\zeta^2)\xi^{-\alpha+2\mathbf{m}}\,, \qquad r^{-}(\xi,\zeta) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \xi^{-\alpha}p^{-}_{\mathbf{m}}(\xi^2)\zeta^{\alpha+2\mathbf{m}}\,.$$

Introduce the $(\mathbf{n+1})\times(\mathbf{n+1})$ matrices

$$\mathcal{A}^{\pm}_{\mathbf{i},\mathbf{j}}=\int\limits_{\Gamma_{\mathbf{i}}}\zeta^{\pm\alpha+2\mathbf{j}}Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,, \tag{5.24}$$

$$\mathcal{B}^{\pm}_{\mathbf{i},\mathbf{j}}=\int\limits_{\Gamma_{\mathbf{i}}}\zeta^{\pm\alpha}p^{\pm}_{\mathbf{j}}(\zeta^2)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}\,. \tag{5.25}$$

Then (5.23) reads as

$$\mathcal{B}^{+}(\mathcal{A}^{-})^{t}=\mathcal{A}^{+}(\mathcal{B}^{-})^{t}\,. \tag{5.26}$$

The difference with the classical case is that for $\alpha\neq 0$ we have four different matrices $\mathcal{A}^{\pm}$, $\mathcal{B}^{\pm}$. It can be shown the conditions $\det\mathcal{A}^{\pm}\neq 0$ are equivalent to the requirement $\langle\kappa+\alpha|\kappa\rangle\neq 0$ which was accepted from the very beginning. So, we can define

$$\mathcal{X}^{\pm}=\left(\mathcal{A}^{\pm}\right)^{-1}\mathcal{B}^{\pm}\,,$$

and the Riemann bilinear relation reads

$$\mathcal{X}^{+}=\left(\mathcal{X}^{-}\right)^{t}\,. \tag{5.27}$$

5.5. **Functional Z^{κ} and $\mathbf{b}^*(\zeta)$. The end of computation.** Let us return to the problem of computing $Z^{\kappa}\big\{\mathbf{b}^*(\zeta)\big(X\big)\big\}$. We have seen that knowledge of singularities of $T(\zeta,\kappa)Z^{\kappa}\left\{\mathbf{b}^*(\zeta,\alpha)(X)\right\}$ leaves us with $\mathbf{n}$ unknowns. Our logic is that the way of fixing them must consist in presenting some normalisation conditions, similar to vanishing of a-periods. These normalisation conditions are
For $\mathbf{m}=\mathbf{0},\cdots,\mathbf{n}$ and any $X\in\mathcal{W}_{\alpha+1,-1}$

$$\int\limits_{\Gamma_{\mathbf{m}}}T(\zeta,\kappa)Z^{\kappa}\Big\{\Big(\mathbf{b}^*(\zeta,\alpha)+\frac{1}{2\pi i}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\left(\overline{D}_{\zeta}\overline{D}_{\xi}\Delta^{-1}_{\zeta}\psi(\zeta/\xi,\alpha)\right)\mathbf{c}(\xi,\alpha)\Big)(X)\Big\} \\ \times Q^{-}(\zeta,\kappa+\alpha)Q^{+}(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0, \tag{5.28}$$

where

$$\overline{D}_{\zeta}f(\zeta)=f(\zeta q)+f(\zeta q^{-1})-2\rho(\zeta)f(\zeta)\,.$$

It is easy to see that the integral in (5.28) is well defined being independent of the way of understanding $\Delta^{-1}_{\zeta}\psi(\zeta/\xi,\alpha)$. This is similar to the previous discussion.

It would be hard to imagine the normalisation conditions (5.28) without the classical analogy with Riemann surfaces. How do we prove (5.28)? The computations are rather complicated, but finally they reduce (5.28) to the following equation. Recall that defining $\mathbf{f}(\zeta,\alpha)$ we first subtracted something from $\mathbf{k}(\zeta,\alpha)$ and then took Δ^{-1}_{ζ}. This was done in order to avoid appearance of transcendental functions. If we do not care about these functions we can define

$$\mathbf{f}_0(\zeta,\alpha)=\Delta^{-1}_{\zeta}\mathbf{k}(\zeta,\alpha)\,.$$

Actually, the only transcendental function which we have to define is $\Delta^{-1}_{\zeta}\psi(\zeta,\alpha)$, and the way of understanding this is irrelevant for what follows. In the reduction relations for $\mathbf{b}^*(\zeta,\alpha)$ (4.16) we had the operator $\mathbf{g}_c(\zeta,\alpha)$. Define $\mathbf{g}_{0,c}(\zeta,\alpha)$ by the same formula replacing $\mathbf{f}(\zeta,\alpha)$ by $\mathbf{f}_0(\zeta,\alpha)$.

Consider the interval $[1,m]$. Introduce an operator

$$\mathbb{A}_c(\zeta)(X_{[1,m]\sqcup c}) = T_{c,[1,m]}(\zeta) q^{\alpha\sigma^3_c}\theta_c\left(X_{[1,m]\sqcup c}\ \theta_c\bigl(T_{c,[1,m]}(\zeta)^{-1}\bigr)\right),$$

where θ signifies the anti-involution

$$\theta(x) = \sigma^2 x^t \sigma^2 \qquad (x\in \mathrm{End}(V)).$$

Then the normalisation condition (5.28) follows from the equation

$$\mathbf{g}_{0,c}(\zeta,\alpha) = -\mathbb{A}_c(\zeta)\mathbf{g}_{0,c}(q^{-1}\zeta,\alpha)\,. \tag{5.29}$$

The proof of this equation is purely algebraic. Actually, reducing our problem to (5.29) closes a long cycle of our work. Long ago we started study of VEV by investigating the reduced qKZ equations. Then we considerably simplified the known solution to reduced qKZ and generalised the problem. Finally, after long computations we reduced the generalised problem of computing Z^κ to (5.29) . But it is easy to see that this equation is intimately related to reduced qKZ equation.

Now suppose we find a function satisfying two requirements:
1. Singular part

$$\zeta^{-\alpha}T(\zeta,\kappa)\bigl(\omega(\zeta,\xi)-\omega_{sym}(\zeta,\xi)\bigr) \text{ is a polynomial in } \zeta^2 \text{ of degree } \mathbf{n}\,. \tag{5.30}$$

2. Normalisation

(5.31)

$$\int\limits_{\Gamma_\mathbf{m}} T(\zeta,\kappa)\left(\omega(\zeta,\xi)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\right)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\frac{d\zeta^2}{\zeta^2}=0\,,$$

$$(\mathbf{m}=\mathbf{0},\cdots,\mathbf{n})\,. \tag{5.32}$$

Then the conditions (5.4) and (5.28) one finds that (5.5) must be satisfied. Indeed, in this situation $T(\zeta,\kappa)Z^\kappa\Bigl\{\mathbf{b}^*(\zeta,\alpha)(X)\Bigr\}$ given by (5.5) satisfies (5.4) and (5.28), and the uniqueness follows from the fact that a function $\zeta^\alpha P(\zeta^2)$ with P being a polynomial of degree $\mathbf{n}$ with vanising deformed a-periods is identically zero if $\langle\kappa+\alpha|\kappa\rangle\neq 0$.

It is not hard to find $\omega(\zeta,\xi)$ satisfying (5.30), (5.31):

$$\omega(\zeta,\xi|\kappa,\alpha) = \frac{4}{T(\zeta,\kappa)T(\xi,\kappa)}v^+(\zeta)^t\mathcal{X}^+v^-(\xi)+\omega_{sym}(\zeta,\xi|\kappa,\alpha)\,, \tag{5.33}$$

$v^\pm(\zeta)$ are vectors with components $v^\pm(\zeta)_\mathbf{j} = \zeta^{\pm\alpha+2\mathbf{j}}$. The uniqueness follows from $\langle\kappa{+}\alpha|\kappa\rangle\neq 0$ as before. Remarkably, the symmetry (5.6) follows from the q-deformed Riemann bilinear identity.

5.6. **Concluding remarks.** Let me summarise the results presented in these lectures and formulate unsolved problems.

We started with the space $\mathcal{W}_{\alpha,0}$ which is the space of spinless operators of the form $q^{2\alpha S(0)}\mathcal{O}$. This space looks rather structureless. However, we were able to describe it as a Fock space created from the primary field $q^{2\alpha S(0)}$ by creation operators $\mathbf{t}^*(\zeta)$,

$\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$. This construction simplifies the complicated problem of computing the functional Z^κ on $\mathcal{W}_{\alpha,0}$. Even in the most simple limit of VEV's the original formulae of Jimbo and Miwa [5] were given by multiple integrals. The formulae by Jimbo and Miwa were generalised to the cases of non-zero magnetic field and temperature using Algebraic Bethe Ansatz in [10, 11]. Our formulae show that evaluation of multiple integrals is possible reducing the computation to two functions $\rho(\zeta)$ and $\omega(\zeta,\xi)$. This is a considerable progress.

For physical application it would be very important to solve the inverse problem: namely, to express operators like, for example, $q^{2\alpha S(0)}\sigma^3_k$ as linear combination of descendants created by $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$. The logic here is similar to the one used at the free fermion point. There we have special basis of operators created by the fermions, and for finding the correlation functions of local spins one has to decompose them in this basis. This inverse problem is not solved yet.

There is another problem whose solution is almost finished. Our construction is very much similar to that of Conformal Field Theory. Namely, we organise the space of local operators in a module generated by certain algebra. The XXZ-model allows a scaling limit which is described by the CFT. the question is: how to relate the action of $\mathbf{t}^*(\zeta)$, $\mathbf{b}^*(\zeta)$, $\mathbf{c}^*(\zeta)$ in this limit with the action of Virasoro algebra? This is the subject of our work in progress.

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